Transfer and teleportation of system-environment entanglement

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We study bidirectional teleportation while explicitly taking into account a mixed environment. This environment initially causes pure dephasing decoherence of the Bell state which assists teleportation. We find that when teleportation is performed in one direction it is accompanied by a transfer of correlations into the teleported qubit state. In the other direction, if no new decoherence process occurs then not only the state of the qubit, but also its correlations with the environment are teleported with unit fidelity. These processes do not depend on the measurement outcome during teleportation and do not differentiate between classical and quantum decoherence. If, on the other hand, the second teleportation step is preceded by decoherence of the Bell state then the situation is much more complicated. Teleportation and transfer of correlations occur simultaneously, yielding different teleported qubit-environment states for different measurement outcomes. These states can differ in the degree of coherence of the teleported qubit, but only for an entangling Bell-state-environment interaction in the first step can they have different amounts of qubit-environment entanglement. In the extreme case, one of the teleported qubit states can be entangled with the environment while the other is separable.

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I. INTRODUCTION

Teleportation [1] is a pivotal example of the consequences and importance of entanglement. Using a two-qubit entangled state one can transport an unknown qubit state onto another qubit. Since fundamentally particles are indistinguishable, this post-teleportation qubit is indistinguishable from the preteleportation one by any measurement. This means that it is the same qubit. It has been also shown that teleportation can be viewed as a primitive subroutine of quantum computation [2].

Since its original discovery [1], teleportation has been extensively studied theoretically. First, more complex teleportation scenarios have been devised [3–5], allowing for teleportation of quantum states of larger ensembles using larger maximally entangled states. Furthermore, the effects of teleportation via a nonmaximally entangled state on the fidelity of teleportation have received much attention [3,6–13]. Here, a particularly important case is related to the loss of coherence [3,9–13], since decoherence is rarely avoidable in realistic qubit realizations. It has been found that only when the average teleportation fidelity over all possible qubit states is larger than $2/3$, the bipartite state used to perform teleportation must be entangled [1,11,14].

Experimental realizations of teleportation were performed quite soon after the theoretical prediction, most notably on optical systems [15–17], but there were also successful experiments performed on NMR [18]. Later realizations on different systems have been demonstrated, such as atoms, ions, and particles [19,20], teleportation between light and matter [21], teleportation of continuous variables [22,23], etc. Simultaneously the distances over which successful teleportation has been demonstrated have been extended until a definitely macroscopic range of teleportation has been obtained [24–26].

Recently, interesting effects resulting from more realistic theoretical treatment of the processes which lead to decoherence have been found. For example, it has been shown that local noise can enhance two-qubit teleportation while it does not increase entanglement of the teleported state [27]. Furthermore, nonlocal memory effects allow for teleportation Fidelity enhancement when teleporting using a mixed entangled state [28]. These results suggest that the nature of the interaction with the environment is important when performing teleportation and the environment could possibly be used to assist faithful teleportation.

Pure dephasing is the dominant decoherence mechanism for many solid state qubits [29–36]. This type of interaction is special, because the eigenstates of the Hamiltonian are separable which allows for an effective formal treatment of the joint system and environment evolution [37–39], which in turn allows straightforward methods to quantify and qualify system-environment entanglement (SEE) to be devised even for mixed states of the environment [37,40–42], such as thermal states. For pure states entanglement and decoherence are unambiguously linked [43,44], but for mixed states, decoherence without entanglement is relatively common [45–50], as is decoherence associated with entanglement generation

[51–53] (most scenarios have never been investigated in the context of SEE). Hence, generalizing notions from studying pure environmental states to mixed states is precarious.

Although plain system decoherence cannot be used to detect SEE, it turns out that this type of entanglement does manifest itself in the evolution of the environment and could in principle be detected by measurements of its state [37]. Furthermore, entanglement manifests itself easily in the evolution of the system once more complicated procedures (involving quantum gates and measurements) are performed [53–56]. This is because the state of the environment measurably differs when entanglement is generated from the separable case and this can in turn influence the evolution of the system. These results, showing how easy it is to multiply methods for detection of SEE via operations and measurements on the qubit alone for pure dephasing, strongly suggest that quantum algorithms, especially those that involve measurements and postmeasurement processing of qubits, are likely to be strongly affected by SEE. Furthermore, since the evolution of the system of interest can be significantly different when the environment entangles with the system, it seems to be a reasonable conjecture that not taking quantum correlations into account when SEE is present will lead to faulty simulations of algorithm operation and may also hinder the effectiveness of error correction codes.

We study bidirectional teleportation [57–62] in the simplest possible scenario when an unknown qubit state is teleported via a Bell state in order to study when SEE or lack thereof is relevant for the operation of the simple, yet nontrivial quantum algorithm. We include a pure dephasing interaction between the Bell state and a mixed environment and keep the degrees of freedom of the environment explicitly while performing teleportation. This allows us to study not only the effect that the environment has on the qubit states during the procedure, but also the effect that the qubits have on the environment. We are especially interested in the behavior of correlations of specific parts of the three-qubit system and the environment, whether quantum or classical.

We find that in the first step of teleportation, the correlations which are initially present between the two-qubit Bell state and the environment are transferred into one qubit while information about the unknown qubit state is teleported. This yields a qubit-environment state in which all of the information about the unknown state is present, but the correlations with the environment make it impossible to read it out from the qubit state alone.

If the teleportation is performed in the other direction without additional decoherence then we observe perfect teleportation of the qubit-environment state. Hence, not only the state of the qubit (which is mixed) is teleported with unit fidelity, but the correlations which were present between the qubit and the environment have been teleported. Note that there is an infinite number of qubit-environment density matrices which yield the same mixed qubit state, so the perfect conveyance of the correlations is by no means obvious for mixed environmental states.

For the two processes described above, the measurement outcome in the teleportation procedure is irrelevant for the post-teleportation state. Furthermore, there is no quantitative or qualitative difference between the situation when the interaction with the environment leads to entanglement or not.

Once the two processes occur simultaneously when we allow new decoherence to take place between teleportations, there is a qualitative change in the results. First, the measurement outcome is no longer irrelevant and there are two distinct qubit-environment states which can be obtained after teleportation. They can differ in qubit coherence, but only if the decoherence in the first step of teleportation is entangling can they also differ in the amount of qubit-environment entanglement. This difference can be arbitrarily large and it is possible for one state to be separable while the other is entangled.

The paper is organized as follows. The basic concept of teleportation is recounted in Sec. II. Teleportation in one direction with the help of a decohered Bell state is studied in Sec. III, while teleportation in the other direction when no additional decoherence process has occurred is studied in Sec. IV. In Sec. V we study the same teleportation process as in Sec. IV, but for the situation when the second teleportation procedure is preceded by decoherence of the Bell state. Examples for Sec. V are provided in Sec. VI. Section VII concludes the paper.

II. TELEPORTATION

The protocol of teleportation of an unknown qubit state [1], regardless of its huge implications for the importance and consequences of entanglement, is particularly straightforward. Three qubits are necessary: qubit A in an unknown quantum state which is to be teleported,

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A, \tag{1}$$

and two qubits B and C, which are initially in a maximally entangled state. Here we will assume that this state is a chosen Bell state,

$$|\Phi_+\rangle_{BC} = \frac{1}{\sqrt{2}}(|00\rangle_{BC} + |11\rangle_{BC}). \tag{2}$$

The procedure itself involves a measurement of qubits A and B in the Bell basis which transfers the information about the coefficients α and β into the state of qubit C, followed by a unitary operation on qubit C, which is dependent on the outcome of the measurement. The unitary operation transforms the postmeasurement state of qubit C into state (1). Note that this procedure faithfully teleports the state from qubit A to qubit C, even if the initial state of qubit A is mixed.

In the following, we will implicitly assume that the measurement outcome on qubits A and B is the Bell state (2). This is because this measurement outcome guarantees the teleportation from qubit A to C without the necessity of performing an additional unitary operation. The choice bears no conceptual consequence on the results presented in the next two sections, as the joint state of the teleported qubit and the environment is the same regardless of the measurement outcome as long as the correct unitary operation on the qubit is performed. Hence, the choice is made strictly for convenience.

III. TELEPORTATION WITH THE HELP OF DECOHERED BELL STATE

Teleportation by means of a nonmaximally entangled state has already been extensively studied, both in the case of pure states [6–8], and in the situation when the decrease in entanglement is related to some form of decoherence [3,9–13]. We will study the situation when the Bell state (2) undergoes pure dephasing due to an interaction with an environment. What is special in our approach is that we will not trace out the environment and instead study the full density matrix of the three qubits and the environment, while assuming that our environment is mixed. This allows us to keep track of what happens with the correlations that have formed throughout the evolution during the teleportation procedure and identify which processes are accompanied by transfer or teleportation of correlations and how faithful they are. The mixedness of the initial state of the environment, on the other hand, allows the study of decoherence processes which are not accompanied by SEE generation, or where quantum and classical correlations coexist.

Pure dephasing is relevant, as it is the dominating decoherence mechanism for many solid state qubits [29–36]. It is also of importance from a strictly theoretical standpoint, since due to the special form of pure-dephasing Hamiltonians it allows for a general treatment of joint system-environment evolutions. This in turn allows for systematic study of such evolutions and consequently pure dephasing evolutions are the only ones for which effective methods for general studies of SEE with mixed initial states of the environment exist [37,40,41]. We comment further on pure dephasing Hamiltonians and evolutions in the Appendix, as well as on if and only if conditions for SEE and the measure of qubit-environment entanglement tailored to pure dephasing evolutions.

We will assume that between initialization of the three qubits in state $|\psi\rangle_A \otimes |\Phi_+\rangle_{BC}$ and the joint measurement of qubits A and B in the Bell basis, there is a time τ during which qubits B and C interact with an environment (E) according to a pure dephasing Hamiltonian of the form

$$\hat{H}_{\text{PD}} = \sum_{i,j=0,1} |ij\rangle_{BCBC}\langle ij| \otimes \hat{V}_{ij}. \tag{3}$$

Here the operators $\hat{V}_{ij}$ act on the subspace of the environment and if the Hamiltonian takes into account the free Hamiltonians of the two qubits and the environment as well as their interaction, they can be written as $\hat{V}_{ij} = \varepsilon_{ij}\mathbb{I}_E + \tilde{V}_{ij} + \hat{H}_E$. Here ε_{ij} is the energy of the two qubit system in pointer state $|ij\rangle_{BC}$, $\tilde{V}_{ij}$ comes from the interaction term, and $\hat{H}_E$ is the free environmental Hamiltonian.

The only assumption made on the pure dephasing Hamiltonian (3) is that its pointer states are separable with respect to the two qubits (so decoherence will be limited to the decay of the off-diagonal elements of the two-qubit density matrix written in the $|ij\rangle_{BC} \equiv |i\rangle_B \otimes |j\rangle_C$ basis, with $i, j = 0, 1$). There are no assumptions made on the environmental operators $\hat{V}_{ij}$, so they can describe both the situation when the qubits interact with different environments as well as the situation when they interact with the same environment. For each qubit to interact with its own environment (assuming that the environments do not interact), The PD Hamiltonian must be a sum of Hamiltonians describing the interaction of each qubit separately [63], $\hat{H}_{\text{PD}} = \hat{H}^B_{\text{PD}} + \hat{H}^C_{\text{PD}}$, with

$$\hat{H}^{B/C}_{\text{PD}} = \sum_{i=0,1} |i\rangle\langle i| \otimes \hat{V}^{B/C}_i, \tag{4}$$

where the operators $\hat{V}^{B/C}_i$ only act on the subspaces corresponding to the environment of a single qubit. Then the environmental operators in Eq. (3) are given by $\hat{V}_{ij} = \hat{V}^B_i + \hat{V}^C_j$.

The evolution operator corresponding to Hamiltonian (3) is given by [40]

$$\hat{U}_{\text{PD}}(\tau) = \sum_{i,j=0,1} |ij\rangle_{BCBC}\langle ij| \otimes \hat{w}_{ij}(\tau), \tag{5}$$

with

$$\hat{w}_{ij}(\tau) = \exp\left(-\frac{i}{\hbar}\hat{V}_{ij}\tau\right) \tag{6}$$

We assume that initially the environment is in a product state with the qubits, and the full initial state of system $ABCE$ is given by

$$\hat{\sigma}(0) = |\psi\rangle_{AA}\langle\psi| \otimes |\Phi_+\rangle_{BCBC}\langle\Phi_+| \otimes \hat{R}(0), \tag{7}$$

where $\hat{R}(0)$ is the initial state of the environment. We do not make any assumptions on the environmental density matrix, which can be pure or mixed, although we will be focusing on mixed environments. This density matrix can describe two separate uncorrelated environments, $\hat{R}(0) = \hat{R}^B(0) \otimes \hat{R}^C(0)$, e.g., when the qubits are far away [64,65], or the same environment.

Using the evolution operator (5) on the initial state (7) we find the state of the qubits and the environment at time τ,

$$\hat{\sigma}(\tau) = |\psi\rangle_{AA}\langle\psi| \otimes \frac{1}{2}\begin{pmatrix} \hat{R}_{00}(\tau) & 0 & 0 & \hat{R}_{01}(\tau) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{R}_{10}(\tau) & 0 & 0 & \hat{R}_{11}(\tau) \end{pmatrix}_{BCE}. \tag{8}$$

In Eq. (8) the density matrix on the right of the tensor product describes the joint state of qubits B, C and the environment [the Hamiltonian (3) has no effect on qubit A]. It is written in a notation which is convenient for interactions leading to pure dephasing [66,67], where the matrix form is used with respect to the pointer basis of the two qubits, $|ij\rangle_{BC}$, while $\hat{R}_{ij}(\tau)$ are environmental matrices with

$$\hat{R}_{ij}(\tau) = \hat{w}_{ii}(\tau)\hat{R}(0)\hat{w}^\dagger_{jj}(\tau). \tag{9}$$

Since $\hat{R}_{00}(\tau)$ and $\hat{R}_{11}(\tau)$ are density matrices, tracing out the environmental degrees of freedom from the matrix (8) yields a dephased Bell state,

$$\hat{\rho}_{BC}(\tau) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & c(\tau) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^*(\tau) & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

with $c(\tau) = \text{Tr}_E \hat{R}_{01}(\tau)$.

We now perform a projective measurement in the Bell basis on qubits A and B for the whole, three qubits and environment, system in state (8). If the outcome is (2), then the postmeasurement state is

$$\hat{\sigma}_{PM}(\tau) = |\Phi_+\rangle_{ABAB}\langle\Phi_+| \otimes \begin{pmatrix} |\alpha|^2\hat{R}_{00}(\tau) & \alpha^*\beta\hat{R}_{01}(\tau) \\ \alpha\beta^*\hat{R}_{10}(\tau) & |\beta|^2\hat{R}_{11}(\tau) \end{pmatrix}_{CE}. \tag{11}$$

If the measurement outcome is a different Bell state, then (after the appropriate unitary operation is performed) we obtain a state analogous to (11), where the only difference is in the Bell state on qubits A and B.

When the postmeasurement state of qubit C and the environment on the right side of the tensor product in Eq. (11) is compared to the premeasurement BCE state [on the right side of the tensor product in Eq. (8)], it is apparent that two processes took place. On one hand, the coefficients of the qubit A state (1) have been teleported to qubit C, but simultaneously the correlations with the environment that were present with qubits B and C have been fully transferred to qubit C.

These correlations are the reason that the teleported state of qubit C is dephased. In fact, the degree of coherence of qubit C is the same as the previous degree of coherence of the Bell state, since the state of qubit C is now given by

$$\hat{\rho}_C^{PM}(\tau) = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta c(\tau) \\ \alpha\beta^* c^*(\tau) & |\beta|^2 \end{pmatrix}, \tag{12}$$

with the same dephasing coefficient $c(\tau)$ as in Eq. (10).

The system-environment correlations which are present in Eqs. (8) and (11) may either be quantum (with an entangled system-environment state) or classical (with a separable system-environment state) if the initial state of the environment is mixed [37,40]. It is important to note here that the nature of the correlations cannot change during teleportation.

The qubit-environment entanglement measure tailored to pure dephasing evolutions, which was introduced in Ref. [41], can be used to quantify the amount of entanglement in the CE state in Eq. (11). We comment in more depth about the applicability of this measure in the Appendix. It can also be used to quantify entanglement of the BCE state of Eq. (8) because during pure dephasing the state of qubits B and C is effectively confined to a two-dimensional subspace. The measure is given by $E_{BCE} = [1 - F(\hat{R}_{00}(\tau), \hat{R}_{11}(\tau))]$ for the dephased Bell state and by

$$E_{CE} = 4|\alpha|^2|\beta|^2[1 - F(\hat{R}_{00}(\tau), \hat{R}_{11}(\tau))] = 4|\alpha|^2|\beta|^2 E_{BCE} \tag{13}$$

for the post-teleportation state. Here the function $F(\hat{\rho}_1, \hat{\rho}_2) = [\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^2$ denotes the fidelity. As seen from Eq. (13) the amount of entanglement transferred depends strongly on the teleported state (1), but teleportation retains the nature of the correlation during transfer. If the state of qubit A before teleportation is either $|0\rangle$ or $|1\rangle$, entanglement will not be transferred, but these are also the only states which will be teleported faithfully. Any superposition state will acquire a part of entanglement with the environment if it is present in state (8) and this fraction is given by $4|\alpha|^2|\beta|^2$. If, on the other hand, the decoherence of the Bell state is separable in its nature then no quantum correlations can be present in the post-teleportation state (qubit-environment entanglement for pure dephasing evolutions is equivalent to the quantum discord [68]).

IV. TELEPORTATION OF DECOHERED STATE

We will now study teleportation from qubit C to qubit A assuming no delay time after the procedure described in the previous section. Hence, the preteleportation state is given by Eq. (11), so qubit C is correlated with the environment while qubits A and B are in Bell state (2). The two-qubit measurement is performed on qubits B and C and we again assume that the outcome is the state (2), for clarity.

After teleportation, the system of three qubits and environment is given by

$$\hat{\sigma}_{PM2}(\tau) = \begin{pmatrix} |\alpha|^2\hat{R}_{00}(\tau) & \alpha^*\beta\hat{R}_{01}(\tau) \\ \alpha\beta^*\hat{R}_{10}(\tau) & |\beta|^2\hat{R}_{11}(\tau) \end{pmatrix}_{AE} \otimes |\Phi_+\rangle_{BCBC}\langle\Phi_+|. \tag{14}$$

A different measurement outcome with appropriate unitary transformation would yield again an analogous state, which would differ only by the Bell state on qubits B and C.

Comparing this state with the preteleportation state (11), we note that the state of qubit A and the environment is exactly the same as the state of qubit C and environment was before teleportation. If the environmental degrees of freedom are traced out of the post-teleportation AE state, we will obtain exactly the preteleportation state of qubit C, which is given by Eq. (12). It is relevant to note here that there is an infinite number of AE states that yield the same dephased state of qubit A. For a mixed environment, such AE states can differ substantially, as there exist both entangled and separable states that lead the same amount of qubit decoherence. The states are not equivalent as entanglement easily manifests itself in, e.g., postmeasurement qubit evolution [54–56], so the exact transfer of correlations is a relevant factor.

Hence, not only the state of qubit C was teleported with unit fidelity to qubit A, but the state of qubit A and its environment is now exactly the same as the initial state of qubit C and the same environment. Qubit-environment correlations have been faithfully teleported during the process, similarly as occurs for pure initial environmental states [58], when the only possible correlations that yield decoherence are quantum.

V. SIMULTANEOUS TRANSFER AND TELEPORTATION OF CORRELATIONS

In this section we would like to study the situation when the teleportation back from qubit C to qubit A is preceded by a decohering of the Bell state on qubits A and B, analogously as in Sec. III. We can justify this choice of qubits to decohere by noting that a joint measurement was just performed on them, which in some qubit realizations means that they had to be brought closer together. However, the level of generality of this modeling of decoherence allows for each qubit to interact with a completely separate environment. The reason why the last qubit does not decohere is pragmatic: no new insight would be obtained, while unnecessary complexity would be introduced.

Hence we start with the $ABCE$ state (11) and allow qubits A and B to undergo pure dephasing for time t. The process is governed by a Hamiltonian with the same structure as Hamiltonian (3), hence the evolution operator (5) and conditional evolution operators (6) also retain the structure. We do not assume that it is the same Hamiltonian, so any operators pertaining to this second process will be labeled with a prime.

Since there are correlations between qubit C and the environment already present in state (11), the newly decohered state cannot be written in product form between parts AB and CE as Eq. (11), nor in a product form between parts ABE and C, equivalently to Eq. (8). In most situations there are at least classical correlations present in the partitions. The state is given by

$$\hat{\sigma}'_{PM}(\tau,t)=\frac{1}{2}\begin{pmatrix}|\alpha|^2\hat{R}^{00}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{00}_{01}(\tau,t) & |\alpha|^2\hat{R}^{01}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{01}_{01}(\tau,t)\\ \alpha\beta^*\hat{R}^{00}_{10}(\tau,t) & |\beta|^2\hat{R}^{00}_{11}(\tau,t) & \alpha\beta^*\hat{R}^{01}_{10}(\tau,t) & |\beta|^2\hat{R}^{01}_{11}(\tau,t)\\ |\alpha|^2\hat{R}^{10}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{10}_{01}(\tau,t) & |\alpha|^2\hat{R}^{11}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{11}_{01}(\tau,t)\\ \alpha\beta^*\hat{R}^{10}_{10}(\tau,t) & |\beta|^2\hat{R}^{10}_{11}(\tau,t) & \alpha\beta^*\hat{R}^{11}_{10}(\tau,t) & |\beta|^2\hat{R}^{11}_{11}(\tau,t)\end{pmatrix}, \quad (15)$$

where the 16 environmental matrices are obtained from Eq. (9) following

$$\hat{R}^{kq}_{ij}(\tau,t)=\hat{w}'_{kk}(t)\hat{R}_{ij}(\tau)\hat{w}'^{\dagger}_{qq}(t). \quad (16)$$

The matrix form in Eq. (15) corresponds to the states of the three qubits, where the basis is arranged in the following order: $\{|000\rangle, |001\rangle, |110\rangle, |111\rangle\}$. The other four elements of the three-qubit basis have been omitted, as all other matrix elements are equal to zero.

Let us now teleport the state of qubit C to qubit A. Contrarily to the results of Secs. III and IV, the measurement outcome on qubits B and C actually matters here. Regardless of the measurement outcome $|\lambda\rangle_{BC}$, the post-teleportation state (after the measurement and appropriate unitary transformation on qubit A) will be of the form

$$\hat{\sigma}'_{PM2}(\tau,t)=\hat{\rho}^{\lambda}_{AE}(\tau,t)\otimes|\lambda\rangle_{BCBC}\langle\lambda|, \quad (17)$$

but the state of qubit A and environment will be given by

$$\hat{\rho}^{\Phi_\pm}_{AE}(\tau,t)=\begin{pmatrix}|\alpha|^2\hat{R}^{00}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{01}_{01}(\tau,t)\\ \alpha\beta^*\hat{R}^{10}_{10}(\tau,t) & |\beta|^2\hat{R}^{11}_{11}(\tau,t)\end{pmatrix}_{AE}, \quad (18)$$

for either outcome $|\Phi_\pm\rangle=(|00\rangle\pm|11\rangle)/\sqrt{2}$, and we get

$$\hat{\rho}^{\Psi_\pm}_{AE}(\tau,t)=\begin{pmatrix}|\alpha|^2\hat{R}^{11}_{00}(\tau,t) & \alpha^*\beta\hat{R}^{10}_{01}(\tau,t)\\ \alpha\beta^*\hat{R}^{01}_{10}(\tau,t) & |\beta|^2\hat{R}^{00}_{11}(\tau,t)\end{pmatrix}_{AE} \quad (19)$$

for either outcome $|\Psi_\pm\rangle=(|01\rangle\pm|10\rangle)/\sqrt{2}$. The states (18) and (19) are obviously different, and although for both states the initial correlations of qubit C with the environment were teleported faithfully [as indicated by the subscript in the matrices $\hat{R}^{kq}_{ij}(\tau,t)$, which always correspond to the matrices in Eq. (11)], different correlations from state (15) are transported into the state (the superscripts pertain to the later decoherence process).

Both states (18) and (19) yield a purely dephased state of qubit A after the environment is traced out, but the degree of coherence can be different. The density matrices describing the state of qubit C alone retain the form of Eq. (12), but with

$$c^{\Phi_\pm}(\tau,t)=\mathrm{Tr}\big(\hat{w}'^{\dagger}_{11}(t)\hat{w}'_{00}(t)\hat{R}_{01}(\tau)\big), \quad (20a)$$

$$c^{\Psi_\pm}(\tau,t)=\mathrm{Tr}\big(\hat{w}'^{\dagger}_{00}(t)\hat{w}'_{11}(t)\hat{R}_{01}(\tau)\big). \quad (20b)$$

The two quantities are obviously the same if the two conditional evolution operators of the later decoherence process are Hermitian and commute, $[\hat{w}'_{00}(t),\hat{w}'_{11}(t)]=0$, $\hat{w}'_{ii}(t)=\hat{w}'^{\dagger}_{ii}(t)$. They are complex conjugates of each other if the matrix $\hat{R}_{01}(\tau)$ is Hermitian.

If there is entanglement in either AE state (18) or (19), it is of the type which can be qualified by the condition of Ref. [37], so the if and only if conditions of separability for states (18) and (19) are

$$\hat{w}'_{00}(t)\hat{R}_{00}(\tau)\hat{w}'^{\dagger}_{00}(t)=\hat{w}'_{11}(t)\hat{R}_{11}(\tau)\hat{w}'^{\dagger}_{11}(t), \quad (21a)$$

$$\hat{w}'_{00}(t)\hat{R}_{11}(\tau)\hat{w}'^{\dagger}_{00}(t)=\hat{w}'_{11}(t)\hat{R}_{00}(\tau)\hat{w}'^{\dagger}_{11}(t), \quad (21b)$$

respectively, while the condition of separability of qubit C and the environment preteleportation and decoherence (11) is $\hat{R}_{00}(\tau)=\hat{R}_{11}(\tau)$. Hence, if there is no entanglement after the first part of teleportation then both states are either entangled or separable, and the amount of entanglement in the two states is also the same [41], as it is quantified by a function analogous to Eq. (13),

$$E_{AE}(\tau,t)=4|\alpha|^2|\beta|^2\big[1-F\big(\hat{R}^{kk}_{00}(\tau,t),\hat{R}^{qq}_{11}(\tau,t)\big)\big], \quad (22)$$

with $k=0$ and $q=1$ for state (18) and $k=1$ and $q=0$ for state (19). Since for a separable state (11), we have $\hat{R}^{00}_{00}(\tau,t)=\hat{R}^{00}_{11}(\tau,t)$ and $\hat{R}^{11}_{00}(\tau,t)=\hat{R}^{11}_{11}(\tau,t)$, the function (22) yields the same outcome for both states. Note that this does not necessarily translate to the same degree of coherence, as discussed in the previous paragraph.

When there is entanglement between qubit C and the environment in state (11) then the conditions of separability for states (18) and (19) differ from each other. The two states can, in this case, not only have a different amount of qubit-environment entanglement, but the situation when one of the states is separable while the other is entangled can be realized. This is a direct consequence of the type of entanglement between the three qubits and the environment present in state (15). The conditions (21) constitute two of the seven nontrivial separability conditions of the first type for an eight-dimensional system and an environment interacting via a pure-dephasing Hamiltonian [40], which are applicable in the case of a density matrix of the form (15). Since this is effectively a 4×4 system, then a further four conditions are automatically fulfilled, and only one nontrivial separability condition of this type is irrelevant in the case of the teleportation process under study. None of the separability conditions of the second type [40] are relevant here, because we are

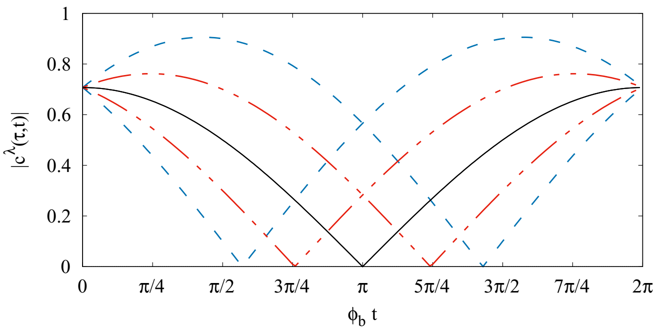

FIG. 1. Exemplary degree of coherence (absolute value) for $\lambda = \Phi_\pm$ and $\lambda = \Psi_\pm$ for a single qubit environment initially in a maximally mixed state, when no entanglement is generated before the first teleportation, as a function of decoherence time before second teleportation. The first decoherence time yields $\varphi_1\tau = \pi/2$. Different curves correspond to different conditional evolution of the environment before the second teleportation with $|x|^2 = 0.1$ (dashed blue lines), $|x|^2 = 0.3$ (dashed-dotted red lines), and $|x|^2 = 0.5$ (solid black lines).

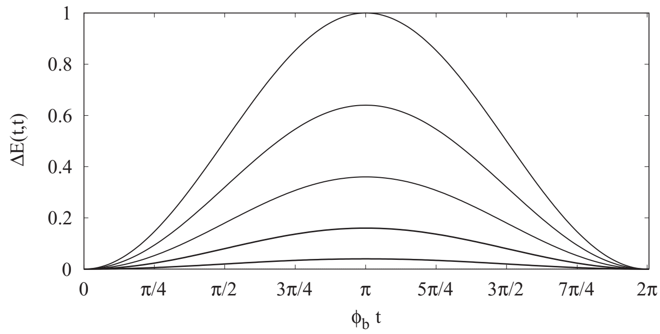

FIG. 2. Evolution of the difference between entanglement present in state (19) and state (18) for a single qubit environment and exemplary pure-dephasing Hamiltonian with $x = y = 1/\sqrt{2}$. Different curves correspond to different initial state of the environment with $c_0 = 0.6, 0.7, 0.8, 0.9, 1$ going from the bottom curve to the top.

dealing with transfer of entanglement to a single qubit and the second type of entanglement has no single-qubit equivalent. We comment on the types and number of nontrivial separability conditions in pure dephasing evolutions for systems larger than a qubit in the Appendix.

VI. EXAMPLE: A SINGLE QUBIT ENVIRONMENT

To illustrate the results of the previous section we will study an exemplary system environment evolution with the smallest possible environment, one composed of a single qubit. We will assume that the initial state of the environment is $\hat{R}(0) = c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1|$, so it is pure for $c_0 = 0, 1$ and maximally mixed for $c_0 = c_1 = 1/2$. We will further assume, for simplicity, that the interaction with the environment is fully asymmetric in both processes, meaning that $\hat{w}_{11}(\tau) = \hat{w}'_{11}(t) = \mathbb{I}_E$.

Let us look at the situation when the first interaction, the one described in Sec. III, yields a separable qubit-environment state at all times τ. This means that the operator $\hat{w}_{00}(\tau)$ must be diagonal in the same basis as the initial state of the environment at all times [37], so it can be written as $\hat{w}_{00}(\tau) = e^{i\varphi_0\tau}|0\rangle\langle 0| + e^{i\varphi_1\tau}|1\rangle\langle 1|$, and we get $\hat{R}_{01}(\tau) = c_0 e^{i\varphi_0\tau}|0\rangle\langle 0| + c_1 e^{i\varphi_1\tau}|1\rangle\langle 1|$. We will write the conditional evolution operator governing the second decoherence process in a general way,

$$\hat{w}'_{00}(t) = e^{i\phi_a t}|a\rangle\langle a| + e^{i\phi_b t}|b\rangle\langle b|. \quad (23)$$

Here $|a\rangle = x|0\rangle + y|1\rangle$ and $|b\rangle = y^*|0\rangle - x^*|1\rangle$ are two orthogonal states that diagonalize the respective part of the Hamiltonian. The parameters x and y, $|x|^2 + |y|^2 = 1$, depend on the interaction Hamiltonian and are used as parameters in this example.

It is now straightforward to find the values of the degree of coherence factors (20), which are given by

$$c^\lambda(\tau, t) = |x|^2\left[c_0 e^{i(\varphi_0\tau \pm \phi_a t)} + c_1 e^{i(\varphi_1\tau \pm \phi_b t)}\right] + |y|^2\left[c_0 e^{i(\varphi_0\tau \pm \phi_b t)} + c_1 e^{i(\varphi_1\tau \pm \phi_a t)}\right], \quad (24)$$

with pluses for $\lambda = \Phi_\pm$ and minuses for $\lambda = \Psi_\pm$. In Fig. 1 we plot the absolute value of $c^\lambda(\tau, t)$ in both cases for a maximally mixed environment (so both qubit-environment states are always separable) for $\varphi_0 = 0$ and $\phi_a = 0$, and for different values of $|x|^2$. We set the first decoherence time so that $\varphi_1\tau = \pi/2$ and the state teleported in the first part of the procedure is dephased, and show the degree of coherence as a function of $\phi_b t$. For $|x|^2 = 0.5$ both curves are the same, but in the other two cases, the degree of coherence of qubit A after the second teleportation strongly depends on the measurement outcome on qubits B and C.

We will now look at a situation when both interactions can lead to entanglement to illustrate the fact that depending on the measurement result in the second teleportation process we can have a qubit which is entangled with its environment or not. To this end, let us assume that both operators $\hat{w}_{00}(\tau)$ and $\hat{w}'_{00}(t)$ are the same and are given by Eq. (23) with $t = \tau$, while the other two conditional evolution operators remain trivial. We then easily find that $\hat{R}^{11}_{00}(t, t) = \hat{R}^{00}_{11}(t, t) = \hat{w}_{00}(t)\hat{R}(0)\hat{w}^\dagger_{00}(t)$, so the separability condition (21b) is fulfilled for all t and there is no qubit-environment entanglement in state (19).

The two environmental operators relevant for entanglement in state (18) are given by $\hat{R}^{00}_{00}(t, t) = \hat{w}_{00}(2t)\hat{R}(0)\hat{w}^\dagger_{00}(2t)$ and $\hat{R}^{00}_{11}(t, t) = \hat{R}(0)$, so the separability condition (21a) is not fulfilled unless $x = 0$ or $y = 0$ outside of discrete points in time. Entanglement measured by the function (22) for $x = y = 1/\sqrt{2}$ is plotted in Fig. 2 for different initial mixedness of the environment and for an equal superposition teleported state (1).

VII. CONCLUSION

In this paper we have studied bidirectional teleportation of a qubit via a maximally entangled Bell state. We took into account a process leading to pure dephasing of the Bell state due to an interaction with an environment, but contrarily to previous works on the subject, we have kept the degrees of freedom of the environment throughout. Thanks to this, we were able to study the behavior of correlations with the environment while the teleportation procedure was operated, which is nontrivial when the environment is initially in a mixed state. In this case decoherence can be the consequence of the buildup of classical correlations instead of entanglement, which is necessary for decoherence induced by an interaction with pure environments.

We have found that for simple procedures when decoherence is a one-time-incident, there is no qualitative difference between the situation when entanglement with the environment is present in the system and the situation when the system-environment state is separable. Correlations, whether quantum or classical, are transported or teleported between qubits with the same fidelity regardless of their nature, similarly as entanglement in the case of pure environments.

Once a second decoherence process is added, the situation becomes very different. First the fidelity of teleportation now depends on the measurement outcome during teleportation for almost all interaction Hamiltonians. Second, there is an interplay between the entangling or separable nature of the first decoherence process and what qubit-environment correlations are possible after the whole bidirectional teleportation. If the first process was separable, the amount of entanglement has to be the same regardless of measurement outcome (even though the amount of qubit coherence does not). If entanglement was present then the amount of entanglement after the second teleportation can be different, and in the extreme case, one of the qubit-environment states after teleportation can be separable while the other is entangled. We illustrate this extreme case in the simplest possible case when the environment is limited to one qubit, and it is sufficient to demonstrate the effect.

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APPENDIX: PURE DEPHASING INTERACTIONS AND EVOLUTIONS

Any system-environment Hamiltonian consists of three parts, $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE}$, where obviously the first and second terms are the free Hamiltonians of the system and environment respectively, while the third describes the interaction. The condition for such a Hamiltonian to be able to only lead to pure dephasing of the system after the degrees of freedom of the environment are traced out is the commutation of the system and interaction Hamiltonians, $[\hat{H}_S, \hat{H}_{SE}] = 0$. This means that there must exist a basis in the system Hilbert space which diagonalizes both Hamiltonians; this is called the pointer basis [43,69] and we will denote it as $|n\rangle$. Hence, if we write the two terms of the Hamiltonian explicitly in the pointer basis,

$$\hat{H}_S = \sum_n \varepsilon_n |n\rangle\langle n|, \tag{A1}$$

$$\hat{H}_{SE} = \sum_n |n\rangle\langle n| \otimes \tilde{V}_n, \tag{A2}$$

a pure dephasing Hamiltonian can be written as

$$\hat{H}_{PD} = \sum_n |n\rangle\langle n| \otimes \hat{V}_n, \tag{A3}$$

with $\hat{V}_n = \varepsilon_n + \tilde{V}_n + \hat{H}_E$. This in turn allows us to find a general form of the evolution operator,

$$\hat{U}_{PD}(t) = \sum_n |n\rangle\langle n| \otimes \hat{w}_n(t), \tag{A4}$$

with

$$\hat{w}_n(t) = e^{-(i/\hbar)\hat{V}_n t}. \tag{A5}$$

Using the evolution above, it is possible to formally write the system-environment density matrix at time t given any initial state. We are interested in the situation when the initial state of the system is pure $|\psi\rangle = \sum_n c_n|n\rangle$, while there are no limitations on the initial state of the environment $\hat{R}(0)$. Then the system-environment state at time t is given by

$$\hat{\sigma}(t) = \sum_{n,m} |n\rangle\langle m| \hat{R}_{nm}(t), \tag{A6}$$

with

$$\hat{R}_{nm}(t) = \hat{w}_n(t)\hat{R}(0)\hat{w}_m^\dagger(t). \tag{A7}$$

The pure dephasing Hamiltonian (3) is precisely of the same type as (A3), but the qubit pointer states $|n\rangle$ have been written using two indices corresponding to the relevant qubits (which undergo decoherence), while the third qubit is not affected. The density matrix (8) is obtained using Eq. (A6) and the matrix notation is used in the subspace of the two qubits, because it is more transparent for small quantum systems.

It is relevant to note that none of the measurements and gates required to perform teleportation qualitatively change the structure of the system-environment density matrix. Hence, even though the states are not obtained strictly through a pure dephasing interaction with an environment, their correlations can nevertheless be quantified using methods specially devised for pure dephasing [37,40,41]. We will shortly reiterate the results presented in those papers below.

System-environment entanglement

When there is no limitation on the size of the system N then, for a state of the form given by Eq. (A6), there are two types of conditions of separability [40]. Conditions of the first type are related to the similarity of conditional evolution of the environment in the case of separability and are of the form

$$\hat{R}_{nn}(t) = \hat{R}_{mm}(t), \tag{A8}$$

while conditions of the second type are more abstract and are given by

$$[\hat{w}_n(t)\hat{w}_m^\dagger(t), \hat{w}_{n'}(t)\hat{w}_{m'}^\dagger(t)] = 0. \tag{A9}$$

There are $N - 1$ independent conditions of the first type and $(N - 1)(N - 2)/2$ conditions of the second type. All of them have to be fulfilled for a state of the form (A6) to be separable, otherwise it is entangled.

In situations when there is only one nonzero off-diagonal element of the density matrix in the system subspace (when the system is a qubit [37] or the evolution takes place in a Hilbert space reduced to two states of the system), then there is only one nontrivial separability condition

$$\hat{R}_{00}(t) = \hat{R}_{11}(t). \tag{A10}$$

In this case, an entanglement measure specifically tailored to pure dephasing evolutions can be used [41], which we use in the article and which is given by Eq. (13) for $c_0 = \alpha$ and $c_1 = \beta$.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[2] D. Gottesman and I. L. Chuang, Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, Nature (London) **402**, 390 (1999).

[3] S. Bandyopadhyay and B. C. Sanders, Quantum teleportation of composite systems via mixed entangled states, Phys. Rev. A **74**, 032310 (2006).

[4] G.-F. Zhang, Thermal entanglement and teleportation in a two-qubit heisenberg chain with dzyaloshinski-moriya anisotropic antisymmetric interaction, Phys. Rev. A **75**, 034304 (2007).

[5] J.-C. Liu, Y.-H. Li, and Y.-Y. Nie, Controlled teleportation of an arbitrary two-particle pure or mixed state by using a five-qubit cluster state, Int. J. Theor. Phys. **49**, 1976 (2010).

[6] P. Agrawal and A. K. Pati, Probabilistic quantum teleportation, Phys. Lett. A **305**, 12 (2002).

[7] F. Yan and T. Yan, Probabilistic teleportation via a non-maximally entangled ghz state, Chin. Sci. Bull. **55**, 902 (2010).

[8] H. Prakash and V. Verma, Minimum assured fidelity and minimum average fidelity in quantum teleportation of single qubit using non-maximally entangled states, Quantum Inf. Process. **11**, 1951 (2012).

[9] G. Bowen and S. Bose, Teleportation as a Depolarizing Quantum Channel, Relative Entropy, and Classical Capacity, Phys. Rev. Lett. **87**, 267901 (2001).

[10] S. Oh, S. Lee, and H.-W. Lee, Fidelity of quantum teleportation through noisy channels, Phys. Rev. A **66**, 022316 (2002).

[11] F. Verstraete and H. Verschelde, Optimal Teleportation with a Mixed State of Two Qubits, Phys. Rev. Lett. **90**, 097901 (2003).

[12] M.-L. Hu, Environment-induced decay of teleportation fidelity of the one-qubit state, Phys. Lett. A **375**, 2140 (2011).

[13] K. Roszak, R. Filip, and T. Novotný, Decoherence control by quantum decoherence itself., Sci. Rep. **5**, 9796 (2015).

[14] L. Aolita, F. de Melo, and L. Davidovich, Open-system dynamics of entanglement:a key issues review, Rep. Prog. Phys. **78**, 042001 (2015).

[15] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Experimental quantum teleportation, Nature (London) **390**, 575 (1997).

[16] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **80**, 1121 (1998).

[17] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Unconditional quantum teleportation, Science **282**, 706 (1998).

[18] M. A. Nielsen, E. Knill, and R. Laflamme, Complete quantum teleportation using nuclear magnetic resonance, Nature (London) **396**, 52 (1998).

[19] M. Riebe, H. Häffner, C. Roos, W. Hänsel, J. Benhelm, G. Lancaster, T. Körber, C. Becher, F. Schmidt-Kaler, D. James *et al.*, Deterministic quantum teleportation with atoms, Nature (London) **429**, 734 (2004).

[20] M. Barrett, J. Chiaverini, T. Schaetz, J. Britton, W. Itano, J. Jost, E. Knill, C. Langer, D. Leibfried, R. Ozeri *et al.*, Deterministic quantum teleportation of atomic qubits, Nature (London) **429**, 737 (2004).

[21] J. F. Sherson, H. Krauter, R. K. Olsson, B. Julsgaard, K. Hammerer, I. Cirac, and E. S. Polzik, Quantum teleportation between light and matter, Nature (London) **443**, 557 (2006).

[22] H. Yonezawa, T. Aoki, and A. Furusawa, Demonstration of a quantum teleportation network for continuous variables, Nature (London) **431**, 430 (2004).

[23] N. Takei, H. Yonezawa, T. Aoki, and A. Furusawa, High-Fidelity Teleportation beyond the No-Cloning Limit and Entanglement Swapping for Continuous Variables, Phys. Rev. Lett. **94**, 220502 (2005).

[24] R. Ursin, T. Jennewein, M. Aspelmeyer, R. Kaltenbaek, M. Lindenthal, P. Walther, and A. Zeilinger, Quantum teleportation across the danube, Nature (London) **430**, 849 (2004).

[25] X.-S. Ma, T. Herbst, T. Scheidl, D. Wang, S. Kropatschek, W. Naylor, B. Wittmann, A. Mech, J. Kofler, E. Anisimova *et al.*, Quantum teleportation over 143 kilometres using active feed-forward, Nature (London) **489**, 269 (2012).

[26] J.-G. Ren, P. Xu, H.-L. Yong, L. Zhang, S.-K. Liao, J. Yin, W.-Y. Liu, W.-Q. Cai, M. Yang, L. Li, K.-X. Yang, X. Han, Y.-Q. Yao, J. Li, H.-Y. Wu, S. Wan, L. Liu, D.-Q. Liu, Y.-W. Kuang, Z.-P. He *et al.*, Ground-to-satellite quantum teleportation, Nature (London) **549**, 70 (2017).

[27] Y. Yeo, Local noise can enhance two-qubit teleportation, Phys. Rev. A **78**, 022334 (2008).

[28] E.-M. Laine, H.-P. Breuer, and J. Piilo, Nonlocal memory effects allow perfect teleportation with mixed states, Sci. Rep. **4**, 4620 (2014).

[29] J. Medford, L. Cywiński, C. Barthel, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Scaling of Dynamical Decoupling for Spin Qubits, Phys. Rev. Lett. **108**, 086802 (2012).

[30] E. Kawakami, T. Jullien, P. Scarlino, D. R. Ward, D. E. Savage, M. G. Lagally, V. V. Dobrovitski, M. Friesen, S. N. Coppersmith, M. A. Eriksson, and L. M. K. Vandersypen, Gate fidelity and coherence of an electron spin in an si/sige quantum dot with micromagnet, Proc. Natl. Acad. Sci. USA **113**, 11738 (2016).

[31] N. Zhao, S.-W. Ho, and R.-B. Liu, Decoherence and dynamical decoupling control of nitrogen vacancy center electron spins in nuclear spin baths, Phys. Rev. B **85**, 115303 (2012).

[32] D. Kwiatkowski and L. Cywiński, Decoherence of two entangled spin qubits coupled to an interacting sparse nuclear spin bath: Application to nitrogen vacancy centers, Phys. Rev. B **98**, 155202 (2018).

[33] F. K. Malinowski, F. Martins, P. D. Nissen, E. Barnes, Ł. Cywiński, M. S. Rudner, S. Fallahi, G. C. Gardner, M. J. Manfra, C. M. Marcus, and F. Kuemmeth, Nuclear spins in gallium arsenide produce noise at discrete frequencies, which can be notch-filtered efficiently to extend coherence times of electron spin qubits to nearly 1ms, Nat. Nanotechnol. **12**, 16 (2017).

[34] G. C. Knee, K. Kakuyanagi, M.-C. Yeh, Y. Matsuzaki, H. Toida, H. Yamaguchi, S. Saito, A. J. Leggett, and W. J. Munro, A strict experimental test of macroscopic realism in a superconducting flux qubit, Nat. Commun. **7**, 13253 (2016).

[35] Y. Wu, L.-P. Yang, M. Gong, Y. Zheng, H. Deng, Z. Yan, Y. Zhao, K. Huang, A. D. Castellano, W. J. Munro, K. Nemoto, D.-N. Zheng, C. P. Sun, Y.-x. Liu, X. Zhu, and L. Lu, An efficient and compact switch for quantum circuits, npj Quantum Inf. **4**, 50 (2018).

[36] S. Touzard, A. Kou, N. E. Frattini, V. V. Sivak, S. Puri, A. Grimm, L. Frunzio, S. Shankar, and M. H. Devoret, Gated Conditional Displacement Readout of Superconducting Qubits, Phys. Rev. Lett. **122**, 080502 (2019).

[37] K. Roszak and L. Cywiński, Characterization and measurement of qubit-environment-entanglement generation during pure dephasing, Phys. Rev. A **92**, 032310 (2015).

[38] H.-B. Chen, C. Gneiting, P.-Y. Lo, Y.-N. Chen, and F. Nori, Simulating Open Quantum Systems with Hamiltonian Ensembles and the Nonclassicality of the Dynamics, Phys. Rev. Lett. **120**, 030403 (2018).

[39] H.-B. Chen, P.-Y. Lo, C. Gneiting, J. Bae, Y.-N. Chen, and F. Nori, Quantifying the nonclassicality of pure dephasing, Nat. Commun. **10**, 3794 (2019).

[40] K. Roszak, Criteria for system-environment entanglement generation for systems of any size in pure-dephasing evolutions, Phys. Rev. A **98**, 052344 (2018).

[41] K. Roszak, Measure of qubit-environment entanglement for pure dephasing evolutions, Phys. Rev. Research **2**, 043062 (2020).

[42] M. Popovic, M. T. Mitchison, and J. Goold, Thermodynamics of decoherence, arXiv:2107.14216.

[43] W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Rev. Mod. Phys. **75**, 715 (2003).

[44] M. A. Schlosshauer, *Decoherence and the Quantum-To-Classical Transition* (Springer Science & Business Media, Berlin, 2007).

[45] S. Hilt and E. Lutz, System-bath entanglement in quantum thermodynamics, Phys. Rev. A **79**, 010101(R) (2009).

[46] J. Helm and W. T. Strunz, Quantum decoherence of two qubits, Phys. Rev. A **80**, 042108 (2009).

[47] J. Helm and W. T. Strunz, Decoherence and entanglement dynamics in fluctuating fields, Phys. Rev. A **81**, 042314 (2010).

[48] A. Pernice and W. T. Strunz, Decoherence and the nature of system-environment correlations, Phys. Rev. A **84**, 062121 (2011).

[49] A. Pernice, J. Helm, and W. T. Strunz, System–environment correlations and non-markovian dynamics, J. Phys. B: At. Mol. Opt. Phys. **45**, 154005 (2012).

[50] R. Lo Franco, B. Bellomo, E. Andersson, and G. Compagno, Revival of quantum correlations without system-environment back-action, Phys. Rev. A **85**, 032318 (2012).

[51] T. Salamon and K. Roszak, Entanglement generation between a charge qubit and its bosonic environment during pure dephasing: Dependence on the environment size, Phys. Rev. A **96**, 032333 (2017).

[52] M. Strzałka, D. Kwiatkowski, L. Cywiński, and K. Roszak, Qubit-environment negativity versus fidelity of conditional environmental states for a nitrogen-vacancy-center spin qubit interacting with a nuclear environment, Phys. Rev. A **102**, 042602 (2020).

[53] M. Strzałka and K. Roszak, Detection of entanglement during pure dephasing evolutions for systems and environments of any size, Phys. Rev. A **104**, 042411 (2021).

[54] K. Roszak, D. Kwiatkowski, and L. Cywiński, How to detect qubit-environment entanglement generated during qubit dephasing, Phys. Rev. A **100**, 022318 (2019).

[55] B. Rzepkowski and K. Roszak, A scheme for direct detection of qubit-environment entanglement generated during qubit pure dephasing, Quantum Inf. Process. **20**, 1 (2020).

[56] K. Roszak and L. Cywiński, Qubit-environment-entanglement generation and the spin echo, Phys. Rev. A **103**, 032208 (2021).

[57] L. Vaidman, Teleportation of quantum states, Phys. Rev. A **49**, 1473 (1994).

[58] S. F. Huelga, J. A. Vaccaro, A. Chefles, and M. B. Plenio, Quantum remote control: Teleportation of unitary operations, Phys. Rev. A **63**, 042303 (2001).

[59] S. F. Huelga, M. B. Plenio, and J. A. Vaccaro, Remote control of restricted sets of operations: Teleportation of angles, Phys. Rev. A **65**, 042316 (2002).

[60] E. O. Kiktenko, A. A. Popov, and A. K. Fedorov, Bidirectional imperfect quantum teleportation with a single bell state, Phys. Rev. A **93**, 062305 (2016).

[61] S. Hassanpour and M. Houshmand, Bidirectional teleportation of a pure epr state by using ghz states, Quantum Inf. Process. **15**, 905 (2016).

[62] A. U. Siddiqui and M. M. Wilde, Quantifying the performance of bidirectional quantum teleportation, arXiv:2010.07905.

[63] K. Roszak and J. K. Korbicz, Entanglement and objectivity in pure dephasing models, Phys. Rev. A **100**, 062127 (2019).

[64] K. Roszak and P. Machnikowski, Complete disentanglement by partial pure dephasing, Phys. Rev. A **73**, 022313 (2006).

[65] F. Galve, A. Mandarino, M. G. A. Paris, C. Benedetti, and R. Zambrini, Microscopic description for the emergence of collective dissipation in extended quantum systems, Sci. Rep. **7**, 42050 (2017).

[66] B.-G. Englert and J. A. Bergou, Quantitative quantum erasure, Opt. Commun. **179**, 337 (2000).

[67] K. Roszak and P. Machnikowski, "Which path" decoherence in quantum dot experiments, Phys. Lett. A **351**, 251 (2006).

[68] K. Roszak and L. Cywiński, Equivalence of qubit-environment entanglement and discord generation via pure dephasing interactions and the resulting consequences, Phys. Rev. A **97**, 012306 (2018).

[69] W. H. Zurek, Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? Phys. Rev. D **24**, 1516 (1981).