

# Full multipartite steering inseparability, genuine multipartite steering, and monogamy for continuous-variable systems

Run Yan Teh <sup>1</sup>, Manuel Gessner,<sup>2,3</sup> Margaret D. Reid,<sup>1</sup> and Matteo Fadel <sup>4</sup>

<sup>1</sup>*Centre for Quantum Science and Technology Theory, Swinburne University of Technology, Melbourne 3122, Australia*

<sup>2</sup>*Laboratoire Kastler Brossel, ENS-Université PSL, Centre National de la Recherche Scientifique, Sorbonne Université, Collège de France, 24 Rue Lhomond, 75005 Paris, France*

<sup>3</sup>*ICFO-Institut de Ciències Fotòniques, The Barcelona Institute of Science and Technology, Avenida Carl Friedrich Gauss 3, 08860 Castelldefels (Barcelona), Spain*

<sup>4</sup>*Department of Physics, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland*



(Received 16 August 2021; accepted 8 December 2021; published 3 January 2022)

We derive inequalities sufficient to detect the genuine  $N$ -partite steering of  $N$  distinct systems. Here, we are careful to distinguish between the concepts of full  $N$ -partite steering inseparability (where steering is confirmed individually for all bipartitions of the  $N$  systems, thus negating the bilocal hidden state model for each bipartition) and genuine  $N$ -partite steering (which excludes all convex combinations of the bilocal hidden state models). Other definitions of multipartite steering are possible and we also derive inequalities to detect a stricter genuine  $N$ -partite steering where only one site needs to be trusted. The inequalities are expressed as variances of quadrature phase amplitudes and thus apply to continuous-variable systems. We show how genuine  $N$ -partite steerable states can be created and detected for the nodes of a network formed from a single-mode squeezed state passed through a sequence of  $N - 1$  beam splitters. A stronger genuine  $N$ -partite steering is created, if one uses two squeezed inputs or  $N$  squeezed inputs. We are able to confirm that genuine tripartite steering (by the above definition and by the stricter definition) has been realized experimentally. Finally, we analyze how bipartite steering and entanglement are distributed among the systems in the tripartite case, illustrating with monogamy inequalities. While we use Gaussian states to benchmark the criteria, the inequalities derived in this paper are not based on the assumption of Gaussian states, which gives an advantage for quantum communication protocols.

DOI: [10.1103/PhysRevA.105.012202](https://doi.org/10.1103/PhysRevA.105.012202)

## I. INTRODUCTION

From the perspective of both fundamental and applied physics, understanding whether  $N$  distinct systems can be genuinely entangled is considered important [1–10]. Svetlichny first showed that a genuine tripartite nonlocality can be shared among three systems [11]. Greenberger, Horne, and Zeilinger (GHZ) [12] and Mermin [13] investigated the nonlocal properties of  $N$  genuinely entangled systems created in an extreme quantum superposition state. Multipartite entangled systems have applications in the field of quantum information. The tripartite-entangled GHZ states were proposed for quantum secret sharing, where two parties must collaborate to uncover a cryptographic key [6]. Usually, the certification of entanglement required to ensure security against eavesdropping involves assumptions about the instruments at the given sites. When entanglement is confirmed by way of violation of a Bell inequality, however, fewer assumptions are needed, resulting in device-independent security [5,14–18]. A quantum network consisting of mutually entangled systems may form the basis for a quantum communication network. Recent papers motivate the use of multipartite Bell nonlocality for device-independent security on a network [19–21].

Einstein-Podolsky-Rosen (EPR) steering is a type of entanglement associated with the nonlocality of the EPR paradox [22]. The concept of EPR steering was motivated by the arguments put forward by Schrödinger in response to the EPR-paradox paper [23–26]. Based on the negation of asymmetric local hidden state (LHS) models, steering is useful for one-sided device-independent quantum security, where one has control over some, but not all, of the devices on the nodes of the network [27–29]. Implementing steering, as opposed to entanglement, thus increases the potential for ultrasecure communication [27,30–32]. Steering has also been proposed as a resource for other applications, including secure quantum teleportation [33], quantum metrology [34], and secret sharing [35]. However, the set of steerable states is a strict subset of the set of entangled states [23]. The detection of  $N$ -partite steering is therefore more challenging and standard witnesses for entanglement will not suffice. An early criterion for steering was an inequality applied to the EPR paradox [24,25,36]. Numerous steering criteria have since been derived, but most refer to the bipartite case [26]. This highlights the need for criteria to confirm steering shared among  $N$  systems, derived with minimal assumptions about the states being measured.

The concept of multipartite steering was introduced by He and Reid [28] and developed for Gaussian states by Kogias *et al.* [37]. Experiments followed [38–43], which motivated

studies of the monogamy of steering [37,43–45]. Marian and Marian have recently derived criteria involving EPR variances that are sufficient to certify multimode steering, and have linked these results to Gaussian states [46]. However, as for multipartite entanglement and nonlocality [10,47–53], different definitions of multipartite steering are possible. This is particularly true for steering, where subtleties enter into the definitions, because different nodes on a network can have different levels of trust in devices.

In this paper, we further address gaps in the literature, by deriving criteria sufficient to detect  $N$ -partite steering. We follow [48,49,54] to provide a treatment where classes of multipartite steering are clearly distinguished. In particular, we consider *full  $N$ -partite steering inseparability* (where there is steering across each of the bipartitions of the  $N$  systems), and distinguish this from the stricter definition, *genuine  $N$ -partite steering* (which excludes convex mixtures of hidden states allowing for bilocality along different bipartitions). Similar to the earlier results of van Loock and Furusawa for the  $N$ -partite entanglement of continuous-variable (CV) systems [10,38,48,49], we envisage that local measurements can be performed on each system and derive inequalities that are only violated if the systems are mutually steerable. The inequalities are in terms of the variances of the local-field quadrature phase amplitudes. We follow the bilocal approach of Svetlichny [11] and Collins *et al.* [55], noting that other approaches might also be considered [50,51,56].

As we show in this paper, the variance criteria are useful to detect genuine multipartite steering for  $N$ -partite Gaussian systems, where one creates Gaussian states for  $N$  field modes from squeezed states and beam splitters. This is relevant given recent CV multipartite entanglement experiments [38,57–62] and Gaussian boson sampling experiments [63], some of which realized networks for very large  $N$ . Three types of  $N$ -partite Gaussian states are considered in this paper: the CV GHZ (and cluster, for  $N = 3$ ), CV EPR, and CV split-squeezed (SS) states. Although we use the  $N$ -partite Gaussian states to benchmark our criteria, the inequalities are valid in principle to detect steering in non-Gaussian systems. This is recognized in the bipartite case where it has been shown that variance inequalities can detect steering for NOON states [64] and for cat states [65,66]. Our paper, in contrast to some previous work [37], is not based on the underlying assumption that the systems are prepared in Gaussian states. This is an important and necessary requirement for one-sided device-independent protocols [27,30–32]. Using values reported in the literature, we apply the inequalities to confirm that genuine tripartite steering has been realized experimentally for optical fields. We also derive inequalities which allow for  $N - 1$  untrusted sites, and deduce that the genuine tripartite steering has been confirmed for  $N = 3$ , with the requirement that only one site needs to be trusted.

We also examine monogamy relations for tripartite systems, where  $N = 3$ . These relations give constraints on the amount of bipartite steering and entanglement between any two of the nodes of the triparty network. We derive a relation that constrains the amount of bipartite entanglement, showing that for the CV GHZ, CV EPR, and CV SS states, although a limited amount of bipartite entanglement is possible, no bipartite steering can be observed as measured by the two-mode Gaussian steering parameter.

The continuous-variable criteria of this paper will apply to systems of large collective spin, in certain limits [67–71]. Furthermore, since the inequalities of this paper are derived from uncertainty relations, the methods presented may be generalized for multipartite spin systems, as done for Bohm’s EPR paradox [72]. Although multiparticle and multimode entanglement and steering can be inferred from spin squeezing [73–78], Fisher information [34,79–81], or (using superselection rules) interference [42,82,83], to demonstrate nonlocality in a strict way requires spatial separation and local measurements [84–89]. Two-particle entanglement and Bell correlations have been studied and reported for separated atoms [90–93], and bipartite EPR steering and genuine  $N$ -partite entanglement have been observed for separated atomic clouds containing several hundred atoms [40,41]. Yet, it remains a challenge to demonstrate  $N$ -partite steering ( $N > 2$ ) for spatially separated atomic systems. The results of this paper may be useful for this purpose.

### Summary of paper

In Sec. II, we summarize the definitions of  $N$ -partite steering, distinguishing between full  $N$ -partite steering inseparability and the more strict definition given by genuine  $N$ -partite steering. In this section, we derive steering inequalities that if violated will reveal the presence of genuine  $N$ -partite steering. These inequalities involve a single set of gain parameters optimized to detect the EPR steering of a single preferred party, and are similar to those considered recently by Marian and Marian in independent work [46].

Continuing, in Sec. III, we focus on tripartite systems. We expand the set of criteria, also deriving inequalities closely related to those of van Loock and Furusawa that have been widely used to study multipartite entanglement [58,60,94]. These inequalities include those with a broader set of gain parameters, which allow certification of genuine tripartite steering once the EPR steering of each party is sufficiently strong (as measured by vanishing conditional inference variances). The work presented in Secs. II and III extends the work of He and Reid [28], who derived inequalities based on a weaker form of genuine  $N$ -partite steering. We confirm that those earlier inequalities (derived in [28,49]) will also certify the stricter form of genuine tripartite steering defined in the present paper. In addition, we derive inequalities that if violated will certify an even stricter form of genuine tripartite steering, giving a method to detect genuine tripartite entanglement and steering where only one node needs to be trusted.

In Sec. IV, we show how the criteria derived in Secs. II and III with suitably optimized gains are useful to detect the genuine  $N$ -partite steering of four types of CV Gaussian states. These are the CV split-squeezed states, the CV EPR states, and the CV GHZ (and cluster) states, created using one, two, and  $N$  squeezed-vacuum states incident on beam splitters. We examine details for  $N = 3$  in Sec. V. Using the criteria, we infer full tripartite steering inseparability and strict genuine tripartite steering from experimental results reported in the literature. Full tripartite steering inseparability has been detected in at least two experiments [38,60], for the correlations generated from CV EPR and cluster states. For CV GHZ and cluster states, the stricter form of genuine tripartite

steering (including where two sites can be untrusted) is also detectable using the inequalities, and we are able to confirm the experimental realization for tripartite CV cluster states in one experiment [60] where values for the van Loock–Furusawa variances were measured. Finally, in Sec. VI, we give an analysis of the monogamy properties of the CV tripartite steerable states. A conclusion is given in Sec. VII.

## II. DEFINITIONS AND CRITERIA FOR $N$ -PARTITE STEERING

### A. Preliminaries

Consider an  $N$ -partite system consisting of  $N$  modes, where each mode labeled  $i = 1, \dots, N$  is described by bosonic annihilation operators  $a_i$  satisfying canonical commutation relations. We introduce the generalized quadrature operators with phase  $\varphi$  as

$$q_i(\varphi) = e^{i\varphi} a_i + e^{-i\varphi} a_i^\dagger, \quad (1)$$

noting we will later define  $x_i = a_i + a_i^\dagger$  and  $p_i = (a_i - a_i^\dagger)/i$  which implies  $\Delta x_i \Delta p_i \geq 1$ .

We use  $\Lambda = A - B$  to denote a bipartition of the  $N$  subsystems into two groups,  $A$  and  $B$ . Proving entanglement for such a bipartition would require us to falsify any separable model of the form [95]

$$\rho_{A-B} = \sum_R P_R \rho_A^{(R)} \rho_B^{(R)} \quad (2)$$

where  $\sum_R P_R = 1$  ( $P_R > 0$ ),  $\rho_{A-B}$  represents a density operator for the combined systems,  $\rho_A^{(R)}$  is an arbitrary density operator for the subsystem  $A$ , and  $\rho_B^{(R)}$  is an arbitrary density operator for the subsystem  $B$ . Here, no assumption of separability is assumed between any subsystems of either  $A$  or  $B$ . However, to confirm *steering* of system  $A$  (by  $B$ ), the falsification is to be achieved without the explicit assumption that  $\rho_B^{(R)}$  would necessarily correspond to a quantum state described by a quantum density operator [24]. To make this distinction symbolically, we denote the density operator  $\rho_A^{(R)}$  by  $\rho_{AQ}^{(R)}$ , but omit the subscript for  $\rho_B^{(R)}$ . Thus, we demonstrate the steering of  $A$  by system  $B$  if we falsify the biseparable local hidden state model symbolized as

$$\rho_{B \rightarrow A} = \sum_R P_R \rho_{AQ}^{(R)} \rho_B^{(R)}. \quad (3)$$

Where more convenient, we will also symbolize as  $\rho_{AQ-B}$ .

As rigorously formalized in [23–25], the definition of steering concerns certain probabilistic models, called LHS models, rather than density operators defined within quantum theory. Such LHS models are local hidden variable models, where extra constraints are given for the local hidden variable states describing the local system at some of the sites. Thus, to determine steering of  $A$  by  $B$ , we negate all models giving the probability distribution for joint measurements at sites  $A$  and  $B$  as

$$P(x_A, x_B | \theta, \phi) = \int \rho(\lambda) d\lambda P_Q(x_A | \lambda, \theta) P(x_B | \lambda, \phi), \quad (4)$$

where  $x_A$  and  $x_B$  are the results of measurements on each system. Here,  $\lambda$  are hidden variables,  $\rho(\lambda)$  is the corresponding

probability density, and  $\theta$  and  $\phi$  are local measurement settings at the sites of systems  $A$  and  $B$ , respectively.  $P(x_A | \lambda, \theta)$  is the probability of an outcome  $x_A$ , given the parameters  $\lambda$  and the settings  $\theta$ . For steering of  $A$ , it is required to negate the model where  $P_Q(x_A | \lambda, \theta)$  is consistent with a local quantum density operator  $\rho_A$  for system  $A$ , as denoted by the subscript  $Q$ . There is thus a constraint on the distributions for  $A$ , so that quantum uncertainty relations are satisfied. If the criteria to negate the LHS models are to be valid, one must be sure to use valid quantum devices and measurements at that site. Hence the quantum sites are referred to as *trusted* sites. Otherwise, the sites are said to be *untrusted*.

In this paper, for notational convenience we use Eq. (3), the meaning of Eqs. (3) and (4) being equivalent. We will denote the bipartition  $A - B$  where the system  $A$  is to be trusted as  $AQ - B$ . Similarly, the bipartition  $A - BQ$  denotes that  $B$  is trusted. If the systems  $A$  and  $B$  comprise more than one subsystem, or mode, then more options of trust are available. Suppose system  $A$  is just one mode labeled  $k$  and system  $B$  is two modes labeled  $l$  and  $m$ . If both subsystems  $l$  and  $m$  are trusted, but system  $k$  is not, then the bipartition is denoted  $k - (lm)Q$ . The bipartition  $k - lm$  where only the site  $l$  is trusted will be denoted  $k - (l)Qm$ .

### B. Steering in one partition

We follow van Loock and Furusawa [10] and introduce the following linear combinations of the operators for the  $N$  systems:

$$u = \sum_{i=1}^N g_i q_i(\varphi_i), \quad v = \sum_{i=1}^N h_i q_i(\chi_i). \quad (5)$$

Here,  $g_i$  and  $h_i$  are real numbers that will be optimized to reduce the variances in  $u$  and  $v$ , as we will later see.

Let us first focus on the factorized description  $\rho_{AQ}^{(R)} \rho_B^{(R)}$ . In this case the variance for the state denoted by  $R$  in the expansion Eq. (3) reads

$$\begin{aligned} (\Delta u)_R^2 &= (\Delta u_A)_{AQ}^2 + (\Delta u_B)_B^2 \\ &\geq (\Delta u_A)_{AQ}^2, \end{aligned} \quad (6)$$

where

$$u_i = \sum_{i \in I} g_i q_i(\varphi_i). \quad (7)$$

Here we use the notation that  $(\Delta u_A)_{AQ}^2$  is the variance with respect to the operators of system  $A$ , evaluated assuming the system is described as a quantum state  $\rho_A^{(R)}$ . The variance associated with a system  $B$  in a hidden variable state is constrained only by the condition  $(\Delta u_B)_B^2 \geq 0$ . Generally, we use the notation  $(\Delta x)^2$  to denote the variance of  $x$ , and  $\Delta x$  to denote the standard deviation.

Moreover, the LHS model symbolized by  $\rho_{AQ} \rho_B$  as defined by Eq. (3) must satisfy the uncertainty relation

$$(\Delta u_A)_{AQ} (\Delta v_A)_{AQ} \geq C_A, \quad (8)$$

where we define

$$C_I = \left| \sum_{i \in I} g_i h_i \sin(\varphi_i - \chi_i) \right| \quad (9)$$

and we use

$$[u_i, v_i] = 2i \sum_{i \in I} g_i h_i \sin(\varphi_i - \chi_i). \quad (10)$$

This arises from the Heisenberg uncertainty relation  $[x_j, p_j] = 2i$ .

Now we assume a LHS description of the form of Eq. (3). For a system in a mixture, the overall variance of  $u$  is [96]

$$(\Delta u)^2 \geq \sum_R P_R (\Delta u)_R^2. \quad (11)$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned} (\Delta u)^2 (\Delta v)^2 &\geq \left[ \sum_R P_R (\Delta u)_R^2 \right] \left[ \sum_R P_R (\Delta v)_R^2 \right] \\ &\geq \left\{ \sum_R P_R (\Delta u)_R (\Delta v)_R \right\}^2. \end{aligned} \quad (12)$$

Thus, using the concavity of the variance, the Cauchy-Schwarz inequality, and the above results, we find for the system described by the LHS model  $\rho_{AQ}\rho_B$  that it is always true that

$$\begin{aligned} (\Delta u)(\Delta v) &\geq \sum_R P_R (\Delta u)_R (\Delta v)_R \\ &\geq \sum_R P_R (\Delta u)_{AQ} (\Delta v)_{AQ} \\ &\geq C_A. \end{aligned} \quad (13)$$

If this condition is violated, then one observes falsification of the LHS model  $\rho_{AQ}\rho_B$ , and therefore steering of  $A$  by  $B$ .

Analogously, the model  $\rho_A\rho_{BQ}$  always implies

$$(\Delta u)(\Delta v) \geq C_B. \quad (14)$$

If this condition is violated, then one observes falsification of the LHS model  $\rho_A\rho_{BQ}$ , and therefore steering of  $B$  by  $A$ .

To summarize our results so far, we detect *one-way steering in a particular direction* of a partition  $\Lambda$  by one of the criteria above. If either one of the two conditions holds, i.e., if

$$(\Delta u)(\Delta v) \geq \max\{C_A, C_B\} \quad (15)$$

is not satisfied, we have observed *one-way steering* in the partition  $\Lambda$ . If, however, we see that both conditions do not hold, i.e., if

$$(\Delta u)(\Delta v) \geq \min\{C_A, C_B\} \quad (16)$$

is false, we have detected *two-way steering* in the partition  $\Lambda$ .

### C. Full steering inseparability

To demonstrate the full  $N$ -partite inseparability of  $N$  systems, it is necessary to prove the failure of each separable model  $\rho_{A-B}$ , for *all* the bipartitions  $\Lambda$  of the  $N$  systems [10]. Full  $N$ -partite steering inseparability is to be defined in a similar way. However, we see that because of the asymmetry in the definition of steering, there is the possibility that steering across a bipartition is in one direction only (“one-way steering”) [97,98]. This would imply that one of the LHS models

$\rho_{A \rightarrow B}$  or  $\rho_{B \rightarrow A}$  is valid, while the other can be negated. A consequence is that different definitions of multipartite steering are possible, as formalized in the following.

#### 1. Definition

We conclude that a system displays *full  $N$ -partite steering inseparability* if one may demonstrate steering for *all* bipartitions  $\Lambda = A - B$  of the  $N$  systems. Specifically, this means demonstrating, for each bipartition, that there is steering at least in one direction; i.e., for each  $\Lambda$ , either all models denoted by  $\rho_{A \rightarrow B}$ , or all models denoted by  $\rho_{B \rightarrow A}$ , or both, can be negated.

We say that we demonstrate *full  $N$ -partite steering two-way inseparability* if, for each bipartition, *all* LHS models  $\rho_{A_\Lambda \rightarrow B_\Lambda}$  and  $\rho_{B_\Lambda \rightarrow A_\Lambda}$  are negated.

#### 2. Criteria

A criterion sufficient to confirm full  $N$ -partite steering inseparability reads as the violation of

$$(\Delta u)(\Delta v) \geq \min_{\Lambda} \max\{C_A, C_B\}, \quad (17)$$

where the minimum includes all bipartitions  $\Lambda = A - B$  of the system (we make sure that  $A - B$  and  $B - A$  are considered as the same partition). Similarly, a violation of the condition

$$(\Delta u)(\Delta v) \geq \min_{\Lambda} \min\{C_A, C_B\} \quad (18)$$

implies full two-way steering inseparability. The proof follows straightforwardly from the definitions. ■

### D. Genuine multipartite steering

#### 1. Definitions

An even stronger condition is given if instead of excluding each bipartition separately, we can exclude also more general LHS models that are constructed from convex combinations of LHS models across all the *different* bipartitions. Formally, this is described in the notation of Sec. II A by a distribution

$$\rho = \sum_{\rho_\Lambda} P_\Lambda \rho_\Lambda. \quad (19)$$

Here,  $\sum_{\Lambda} P_\Lambda = 1$ ,  $P_\Lambda > 0$ , and each  $\rho_\Lambda$  describes an LHS model for at least one (both) direction(s) of the partition  $\Lambda$  in order to exclude genuine multipartite (two-way) steering. We conclude that we demonstrate genuine  $N$ -partite steering if the above model is negated.

Here, we already have two definitions, and as we discuss in Sec. II E and below, other definitions are also possible. In this paper, we will adopt the stricter of the above definitions, and refer to this as Definition 1 for genuine tripartite steering. This definition is based on the concept to ensure steering across all bipartitions, in both directions.

*Definition 1.* We say that a system displays *genuine  $N$ -partite steering* if the above model, Eq. (19), can be negated, where  $\rho_\Lambda$  denotes the LHS model in *both* directions ( $A \rightarrow B$  and  $B \rightarrow A$ ). Specifically, we negate that the correlations can be modeled by a theory that expresses the joint probability for



a set of outcomes  $\mathbf{x}_I = (x_1, \dots, x_N)$  as

$$P(\mathbf{x}_I|\mathbf{a}) = \sum_{\lambda} P_{\lambda} P(\mathbf{x}_I|\lambda, \mathbf{a}). \quad (20)$$

Here,  $x_i$  is an outcome of a measurement performed on the subsystem  $i$ , and  $\mathbf{a}$  is a set of numbers that denotes the measurement settings for each subsystem. The  $\lambda$  indexes all the LHS models symbolized by  $AQ - B$  and  $BQ - A$  for each bipartition  $\Lambda = A - B$ . Here,  $P(\mathbf{x}_I|\lambda, \mathbf{a})$  is the probability for outcomes  $\mathbf{x}_I = (x_1, \dots, x_N)$ , given the system is in the LHS state denoted by  $\lambda$  and with measurement settings  $\mathbf{a}$ .  $P_{\lambda} \geq 0$  is the probability for the LHS state  $\lambda$ , and  $\sum_{\lambda} P_{\lambda} = 1$ .

*Definition 2.* Genuine  $N$ -partite one-way steering is confirmed if the above model is negated, where for each bipartition  $\Lambda$ , we consider only one of the directions for steering, i.e., only one of  $AQ - B$  and  $BQ - A$  is included in the convex combination.

*Definition 3 (one trusted site).* We may propose as a very strict yet convincing definition of genuine  $N$ -partite steering that one negates all LHS models (and their convex combinations) where we consider that only one site can be trusted. In the tripartite case, we would seek to negate  $kQ - lm$ , and also the models  $k - (lQ)m$  and  $k - l(mQ)$  (and all convex combinations). This definition is stricter than the Definition 1, which only requires negation of  $kQ - lm$ ,  $k - (lmQ)$ . The negation of  $kQ - lm$ ,  $k - (lQ)m$  and  $k - l(mQ)$  implies negation of  $kQ - lm$ ,  $k - (lmQ)$ , and hence genuine  $N$ -partite steering by Definition 3 implies genuine  $N$ -partite steering according to Definition 1. Definition 1, that requires negation of  $k - (lmQ)$ , does not exclude that there is steering between the system  $l$  and  $m$ . This is because in this case the LHS model has two trusted sites  $l$  and  $m$ . By contrast, Definition 3 requires negation of  $k - l(mQ)$  and  $k - (lQ)m$ , which excludes steering for the combined systems  $lm$ .

## 2. Criteria

Let us first consider genuine multipartite steering (Definition 2) and use methods as above, together with Eq. (15), to obtain the condition

$$\begin{aligned} (\Delta u)(\Delta v) &\geq \sum_{\rho_{\Lambda}} P_{\Lambda} \max\{C_A^{(\Lambda)}, C_B^{(\Lambda)}\} \\ &\geq \min_{\Lambda} \max\{C_A^{(\Lambda)}, C_B^{(\Lambda)}\} \end{aligned} \quad (21)$$

where  $C_A^{(\Lambda)}$  and  $C_B^{(\Lambda)}$  are the values of  $C_A$  and  $C_B$  for the bipartition  $\Lambda$ . Analogously, we obtain the condition for genuine multipartite two-way steering (Definition 1) from Eq. (16) as

$$(\Delta u)(\Delta v) \geq \min_{\Lambda} \min\{C_A^{(\Lambda)}, C_B^{(\Lambda)}\}. \quad (22)$$

The definition used in Eq. (4) with respect to the untrusted sites  $l$  and  $m$  is along the lines introduced by Svetlichny [11] and Collins *et al.* [55], in relation to genuine tripartite nonlocality. There is no assumption of locality made between the two systems,  $l$  and  $m$ , and indeed the combined bipartite system denoted  $lm$  may be nonlocal. The Svetlichny model corresponds to Eq. (4) but without the assumption of a trusted quantum state for  $k$ , and is referred to as a *bilocal* model.

## E. Discussion of definitions

Recent analyses indicate that for consistency with operational definitions of genuine multipartite nonlocality, a weaker definition of genuine tripartite nonlocality corresponding to a stricter subset of models is preferable [50,51,56]. This definition takes into account no-signaling and time ordering between measurements made by the untrusted parties. While this is an important issue that may lead to more sensitive criteria, we do not address this in this paper. We derive criteria sufficient to negate the Svetlichny-type models [Eq. (4)], noting that such criteria will also be sufficient to rule out the stricter subset.

Other definitions may also become applicable, where we anticipate that for future applications, there is a strategy for trust that means not all LHS models will be relevant. A question relevant to secret sharing applications is whether the steering of at least one of the single systems requires *all* of the remaining  $N - 1$  parties. Following [54], this value  $N - 1$  is called the *depth of steering parties*. It can be shown that genuine  $N$ -partite steering as given by Definition 1 is a necessary condition for  $N$  systems to have a depth of steering parties of  $N - 1$ .

If we anticipate application of the steerable states to scenarios with  $N = 3$  where there will always be at least *two trusted sites among three*, then the relevant LHS models that describe separability for the system are  $(kQ) - (lQ)m$ ,  $(kQ) - l(mQ)$ , and  $k - (lmQ)$  ( $k = 1, 2, 3$ ). Since negation of  $(kQ) - lm$  implies negation of  $(kQ) - (mQ)l$  and  $(kQ) - m(lQ)$ , we see that the negation of Eq. (20) will imply negation of all these LHS models, based on two trusted sites. Hence, the criteria derived in this paper according to Definition 1 will negate this type of genuine tripartite steering. This is useful, for example, if there are two fixed trusted sites on a network. Definitions based on sites with fixed trust have been given in [39].

More generally, Definition 1 for genuine  $N$ -partite steering will negate all LHS models (and convex combinations), where  $N - 1$  of the subsystems are trusted. However, if we restrict to just *one* trusted site on a network, we would wish to negate the LHS models  $kQ - lm$ ,  $k - (lQ)m$ , and  $k - l(mQ)$ . This then requires Definition 3. The criteria derived in this paper according to Definition 1 do *not* detect this type of genuine tripartite steering.

It is also possible to consider negation of the LHS models (and convex mixtures of them) relevant to secret sharing:  $kQ - lm$ . This is where one does not trust the two collaborating (steering) parties  $l$  and  $m$ , which gives the alternative definition (Definition 2) of genuine tripartite steering, referred to in [28]. The criteria derived in this paper will negate all such models, and therefore confirm genuine tripartite steering according to this earlier definition, which we call Definition 2.

Definition 3 gives a strict definition of genuine  $N$ -partite steering. In the tripartite case, this implies to negate the model  $k - (lQ)m$ , since the bipartite system composed of systems  $l$  and  $m$  (where only  $l$  is trusted) can show steering. Definition 1 of genuine  $N$ -partite steering does *not* negate this model, and hence may not exclude that there is steering among  $N - 1$  subsystems. Criteria for genuine tripartite steering with Definition 3 will also be presented in this paper, in Sec. III D.

### F. Steering, security against an eavesdropper, and the EPR paradox

One may demonstrate the EPR paradox [22] if we are able to select two observables  $\hat{O}_B^X$  and  $\hat{O}_B^P$  of  $B$ , such that [36]

$$\mathcal{S}_{A|B} < 1. \quad (23)$$

Here  $\mathcal{S}_{A|B} = \Delta(x_A - O_B^X)\Delta(p_A - O_B^P)$  and  $x_A$  and  $p_A$  are observables for system  $A$ , such that the uncertainty relation gives  $\Delta x_A \Delta p_A \geq 1$ . The observables  $O_B^X$  and  $O_B^P$  are general, although in this paper, we will use linear combinations of quadrature phase amplitudes. The condition is also sufficient to demonstrate steering of system  $A$ , by the steering parties of a distinct system  $B$  [24,25].

Consider  $u$  and  $v$  as defined in Eq. (5), where we have three systems identified as  $k$ ,  $l$ , and  $m$ , so that  $i = 1, 2, 3$  corresponds to  $i = k, l, m$ . Let  $h_k = g_k = 1$ . Denoting the values of  $u$  and  $v$  in this case by  $u_k$  and  $v_k$ , so that

$$\begin{aligned} u_k &= x_k + h_l x_l + h_m x_m, \\ v_k &= p_k + g_l p_l + g_m p_m, \end{aligned} \quad (24)$$

then an inference variance product is defined

$$S_{k|lm} = \Delta u_k \Delta v_k. \quad (25)$$

Here we use the notation  $S$  instead of  $\mathcal{S}$ , to make clear we have specified the observables  $O_B^X$  and  $O_B^P$  to be used by the steering parties. The violation of the inequality

$$S_{k|lm} \geq 1 \quad (26)$$

then implies an EPR paradox, where the ‘‘elements of reality’’ referred to in the EPR argument relate to the system  $k$ . This constitutes a steering of system  $k$  by the systems  $l$  and  $m$ , because the local hidden state model of Eq. (4) is falsified. This condition is readily generalized to  $N$  systems.

In the setup to measure  $S_{k|lm} < 1$ , the observers at the combined systems  $l$  and  $m$  are in some way (either alone or together) making measurements at their sites, in order to predict or infer the results of measurements made by an observer of system  $k$ . An EPR steering paradox is obtained when the errors in the inference, given by  $\Delta u_k$  and  $\Delta v_k$ , have a product that goes below the value of the Heisenberg uncertainty bound for system  $k$ , because a local state with these variances in  $x_k$  and  $p_k$  is not possible according to quantum mechanics.

Full tripartite steering inseparability is a necessary condition for genuine tripartite steering and is the key to some important applications. We consider a quantum secret sharing scenario, where two observers 2 and 3 collaborate to infer the result of a measurement of the spin of a third party, 1, for the purpose of sharing a secret key [6,35,99–101]. The observers later collaborate via public channels, to confirm entanglement between the groups, thus determining if there has been intervention by an eavesdropper. Extra security is possible, if steering of the third system by the two parties 2 and 3 is confirmed [28]. This is because then there are minimal assumptions made about the steering parties, while the station of the third party 1 is kept secure [27,30].

Thus, if one is able to demonstrate that for certain states there is steering across all bipartitions, we have a system that can be used for this purpose in a symmetrical fashion among the different observers and sites. Tripartite steering

inseparability does not necessarily ensure however that the steering cannot be generated by mixing states where there is steering between just two parties (refer to [48,49] for similar discussions in relation to entanglement). The desirable feature to ensure that two observers are required to observe the steering, as opposed to one, can be confirmed for a particular bipartition, in principle, by negating steering of the third party 1 by 2 (or 3) [102].

### G. Sum inequalities

Previous papers on multipartite entanglement have considered inequalities involving the sum of the variances,  $(\Delta u)^2 + (\Delta v)^2$ , where  $u$  and  $v$  are defined by Eq. (5) [10]. Such sum criteria can be derived from the above criteria 1 and 2, using that  $x^2 + y^2 \geq 2xy$  for all real  $x$  and  $y$ . Specifically, if the criterion involving the products is of the form  $\Delta u \Delta v \geq I$  where  $I$  is an expression involving  $g_i$  and  $h_i$ , then we obtain the criterion

$$\frac{(\Delta u)^2 + (\Delta v)^2}{2} \geq \Delta u \Delta v \geq I \quad (27)$$

for the sum of the variances. We see however from this relation that the violation of sum criterion Eq. (27) will always imply violation of the product inequality  $\Delta u \Delta v \geq I$ .

## III. CRITERIA FOR CONTINUOUS-VARIABLE TRIPARTITE STEERING

### A. Single inference inequalities

For concreteness, we consider three systems and use the results of the previous section to obtain product inequalities for the certification of genuine tripartite steering. One may define the linear combination [10]

$$\begin{aligned} u &= h_1 x_1 + h_2 x_2 + h_3 x_3, \\ v &= g_1 p_1 + g_2 p_2 + g_3 p_3 \end{aligned} \quad (28)$$

where  $h_j$  and  $g_j$  ( $j = 1, 2, 3$ ) are real numbers. We note that the phase of the quadratures  $x_j$  and  $p_j$  can be adjusted independently at each site and we may also define, for example,

$$\begin{aligned} u' &= h_1 x_1 - h_2 p_2 + h_3 x_3, \\ v' &= g_1 p_1 + g_2 x_2 + g_3 p_3. \end{aligned} \quad (29)$$

Assuming the Heisenberg uncertainty relation  $\Delta x_j \Delta p_j \geq 1$  for each  $j = 1, 2, 3$ , the separability assumption of Eq. (2) written as ( $k, l, m \in \{1, 2, 3\}$  and  $k \neq l \neq m$ )

$$\rho_{lm,k} = \sum_R P_R \rho_{lm,Q}^R \rho_{k,Q}^R \quad (30)$$

implies [49]

$$\Delta u \Delta v \geq |h_k g_k| + |h_l g_l + h_m g_m|. \quad (31)$$

Similarly, based on the uncertainty relation  $\Delta(-h_2 p_2 + h_m x_m) \Delta(g_2 x_2 + g_m p_m) \geq |g_2 h_2 + g_m h_m|$ , the separability assumption implies [49]

$$\Delta u' \Delta v' \geq |h_k g_k| + |h_l g_l + h_m g_m|. \quad (32)$$

This brings us to inequalities for steering.

*Lemma 1.* Violation of

$$\Delta u \Delta v \geq |g_k h_k| \quad (33)$$

implies EPR steering of system  $k$  by the systems  $l$  and  $m$ . The violation of

$$\Delta u \Delta v \geq |g_l h_l + g_m h_m| \quad (34)$$

implies EPR steering of systems  $l$  and  $m$ , by system  $k$ . The violation of the inequality

$$\Delta u \Delta v \geq \min\{|g_k h_k|, |g_l h_l + g_m h_m|\} \quad (35)$$

thus implies two-way EPR steering across the bipartition  $A|B$  where  $A = \{k\}$  and  $B = \{l, m\}$ . Here,  $l \neq k \neq m$  and  $l, k, m \in \{1, 2, 3\}$ . In the steering of system  $k$ , the observers of the combined systems  $l$  and  $m$  infer values for the  $x_k$  and  $p_k$  of system  $k$ , in order to violate the inequality Eq. (33). In the steering of combined systems  $l$  and  $m$ , the observer at  $k$  infers values for the  $h_l x_l + h_m x_m$  and  $g_l p_l + g_m p_m$  of the combined systems  $l$  and  $m$ , in order to violate the inequality Eq. (34). The two-way steering can be established by selecting different values of  $g_i$  and  $h_i$  for *each* direction of steering. However, here, we are interested to rule out all bipartitions with a *single* inequality. In this way, one can immediately also rule out the mixtures associated with different bipartitions and LHS models, and hence deduce genuine tripartite steering. We note the same result will apply to  $u'$  and  $v'$  defined by Eq. (29).

*Proof.* The proof of the lemma follows from Eq. (18) taking  $C_A = |g_k h_k|$  and  $C_B = |g_l h_l + g_m h_m|$ . ■

Now we arrive at conditions sufficient for genuine tripartite steering, using Definition 1 of Sec. IID.

*Criterion 1.* Violation of the inequality

$$S_3 \equiv \Delta u \Delta v \geq \min\{|g_1 h_1|, |g_2 h_2 + g_3 h_3|, |g_2 h_2|, |g_1 h_1 + g_3 h_3|, |g_3 h_3|, |g_1 h_1 + g_2 h_2|\} \quad (36)$$

is sufficient to confirm full two-way tripartite steering inseparability and also to confirm genuine tripartite steering. An identical result holds for  $\Delta u' \Delta v'$ . The proof follows from Lemma 1 and Eq. (22). ■

Letting  $g_1 = h_1 = 1$ ,  $g_2 = -h_2 = 1/\sqrt{2}$ , and  $g_3 = -h_3 = 1/\sqrt{2}$ , Criterion 1 implies that observation of

$$S_3 = \Delta \left[ x_1 - \frac{(x_2 + x_3)}{\sqrt{2}} \right] \Delta \left[ p_1 + \frac{(p_2 + p_3)}{\sqrt{2}} \right] < 0.5 \quad (37)$$

is sufficient to confirm genuine tripartite steering. This inequality has been used previously to certify genuine tripartite steering, in the work of [28,38,49]. The definition of genuine tripartite steering in those works however was based on Definition 2 (refer to Sec. IID). The results of this paper show that the inequality Eq. (37) [and the associated criterion for the sum of the variances, see Eq. (27)] also implies stronger genuine tripartite steering, defined according to Definition 1 (Sec. IID). Generally, where  $|g_l h_l + g_m h_m| \geq |g_k h_k|$ , Criterion 1 reduces to  $\Delta u \Delta v \geq \min\{|g_1 h_1|, |g_2 h_2|, |g_3 h_3|\}$ , which equates to that derived using Definition 2.

*Criterion 2.* Violation of the inequality

$$S_3 \equiv \Delta u \Delta v \geq \min\{\max\{|g_1 h_1|, |g_2 h_2 + g_3 h_3|\}, \max\{|g_2 h_2|, |g_1 h_1 + g_3 h_3|\}, \max\{|g_3 h_3|, |g_1 h_1 + g_2 h_2|\}\} \quad (38)$$

is sufficient to confirm full tripartite steering inseparability. The proof is given as for Eq. (21). ■

## B. Average-variance inequalities

The above approach uses a *single* inequality to confirm  $N$ -partite steering, and hence there is no optimization of the gains  $g_i$  and  $h_i$  for each bipartition. Another approach introduced in [28] is to take averages over the variances associated with each bipartition, thus allowing one to individually optimize gains for each bipartition. The approach also allows one to see that if it is possible to show sufficient steering of *each* system  $k = 1, 2, 3$ , so that each of the EPR variances  $S_{k|lm}$  becomes sufficiently small, then genuine tripartite steering must follow.

This brings us to the following criterion to certify genuine tripartite steering.

*Criterion 3.* We define

$$S_{k|lm} \equiv \Delta(x_k + h_{k,l} x_l + h_{k,m} x_m) \Delta(p_k + g_{k,l} p_l + g_{k,m} p_m) \quad (39)$$

where  $h_{k,l}$ ,  $h_{k,m}$ ,  $g_{k,l}$ , and  $g_{k,m}$  are real constants. Here,  $l \neq k \neq m$  and  $l, k, m \in \{1, 2, 3\}$ . Violation of the following inequality will certify genuine tripartite steering among the three systems:

$$S_{1|23} + S_{2|13} + S_{3|12} \geq \min(1, |g_{2,1} h_{2,1} + g_{2,3} h_{2,3}|, |g_{3,2} h_{3,2} + g_{3,1} h_{3,1}|, |g_{1,2} h_{1,2} + g_{1,3} h_{1,3}|). \quad (40)$$

Provided  $|g_{k,l} h_{k,l} + g_{k,m} h_{k,m}| \geq 1$ , for each  $k$  where  $k \neq l \neq m$ , this simplifies to the inequality

$$S_{1|23} + S_{2|13} + S_{3|12} \geq 1. \quad (41)$$

*Proof.* We consider that the system is described by mixtures of the type

$$\rho_{\text{mix}} = P'_1 \rho_{1Q-23} + P''_1 \rho_{1-(23Q)} + P'_2 \rho_{2Q-13} + P''_2 \rho_{2-(13Q)} + P'_3 \rho_{3Q-12} + P''_3 \rho_{3-(12Q)}, \quad (42)$$

where we use the abbreviated notation for LHS mixtures, given in Sec. II. Here,  $P'_k$ ,  $P''_k$ , and  $P_k$  are probabilities, such that  $P_k = P'_k + P''_k$  and  $P_1 + P_2 + P_3 = 1$ . Using Definition 1 of Sec. IID for genuine tripartite steering, we wish to negate all such models  $\rho_{\text{mix}}$ . First, we note that if  $S_{k|lm} < 1$ , we certify steering of system  $k$  (Lemma 1). If  $S_{k|lm} < |g_{k,l} h_{k,l} + g_{k,m} h_{k,m}|$ , then we certify steering of the combined systems  $lm$ . This is because the LHS models (denoted  $\rho_{k-(lmQ)}$  and  $\rho_{kQ-lm}$ ) associated with that bipartition  $k - lm$  imply  $S_{k|lm} \geq 1$  and  $S_{lm|k} \geq |g_l h_l + g_m h_m|$ , respectively. More generally, each LHS model associated with the bipartition  $k - lm$  implies

$$S_{k|lm} \geq \min\{1, |g_{k,l} h_{k,l} + g_{k,m} h_{k,m}|\}. \quad (43)$$

This is also true if for any particular  $k$  we define

$$S_{k|lm} \equiv \Delta(x_k - h_{k,l} p_l + h_{k,m} x_m) \Delta(p_k + g_{k,l} x_l + g_{k,m} p_m) \quad (44)$$

or as

$$S_{k|lm} \equiv \Delta(x_k - g_{k,l}p_l + g_{k,m}p_m)\Delta(-p_k + h_{k,l}x_l + h_{k,m}x_m) \quad (45)$$

as we see from the results of Eqs. (29) and (32). Abbreviating the notation to define  $S_k \equiv S_{k|lm}$ , we assume the system is described by (42) and we use Eq. (12). We see that the value of  $S_1$  would be taken as the average over the three different bipartitions  $k - lm$ , and similarly for  $S_2$  and  $S_3$ , which leads to the inequality

$$\begin{aligned} S_1 + S_2 + S_3 \geq & P_1 S_{1,1} + P_2 S_{1,2} + P_3 S_{1,3} \\ & + P_1 S_{2,1} + P_2 S_{2,2} + P_3 S_{2,3} \\ & + P_1 S_{3,1} + P_2 S_{3,2} + P_3 S_{3,3}. \end{aligned} \quad (46)$$

Here, we denote  $S_{k',k}$  as the value of  $S_{k'}$  for the bipartition  $k - lm$ . The system will be in a given bipartition with a fixed probability,  $P_i$ . Recognizing each  $S_{k',k}$  to be positive, we can therefore write

$$S_1 + S_2 + S_3 \geq P_1 S_{1,1} + P_2 S_{2,2} + P_3 S_{3,3}. \quad (47)$$

If the system is in the bipartition labeled 1, i.e., bipartition 1 - 23, then  $S_{1,1} \geq \min(1, |g_{1,2}h_{1,2} + g_{1,3}h_{1,3}|)$ . Applying the conditions on the  $S_{k,k}$  from Lemma 1, this becomes

$$\begin{aligned} S_1 + S_2 + S_3 \geq & P_1 \min(1, |g_{1,2}h_{1,2} + g_{1,3}h_{1,3}|) \\ & + P_2 \min(1, |g_{2,1}h_{2,1} + g_{2,3}h_{2,3}|) \\ & + P_3 \min(1, |g_{3,2}h_{3,2} + g_{3,1}h_{3,1}|), \end{aligned} \quad (48)$$

which leads to

$$\begin{aligned} S_1 + S_2 + S_3 \geq & \min(1, |g_{2,1}h_{2,1} + g_{2,3}h_{2,3}|, \\ & |g_{3,2}h_{3,2} + g_{3,1}h_{3,1}|, |g_{1,2}h_{1,2} + g_{1,3}h_{1,3}|). \end{aligned} \quad (49)$$

We note that provided  $|g_{k,l}h_{k,l} + g_{k,m}h_{k,m}| \geq 1$ , the LHS bipartition  $k - lm$  implies  $S_k \geq 1$ . Here we use that for the bipartition  $k - lm$ , for both LHS models  $S_{k,k} \geq 1$ , provided  $|g_{k,l}h_{k,l} + g_{k,m}h_{k,m}| \geq 1$ . ■

Criterion 3 justifies the use of

$$S_1 + S_2 + S_3 \geq 1 \quad (50)$$

where we take  $|g_{k,l}h_{k,l} + g_{k,m}h_{k,m}| \geq 1$  for each  $k$ . The condition  $S_1 + S_2 + S_3 < 1$  was stated as sufficient to detect genuine tripartite steering in [28], based on Definition 2 of genuine tripartite steering. We now see that this condition also holds as a criterion, according to the stricter Definition 1 used in this paper.

We note that the phases of  $u$  and  $v$  can be adjusted according to Eq. (29) and Eq. (44) or Eq. (45). To be explicit, in Sec. V A 4 we will define  $S_{1|23}$  and  $S_{3|12}$  according to Eq. (44) and  $S_{2|13}$  according to Eq. (45). Criterion 3 also applies for this choice of phase. It is also possible to extend Criterion 3, by deriving an inequality that keeps more terms at the step Eq. (46) in the derivation. However, this gave no advantage for the states examined in this paper (refer to Supplemental Material [103]).

### C. Van Loock–Furusawa-type steering criteria

The work of van Loock and Furusawa motivated experimental measurements of sums of variances, which enabled detection of genuine multipartite entanglement. The application to steering of these inequalities was considered by Teh and Reid [49], using Definition 2 for genuine multipartite steering. We are thus motivated to examine these criteria using the stricter Definition 1.

In their paper [10], van Loock and Furusawa consider quantities

$$\begin{aligned} B_I & \equiv [\Delta(x_1 - x_2)]^2 + [\Delta(p_1 + p_2 + g_3 p_3)]^2, \\ B_{II} & \equiv [\Delta(x_2 - x_3)]^2 + [\Delta(g_1 p_1 + p_2 + p_3)]^2, \\ B_{III} & \equiv [\Delta(x_1 - x_3)]^2 + [\Delta(p_1 + g_2 p_2 + p_3)]^2 \end{aligned} \quad (51)$$

defined for arbitrary real parameters  $g_1$ ,  $g_2$ , and  $g_3$ . They consider the biseparable bipartitions denoted  $\rho_{kQ-(lm)Q}$  in our notation, for which the following uncertainty relation holds:

$$\Delta u \Delta v \geq |h_k g_k| + |h_l g_l + h_m g_m|. \quad (52)$$

They then use Eq. (52), to show that inequality  $B_I \geq 4$  is implied by both the biseparable states  $\rho_{(13)Q,2Q}$  and  $\rho_{(23Q),1Q}$ , which assume separability between systems 1 and 2. Similarly, a second inequality  $B_{II} \geq 4$  is implied by the biseparable states  $\rho_{(13Q),2Q}$  and  $\rho_{(12Q),3Q}$ , while a third inequality  $B_{III} \geq 4$  follows from biseparable states  $\rho_{(12Q),3Q}$  and  $\rho_{(23Q),1Q}$ . Van Loock and Furusawa derived a condition for full tripartite inseparability, based on the violation of two of the above inequalities.

Here, we will prove a similar result for steering. We follow [49] and define the product inequalities that were used for multipartite entanglement:

$$\begin{aligned} S_I & \equiv \Delta(x_1 - x_2)\Delta(p_1 + p_2 + g_3 p_3), \\ S_{II} & \equiv \Delta(x_2 - x_3)\Delta(g_1 p_1 + p_2 + p_3), \\ S_{III} & \equiv \Delta(x_1 - x_3)\Delta(p_1 + g_2 p_2 + p_3). \end{aligned} \quad (53)$$

First, following the proofs of [10,49], we note that the inequality  $S_I \geq 2$  is implied by the biseparable states associated with bipartitions 13 - 2 and 23 - 1. Similarly, the second inequality  $S_{II} \geq 2$  is implied by the biseparable states associated with bipartitions 13 - 2 and 12 - 3, and the third inequality  $S_{III} \geq 2$  is implied by separable states over bipartitions 12 - 3 and 23 - 1. This means that the violation of any two of the inequalities  $S_I \geq 2$ ,  $S_{II} \geq 2$ , and  $S_{III} \geq 2$  is sufficient to prove full tripartite inseparability.

We now extend the result for tripartite steering.

*Criterion 4.* The violation of any two of the inequalities

$$S_I \geq 1, S_{II} \geq 1, S_{III} \geq 1 \quad (54)$$

implies full tripartite two-way steering inseparability.

*Proof.* Here, we use the results with  $h_3 = 0$ ,  $h_1 = g_1 = g_2 = 1$ , and  $h_2 = -1$  to show that the LHS models  $\rho_{1Q}\rho_{23}$  and  $\rho_{1}\rho_{23Q}$  imply  $S_I \geq 1$ , and similarly the LHS models  $\rho_{2Q}\rho_{13}$  and  $\rho_{2}\rho_{13,Q}$  imply

$$S_I \geq 1. \quad (55)$$

This follows from Lemma 1: Considering bipartition  $\{k - lm\}$  where  $k = 1$ ,  $l = 2$ , and  $m = 3$ , we find that violation of the inequality  $\Delta u \Delta v \geq 1$  implies two-way steering across the



bipartition  $\{1 - 23\}$ , and similarly across bipartition  $\{2 - 13\}$ . Similarly, we see that LHS models  $\rho_{3Q}\rho_{12}$  and  $\rho_{3\rho_{12,Q}}$  (and the LHS models  $\rho_{2Q}\rho_{13}$  and  $\rho_{2\rho_{13,Q}}$ ) imply

$$S_{II} \geq 1. \quad (56)$$

The LHS models  $\rho_{3Q}\rho_{12}$  and  $\rho_{3\rho_{12,Q}}$  (and the LHS models  $\rho_{1Q}\rho_{23}$  and  $\rho_{1\rho_{23,Q}}$ ) imply

$$S_{III} \geq 1. \quad (57)$$

Hence, if two inequalities are violated, all three bipartitions show two-way steering, and the result follows. ■

*Criterion 5.* We confirm genuine tripartite steering, if the inequality

$$S_I + S_{II} + S_{III} \geq 2 \quad (58)$$

is violated.

*Proof.* For  $N = 3$  parties, there are three bipartitions, and three corresponding biseparable states  $\rho_{1,23}$ ,  $\rho_{2,13}$ , and  $\rho_{3,12}$  that we index by  $k = 1, 2, 3$ , respectively. Consider any mixture of the form Eq. (42). We consider the three types of bipartitions, grouping the two associated LHS models together. We use that for mixtures  $\rho_{\text{mix}} = \sum P_R \rho^{(R)}$ , the result Eq. (12) follows, where here the subscript denotes that the averages are over the state  $\rho^{(R)}$ . We can then write

$$\begin{aligned} S_I &\geq P_1 S_{I,1} + P_2 S_{I,2} + P_3 S_{I,3} \\ &\geq P_1 S_{I,1} + P_2 S_{I,2} \geq P_1 + P_2 \end{aligned} \quad (59)$$

where here we define  $S_{I,k}$  as the expected value of  $S_I$  for the bipartition  $\rho_{k,lm}$  with probability  $P_k$ . This uses that we know from the proof of Criterion 4 that both of the LHS models  $\rho_{1Q}\rho_{23}$  and  $\rho_{1\rho_{23Q}}$  with bipartition  $k = 1$  will satisfy  $S_I \geq 1$ , and also  $S_{III} \geq 1$ . Similarly, both LHS models  $\rho_{2Q}\rho_{13}$  and  $\rho_{2\rho_{13,Q}}$  satisfy  $S_I \geq 1$  and  $S_{II} \geq 1$ , and both LHS models  $\rho_{3Q}\rho_{12}$  and  $\rho_{3\rho_{12,Q}}$  satisfy  $S_{II} \geq 1$  and  $S_{III} \geq 1$ . Hence, for any mixture  $S_I \geq P_1 + P_2$ . Similarly,  $S_{II} \geq P_2 + P_3$  and  $S_{III} \geq P_1 + P_3$ . Then we see that since  $\sum_{k=1}^3 P_k = 1$ , for any mixture it must be true that  $S_I + S_{II} + S_{III} \geq 2$ . ■

Immediately, from the result Eq. (27), one arrives at the corresponding criteria for the original van Loock–Furusawa inequalities, involving summations. We choose to write these explicitly, in the event this may be useful, because the quantities have been experimentally reported.

*Criterion 4b.* The violation of any two of the inequalities

$$B_I \geq 2, \quad B_{II} \geq 2, \quad B_{III} \geq 2 \quad (60)$$

implies full tripartite two-way steering. The proof follows from Criterion 4 and the result Eq. (27).

*Criterion 5b.* We confirm genuine tripartite steering, if the inequality

$$B_I + B_{II} + B_{III} \geq 4 \quad (61)$$

is violated, where  $B_I \geq 2$ ,  $B_{II} \geq 2$  and  $B_{III} \geq 2$  are the van Loock–Furusawa steering inequalities, Eq. (60). The proof follows from Criterion 5 and the result Eq. (27).

This result confirms the validity of the criteria given as (35) and Result (4) stated in [49] and [28], respectively (based on Definition 2), for Definition 1, used in this paper.

We also note criteria involving just two of the van Loock–Furusawa inequalities. These inequalities are an adaption of the similar criterion derived for entanglement in [48,49].

*Criterion 5c.* We can confirm genuine tripartite steering, if with  $g_1 = g_2 = g_3 = 1$  the inequality

$$B_I + S_{II} < 1 \quad (62)$$

is satisfied (or  $S_I + S_{III} < 1$ , or  $S_{II} + S_{III} < 1$ ).

*Proof.* This follows from the proof of Criterion 5.

*Criterion 6c.* We can confirm genuine tripartite steering, if with  $g_1 = g_2 = g_3 = 1$  the inequality

$$B_I + B_{II} < 2 \quad (63)$$

is satisfied (or  $B_I + B_{III} < 2$ , or  $B_{II} + B_{III} < 2$ ).

*Proof.* This follows from Criterion 5c.

#### D. Strict genuine tripartite steering: Definition 3

We now give criteria for the stricter definition of genuine tripartite steering, discussed in Sec. II E. This definition allows the inference of genuine tripartite steering with only one trusted site, and negates all LHS models that allow steering among two parties.

*Criterion 7.* We confirm genuine tripartite steering by Definition 3, if either of the following inequalities is violated with  $g_i = 1$  ( $i = 1, 2, 3$ ):

$$S_I + S_{II} + S_{III} \geq 2 \quad (64)$$

or  $B_I + B_{II} + B_{III} \geq 4$ . Similarly, we conclude genuine tripartite steering by Definition 3 if any one of  $S_I + S_{II} \geq 1$ ,  $S_{II} + S_{III} \geq 1$ , and  $S_I + S_{III} \geq 1$  (or  $B_I + B_{II} \geq 2$ ,  $B_{II} + B_{III} \geq 2$ , and  $B_I + B_{III} \geq 2$ ) is violated, with  $g_i = 1$  ( $i = 1, 2, 3$ ).

*Proof.* For  $N = 3$  parties, there are three bipartitions, and three corresponding biseparable states  $\rho_{1,23}$ ,  $\rho_{2,13}$ , and  $\rho_{3,12}$  that we index by  $k = 1, 2, 3$ , respectively. For Definition 3, the relevant LHS models for  $\rho_{1,23}$  ( $k = 1$ ) are  $\rho_{1Q}\rho_{23}$ ,  $\rho_{1\rho_{2(3Q)}}$  and  $\rho_{1\rho_{(2Q)3}}$ , and similarly for each  $k = 2$  and  $3$ . As for the proof of Criterion 5, we consider any convex mixture of these LHS states  $\rho_{\text{mix}} = \sum_k P_k \rho^{(k)}$  where  $P_k \rho^{(k)} = P_{k,k} \rho_{kQ} \rho_{lm} + P_{k,l} \rho_k \rho_{(lQ)m} + P_{k,m} \rho_k \rho_{l(mQ)}$  and  $P_{k,k} + P_{k,l} + P_{k,m} = P_k$ ,  $\sum_k P_k = 1$ , with each  $P_{k,i} \geq 0$ . From Eq. (31), the uncertainty relation for  $\rho_{kQ} \rho_{(lm)Q}$  is

$$\Delta u \Delta v \geq |h_k g_k| + |h_l g_l + h_m g_m|. \quad (65)$$

For  $\rho_{1Q}\rho_{23}$ , using  $h_1 = 1$ ,  $h_2 = -1$ ,  $h_3 = 0$ , and  $g_1 = g_2 = g_3 = 1$ , the relation becomes

$$\Delta(x_1 - x_2) \Delta(p_1 + p_2 + p_3) \geq |h_1 g_1| = 1, \quad (66)$$

implying  $S_I \geq 1$ . For  $\rho_{1\rho_{(2Q)3}}$ , using  $h_1 = 1$ ,  $h_2 = -1$ ,  $h_3 = 0$ , and  $g_1 = g_2 = g_3 = 1$ , we see that

$$\Delta(x_1 - x_2) \Delta(p_1 + p_2 + p_3) \geq |h_2 g_2| = 1, \quad (67)$$

i.e.,  $S_I \geq 1$ . However, for  $\rho_{1\rho_{2(3Q)}}$ , using  $h_1 = 1$ ,  $h_2 = 1$ ,  $h_3 = 0$ , and  $g_1 = g_2 = g_3 = 1$ ,

$$\Delta(x_1 - x_2) \Delta(p_1 + p_2 + p_3) \geq |h_3 g_3| = 0, \quad (68)$$

i.e.,  $S_I \geq 0$ . Proceeding similarly, for  $\rho_{1Q}\rho_{23}$ , using  $h_1 = 1$ ,  $h_3 = -1$ ,  $h_2 = 0$ , and  $g_1 = g_2 = g_3 = 1$ ,

$$\Delta(x_1 - x_3) \Delta(p_1 + p_2 + p_3) \geq |h_1 g_1|, \quad (69)$$

i.e.,  $S_{\text{III}} \geq 1$ . Continuing, we find for  $\rho_1 \rho_{(2Q)3}$  that  $S_{\text{III}} \geq 0$ , and for  $\rho_1 \rho_{2(3Q)}$  that  $S_{\text{III}} \geq 1$ . Proceeding similarly, for  $\rho_1 \rho_{23}$ , using  $h_2 = 1$ ,  $h_3 = -1$ ,  $h_1 = 0$ , and  $g_1 = g_2 = g_3 = 1$ , we see from

$$\Delta(x_2 - x_3)\Delta(p_1 + p_2 + p_3) \geq |h_1 g_1| \quad (70)$$

that  $S_{\text{II}} \geq 0$ . For  $\rho_1 \rho_{(2Q)3}$ , we find  $S_{\text{II}} \geq 1$ , and for  $\rho_1 \rho_{2(3Q)}$ , we see that  $S_{\text{III}} \geq 1$ . In summary, for  $\rho_1 \rho_{23}$  and  $\rho_1 \rho_{(2Q)3}$ ,  $S_{\text{I}} \geq 1$ ; for  $\rho_1 \rho_{23}$  and  $\rho_1 \rho_{2(3Q)}$ ,  $S_{\text{III}} \geq 1$ ; for  $\rho_1 \rho_{(2Q)3}$  and  $\rho_1 \rho_{2(3Q)}$ ,  $S_{\text{II}} \geq 1$ .

Similarly, for  $\rho_2 \rho_{13}$  and  $\rho_2 \rho_{(1Q)3}$ ,  $S_{\text{I}} \geq 1$ ; for  $\rho_2 \rho_{13}$  and  $\rho_2 \rho_{1(3Q)}$ ,  $S_{\text{II}} \geq 1$ ; for  $\rho_2 \rho_{(1Q)3}$  and  $\rho_2 \rho_{1(3Q)}$ ,  $S_{\text{III}} \geq 1$ . For  $\rho_3 \rho_{21}$  and  $\rho_3 \rho_{(2Q)1}$ ,  $S_{\text{II}} \geq 1$ ; for  $\rho_3 \rho_{21}$  and  $\rho_3 \rho_{2(1Q)}$ ,  $S_{\text{III}} \geq 1$ ; for  $\rho_3 \rho_{(2Q)1}$  and  $\rho_3 \rho_{2(1Q)}$ ,  $S_{\text{I}} \geq 1$ . Hence, for  $\rho_{\text{mix}}$ , using that for mixtures the variance is given according to Eq. (11), we find

$$\begin{aligned} S_{\text{I}} &\geq P_{1,1} + P_{1,2} + P_{2,1} + P_{2,2} + P_{3,2} + P_{3,1}, \\ S_{\text{II}} &\geq P_{1,2} + P_{1,3} + P_{2,3} + P_{2,2} + P_{3,3} + P_{3,2}, \\ S_{\text{III}} &\geq P_{1,1} + P_{1,3} + P_{2,3} + P_{2,1} + P_{3,3} + P_{3,1}. \end{aligned} \quad (71)$$

Hence,

$$\begin{aligned} S_{\text{I}} + S_{\text{II}} + S_{\text{III}} &\geq 2P_{1,1} + 2P_{1,2} + 2P_{1,3} \\ &\quad + 2P_{2,2} + 2P_{2,1} + 2P_{2,3} \\ &\quad + 2P_{3,3} + 2P_{3,2} + 2P_{3,1} \geq 2. \end{aligned} \quad (72)$$

The proof of the second inequality follows from Eq. (27). We also see that

$$\begin{aligned} S_{\text{I}} + S_{\text{II}} &\geq P_{1,1} + 2P_{1,2} + P_{1,3} + 2P_{2,2} + P_{2,1} + P_{2,3} \\ &\quad + P_{3,3} + 2P_{3,2} + P_{3,1} \geq 1, \end{aligned} \quad (73)$$

and similarly,  $S_{\text{I}} + S_{\text{III}} \geq 1$  and  $S_{\text{II}} + S_{\text{III}} \geq 1$ . The results for the sum inequalities  $B_{\text{I}}$ ,  $B_{\text{II}}$ , and  $B_{\text{III}}$  follow from Eq. (27). ■

#### IV. GENUINE $N$ -PARTITE STEERABLE STATES

We now give an analysis of how to generate and detect genuine  $N$ -partite steering. We consider three types of states that we refer to as the CV EPR, CV split-squeezed, and CV GHZ states. These are generated by a network of  $N - 1$  beam splitters, using two, one, and  $N$  squeezed-vacuum input states, respectively.

##### A. CV EPR state

We begin with the CV EPR state. This state has been used to generate genuine tripartite entanglement [60], and can be generated following the schemes suggested in [10,49]. The setup for  $N = 3$  is illustrated in Fig. 1. We will show that the schemes also produce a genuinely  $N$ -partite steerable state that can be detected using Criterion 1, derived in the previous section.

Two orthogonally squeezed inputs are placed through a 50 : 50 beam splitter BS1, to produce EPR entangled fields with boson operators  $a_1$  and  $a'_2$  at the two outputs. We show this by writing the output fields as

$$\begin{aligned} a_1 &= \sqrt{R_1} a_1^{(\text{in})} + \sqrt{T_1} a_2^{(\text{in})}, \\ a'_2 &= \sqrt{T_1} a_1^{(\text{in})} - \sqrt{R_1} a_2^{(\text{in})} \end{aligned} \quad (74)$$

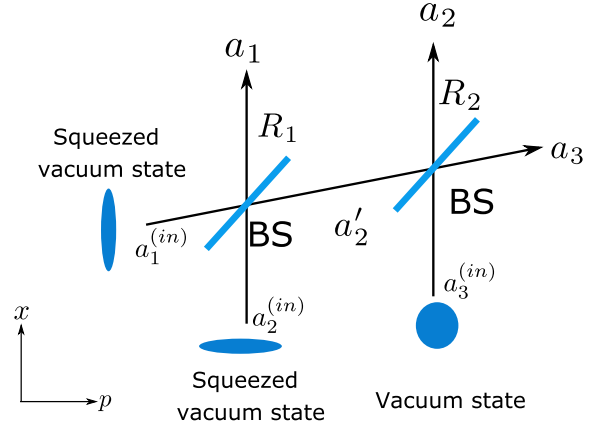


FIG. 1. Generation of the tripartite-entangled CV EPR state. The configuration uses two squeezed-vacuum inputs  $a_1^{(\text{in})}$  and  $a_2^{(\text{in})}$ , and two beam splitters (BS) with reflectivities  $R_1 = R_2 = 1/2$ . The  $x_i$  and  $p_i$  are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes, denoted by  $a_i$  ( $i = 1, 2, 3$ ).

where  $R_1 + T_1 = 1$ ,  $R_1$  being the reflectivity of the beam splitter, and  $a_1^{(\text{in})}$  and  $a_2^{(\text{in})}$  are the inputs to the beam splitter (Fig. 1). This gives  $a_1^{(\text{in})} = \sqrt{R_1} a_1 + \sqrt{T_1} a'_2$  and  $a_2^{(\text{in})} = \sqrt{T_1} a_1 + \sqrt{R_1} a'_2$ . Thus,  $\sqrt{T_1} x_1 - \sqrt{R_1} x'_2 = x_2^{(\text{in})}$  and  $\sqrt{R_1} p_1 + \sqrt{T_1} p'_2 = p_1^{(\text{in})}$ , where  $x_j$ ,  $x'_j$  and  $p_j$ ,  $p'_j$  are the quadratures of the fields  $a_j$  and  $a'_j$  respectively, and  $x_j^{(\text{in})}$  and  $p_j^{(\text{in})}$  are the quadratures of the field  $a_j^{(\text{in})}$ . If the input  $a_2^{(\text{in})}$  is a squeezed input with  $\Delta^2 x_2^{(\text{in})} = e^{-2r}$ , then

$$\Delta^2(\sqrt{T_1} x_1 - \sqrt{R_1} x'_2) = \Delta^2 x_2^{(\text{in})} = e^{-2r} \quad (75)$$

where we use the notation  $\Delta^2 x \equiv (\Delta x)^2$  to simplify the use of brackets. If the input  $a_1^{(\text{in})}$  is squeezed in  $p$ , so that  $\Delta^2 p_1^{(\text{in})} = e^{-2r}$ , then

$$\Delta^2(\sqrt{R_1} p_1 + \sqrt{T_1} p'_2) = e^{-2r}. \quad (76)$$

Choosing  $R_1 = \frac{1}{2}$ , these fields satisfy

$$\begin{aligned} \Delta^2(x_1 - x'_2) &= 2e^{-2r}, \\ \Delta^2(p_1 + p'_2) &= 2e^{-2r} \end{aligned} \quad (77)$$

where  $x_i$  and  $p_i$  are the quadratures associated with each mode  $a_i$ . Entanglement is detected when [104]

$$\Delta(x_1 - x'_1)\Delta(p_1 + p'_1) < 2, \quad (78)$$

implying the fields to be entangled for all  $r > 0$ .

More generally, to investigate the EPR steering correlations as in [36], we find

$$\begin{aligned} \Delta^2(x_1 - g_{x,s} x'_2) &= g_{x,s}^2 (T_1 e^{2r} + R_1 e^{-2r}) + R_1 e^{2r} + T_1 e^{-2r} \\ &\quad - 2\sqrt{R_1 T_1} g_{x,s} (e^{2r} - e^{-2r}) \end{aligned}$$

which is minimum for

$$g_{x,s} = \frac{\sqrt{R_1 T_1} (e^{2r} - e^{-2r})}{(T_1 e^{2r} + R_1 e^{-2r})}. \quad (79)$$

Similarly,

$$\Delta^2(p_1 + g_{p,s} p'_2) = g_{p,s}^2 (T_1 e^{-2r} + R_1 e^{2r}) + R_1 e^{-2r} + T_1 e^{2r}$$

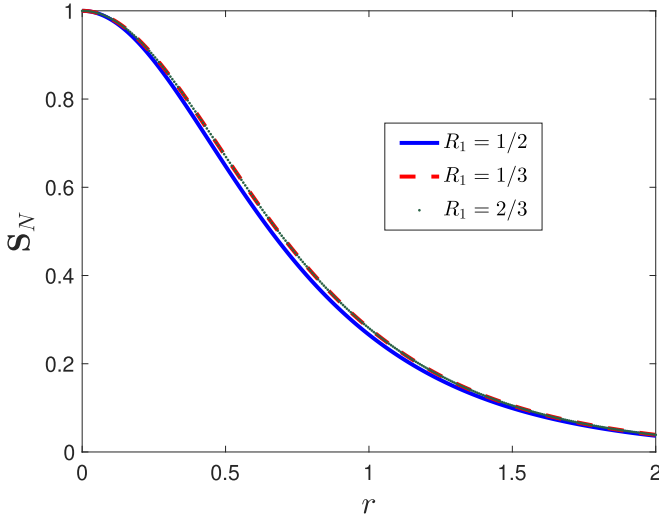


FIG. 2. Genuine  $N$ -partite steering for the CV EPR state: The plots shows the optimal steering parameter  $S_N$  defined in Eqs. (82) and (92) as a function of the squeezing parameter  $r$  and for different reflectivities  $R_1$ . Steering of system 1 is obtained when  $S_N \equiv S_{1|2,\dots,N} < 1$ . It is evident that genuine  $N$ -partite steering is obtained for sufficiently large  $r$ , implying mutual steering between all subsystems, as given by Criterion 1b and Eq. (95).

$$-2\sqrt{R_1 T_1} g_{p,s} (e^{2r} - e^{-2r})$$

which is minimum for

$$g_{p,s} = \frac{\sqrt{R_1 T_1} (e^{2r} - e^{-2r})}{(R_1 e^{2r} + T_1 e^{-2r})}. \quad (80)$$

This gives

$$\begin{aligned} \Delta^2(x_1 - g_{x,s} x'_2) &= \frac{1}{(1 - R_1)e^{2r} + R_1 e^{-2r}}, \\ \Delta^2(p_1 + g_{p,s} p'_2) &= \frac{1}{R_1 e^{2r} + (1 - R_1)e^{-2r}}. \end{aligned} \quad (81)$$

For all values of beam splitter reflectivity  $R_1$ , there is EPR steering whenever  $r > 0$ , and perfect EPR correlation as the variances become zero, as  $r \rightarrow \infty$ . The optimal EPR steering product as defined by Eq. (25) for the two output modes is

$$S_2 \equiv S_{1|2} = \Delta u_1 \Delta v_1 = \Delta(x_1 - g_{x,s} x'_2) \Delta(p_1 + g_{p,s} p'_2) \quad (82)$$

where here for Eq. (25) we identify  $h_2 = -g_{x,s}$  and  $g_2 = g_{p,s}$ . We plot  $S_2$  in Fig. 2 for the optimal choice of gains  $g_{x,s}$  and  $g_{p,s}$ , for various values of reflectivity  $R_1$ . One may prove by differentiation that the optimal choice to minimize  $S_2$  is  $R_1 = 1/2$ . The optimal steering product then becomes

$$S_2 \equiv S_{1|2} = \Delta u_1 \Delta v_1 = \frac{1}{\cosh 2r} \quad (83)$$

for optimal gains given by

$$g_{x,s} = g_{p,s} = \tanh 2r. \quad (84)$$

In Fig. 3, we plot the optimal gains versus  $r$ .

As shown in [10,49], to produce tripartite entangled fields, the field 2 is then split using a beam splitter labeled BS2, with a vacuum input  $a_3^{(in)}$ , to give two new outputs  $a_2 = (\sqrt{R_2} a'_2 + \sqrt{T_2} a_3^{(in)})$  and  $a_3 = (\sqrt{T_2} a'_2 - \sqrt{R_2} a_3^{(in)})$ , where  $R_2 + T_2 = 1$ .

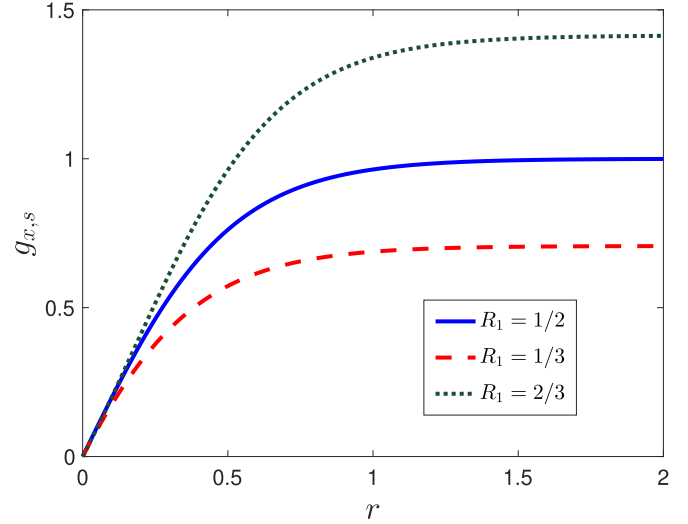


FIG. 3. Optimal gains: The gains  $g_{x,s}$  required for the steering in Fig. 2 as given by Eq. (79), as a function of the squeezing parameter  $r$  for different reflectivities  $R_1$ . The plots for  $g_{p,s}$  as in Eq. (80) are obtained on replacing  $R_1$  with  $T_1$ .

The setup for  $N = 3$  is shown in Fig. 1. We see that  $a'_2 = \sqrt{R_2} a_2 + \sqrt{T_2} a_3$ . Thus

$$\begin{aligned} x'_2 &= \sqrt{R_2} x_2 + \sqrt{T_2} x_3, \\ p'_2 &= \sqrt{R_2} p_2 + \sqrt{T_2} p_3 \end{aligned} \quad (85)$$

where  $x_j$  and  $p_j$  are the quadratures of the field  $a_j$ . Taking  $R_1 = \frac{1}{2}$ , from Eq. (77) we see that  $\Delta^2[x_1 - \frac{1}{\sqrt{2}}(x_2 + x_3)] = 2e^{-2r}$  and  $\Delta^2[p_1 + \frac{1}{\sqrt{2}}(p_2 + p_3)] = 2e^{-2r}$ , implying that the variances are zero for large  $r$ . The output fields satisfy the condition

$$\Delta \left[ x_1 - \frac{1}{\sqrt{2}}(x_2 + x_3) \right] \Delta \left[ p_1 + \frac{1}{\sqrt{2}}(p_2 + p_3) \right] < 1 \quad (86)$$

for genuine tripartite entanglement given by Eq. (17) of [49], with equal gains for the second and third modes. The product becomes zero, indicating maximum EPR entanglement, in the limit of large  $r$ .

However, we are interested to examine the tripartite EPR steering. On taking  $R_2 = \frac{1}{2}$ , from Eq. (81) we see on substituting Eq. (85) that

$$\begin{aligned} \Delta^2 \left[ x_1 - \frac{g_{x,s}}{\sqrt{2}}(x_2 + x_3) \right] &= \frac{1}{\cosh 2r}, \\ \Delta^2 \left[ p_1 + \frac{g_{p,s}}{\sqrt{2}}(p_2 + p_3) \right] &= \frac{1}{\cosh 2r}. \end{aligned} \quad (87)$$

Hence the steering product defined by Eq. (25) for the three output modes becomes

$$S_3 \equiv S_{1|23} = \Delta u_1 \Delta v_1 = \frac{1}{\cosh 2r} \quad (88)$$

where here for Eq. (25) we identify  $h_2 = h_3 = -g_{x,s}/\sqrt{2}$  and  $g_2 = g_3 = g_{p,s}/\sqrt{2}$ . There is steering of system 1 for all values of  $r$ , as we see by examining the steering condition Eq. (26). We will also see that Criterion 1 for genuine tripartite steering is satisfied for  $r > 0.76$ .

Continuing, it is possible to select the reflectivities of the string of beam splitters so that  $x'_2 = \frac{1}{\sqrt{N}}(\sum_i^{N-1} x_i)$  and  $p'_2 = \frac{1}{\sqrt{N}}(\sum_i^{N-1} p_i)$ . In this case, on substituting in Eq. (77), we obtain

$$\Delta^2 \left[ x_1 - \frac{1}{\sqrt{N-1}}(x_2 + x_3 + \dots x_N) \right] = 2e^{-2r},$$

$$\Delta^2 \left[ p_1 + \frac{1}{\sqrt{N-1}}(p_2 + p_3 + \dots p_N) \right] = 2e^{-2r}. \quad (89)$$

The criterion

$$S_N = \Delta u \Delta v < \frac{1}{N-1}, \quad (90)$$

defined for  $u$  and  $v$  with  $g_1 = h_1 = 1$  and  $g_i = g$  and  $h_i = h$  for  $i > 1$ , is sufficient to confirm genuine  $N$ -partite entanglement, as shown in [10,49]. This criterion is clearly satisfied for large  $r$ .

To analyze genuine  $N$ -partite steering, we use Eq. (81) to obtain

$$\Delta^2 \left[ x_1 - \frac{g_{x,s}}{\sqrt{N-1}}(x_2 + x_3 + \dots x_N) \right] = \frac{1}{\cosh 2r},$$

$$\Delta^2 \left[ p_1 + \frac{g_{p,s}}{\sqrt{N-1}}(p_2 + p_3 + \dots p_N) \right] = \frac{1}{\cosh 2r}. \quad (91)$$

Hence the steering product as defined by Eq. (25) for the  $N$  output modes becomes

$$S_N \equiv S_{1|2,\dots,N} = \Delta u_1 \Delta v_1 = \frac{1}{\cosh 2r} \quad (92)$$

where here for Eq. (25) we identify for  $i > 1$  that  $h_i = -g_{x,s}/\sqrt{N-1}$  and  $g_i = g_{p,s}/\sqrt{N-1}$ . There is steering of system 1 for all values of  $r$ , as we see by examining the steering condition Eq. (26). We also see that Criterion 1 as extended for genuine  $N$ -partite steering below is satisfied for large  $r$ . We consider the state created by selecting for the beam splitters,  $R_{N-1} = 1/2$ ,  $R_{N-2} = \frac{1}{3}$ ,  $R_{N-r} = \frac{1}{r+1}$  for  $r < N-1$ , as explained in [10,49], with  $R_1 = 1/2$ . The state produced shows genuine  $N$ -partite steering.

Criterion 1 can be generalized to  $N$  parties, to give the following criterion.

*Criterion 1b.* Selecting  $u_1 = x_1 + h(x_2 + x_3 + \dots x_N)$  and  $v_1 = p_1 + g(p_2 + p_3 + \dots p_N)$ , we see that the corresponding gains for Criterion 1 are  $g_1 = h_1 = 1$  and  $g_i = g$  and  $h_i = h$  ( $i > 1$ ). Since  $0 \leq g, -h \leq 1$ , one simultaneously confirms two-way steering along all bipartitions if with this choice of  $u$  and  $v$ , one can confirm

$$S_N \equiv \Delta u_1 \Delta v_1$$

$$< \min\{1, |(N-1)gh|; |gh|, |1 + (N-2)gh|;$$

$$\times |2gh|, |1 + (N-3)gh|; |3gh|, |1 + (N-4)gh|; \dots;$$

$$\times |1 + gh|, |(N-2)gh|; |1 + 2gh|, |(N-3)gh|; \dots\}$$

$$< \min\{|gh|, |1 - (N-2)gh|\}. \quad (93)$$

Where the gains are  $g_i = \frac{g_{p,s}}{\sqrt{N-1}}$  and  $h_i = -\frac{g_{x,s}}{\sqrt{N-1}}$ , so that  $u_1 = x_1 - \frac{g_{x,s}}{\sqrt{N-1}}(x_2 + x_3 + \dots x_N)$  and  $v_1 = p_1 +$

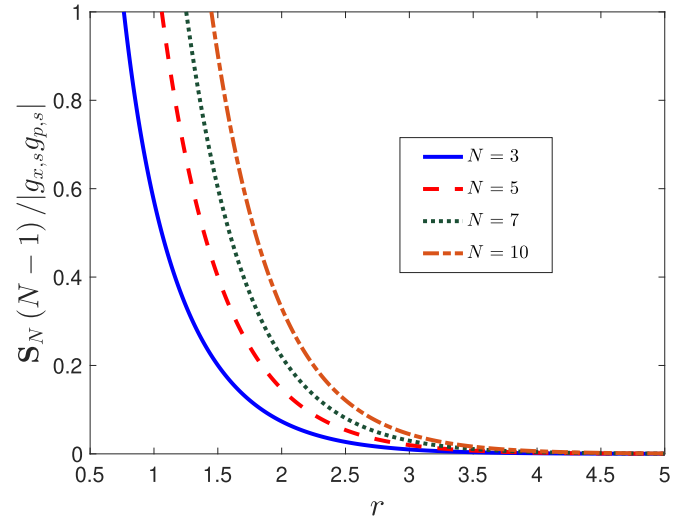


FIG. 4. Genuine  $N$ -partite steering for the CV EPR state: The value of  $S_N$  divided by the bound  $|g_{x,s}g_{p,s}|/(N-1)$  as provided in Criterion 1b, for the CV EPR state. The analytical expression for  $S_N$  and the optimal gains  $g_{x,s}$  and  $g_{p,s}$  are given by Eq. (84). When the value is less than 1, there is genuine  $N$ -partite steering according to Criterion 1b, given by Eq. (94).

$\frac{g_{p,s}}{\sqrt{N-1}}(p_2 + p_3 + \dots p_N)$ , this reduces to

$$S_N \equiv \Delta u_1 \Delta v_1 < \min \left\{ \frac{g_{x,s}g_{p,s}}{N-1}, 1 - \frac{N-2}{N-1}g_{x,s}g_{p,s} \right\}. \quad (94)$$

This inequality therefore gives a criterion for genuine  $N$ -partite steering. With the optimal choice of gains, the inequality becomes  $S_N < \frac{\tanh^2 2r}{N-1}$ , since the second term is greater.

For the choice of gains  $g_{x,s}$  and  $g_{p,s}$  given by Eqs. (79) and (80) that optimize for steering of system 1, the value of  $S_N$  is plotted in Fig. 2. The criterion  $S_N < 1$  is clearly satisfied for all  $r > 0$ , and as  $r \rightarrow \infty$ ,  $S_N \rightarrow 0$ , implying maximal EPR steering of system 1. Using the expressions in Eq. (89) and the inequality Eq. (94), the corresponding condition on  $r$  for genuine  $N$ -partite steering according to Criterion 1b with this choice of gains is given by

$$S_N \equiv S_{1|2,\dots,N} = \Delta u_1 \Delta v_1 = \frac{1}{\cosh 2r} < \frac{\tanh^2 2r}{N-1}, \quad (95)$$

which is satisfied for  $r > 0.76$  for  $N = 3$ . The normalized value given by  $S_N$  divided by the bound  $|g_{x,s}g_{p,s}|/(N-1)$  is plotted in Fig. 4. The minimum squeezing parameter required to show genuine steering according to Eq. (95) satisfies  $\cosh 2r > \frac{(N-1) + \sqrt{(N-1)^2 + 4}}{2}$ . We note that it is likely genuine  $N$ -partite steering can be detected for smaller  $r$  values if the gains are chosen to optimize the inequality, rather than to optimize for the steering of system 1.

## B. CV split-squeezed state

Genuine  $N$ -partite entanglement and steering can also be generated from a network with just one *single* squeezed-state vacuum input (Fig. 5). This is possible because the two outputs of a beam splitter with a single squeezed-vacuum input



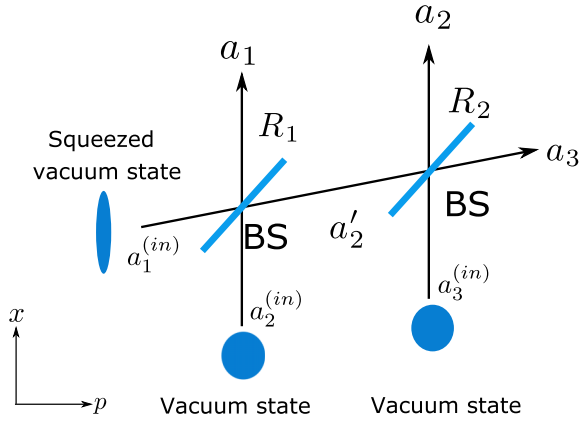


FIG. 5. Generation of the tripartite-entangled CV split-squeezed state. The configuration uses a single squeezed-vacuum input  $a_1^{(in)}$  and two beam splitters (BS) with reflectivities  $R_1 = 1/2$  and  $R_2 = 1/2$ . The  $x_i$  and  $p_i$  are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes  $i$  ( $i = 1, 2, 3$ ).

are EPR correlated [36]. A high degree of squeezing  $r$  is required for the input, however, in order to generate a feasible amount of multipartite entanglement.

We calculate the steering correlations explicitly. We first consider the bipartite steering created where one squeezed input is placed through a beam splitter BS1 with inputs  $a_1^{(in)}$  and  $a_2^{(in)}$ . This produces EPR entangled fields with boson operators  $a_1$  and  $a'_2$  at the two outputs. Using the procedure from Sec. IV A, if  $a_2^{(in)}$  is a vacuum then  $\Delta^2 x_2^{(in)} = 1$ , implying

$$\Delta^2(\sqrt{T_1}x_1 - \sqrt{R_1}x'_2) = \Delta^2 x_2^{(in)} = 1. \quad (96)$$

If  $a_1^{(in)}$  is squeezed in  $p$ , so that  $\Delta^2 p_1^{(in)} = e^{-2r}$ , we have

$$\Delta^2(\sqrt{R_1}p_1 + \sqrt{T_1}p'_2) = e^{-2r}. \quad (97)$$

Choosing  $R_1 = \frac{1}{2}$ , these fields satisfy

$$\begin{aligned} \Delta^2(x_1 - x'_2) &= 2, \\ \Delta^2(p_1 + p'_2) &= 2e^{-2r} \end{aligned} \quad (98)$$

where  $x_i$  and  $p_i$  are the quadratures associated with each mode  $a_i$ . Entanglement is detected when  $\Delta(x_1 - x'_2)\Delta(p_1 + p'_2) < 2$  [104] so that the fields are entangled for all  $r > 0$ .

We next examine the EPR steering between the two modes. For arbitrary  $R_1$ , we obtain

$$\begin{aligned} \Delta^2(x_1 - g_{x,s}x'_2) &= g_{x,s}^2(T_1 e^{2r} + R_1) + R_1 e^{2r} + T_1 \\ &\quad - 2\sqrt{R_1 T_1} g_{x,s}(e^{2r} - 1) \end{aligned} \quad (99)$$

which is minimum for

$$g_{x,s} = \frac{\sqrt{R_1 T_1}(e^{2r} - 1)}{(T_1 e^{2r} + R_1)}. \quad (100)$$

Similarly,

$$\begin{aligned} \Delta^2(p_1 + g_{p,s}p'_2) &= g_{p,s}^2(T_1 e^{-2r} + R_1) + R_1 e^{-2r} + T_1 \\ &\quad - 2\sqrt{R_1 T_1} g_{p,s}(1 - e^{-2r}) \end{aligned} \quad (101)$$

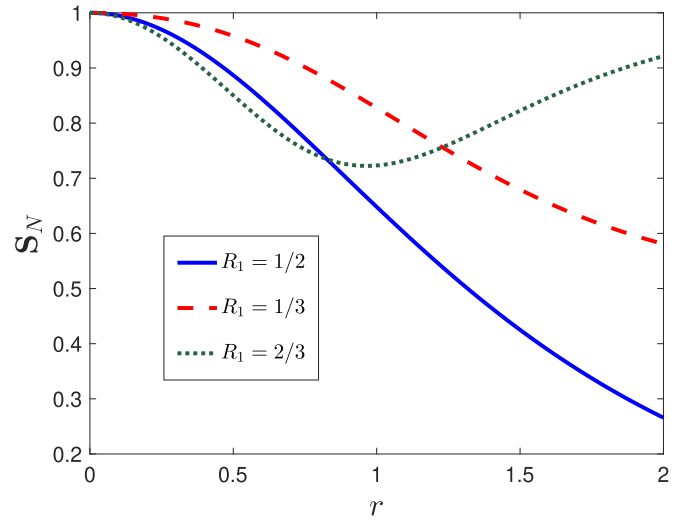


FIG. 6. Genuine  $N$ -partite steering for the CV SS state: The plots shows the optimal steering parameter  $S_N$  defined in Eqs. (105) and (111) as a function of the squeezing parameter  $r$  and for different reflectivities  $R_1$ . Steering of system 1 is obtained when  $S_N \equiv S_{1|2,\dots,N} < 1$ . It is evident that genuine  $N$ -partite steering is obtained for sufficiently large  $r$ , implying mutual steering between all subsystems, as given by Criterion 1b. The blue solid line is the result given by Eq. (111).

which is minimum for

$$g_{p,s} = \frac{\sqrt{R_1 T_1}(1 - e^{-2r})}{(R_1 + T_1 e^{-2r})}. \quad (102)$$

This gives

$$\Delta^2(x_1 - g_{x,s}x'_2) = \frac{e^{2r}}{(1 - R_1)e^{2r} + R_1}, \quad (103)$$

which implies that for  $r = 0$ ,  $\Delta^2(x_1 - g_{x,s}x'_2) = 1$ . Similarly,

$$\Delta^2(p_1 + g_{p,s}p'_2) = \frac{e^{-2r}}{R_1 + (1 - R_1)e^{-2r}}, \quad (104)$$

which is 1 for  $r = 0$  and for large  $r$  becomes zero. This is true for all values of reflectivity  $R_1$ . Hence, the optimal steering product  $S_2 = \Delta(x_1 - g_{x,s}x'_2)\Delta(p_1 + g_{p,s}p'_2)$  defined by Eq. (25) for the two output modes is given by

$$S_2^2 \equiv (S_{1|2})^2 = \frac{1}{1 + 4R_1(1 - R_1) \sinh^2 r}. \quad (105)$$

As long as  $R_1 \neq 0$ ,  $S_2 \rightarrow 0$  for large  $r$ .

We plot the bipartite steering product  $S_2$  in Figs. 6 and 7, versus  $r$ . One may prove by differentiation for each fixed  $r$  that the optimal choice to minimize  $S_2$  is  $R_1 = 1/2$ , which gives

$$g_{x,s} = g_{p,s} = \frac{1 - e^{-2r}}{1 + e^{-2r}} \quad (106)$$

and

$$\begin{aligned} \Delta^2(x_1 - g_{x,s}x'_2) &= \frac{2}{1 + e^{-2r}}, \\ \Delta^2(p_1 + g_{p,s}p'_2) &= \frac{2e^{-2r}}{1 + e^{-2r}}. \end{aligned} \quad (107)$$

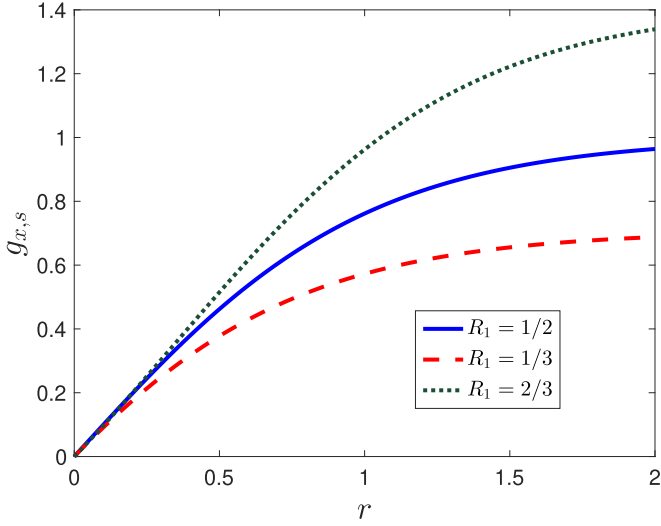


FIG. 7. Optimal gains: The gains  $g_{x,s}$  required for the steering in Fig. 6 as given in Eq. (79), as a function of the squeezing parameter  $r$ , for different reflectivities  $R_1$ . The plots for  $g_{p,s}$  are obtained on replacing  $R_1$  with  $T_1$ .

This leads to  $\mathbf{S}_2 = \frac{1}{\cosh r}$ , indicating steering of system 1 for all  $r$ , as given by the steering condition Eq. (26). We note that although  $\mathbf{S}_2$  vanishes for significantly large  $r$ , the value of  $\mathbf{S}_2$  remains close to 1 for the experimental achievable set of squeezing parameters,  $r < 2$ . This is particularly true where the reflectivity deviates from the ideal value of  $R_1 = 1/2$ .

To produce tripartite entangled fields, the field  $a'_2$  is then split using a beam splitter labeled BS2, with a vacuum input  $a_3^{(in)}$ , to give two new outputs  $a_2 = (\sqrt{R_2}a'_2 + \sqrt{T_2}a_3^{(in)})$  and  $a_3 = (\sqrt{T_2}a'_2 - \sqrt{R_2}a_3^{(in)})$ , where  $R_2^2 + T_2^2 = 1$ , as above. This gives  $a'_2 = \sqrt{R_2}a_2 + \sqrt{T_2}a_3$ . Taking  $R_2 = \frac{1}{2}$ , we see that

$$\begin{aligned} \Delta^2 \left[ x_1 - \frac{\sqrt{R_1}}{\sqrt{2T_1}}(x_2 + x_3) \right] &= 1/T_1, \\ \Delta^2 \left[ p_1 + \frac{\sqrt{T_1}}{\sqrt{2R_1}}(p_2 + p_3) \right] &= e^{-2r}/R_1. \end{aligned} \quad (108)$$

The product is  $e^{-2r}/R_1 T_1$  which for a given  $r$  minimizes for  $R_1 = T_1$ , i.e., for  $R_1 = \frac{1}{2}$ . The output fields  $a_1$ ,  $a_2$ , and  $a_3$  satisfy the condition Eq. (86) for genuine tripartite entanglement, with equal gains for the second and third modes.

Continuing, it is possible to select the reflectivities of the string of beam splitters so that we obtain

$$\begin{aligned} \Delta^2 \left[ x_1 - \frac{1}{\sqrt{N-1}}(x_2 + x_3 + \dots x_N) \right] &= 2, \\ \Delta^2 \left[ p_1 + \frac{1}{\sqrt{N-1}}(p_2 + p_3 + \dots p_N) \right] &= 2e^{-2r}. \end{aligned} \quad (109)$$

This is done by selecting for the beam splitters,  $R_{N-1} = 1/2$ ,  $R_{N-2} = \frac{1}{3}$ ,  $R_{N-r} = \frac{1}{r+1}$  for  $r < N-1$ , as explained in [10,49], with  $R_1 = 1/2$ . The outputs satisfy the condition for genuine  $N$ -partite entanglement given by Eq. (90).

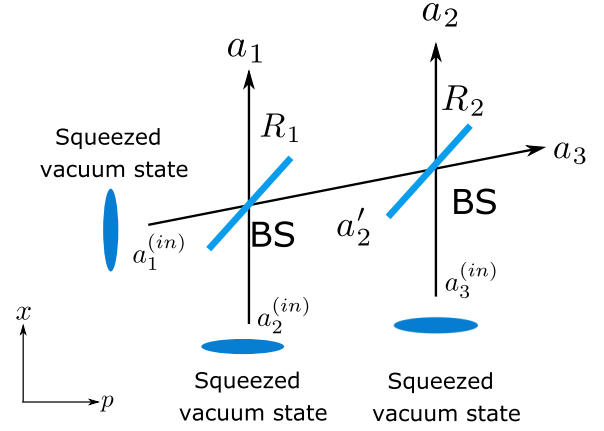


FIG. 8. Generation of the tripartite-entangled CV GHZ state. The standard configuration uses three squeezed-vacuum inputs and two beam splitters (BS) with reflectivities  $R_1 = 1/3$  and  $R_2 = 1/2$ . The  $x_i$  and  $p_i$  are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes  $i$  ( $i = 1, 2, 3$ ).

Similarly, for the same string of beam splitters, we obtain

$$\begin{aligned} \Delta^2 \left[ x_1 - \frac{g_{x,s}}{\sqrt{N-1}}(x_2 + x_3 + \dots x_N) \right] &= \frac{2}{(1 + e^{-2r})}, \\ \Delta^2 \left[ p_1 + \frac{g_{p,s}}{\sqrt{N-1}}(p_2 + p_3 + \dots p_N) \right] &= \frac{2e^{-2r}}{(1 + e^{-2r})}. \end{aligned} \quad (110)$$

The value of the steering product defined by Eq. (25) for the  $N$  output modes becomes

$$\mathbf{S}_N \equiv S_{1\{2,\dots,N\}} = \Delta u_1 \Delta v_1 = \frac{1}{\cosh r}. \quad (111)$$

There is steering of system 1 for all values of  $r$ , as we see by examining the steering condition Eq. (26). A criterion to reveal genuine  $N$ -partite steering is given by Criterion 1b, as above. The product  $\mathbf{S}_N$  is plotted for the choice of gains given by Eqs. (100) and (102) in Fig. 6. These gains are optimized to detect the steering of system 1, i.e., to minimize the value of  $\mathbf{S}_N$  rather than to optimize violation of the inequality of Criterion 1b. We see that genuine  $N$ -partite steering is possible for sufficiently large  $r$ , despite that only one squeezed-vacuum state has been used to generate the output fields. In order to achieve the condition given by Criterion 1b, we require

$$\Delta u \Delta v = \frac{2e^{-r}}{(1 + e^{-2r})} < \frac{|g_{x,s}g_{p,s}|}{2}.$$

This leads to the inequality  $\cosh^2 r - (N-1)\cosh r - 1 < 0$ . For  $N = 3$ , this reduces to  $r > 1.53$ .

### C. CV GHZ states

The CV GHZ state is generated by combining  $N$  squeezed-vacuum states at the inputs of the  $N-1$  beam splitters [10,105]. The first mode is squeezed in the direction  $p$  orthogonal to the direction of squeezing  $x$  of the remaining modes. The variances are given as  $\Delta^2 X = e^{-2r}$ , where  $X$  is the squeezed quadrature (Fig. 8).

The final output variances are such that [10]

$$\Delta^2(x_i - x_j) = 2e^{-2r} \quad (112)$$

for any  $i \neq j$  and

$$\Delta^2(p_1 + p_2 + p_3 + \dots p_N) = Ne^{-2r} \quad (113)$$

which is the special case given by Eq. (35) of [10] with  $g^{(N)} = 1$ . Steering of system  $i$  is detected on selecting  $u = x_i - x_j$  and  $v = p_1 + p_2 + p_3 + \dots p_N$  and observing  $\Delta u \Delta v < 1$ , as we see by examining the steering condition Eq. (26). Clearly, this is obtained for the expressions of  $u$  and  $v$ , for large  $r$ . For smaller  $r$ , a different choice of gain is optimal, to allow steering for all  $r$ .

We now give a more detailed analysis. For just two modes, the correlations are identical to those given in Sec. IV A, for the CV EPR state. Explicitly, from Eqs. (75) and (76), we get  $\Delta^2(x_1 - \frac{1}{\sqrt{2}}x_2) = \frac{3}{2}e^{-2r}$  and  $\Delta^2(p_1 + \sqrt{2}p_2) = 3e^{-2r}$  for  $R_1 = 1/3$  and  $T_1 = 2/3$ . Following the same procedure, using Eq. (85), the moments in the tripartite case are

$$\begin{aligned} \Delta^2[x_1 - \frac{1}{2}(x_2 + x_3)] &= \frac{3}{2}e^{-2r}, \\ \Delta^2(p_1 + p_2 + p_3) &= 3e^{-2r}. \end{aligned} \quad (114)$$

More generally, we allow different gains, and consider  $\Delta^2[x_1 + h(x_2 + x_3)]$  and  $\Delta^2[p_1 + g(p_2 + p_3)]$ . Generalizing to the  $N$ -partite CV GHZ state, there will be  $N - 1$  beam splitters with reflectivities  $R_1 = 1/N$ ,  $R_2 = 1/(N - 1)$ ,  $\dots$ ,  $R_{N-1} = 1/2$ . The variances are

$$\begin{aligned} \Delta^2 \left[ x_1 + h \left( \sum_{j=2}^N x_j \right) \right] &= \frac{1}{N} [h(N-1) + 1]^2 \Delta^2 x_1^{(in)} \\ &\quad + \frac{(N-1)}{N} (h-1)^2 \Delta^2 x_2^{(in)}, \\ \Delta^2 \left[ p_1 + g \left( \sum_{j=2}^N p_j \right) \right] &= \frac{1}{N} [g(N-1) + 1]^2 \Delta^2 p_1^{(in)} \\ &\quad + \frac{(N-1)}{N} (g-1)^2 \Delta^2 p_2^{(in)} \end{aligned} \quad (115)$$

where  $\Delta^2 x_1^{(in)} = \Delta^2 p_2^{(in)} = e^{2r}$  and  $\Delta^2 x_2^{(in)} = \Delta^2 p_1^{(in)} = e^{-2r}$ , as provided in Eq. (A5) of [49]. The optimal gains  $g$  and  $h$ , on differentiation, are

$$\begin{aligned} h &= -\frac{\Delta^2 x_1^{(in)} - \Delta^2 x_2^{(in)}}{\Delta^2 x_2^{(in)} + (N-1)\Delta^2 x_1^{(in)}}, \\ g &= -\frac{\Delta^2 p_1^{(in)} - \Delta^2 p_2^{(in)}}{\Delta^2 p_2^{(in)} + (N-1)\Delta^2 p_1^{(in)}}. \end{aligned} \quad (116)$$

For large  $r$ , the optimal values become  $g \rightarrow 1$  and  $h \rightarrow -1/(N-1)$ . These optimal gains for different  $N$  are plotted in Figs. 9 and 10 versus  $r$ .

The expressions in Eqs. (115) and (116) give

$$\begin{aligned} \Delta^2[x_1 + h(x_2 + \dots + x_N)] &= \frac{N}{e^{-2r} + (N-1)e^{2r}}, \\ \Delta^2[p_1 + g(p_2 + \dots + p_N)] &= \frac{N}{e^{2r} + (N-1)e^{-2r}}. \end{aligned} \quad (117)$$

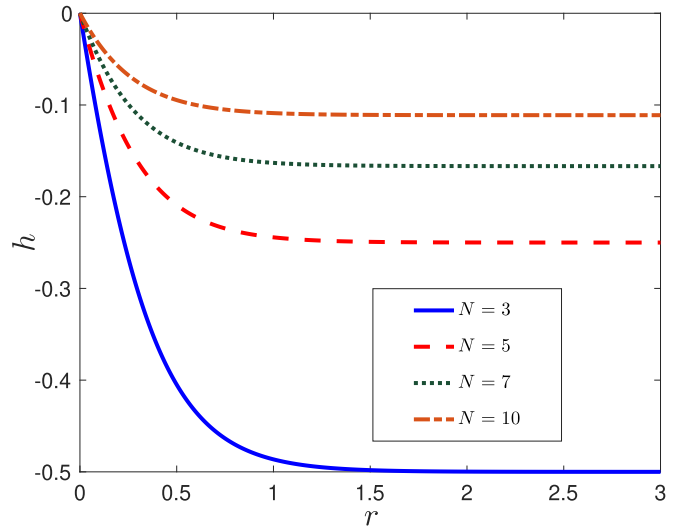


FIG. 9. Optimal gains: The optimal gains  $h$  that minimize  $\Delta u \Delta v$  in Eq. (115) for CV GHZ states. The analytical expressions for these gains are given by Eq. (116).

Immediately, we see that the criterion  $S_N = \Delta u \Delta v < \frac{1}{N-1}$  as given by Eq. (90) and proved in [10,49] for  $N$ -partite entanglement is satisfied for large  $r$ .

To examine the steering, we find that

$$\begin{aligned} \mathbf{S}_N &\equiv S_{1|2,\dots,N} \\ &= \frac{N}{\sqrt{N^2 + 4(N-1)\sinh^2 2r}}. \end{aligned} \quad (118)$$

The analytical expression for  $\mathbf{S}_N$  is plotted in Fig. 11. We see using the condition  $S_N < 1$  of Eq. (26) that steering of system 1 is possible for all values of  $r$ . As  $N$  becomes large,  $\mathbf{S}_N \rightarrow \sqrt{N}/(2\sinh 2r)$ , indicating genuine  $N$ -partite steering for sufficiently large  $r$ , by Criterion 1b. The following condi-

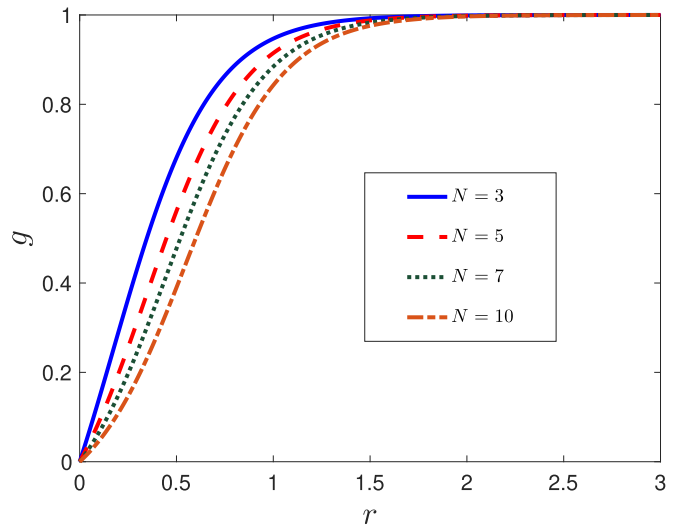


FIG. 10. Optimal gains: The optimal gains  $g$  that minimize  $\Delta u \Delta v$  in (115) for CV GHZ states. The analytical expressions for these gains are given by Eq. (116).

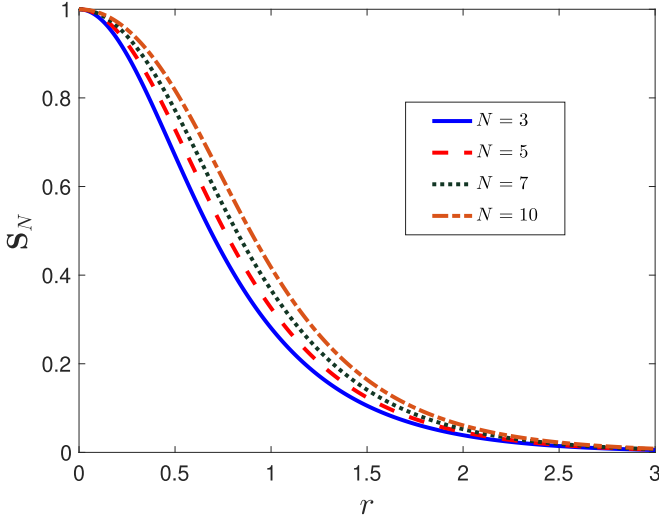


FIG. 11. Genuine  $N$ -partite steering for the CV GHZ state: The value of  $S_N$  as a function of the squeezing parameter  $r$ , for CV GHZ states. The analytical expression for  $S_N$  is given by Eq. (118). Steering of system 1 is obtained when  $S_N \equiv S_{1|2,\dots,N} < 1$ . It is evident that genuine  $N$ -partite steering is obtained for sufficiently large  $r$ , implying mutual steering between all subsystems, as given by Criterion 1b.

tion is sufficient to reveal genuine  $N$ -partite steering:

$$S_N \equiv \Delta u_1 \Delta v_1 < \min\{|gh|, |1 - (N-2)|gh|\}. \quad (119)$$

In the limit of large  $r$ , this reduces to  $S_N < \frac{1}{N-1}$ , which is clearly satisfied for large  $r$ .

More generally, using the analytical expression for  $S_N$  and the optimal gains  $g$  and  $h$  given by Eqs. (118) and (116), we plot the value of  $S_N$  divided by the bound provided in Criterion 1b, given by Eq. (119). As shown in Fig. 12, we see that genuine  $N$ -partite steering is possible for large  $N$ . The minimum squeezing parameter required to show this steering satisfies the inequality  $\sinh^2 2r \geq (N^2(N-1) + N^2\sqrt{(N-1)^2 + 4})/8$  in agreement with the results in Fig. 12. Here, the CV GHZ state does not for a fixed  $r$  give a smaller value  $S_N$  than that obtained with the CV EPR state, which has only two squeezed inputs. However,  $S_N$  is an asymmetric parameter measuring steering of one mode only. The advantage is that, unlike the CV EPR state, the GHZ state has symmetry with respect to all modes.

On the other hand, the work of van Loock and Furusawa [10] reveals that asymmetric squeezing strengths can give stronger quantum correlations for the CV GHZ state. The different squeezing strengths are related by the expression [see Eq. (34) in [10]]

$$e^{\pm 2r_1} = (N-1) \sinh 2r_2 \left( \sqrt{1 + \frac{1}{(N-1)^2 \sinh^2 2r_2}} \pm 1 \right) \quad (120)$$

where  $r_1$  is the input squeeze parameter for mode 1 and  $r_j = r_2$  for  $j \neq 1$ . We confirm this leads to a steering value  $S_3$  improved over the CV EPR case, for the same average squeezing value, and relative to the squeezing parameter  $r_2$  of the second and third modes. For general  $N$ , following as

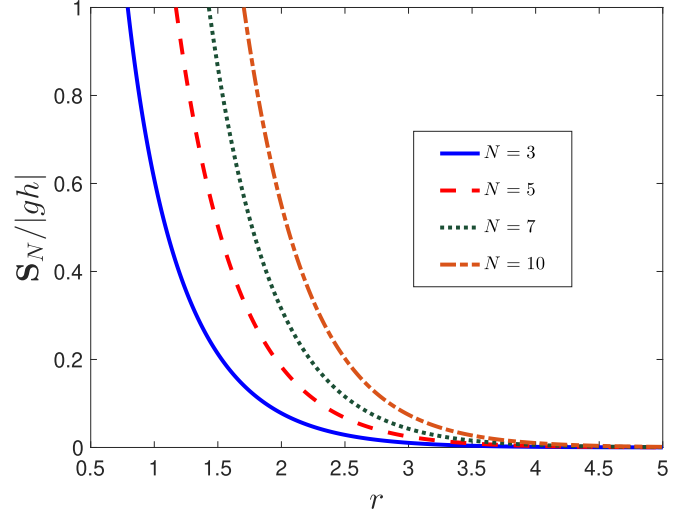


FIG. 12. Genuine  $N$ -partite steering for the CV GHZ state: The value of  $S_N$  divided by the bound  $|gh|$  as provided in Criterion 1b, for CV GHZ states. The analytical expression for  $S_N$  and the optimal gains  $g$  and  $h$  are given by Eqs. (118) and (116), respectively. When the value is less than 1, there is genuine  $N$ -partite steering according to Criterion 1b given by Eq. (119).

above, we use

$$\begin{aligned} \Delta^2[x_1 + h(x_2 + \dots + x_N)] &= \frac{N}{e^{-2r_2} + (N-1)e^{2r_1}}, \\ \Delta^2[p_1 + g(p_2 + \dots + p_N)] &= \frac{N}{e^{2r_2} + (N-1)e^{-2r_1}} \end{aligned} \quad (121)$$

where the  $g$  and  $h$  are given by Eq. (116). This leads to

$$S_N = \frac{N}{\sqrt{(N-1)^2 \sinh^2 2r_2 + 1} + (N-1) \cosh 2r_2}. \quad (122)$$

For large  $N$ , the limit becomes  $S_N \rightarrow e^{-2r_2}$ , which is improved on the CV EPR and CV SS states. The CV GHZ states require a single very strong squeezing at the first input  $r_1$ , but for less squeezing in the  $N-1$  inputs, a much higher steering as measured by  $S_N$  is possible.

In Fig. 13,  $S_N$  is plotted for the CV GHZ state with two squeezing strengths, for  $N = 3, 10$ , and  $100$ . Also, plotted are  $S_3$  for CV EPR and CV SS states.

## V. GENUINE TRIPARTITE STEERABLE STATES

We now consider in more detail the steering for the three types of states considered in the last section, where  $N = 3$ . Examples for  $N = 4$  are given in the Supplemental Material [103]. It is useful to first summarize criteria that detect steering across *particular* bipartitions. Here, we consider  $u = h_1 x_1 + h_2 x_2 + h_3 x_3$  and  $v = g_1 p_1 + g_2 p_2 + g_3 p_3$ . Using Lemma 1, we detect steering of mode  $k$  by  $lm$  if ( $k \neq l \neq m$ )  $S_{k|lm} < 1$  where

$$S_{k|lm} = \frac{\Delta(h_k x_k + h_l x_l + h_m x_m) \Delta(g_k p_k + g_l p_l + g_m p_m)}{|g_k h_k|}. \quad (123)$$



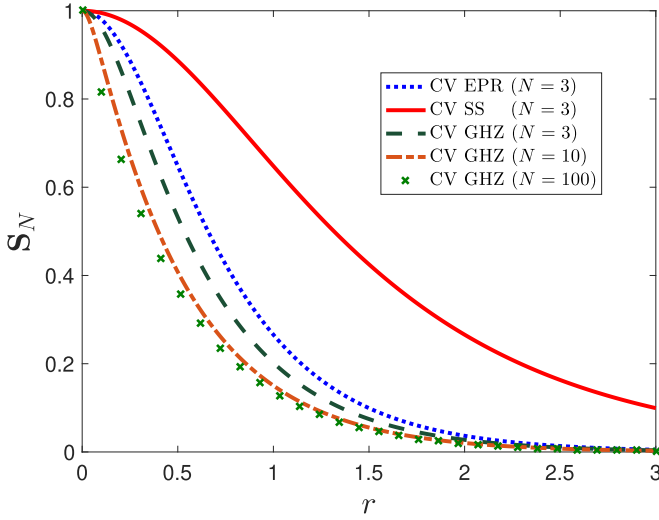


FIG. 13. Genuine  $N$ -partite steering for the CV GHZ state with asymmetric squeezing strengths. The steering product  $\mathbf{S}_3$  is also shown for CV EPR and CV SS states. The value of  $\mathbf{S}_N$  is plotted for the CV GHZ state with two squeezing strengths, for  $N = 3, 10$ , and  $100$ . The two squeezing strengths for the GHZ state are related by Eq. (120). In that case,  $r$  as plotted corresponds to  $r_2 = r_3$ . The values of  $\mathbf{S}_N \equiv S_{1|2,\dots,N}$  given by the green crosses are very close to  $e^{-2r}$ , as predicted by the large  $N$  limit of  $e^{-2r_2}$ . Steering of system 1 is obtained when  $\mathbf{S}_N < 1$ . Genuine  $N$ -partite steering which implies mutual steering of all subsystems is obtained for large  $r$ .

Thus we consider minimizing  $\Delta(x_k + \tilde{h}_l x_l + \tilde{h}_m x_m) \Delta(p_k + \tilde{g}_l p_l + \tilde{g}_m p_m)$  where  $\tilde{g}_i = g_i/g_k$  and  $\tilde{h}_i = h_i/h_k$ . This example is relevant for the application of secret sharing where the collaborators  $l$  and  $m$  cannot be trusted. In order to detect the steering across the bipartitions, one can select different optimal choices of  $\tilde{g}_i$  and  $\tilde{h}_i$  for each bipartition,  $k - lm$ . Similarly, from Lemma 1, we detect steering of the combined systems  $l$  and  $m$ , if  $S_{lm|k} < 1$  where

$$S_{lm|k} = \frac{\Delta(h_k x_k + h_l x_l + h_m x_m) \Delta(g_k p_k + g_l p_l + g_m p_m)}{|g_l h_l + g_m h_m|}. \quad (124)$$

We note the generalization of the definition here of the steering product given in Eq. (25) where there is the steering of two systems. We also note that in this case, as the variance product  $S_{k|lm}$  goes to zero, so too will  $S_{lm|k}$ .

### A. CV GHZ and cluster states

#### 1. Genuine tripartite steering using a single inequality

The inequality of Criterion 1 is useful to detect the genuine tripartite steering of the CV GHZ state. For a single choice of  $g_i$  and  $h_i$  defining  $u = h_1 x_1 + h_2 x_2 + h_3 x_3$  and  $v = g_1 p_1 + g_2 p_2 + g_3 p_3$ , we wish to violate Eq. (36). Defining  $\mathbf{S}_3 = \Delta u \Delta v$ , we test each of the six inequalities  $\mathbf{S}_3 < \mathcal{B}$  provided by considering the right side of the inequality, where

$$\mathcal{B} \in \{|g_1 h_1|, |g_2 h_2 + g_3 h_3|, |g_2 h_2|, |g_1 h_1 + g_3 h_3|, |g_3 h_3|, |g_1 h_1 + g_2 h_2|\}. \quad (125)$$

TABLE I. Values of the gains  $h$  and  $g$  that minimize the variance product in criterion Eq. (126).

$r$	CV GHZ	
	$h$	$g$
0	0	0
0.25	-0.27	0.36
0.50	-0.40	0.68
0.75	-0.46	0.86
1.00	-0.49	0.95
1.50	-0.50	0.99
2.00	-0.50	1.00

If each inequality  $\mathbf{S}_3 < \mathcal{B}$  is satisfied, we indicate steering across one of the bipartitions in a certain direction. Genuine tripartite steering is confirmed if *all* six possibilities are verified simultaneously, using a single set of gains, e.g., from the single inequality Eq. (36).

We select  $g_1 = h_1 = 1$ ,  $g_2 = g_3 = g$ , and  $h_2 = h_3 = h$  and optimize for  $g$  and  $h$  by minimizing  $\mathbf{S}_3$  with respect to these gains. Criterion 1 given by Eq. (36) is then reduced to the violation of

$$\mathbf{S}_3 \geq \min\{\mathcal{B}_{1|23}, \mathcal{B}_{23|1}, \mathcal{B}_{2|13}, \mathcal{B}_{13|2}, \mathcal{B}_{3|12}, \mathcal{B}_{12|3}\}, \quad (126)$$

where in this particular case  $\mathcal{B}_{1|23} = 1$ ,  $\mathcal{B}_{23|1} = 2|gh|$ ,  $\mathcal{B}_{2|13} = \mathcal{B}_{3|12} = |gh|$ , and  $\mathcal{B}_{13|2} = \mathcal{B}_{12|3} = |1 + gh|$ . By differentiating  $\mathbf{S}_3$  with respect to  $g$  and  $h$ , we select the optimal gains given by the analytical expressions Eq. (116). The numerical values of optimal gains as a function of the squeezing parameter  $r$  are tabulated in Table I. The value of  $\mathbf{S}_3$  as a function of the squeezing parameter  $r$  is given by Eq. (118). We calculate that genuine tripartite steering is detectable using Criterion 1 when  $r$  is sufficiently large ( $r > 0.8$ ). The choice of gains used here may not be the optimal to observe genuine tripartite steering for a fixed  $r$ , but nonetheless ensures steering of system 1 (or system 1 combined with 2 or 3) for all  $r$  values.

In Fig. 14, we plot  $\mathbf{S}_3$  for a CV GHZ state with two squeezing strengths, as given by Eq. (122). We also plot the different bounds on the right side of the inequality Eq. (126). The values of  $g$  and  $h$  are given by Table II. We take  $r_2 = r_3 = r$  and obtain a minimum  $r_2 = r_3$  of 0.633 to observe steering

TABLE II. Values of the gains  $h$  and  $g$ , as given by the analytical expressions in Eq. (116), that minimize the variance product  $\mathbf{S}_3$ . Here, the squeezing strength  $r_2 = r$  and  $r_1$  is related to  $r_2$  by the relation Eq. (120). For large  $r$ , the gains are identical to those in the case where there is only one squeezing strength.

$r_2 = r$	CV GHZ	
	$h$	$g$
0	0	0
0.25	-0.34	0.51
0.50	-0.45	0.80
0.75	-0.48	0.97
1.00	-0.49	0.99
1.50	-0.50	1.00
2.00	-0.50	1.00

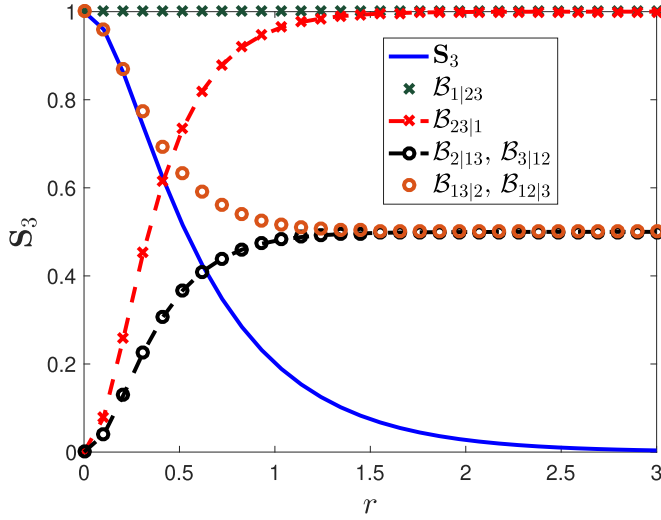


FIG. 14. The value of  $S_3 \equiv S_{1|2} = \Delta u \Delta v$  as a function of the squeezing parameter  $r$ , for the CV GHZ state, for  $g_1 = h_1 = 1$ , and  $g_2 = g_3 = g$  and  $h_2 = h_3 = h$ . The results here correspond to the case where two squeezing strengths are used, with  $r_2 = r$  and  $r_1$  is related to  $r_2$  by the relation Eq. (120). The gains  $g$  and  $h$  as a function of  $r$  are given by the analytical expressions in Eq. (116). The blue solid line corresponds to the value of  $S_3$ , while other lines correspond to different bounds on the right side of the inequality Eq. (126). The gains are optimized to minimize  $S_3$ , and hence to optimize the observation of the steering of system 1.

in all bipartitions. This corresponds to  $r_1 = 0.95$  using the relation Eq. (120).

## 2. Full tripartite steering inseparability

Next, we minimize the quantities  $S_{k|lm}$  and  $S_{lm|k}$  as defined in Eqs. (123) and (124), for each bipartition. For the steering of mode 1 by 2 and 3, we minimize the quantity  $S_{1|23}$  by optimizing the gains  $\tilde{h}_2, \tilde{h}_3, \tilde{g}_2$ , and  $\tilde{g}_3$ . On the other hand, for the steering of modes 23 by 1, we minimize the quantity  $S_{23|1}$ , with an independent choice of gains. For the steering of mode 2 by 1 and 3, we minimize  $S_{2|13}$  by optimizing the gains  $\tilde{h}_1, \tilde{h}_3, \tilde{g}_1$ , and  $\tilde{g}_3$ . The quantity  $S_{13|2}$  is independently minimized for the steering of modes 1 and 3, by 2. We proceed similarly, for  $S_{3|12}$  and  $S_{12|3}$ .

The gains are optimized independently in each case. All the optimal gains are numerically computed using the *fminsearch* function in MATLAB. The maximum number of iterations that calls the *fminsearch* function is chosen to be  $10^6$ . The tolerance of this MATLAB function is chosen to be  $10^{-6}$ , where no further iteration is taken if the quantity to be minimized is smaller than  $10^{-6}$  from one iteration to the next. The gains are given in Tables III and IV.

Figure 15 shows the results for the steering across the bipartitions for the CV GHZ state with a single squeezing strength  $r$ . For all  $r$ , two-way steering can be detected across each bipartition, using the relevant inequalities with the gains given in Tables III and IV. For large  $r$ , the inferences improve with the correct choice of gains, and all the relevant variances become zero. This gives a method to confirm full tripartite

TABLE III. The optimal gains for the CV GHZ state. The optimal gains for  $S_{3|12}$  are identical to the gains for  $S_{2|13}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	Gains for $S_{1 23}$				Gains for $S_{2 13}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-0.27	0.36	-0.27	0.36	-0.27	0.36	-0.27	0.36
0.50	-0.40	0.68	-0.40	0.68	-0.40	0.68	-0.40	0.68
0.75	-0.46	0.86	-0.46	0.86	-0.46	0.86	-0.46	0.86
1.00	-0.49	0.95	-0.49	0.95	-0.49	0.95	-0.49	0.95
1.50	-0.50	0.99	-0.50	0.99	-0.50	0.99	-0.50	0.99
2.00	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00

two-way steering inseparability. The results for asymmetric squeezing strengths  $r_1$  and  $r_2$  are also given.

## 3. Behavior in the highly squeezed limit

Most interesting is the limit of large  $r$  where the correlations become ideal, implying zero variances, so that  $S_{k|lm} \rightarrow 0$  and  $S_{lm|k} \rightarrow 0$ . For the standard CV GHZ state [105] and for the CV GHZ state with asymmetric squeezing strengths [10], the optimal gain coefficients (refer to Tables III–VI) are such that the steering criterion, for the steering of system  $k$ , becomes

$$\Delta \left[ x_k - \frac{(x_l + x_m)}{2} \right] \Delta [p_k + p_l + p_m] < 1 \quad (127)$$

for each  $k = 1, 2, 3$  (recalling  $k \neq l \neq m$ , where  $k, l, m \in \{1, 2, 3\}$ ). For the standard CV GHZ state, we may also use  $\Delta[x_k - x_l] \Delta[p_k + p_l + p_m] < 1$ , or  $\Delta[x_k - x_m] \Delta[p_k + p_l + p_m] < 1$  [105]. For large  $r$ , the steering of system  $lm$  by  $k$  is optimized by the same choice of gains. Noting that in this limit  $|g_l h_l + g_m h_m| = 1$ , we see that in fact the same inequality Eq. (127) detects steering of  $lm$  by  $k$ , and therefore detects two-way steering across the bipartition  $k - lm$ .

## 4. Genuine tripartite steering in a tripartite cluster state

We also compare with the results presented by Wang *et al.* [106]. Here, the authors fixed  $R_1 = 2/3$  and varied  $R_2$  to optimize the Gaussian steering parameter  $\mathcal{G}^{A \rightarrow B}$  [37]. In particular, their analysis for  $R_1 = 2/3$  and  $R_2 = 1/2$  corresponds to that for a tripartite unweighted cluster state. This is of interest here, as this corresponds to where no two modes can

TABLE IV. The optimal gains for the CV GHZ state. The optimal gains for  $S_{12|3}$  are identical to the gains for  $S_{13|2}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	Gains for $S_{23 1}$				Gains for $S_{13 2}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-1.37	1.87	-1.37	1.87	-1.37	1.87	-1.37	1.87
0.50	-0.73	1.23	-0.73	1.23	-0.73	1.23	-0.73	1.23
0.75	-0.58	1.08	-0.58	1.08	-0.58	1.08	-0.58	1.08
1.00	-0.53	1.03	-0.53	1.03	-0.53	1.03	-0.53	1.03
1.50	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00
2.00	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00

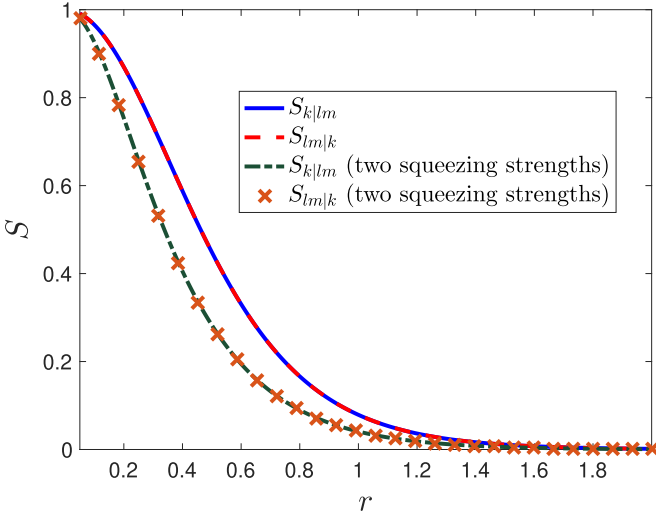


FIG. 15. The steering across each bipartition as a function of squeezing parameter  $r$ , for the CV tripartite GHZ state. Here  $k \neq l \neq m$ . There is symmetry between all three systems and the plots hold for each  $k = 1, 2$ , and  $3$ . The values of  $S = \Delta u \Delta v$  for  $S_{k|lm}$  and  $S_{lm|k}$  are determined by the choice of gains in Tables III and IV. Here,  $S_{k|lm}$  and  $S_{lm|k}$  coincide. The gains for the case with two squeezing strengths are tabulated in Tables V and VI.

steer one another, but where steering requires all three modes, in line with the notion of genuine tripartite steering. They confirm full tripartite steering inseparability (as we define it) for this state, based on the assumption of a Gaussian state, using the Gaussian parameter  $\mathcal{G}^{A \rightarrow B}$ . Here, we give a method sufficient to confirm tripartite steering inseparability and genuine tripartite steering, *without* the assumption of Gaussian states. A summary of the correlations for this state is given in the Supplemental Material [103].

Before considering genuine tripartite steering for the cluster state, we investigate the steering across all possible bipartitions separately. For the bipartition 1 – 23, in our approach, we consider

$$\begin{aligned} u'_1 &= h'_{1,1}x_1 - h'_{1,2}p_2 + h'_{1,3}x_3, \\ v'_1 &= g'_{1,1}p_1 + g'_{1,2}x_2 + g'_{1,3}p_3. \end{aligned} \quad (128)$$

TABLE V. The optimal gains for the CV GHZ state with two squeezing strengths. Here,  $r = r_2$  while  $r_1$  is related to  $r_2$  by Eq. (120). The optimal gains for  $S_{3|12}$  are identical to the gains for  $S_{2|13}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	Gains for $S_{1 23}$				Gains for $S_{2 13}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-0.34	0.51	-0.34	0.51	-0.34	0.51	-0.34	0.51
0.50	-0.45	0.80	-0.45	0.80	-0.45	0.80	-0.45	0.80
0.75	-0.48	0.93	-0.48	0.93	-0.48	0.93	-0.48	0.93
1.00	-0.49	0.97	-0.49	0.97	-0.49	0.97	-0.49	0.97
1.50	-0.49	0.97	-0.49	0.97	-0.49	0.97	-0.49	0.97
2.00	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00

TABLE VI. The optimal gains for the CV GHZ state with two squeezing strengths. Here,  $r = r_2$  while  $r_1$  is related to  $r_2$  by Eq. (120). The optimal gains for  $S_{12|3}$  are identical to the gains for  $S_{13|2}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	Gains for $S_{23 1}$				Gains for $S_{13 2}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-0.98	1.48	-0.98	1.48	-0.98	1.48	-0.98	1.48
0.50	-0.62	1.12	-0.62	1.12	-0.62	1.12	-0.62	1.12
0.75	-0.54	1.04	-0.54	1.04	-0.54	1.04	-0.54	1.04
1.00	-0.51	1.01	-0.51	1.01	-0.51	1.01	-0.51	1.01
1.50	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00
2.00	-0.50	1.00	-0.50	1.00	-0.50	1.00	-0.50	1.00

As above, we analyze the steering for different bipartitions by considering the quantity  $S'_{1P} \equiv \Delta u'_1 \Delta v'_1$ . Full details are given in the Supplemental Material [103]. We obtain two-way steering along this bipartition if  $S'_{1P} < \mathcal{B}_{1|23}$  and  $S'_{1P} < \mathcal{B}_{23|1}$  where the bounds are  $\mathcal{B}_{1|23} = |g'_{1,1}h'_{1,1}|$  and  $\mathcal{B}_{23|1} = |g'_{1,2}h'_{1,2} + g'_{1,3}h'_{1,3}|$ . These conditions become  $S_{1|23} < 1$  and  $S_{23|1} < 1$ , on defining the steering parameters as  $S_{1|23} = S'_{1P}/\mathcal{B}_{1|23}$  and  $S_{23|1} = S'_{1P}/\mathcal{B}_{23|1}$ .

It has been shown in the work of Wang *et al.* [106] that a choice of  $h'_{1,1} = g'_{1,2} = -\sqrt{(1-R_1)}/\sqrt{R_1R_2} = -1$ ,  $h'_{1,3} = -\sqrt{(1-R_2)}/\sqrt{R_2} = -1$ ,  $g'_{1,1} = -h'_{1,2} = 1$ , and  $g'_{1,3} = 0$  will lead to  $S'_{1P} \rightarrow 0$  for a large squeezing parameter  $r$ . Here, we have taken  $R_2 = 1/2$  and  $R_1 = 2/3$ . These gains imply that  $\min\{|g'_{1,1}h'_{1,1}|, |g'_{1,2}h'_{1,2} + g'_{1,3}h'_{1,3}|\}$  is 1. The analytical expression for  $S'_{1P}$  is  $S'_{1P} = \sqrt{6}e^{-2r}$ . For large  $r$ , we see that the steering inequalities  $S'_{1P} < \mathcal{B}_{1|23}$  and  $S'_{1P} < \mathcal{B}_{23|1}$  are both satisfied for this choice of gains. Both bounds  $\mathcal{B}_{1|23}$  and  $\mathcal{B}_{23|1}$  are 1 for this choice of gains.

Similarly, there is two-way steering along the bipartition 2 – 13 if  $S'_{2P} < \mathcal{B}_{2|13}$  and  $S'_{2P} < \mathcal{B}_{13|2}$ . Here,  $S'_{2P} = \Delta u'_2 \Delta v'_2$ , where

$$\begin{aligned} u'_2 &= h'_{2,1}x_1 - h'_{2,2}p_2 + h'_{2,3}x_3, \\ v'_2 &= g'_{2,1}p_1 + g'_{2,2}x_2 + g'_{2,3}p_3. \end{aligned} \quad (129)$$

With the choice of  $h'_{2,1} = g'_{2,2} = -\sqrt{(1-R_1)}/\sqrt{R_1R_2} = -1$ ,  $h'_{2,3} = -\sqrt{(1-R_2)}/\sqrt{R_2} = -1$ ,  $g'_{2,1} = -h'_{2,2} = 1$ , and  $g'_{2,3} = 0$ ,  $S'_{2P} = \sqrt{6}e^{-2r} \rightarrow 0$  for a large squeezing parameter  $r$ . These gains imply that  $\min\{|g'_{2,2}h'_{2,2}|, |g'_{2,1}h'_{2,1} + g'_{2,3}h'_{2,3}|\}$  is 1. We note that  $S'_{1P}$  and  $S'_{2P}$  have the same analytical expression. The steering conditions for this bipartition become  $S_{2|13} < 1$  and  $S_{13|2} < 1$ , on defining  $S_{2|13} = S'_{2P}/\mathcal{B}_{2|13}$  and  $S_{13|2} = S'_{2P}/\mathcal{B}_{13|2}$  where  $\mathcal{B}_{2|13} = |g'_{2,2}h'_{2,2}|$  and  $\mathcal{B}_{13|2} = |g'_{2,1}h'_{2,1} + g'_{2,3}h'_{2,3}|$ . Both bounds are 1 for this choice of gains.

Finally, in order to demonstrate steering  $12 \rightarrow 3$  and  $3 \rightarrow 12$ , the inequalities  $S'_{3P} < \mathcal{B}_{3|12}$  and  $S'_{3P} < \mathcal{B}_{12|3}$  are used, where  $S'_{3P} = \Delta u'_3 \Delta v'_3$  with

$$\begin{aligned} u'_3 &= h'_{3,1}x_1 - h'_{3,2}p_2 + h'_{3,3}x_3, \\ v'_3 &= g'_{3,1}p_1 + g'_{3,2}x_2 + g'_{3,3}p_3, \end{aligned} \quad (130)$$

and  $\mathcal{B}_{3|12} = |g'_{3,3}h'_{3,3}|$  and  $\mathcal{B}_{12|3} = |g'_{3,1}h'_{3,1} + g'_{3,2}h'_{3,2}|$ . When the gains  $h'_{3,1} = -\sqrt{(1-R_1)}/\sqrt{R_1R_2} = -1$ ,  $g'_{3,2} = h'_{3,3} =$

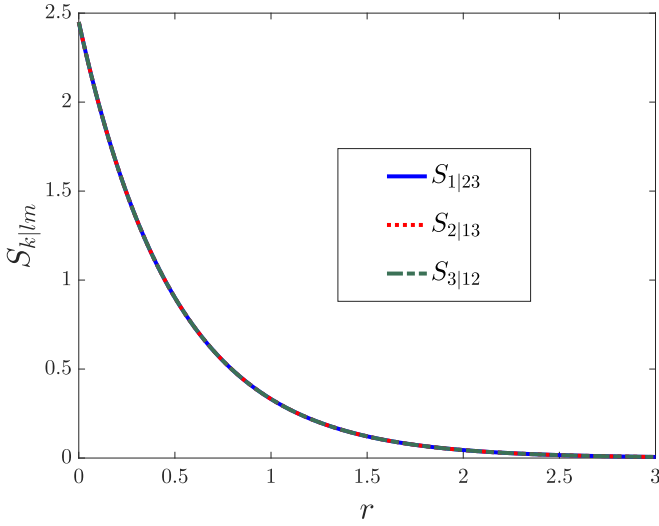


FIG. 16. Steering parameter  $S_{k|lm}$  as a function of the squeezing parameter  $r$ , for the bipartitions of the cluster state considered by Wang *et al.* [106]. Here,  $R_2 = 1/2$  and  $R_1 = 2/3$ . Steering occurs when  $S_{k|lm} < 1$ . Here, all three lines coincide, as expected from symmetry. There is two-way steering across all possible bipartitions for  $r \geq 0.44$ . This occurs when  $S_{k|lm}$  is below 1.

$-\sqrt{(1-R_2)}/\sqrt{R_2} = -1$ ,  $-h'_{3,2} = g'_{3,3} = 1$ , and  $g'_{3,1} = 0$  can be used, the analytical expression for  $S'_{3P}$  is  $S'_{3P} = \sqrt{6}e^{-2r}$ . Also, we find  $\min\{|g'_{3,3}h'_{3,3}|, |g'_{3,1}h'_{3,1} + g'_{3,2}h'_{3,2}|\} = 1$ . The steering inequalities can be expressed as  $S_{3|12} < 1$  and  $S_{12|3} < 1$  where we define  $S_{3|12} = S'_{3P}/\mathcal{B}_{3|12}$  and  $S_{12|3} = S'_{3P}/\mathcal{B}_{12|3}$ . Both bounds are 1 for this choice of gains.

The steering results for the cluster state are plotted in Fig. 16, for a varying squeezing parameter  $r$ . This allows the determination of the squeezing parameter  $r$  required to detect two-way steering across each bipartition. In fact, we see that there is two-way steering across all possible bipartitions for  $r \geq 0.44$ . This confirms that full tripartite steering inseparability can be detected for large  $r$ , using the inequalities with the given choice of gains. We note that we have not optimized the gains, and therefore this may not be the optimal criterion.

Having demonstrated steering across each bipartition, and hence full tripartite steering inseparability, we are also able to demonstrate genuine tripartite steering for this cluster state using Criterion 3. On examining the gains  $h'_{i,j}$  and  $g'_{i,j}$  selected for the  $u'_k$  and  $v'_k$  used in the definitions of  $S_{k|lm}$  above, we see that  $S_{k|lm}$  are identical to  $S_{k|lm}$  defined according to Eqs. (44) and (45), provided we choose  $h_{1,2} = 1$ ,  $h_{1,3} = 1$ ,  $g_{1,2} = -1$ ,  $g_{1,3} = 0$ ,  $h_{2,1} = 1$ ,  $h_{2,3} = 1$ ,  $g_{2,1} = -1$ ,  $g_{2,3} = 0$ ,  $h_{3,1} = 1$ ,  $h_{3,2} = 1$ ,  $g_{3,1} = 0$ , and  $g_{3,2} = -1$  in the definitions of  $S_{k|lm}$ . For those gains, all the relevant bounds on the right side of the inequality of Criterion 3 are 1. The inequality of Criterion 3 reduces to

$$S'_1 + S'_2 + S'_3 \geq 1 \quad (131)$$

where  $S'_k = S_{k|lm}$  are given by Eqs. (44) and (45). The analytical expressions for  $S'_1$ ,  $S'_2$ , and  $S'_3$  are solved above, as  $S'_1 = S'_2 = S'_3 = \sqrt{6}e^{-2r}$ , respectively. Using these expressions, we investigate genuine tripartite steering based on Eq. (131), as

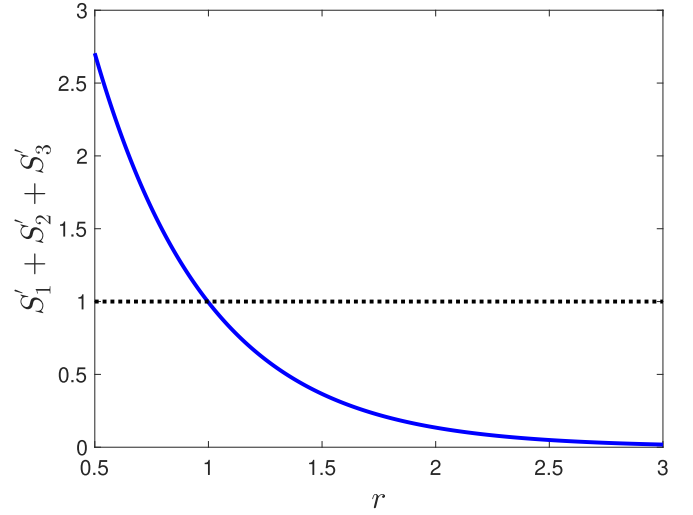


FIG. 17. Genuine tripartite steering as a function of the squeezing parameter  $r$ , using the criterion given by violation of the inequality Eq. (131), for the cluster state studied by Wang *et al.* [106]. The values of  $S'_1 + S'_2 + S'_3$  are given by the blue solid line. The black dotted line corresponds to the bound for the steering inequality. There is genuine tripartite steering for  $r > 1$ .

a function of squeezing parameter. The result is plotted in Fig. 17 and genuine tripartite steering is possible for  $r > 1$ .

## B. CV EPR state

### 1. Genuine tripartite steering with a single inequality

To investigate genuine tripartite steering for the CV EPR state, Criterion 1 given by Eq. (126) can be used. We first select  $g_1 = h_1 = 1$ ,  $g_1 = g_2 = g$ , and  $h_1 = h_2 = h$  and optimize for  $g$  and  $h$  by minimizing  $S$  with respect to these gains. This optimization has been carried out in Sec. IV A. The optimal values are  $g = g_{x,s}/\sqrt{2}$  and  $h = g_{p,s}/\sqrt{2}$  where  $g_{x,s}$  and  $g_{p,s}$  are given by Eqs. (79) and (80). The value of  $S_3$  as a function of the squeezing parameter  $r$  is plotted in Figs. 2 and 18, relative to the bounds of Eq. (125). We see that for this choice of gains, there is steering of system 1 for all  $r$  values.

Genuine tripartite steering is detectable with Criterion 1, for sufficiently large  $r > 0.76$ . The experiment of Walk *et al.* [31] reports a maximum squeezing of  $-6.5$  dB, corresponding to a squeeze parameter of  $r = 0.75$ . This suggests that detection of genuine tripartite steering may be feasible using this approach in the near future.

### 2. Full tripartite steering inseparability

Next, we examine full tripartite steering inseparability. We numerically compute gains that minimize  $S_{k|lm}$  and  $S_{lm|k}$  [Eqs. (123) and (124), respectively] for each bipartition. The values of these gains are given in Tables VII and VIII. In particular, the gains for  $S_{1|23}$  are as above. As for the CV GHZ state, there is two-way steering for all the bipartitions, and hence full tripartite steering inseparability, for all values of  $r$ . This is evident in Fig. 19. We note that in the limit of large  $r$ , the correlations become ideal, implying zero variances, so that  $S_{k|lm} \rightarrow 0$  and  $S_{lm|k} \rightarrow 0$ .



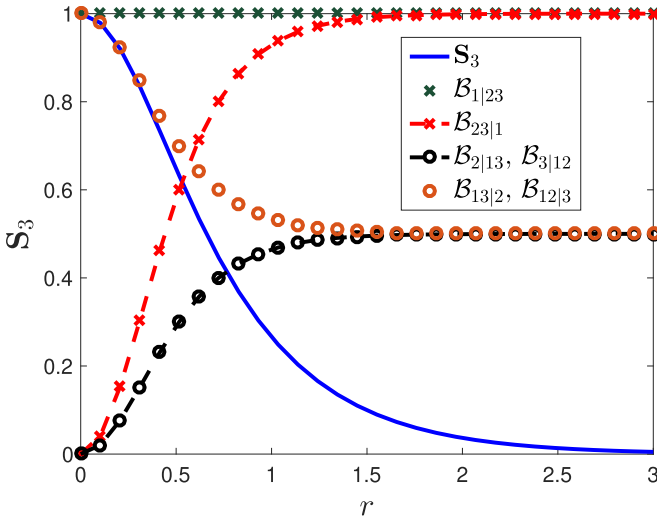


FIG. 18. The value of  $S_3 = \Delta u \Delta v$  as a function of the squeezing parameter  $r$ , for the CV EPR state given by Fig. 1. The gains as a function of  $r$  are given by the analytical expressions in Eqs. (79) and (80), which here are optimized to enhance observation of steering of system 1. The blue solid line corresponds to the value of  $S_3$ . The remaining lines correspond to different bounds on the right side of the inequality Eq. (126). When  $S$  is smaller than all bounds, there is genuine tripartite steering. This is obtained for  $r > 0.76$ .

### 3. Behavior in the highly squeezed limit

In the limit of large  $r$  where the correlations become ideal and the variances small, the steering of system  $lm$  by  $k$  is optimized by the same choice of gains as the steering of  $k$  by  $lm$ . The optimal gain coefficients are such that the best steering criterion for the steering of system  $k = 1$  is, in the limit of large  $r$ ,

$$\Delta \left[ x_k - \frac{(x_l + x_m)}{\sqrt{2}} \right] \Delta \left[ p_k + \frac{(p_l + p_m)}{\sqrt{2}} \right] < 1, \quad (132)$$

as expected from the analysis of Sec. IV. Using that  $|g_l h_l + g_m h_m| = 1$ , we see that the same inequality Eq. (132) confirms steering of  $lm$  by  $k$ , and therefore detects two-way steering across the bipartition  $k - lm$ .

TABLE VII. The optimal gains for the CV EPR state. The optimal gains for  $S_{3|12}$  are identical to the gains for  $S_{2|13}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	$S_{1 23}$				$S_{2 13}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-0.33	0.33	-0.33	0.33	-0.35	0.35	0.06	0.06
0.50	-0.54	0.54	-0.54	0.54	-0.65	0.65	0.21	0.21
0.75	-0.64	0.64	-0.64	0.64	-0.90	0.90	0.40	0.40
1.00	-0.68	0.68	-0.68	0.68	-1.08	1.08	0.58	0.58
1.50	-0.70	0.70	-0.70	0.70	-1.28	1.28	0.82	0.82
2.00	-0.71	0.71	-0.71	0.71	-1.36	1.36	0.93	0.93

TABLE VIII. The optimal gains for the CV EPR state. The optimal gains for  $S_{13|2}$  are identical to the gains for  $S_{12|3}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	$S_{23 1}$				$S_{13 2}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-1.53	1.53	-1.53	1.53	-2.98	2.98	0.52	0.52
0.50	-0.93	0.93	-0.93	0.93	-1.71	1.71	0.56	0.56
0.75	-0.78	0.78	-0.78	0.78	-1.40	1.40	0.63	0.63
1.00	-0.73	0.73	-0.73	0.73	-1.31	1.31	0.70	0.70
1.50	-0.71	0.71	-0.71	0.71	-1.32	1.32	0.85	0.85
2.00	-0.71	0.71	-0.71	0.71	-1.37	1.37	0.93	0.93

### 4. Experimental observation

The observation of steering of a mode  $k$  using the criterion Eq. (132) has been reported in the experimental system of Armstrong *et al.* [38], using the setup of Fig. 1 for EPR states. They reported steering of each mode in a tripartite system using the violation of  $S_{k|lm} \geq 1$ , where  $g_k = h_k = 1$  for the steered mode. This corresponds to a realization of full tripartite steering inseparability, according to the definition given in Sec. II C. They found agreement with the theoretical predictions, with  $S_{1|23} = 0.78$  and  $S_{2|13} = S_{3|12} = 0.87$  (refer to Table I in the supplementary information of [38]). From the

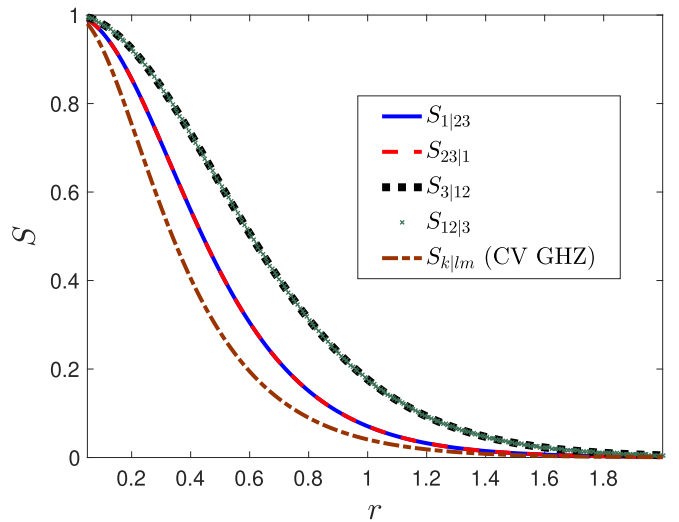


FIG. 19. The steering for each bipartition as a function of squeezing parameter  $r$ , for the CV EPR state given by Fig. 1. Here,  $S = \Delta u \Delta v$  with the choice of gains  $g$  and  $h$  given in Tables VII and VIII, for the various quantities  $S = S_{k|lm}$  or  $S_{km|l}$  indicated. A value of  $S$  less than 1 implies steering. We see that there is two-way steering across each bipartition. There is symmetry with respect to subsystems 2 and 3 (refer to Fig. 1) and hence the plots for  $S_{2|13}$  and  $S_{3|12}$  are identical to those of  $S_{3|12}$  and  $S_{12|3}$ . For comparison, we also plot (brown dotted line) the value of  $S_{k|lm}$ , which is identical for all values of  $k$  (and  $l \neq m \neq k$ ), for the CV GHZ state with two squeezing parameters.

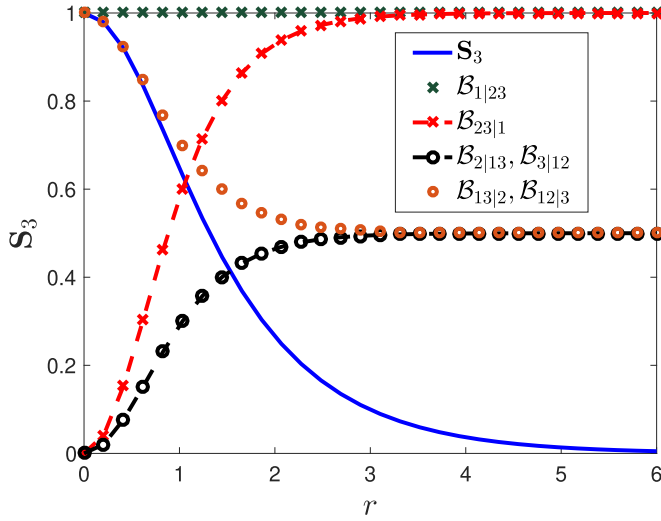


FIG. 20. The value of  $S_3 = \Delta u \Delta v$  as a function of the squeezing parameter  $r$ , for the CV split-squeezed state. The gains as a function of  $r$  are given by the analytical expressions in Eqs. (100) and (102), which are optimized for the observation of steering of system 1 (refer to Fig. 5). The blue solid line corresponds to the value of  $S_3$ . The remaining lines correspond to different bounds on the right side of the inequality Eq. (126). When  $S_3$  is smaller than all the bounds, there is genuine tripartite steering. This is obtained for  $r > 1.53$ .

gains provided in the same table of their paper, we obtain

$$S_{23|1} = \frac{\Delta(h_2 x_2 + h_3 x_3 + x_1) \Delta(g_2 p_2 + g_3 p_3 + p_1)}{|g_2 h_2 + g_3 h_3|} = 0.95, \quad (133)$$

which indicates two-way steering for the bipartition 1 – 23. However, we estimate  $S_{13|2} = S_{12|3} = 1.73 > 1$ , which does not satisfy the requirement for full tripartite two-way steering inseparability. The estimated maximum squeeze parameter for the experiment was  $r \sim 0.47$  based on a noise suppression of  $-4.1$  dB. The experiment was also constructed for up to eight modes, with the steering of each mode of the eight mode system observed.

### C. CV split squeezed state

The CV SS state arises from a single squeezed input, as in Fig. 5. Full tripartite inseparability can be detected using criteria for entanglement, as explained in [70,71]. Here, however, we optimize the correlations further, by selecting a *different* choice of beam splitter reflectivity, as in Fig. 5.

#### 1. Genuine tripartite steering using a single inequality

From Fig. 20, we see that genuine tripartite steering is detectable using Criterion 1 for  $r > 1.56$ . This is expected from the results of Sec. IV. Here, we use Criterion 1 as given by the inequality Eq. (126), where we select  $g_1 = h_1 = 1$ ,  $g_1 = g_2 = g$ , and  $h_1 = h_2 = h$  and optimize for  $g$  and  $h$ . The optimal gains are derived in Sec. IV. With only one squeezed input, the amount of steering for a given  $r$  is reduced, but nonetheless perfect steering is possible for large  $r$ .

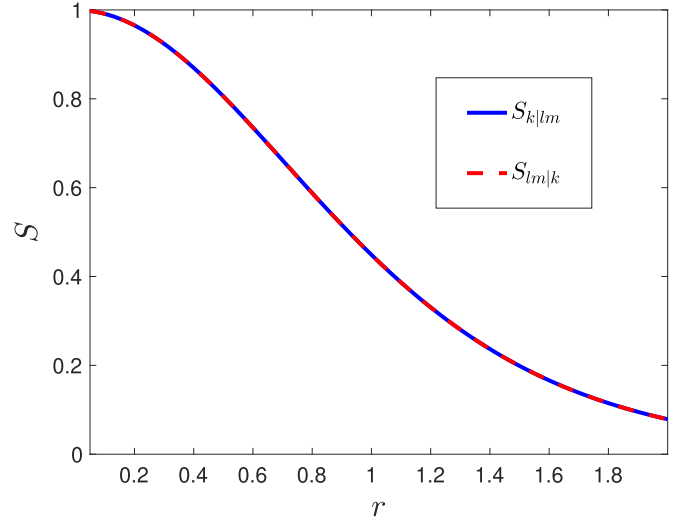


FIG. 21. The steering for each bipartition as a function of squeezing parameter  $r$ , for the CV tripartite SS state. Here,  $S = S_{k|lm}$  or  $S_{lm|k}$  as indicated. Here, the lines coincide. We numerically compute  $S_{k|lm}$  and  $S_{lm|k}$ , as defined in Eqs. (123) and (124), with the choice of gains provided in Tables IX and X. Here  $k \neq l \neq m$  and  $k = 1, 2, \text{ or } 3$ .  $S < 1$  implies steering. Steering is observed in both directions across each bipartition.

#### 2. Full tripartite steering inseparability

Full tripartite steering inseparability can be detected, for all  $r$ . This is seen from Fig. 21. We numerically obtain the gains that minimize  $S_{k|lm}$  and  $S_{lm|k}$ , as given in Eqs. (123) and (124), respectively, for each bipartition. When  $S_{k|lm}, S_{lm|k} < 1$ , there is steering for the corresponding bipartition. We see from the figure that there is two-way steering across all bipartitions. Hence, it is possible to detect full tripartite two-way steering inseparability.

### D. Experimental genuine tripartite steering using the van Loock–Furusawa-type inequalities

We are able to apply the criteria derived in this paper to confirm from experimental data the realization of genuine tripartite steering for cluster states. The criteria that we use are based on the van Loock–Furusawa variances, which have been measured experimentally.

TABLE IX. The optimal gains for the CV SS state. The optimal gains for  $S_{3|12}$  are identical to the gains for  $S_{2|13}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	$S_{1 23}$				$S_{2 13}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-0.15	0.18	-0.15	0.18	-0.15	0.18	-0.15	0.18
0.50	-0.27	0.36	-0.27	0.36	-0.27	0.36	-0.27	0.36
0.75	-0.35	0.54	-0.35	0.54	-0.35	0.54	-0.35	0.54
1.00	-0.40	0.68	-0.40	0.68	-0.40	0.68	-0.40	0.68
1.50	-0.46	0.86	-0.46	0.86	-0.46	0.86	-0.46	0.86
2.00	-0.49	0.95	-0.49	0.95	-0.49	0.95	-0.49	0.95

TABLE X. The optimal gains for the CV SS state. The optimal gains for  $S_{13|2}$  are identical to the gains for  $S_{12|3}$  with  $\tilde{h}_3 = \tilde{h}_2$  and  $\tilde{g}_3 = \tilde{g}_2$ .

$r$	$S_{23 1}$				$S_{13 2}$			
	$\tilde{h}_2$	$\tilde{g}_2$	$\tilde{h}_3$	$\tilde{g}_3$	$\tilde{h}_1$	$\tilde{g}_1$	$\tilde{h}_3$	$\tilde{g}_3$
0.25	-2.81	3.31	-2.81	3.31	-2.81	3.31	-2.81	3.31
0.50	-1.37	1.87	-1.37	1.87	-1.37	1.87	-1.37	1.87
0.75	-0.93	1.43	-0.93	1.43	-0.93	1.43	-0.93	1.43
1.00	-0.73	1.23	-0.73	1.23	-0.73	1.23	-0.73	1.23
1.50	-0.58	1.08	-0.58	1.08	-0.58	1.08	-0.58	1.08
2.00	-0.53	1.03	-0.53	1.03	-0.53	1.03	-0.53	1.03

First, we note from the form of the inequality of Criterion 1 and from the symmetry of the CV GHZ state that in the limit of large  $r$ , the CV GHZ state will give a zero value for the van Loock–Furusawa quantities  $B_i$  and  $S_i$ . These quantities were defined by the van Loock–Furusawa-type Criteria 4 and 5 in Sec. III C. This is also seen directly, from the predictions Eqs. (112) and (113) for the GHZ variances. The CV EPR state will also give, in the limit of large  $r$ , a zero for the product  $S_i$ , as has been shown in [10,49] where similar inequalities were used to detect genuine tripartite entanglement and steering. Hence, the van Loock–Furusawa-type steering inequalities of Criteria 4 and 5 can also be used to detect full tripartite steering inseparability and genuine tripartite steering.

This is useful for interpreting the level of steering generated in previous CV experiments, which measure the van Loock–Furusawa inequalities. Previous experiments report full multipartite inseparability [60]. In the experiment of Armstrong *et al.* [60], the measured variances in the tripartite case are  $B_I$  and  $B_{II}$  of Eq. (51). They reported  $B_I = B_{II} = 0.14 < 2$  (note a different scaling of quadrature amplitudes) for the CV EPR state. This implies an experimental confirmation of full tripartite two-way steering inseparability according to Criterion 4b.

In order to demonstrate genuine tripartite steering, Criterion 5b can be used. However,  $B_{III}$  was not directly measured in the experiment [60]. Here, the inequality of Criterion 6c involving just  $B_I$  and  $B_{II}$  is useful. Armstrong *et al.* [60] measured the van Loock–Furusawa entanglement inequalities for a CV cluster state. In the following, we apply van Loock–Furusawa-type inequalities of Sec. III C to show genuine tripartite steering for the cluster state. The variances measured in the experiment are  $B'_I \equiv \Delta^2(p_1 - x_2) + \Delta^2(p_2 - x_1 - x_3)$  and  $B'_{II} \equiv \Delta^2(p_3 - x_2) + \Delta^2(p_2 - x_1 - x_3)$ .

Following the proofs given for Criteria 5 and 6c, and the result Eq. (27), we see that the violation of the inequality

$$B'_I + B'_{II} \geq 2P_1 + 2P_2 + 2P_3 \geq 2 \quad (134)$$

implies genuine tripartite steering. Following the proof for Criterion 7, we see that the same violation also implies genuine tripartite steering by the stricter Definition 3. In the experiment, Armstrong *et al.* [60] obtained  $\Delta^2(p_1 - x_2) + \Delta^2(p_2 - x_1 - x_3) = 0.12$  and  $\Delta^2(p_3 - x_2) + \Delta^2(p_2 - x_1 - x_3) = 0.18$ , which violates the above inequality and hence

demonstrates *experimentally* genuine tripartite steering, by Definitions 1 and 3.

We note that the CV GHZ will satisfy Criterion 7 of Sec. III D for large  $r$ . This follows from the predictions Eqs. (112) and (113) for the variances. This gives an avenue to generate and detect the strict form of genuine tripartite steering (Definition 3) for these states, which applies to networks of only one trusted site. The extension to  $N$  systems would seem straightforward.

## VI. MONOGAMY RELATIONS

In this section, we summarize how the bipartite entanglement and steering are distributed among the subsystems of the tripartite steerable states. It is known that for three qubit systems, the bipartite entanglement between any two of them is limited by monogamy relations [107]. This result can be extended to  $N$ -qubit systems [108] and to nonlocality and steering [44,109–115]. The monogamy for higher-dimensional systems is more complex, and has been investigated for CV systems [44,116,117].

We first examine the distribution of bipartite steering among the three systems created in CV GHZ, CV EPR, and CV SS states. For steering, it is known that [44]

$$\mathcal{S}_{A|B}\mathcal{S}_{A|C} \geq 1 \quad (135)$$

where  $\mathcal{S}_{i|j}$  is defined in Sec. II F for an arbitrary observable of the steering parties. In this paper, the observables taken for the steering parties are linear combinations of quadrature phase amplitudes, in which case we write  $S \equiv \mathcal{S}$ . The steering parameter is then  $S_{k|l} = \Delta(x_k - h_{kl}x_l)\Delta(p_k + g_{kl}p_l)$  where the gains  $h_{kl}$  and  $g_{kl}$  are optimized to minimize the value of  $S_{k|l}$ . Regardless of the choice of gains, however, steering is obtained when  $S_{k|l} < 1$  [25,36]. For the choice of optimal gains,  $S_{k|l} < 1$  becomes a necessary and sufficient condition to demonstrate steering for Gaussian states and measurements [23]. The specific monogamy inequality

$$S_{k|l}S_{k|m} \geq \max\{1, S_{k|lm}^2\}, \quad (136)$$

where  $S_{k|lm}$  is defined with the optimal linear gains  $h_{k(lm)}$  and  $g_{k(lm)}$  that minimize the value of  $S_{k|lm}$ , follows from the definitions and Eq. (135) (without the assumption of Gaussian states [38,44,118]). The relation has been verified experimentally for the CV EPR state [38,119]. Where there is collective steering such that  $S_{1|23} < 1$ , the monogamy relation gives  $S_{1|2}S_{1|3} \geq 1$ .

We see from the monogamy relation Eq. (136) that where there is symmetry with respect to fields  $l$  and  $m$ , so that  $S_{k|l} = S_{k|m}$ , then  $S_{k|l} \geq 1$  and  $S_{k|m} \geq 1$ . Thus for the CV GHZ state, which has three symmetrical modes, there can be no bipartite steering as witnessed by  $S_{k|l} < 1$ , for any system mode  $k = 1, 2, 3$  (Fig. 22). This implies there is no such steering of  $k$  by just one of the other single subsystems  $l$  or  $m$ . There is however maximal steering of any mode  $k$  if one considers collectively both of the other modes, since  $S_{k|lm} \rightarrow 0$  for large  $r$ . The steering witness  $S_{k|lm}$  refers to the error in the estimate of  $x$  and  $p$  of system  $k$  by systems  $l$  and  $m$ , and hence this property is of value for CV secret sharing.

For the CV EPR system, there is symmetry between the modes 2 and 3, as depicted in Fig. 1. Therefore,  $S_{1|2} = S_{1|3} \geq$

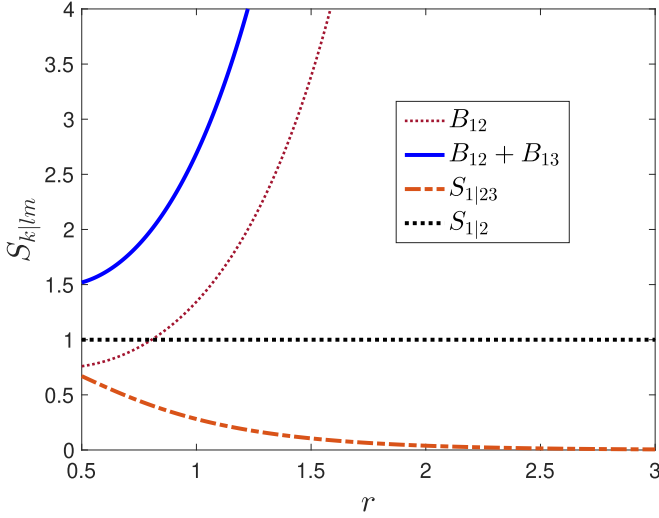


FIG. 22. Steering and entanglement monogamy for the CV GHZ state depicted in Fig. 8. Here, all modes are symmetric, e.g.,  $S_{1|23} = S_{2|13} = S_{3|12}$  and  $S_{1|2} = S_{1|3}$ . Steering of system  $k$  by both steering parties  $l$  and  $m$  is optimal for large  $r$ , as  $S_{k|lm} \rightarrow 0$ , given by the dash-dotted brown line. However, no bipartite steering as measurable by the criterion  $S_{ij} < 1$  is possible for any mode  $i$ . Similarly, there is no DGCZ bipartite entanglement measurable for large  $r$ , as  $B_{ij} > 1$  in this limit. We observe that the steering monogamy relation Eq. (136) is saturated for all  $k$ , with  $S_{k|l}S_{k|m} = 1$  (black dotted line). The entanglement monogamy inequalities Eqs. (139) and (140) are satisfied.

1, as evident in Fig. 23. We see however that  $S_{1|23} \rightarrow 0$  for large  $r$ , implying that mode 1 steered collectively by 2 and 3 is useful for secret sharing.

The CV split-squeezed state also has symmetry between the modes 2 and 3, as depicted in Fig. 5. Again, we obtain

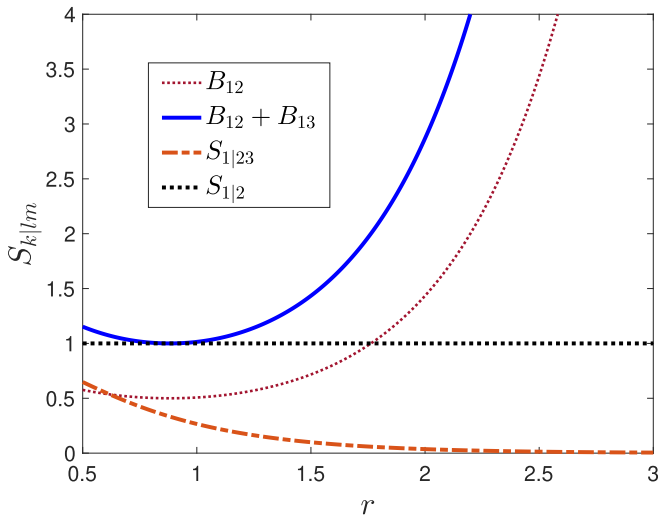


FIG. 23. Steering and entanglement monogamy for the CV EPR state depicted in Fig. 1. Modes 2 and 3 are symmetric, e.g.,  $S_{1|2} = S_{1|3}$ . Steering of system 1 by both 2 and 3 is optimal for large  $r$ , but no bipartite steering measured by  $S_{1|2} < 1$  is possible. The steering monogamy relation Eq. (136) is saturated for  $k = 1$ , with  $S_{1|2}S_{1|3} = 1$ . A similar result is obtained for  $k = 3$ .

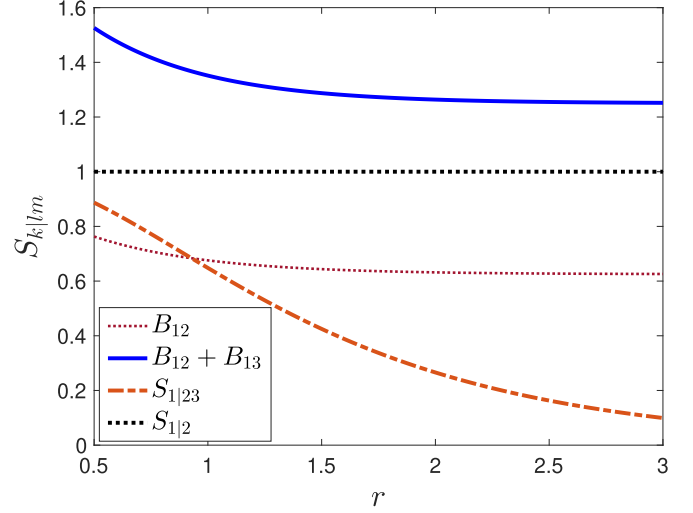


FIG. 24. Steering and entanglement monogamy for the CV SS state depicted in Fig. 5. Notation is as for Fig. 22. Although bipartite steering according to  $S_{1|ij}$  is not possible, we observe bipartite entanglement  $B_{12} = B_{13} \sim 0.63$  for large  $r$ . The steering monogamy relation Eq. (136) is saturated for  $k = 1$ , with  $S_{1|2}S_{1|3} = 1$ . A similar result is obtained for  $k = 3$ .

$S_{1|23} \rightarrow 0$  for large  $r$ , and see that  $S_{1|2} = S_{1|3} \geq 1$ . We show this relation in Fig. 24.

Further monogamy relations have been derived for Gaussian systems [113–115], using the Gaussian steering quantifier  $\mathcal{G}^{B \rightarrow A}$  [37] that quantifies the steerability of mode  $A$  by  $B$ . For two-mode Gaussian steering, the measure of Gaussian steering  $\mathcal{G}^{B \rightarrow A}$  may be mapped from  $S_{A|B}$ , since  $S_{A|B} < 1$  is a necessary and sufficient condition for steering [23]. Explicitly, the steerability of mode  $A$  by  $B$  by the steering parameter  $S_{A|B}$  and the Gaussian steering quantifier  $\mathcal{G}^{B \rightarrow A}$  are related by the expression

$$S_{A|B} = e^{-2\mathcal{G}^{B \rightarrow A}}. \quad (137)$$

We note that  $S_{A|B} = 0$  corresponds to maximum steering. In the present paper, we restrict study to the condition  $S_{A|B} < 1$  which confirms steering without the assumption of Gaussian states, giving an advantage for applications relating to secure quantum communication. However, the Gaussian monogamy relations derived in [37] and verified experimentally [43] will hold for the CV systems we examine in this paper, which are examples of Gaussian states.

We now turn to examine the distribution of bipartite entanglement among the tripartite steerable systems. Rosales-Zárate *et al.* derived four monogamy inequalities for entanglement [118]. A monogamy relation exists for the bipartite CV variance

$$B_{ij} \equiv \frac{1}{4} \{ [\Delta(X_i - X_j)]^2 + [\Delta(P_i + P_j)]^2 \} \quad (138)$$

of Duan, Giedke, Cirac, and Zoller (DGCZ) [120]. The criterion  $B_{ij} < 1$  is sufficient to confirm entanglement between states  $i$  and  $j$  [120]. We note  $B_{12}$  is given by  $B_I$  of Eq. (51) with  $g_3 = 0$ , and  $B_{13}$  is  $B_{III}$  in Eq. (51) with  $g_2 = 0$  (apart from a renormalization factor). For any tripartite state of systems



labeled 1, 2, and 3, the monogamy inequalities

$$B_{12} + B_{13} \geq 1 \quad (139)$$

and

$$B_{12} + B_{13} \geq \max\{1, S_{1|23}\} \quad (140)$$

will always hold [118].

As pointed out by Rosales-Zárate *et al.* [118], the monogamy relation Eq. (139) implies that where there is symmetry between modes 2 and 3 (so that  $B_{12} = B_{13}$ ), it follows that  $B_{12} \geq 0.5$  and  $B_{13} \geq 0.5$ . Maximum bipartite entanglement ( $B_{12} = 0$ ,  $B_{13} = 0$ ) is not possible. Since the CV GHZ is fully symmetric for all three modes, this limit applies to all DGCZ bipartite entanglement for the CV GHZ state, as observed in Fig. 22. In fact, for large  $r$  there is no DGCZ bipartite entanglement for the CV GHZ state. The constraint  $B_{12}, B_{13} \geq 0.5$  also applies to the two symmetric modes of the CV EPR and CV SS states (Figs. 23 and 24). The optimal limit  $B_{12} = B_{13} = 0.5$  is not obtained for any of the three CV states, because of the enhanced fluctuations arising from the antisqueezed quadrature of the highly squeezed inputs. The CV SS state however has fewer squeezed inputs, and Fig. 24 shows that for this case,  $B_{12} = B_{13} \sim 0.625$  for large  $r$ . The monogamy relations Eqs. (139) and (140) plotted in Fig. 23 agree with the results presented by Rosales-Zárate *et al.* (Fig. 3 with the parameter  $\eta_0 = 0.5$  corresponds to the CV EPR state). Rosales-Zárate *et al.* showed how the value  $B_{12} = B_{13} = 0.5$  can be obtained for the CV EPR state, if mode 1 is attenuated to reduce the increased vacuum fluctuations entering from the squeezed input. The full calculations are given in the Supplemental Material [103].

A more general version of the inequality Eq. (139) can be given in terms of the entanglement parameter

$$S_{kl} = \Delta(x_k - h_{kl}x_l)\Delta(p_k + g_{kl}p_l)/(1 + h_{kl}g_{kl}). \quad (141)$$

Here,  $g_{ij}$  and  $h_{ij}$  are gain factors, selected to minimize  $S_{kl}$ . The condition  $S_{ij} < 1$  (for any  $g_{ij}, h_{ij}$ ) confirms entanglement between modes  $i$  and  $j$  without the assumption of Gaussian states, as proved by Giovannetti, Mancini, Vitali, and Tombesi [104]. The monogamy relation

$$S_{kl}S_{km} \geq \frac{\max\{1, S_{k|lm}^2\}}{(1 + h_{kl}g_{kl})(1 + h_{km}g_{km})} \quad (142)$$

holds for all states. The proof of the inequality Eq. (142) extends the work of [118] and is given in the Supplemental Material [103]. This inequality is *equivalent* to the steering inequality, Eq. (136). The criterion  $S_{ij} < 1$  with optimal gains was shown equivalent to the Simon-Peres condition for entanglement, provided there is symmetry between  $X$  and  $P$  such that the correlation matrix elements satisfy  $\langle X_1, X_2 \rangle = -\langle P_1, P_2 \rangle$  in which case  $h_{ij} = g_{ij}$  [121]. For two-mode Gaussian systems  $i$  and  $j$ , the Simon-Peres condition is necessary and sufficient to confirm entanglement [122,123]. The monogamy relation for this special case was derived in [118]. By considering the variance expressions for observables  $\hat{X} = \alpha_1\hat{x}_1 - \alpha_2\hat{x}_2$  and  $\hat{P} = \beta_1\hat{p}_1 + \beta_2\hat{p}_2$ , which involve an extra parameter, Marian and Marian generalized the results

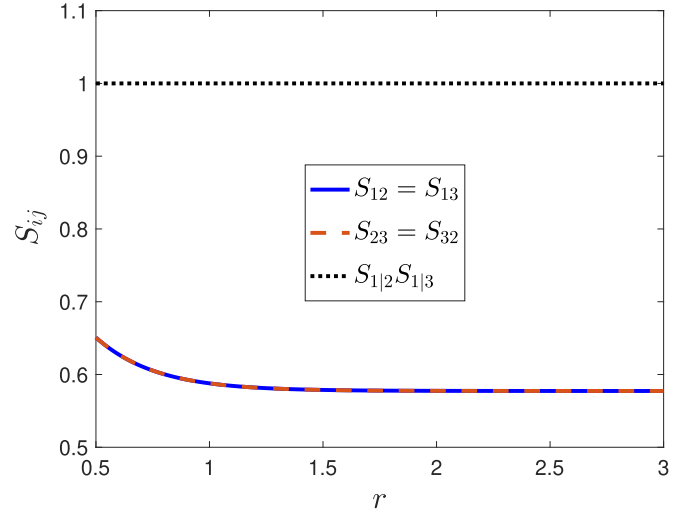


FIG. 25. Bipartite entanglement and entanglement monogamy for the CV GHZ state depicted in Fig. 8. The system is symmetric with respect to all modes. Entanglement between  $i$  and  $j$  is confirmed when  $S_{ij} < 1$ . Bipartite entanglement is observed between all pairs of parties. The values of  $S_{1|2}S_{1|3}$  are given by the upper black dashed line. Saturation of the generalized monogamy entanglement inequality Eq. (142) is observed for all  $k$ .

to show the how EPR variance criteria reduce to the Simon-Peres condition for the two-mode Gaussian states [123].

In Figs. 25–27, we see bipartite entanglement  $S_{kl} < 1$  to be possible for all three states (the CV GHZ, CV EPR, and CV SS states). This extends the results of [118]. The bipartite entanglement in these figures is numerically computed where  $S_{kl}$  is minimized with the optimal gains  $h_{kl}$  and  $g_{kl}$ , using the `fminsearch` function in MATLAB. We found that  $S_{kl} = S_{lk}$  with

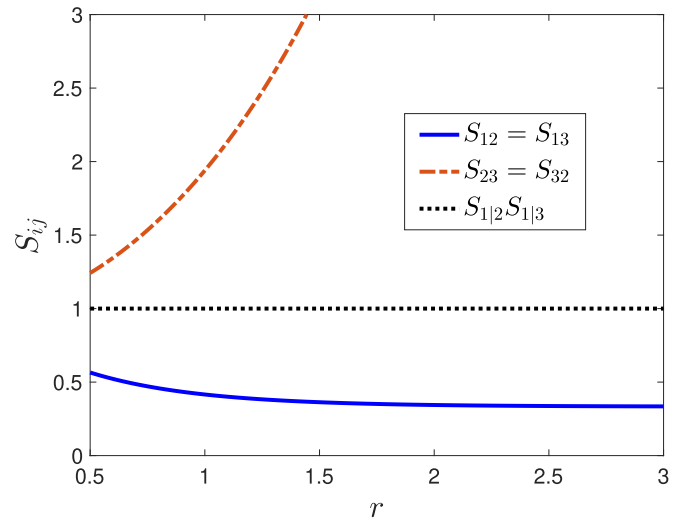


FIG. 26. Bipartite entanglement and entanglement monogamy for the CV EPR state depicted in Fig. 1. Entanglement between  $i$  and  $j$  is observed when  $S_{ij} < 1$ . The values of  $S_{1|2}S_{1|3}$  are given by the upper black dashed line, and correspond to saturation of the generalized monogamy entanglement inequality Eq. (142) for  $k = 1$ . Here, there is bipartite entanglement detectable between systems 1 and 2, and between systems 1 and 3.

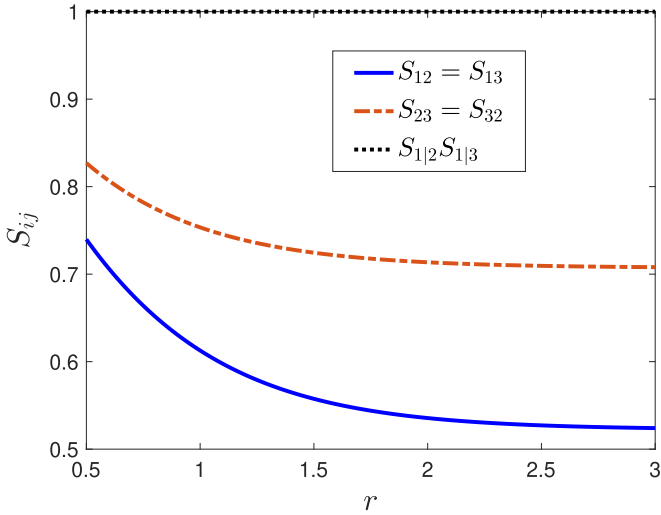


FIG. 27. Bipartite entanglement and entanglement monogamy for the CV SS state depicted in Fig. 5. Entanglement between  $i$  and  $j$  is observed when  $S_{ij} < 1$ . Here, bipartite entanglement is detectable between all pairs of systems. The values of  $S_{1|2}S_{1|3}$  are given by the upper black dashed line, and correspond to saturation of the generalized monogamy entanglement inequality Eq. (142) for  $k = 1$ .

the gains satisfying the relation  $h_{kl} = 1/h_{lk}$  and  $g_{kl} = 1/g_{lk}$ , as proved in He *et al.* [33]. The relation Eq. (142) confirms it is possible to obtain  $S_{ij} < 1$ , even in the presence of symmetric modes. However, the optimal EPR-correlated bipartite states have  $h_{1j} = g_{1j} = 1$ , in which case for symmetrical fields where  $S_{1i} = S_{1j}$ , we will find  $S_{1j} \geq 0.5$ . In fact, we observe the approximate relation  $S_{ij} \gtrsim 0.5$  for each of the three types of CV states.

For the states in this paper,  $S_{1|23}$  is equal to or smaller than 1, and the generalized monogamy inequality becomes  $S_{12}S_{13} \geq 1/[(1 + h_{12}g_{12})(1 + h_{13}g_{13})]$ . A saturation of the generalized entanglement monogamy inequality is then observed when  $S_{1|2}S_{1|3} = 1$ , this being equivalent to the saturation of the steering monogamy relation Eq. (136). The authors of [118] observed that while there is no saturation of the monogamy relations Eq. (139) for the CV EPR and CV GHZ states, saturation of the generalized monogamy relation Eq. (142) can be observed for CV EPR states. However, that result was limited by the restriction  $h_{ij} = g_{ij}$ . Here, we extend this result, by examining the CV GHZ and CV SS state, as given in Figs. 25 and 27. These figures clearly show saturation of the generalized monogamy relation Eq. (27) to be possible for each of these states, for all  $r$ .

## VII. CONCLUSION

In this paper we have examined the concept of multipartite steering for multimode CV systems, deriving inequalities that if violated signify the presence of genuine  $N$ -partite steering. In contrast with much previous work, the inequalities are derived without the assumption of Gaussian states. This gives an advantage for protocols which rely on the rigorous confirmation of multipartite entanglement or steering (e.g., quantum key distribution).

Our classification of  $N$ -partite steering carefully distinguishes between full tripartite steering inseparability and genuine tripartite steering. In fact, because of the asymmetry of the steering correlation with respect to parties which may have different levels of trust, a multitude of definitions is possible. Here, we use a conceptual definition based on the requirement to have steering both ways along each of the different bipartitions of the  $N$  systems. However, alternative definitions are likely to be useful, especially where the  $N$  systems are nodes of a network with fixed levels of trust. In this paper, we also consider one such alternative definition (Definition 3) of genuine multipartite steering, motivated by the example of networks with only one trusted subsystem. We derive inequalities to detect this strict type of steering.

In Sec. IV, strategies are identified to generate genuine multipartite steering based on the coherent beam splitter mixing of one, two, and  $N$  squeezed beams with vacuum modes. We examine the CV GHZ, CV EPR, and CV split-squeezed states that use  $N$ , two, and one squeezed beams as inputs for a network of  $N$  subsystems, respectively. It is shown that the genuine  $N$ -partite steering for each of these states can be detected by the inequalities derived in this paper, provided the squeezing of the inputs is sufficiently large. Specific examples for  $N = 3$  are given in Sec. V, and the genuine  $N$ -partite steering of a tripartite CV cluster state is also analyzed. We also show that these systems predict the strict form of genuine multipartite steering given by Definition 3, and that this type of steering can be detected by a van Loock–Furusawa-type inequality.

Using the inequalities derived in this paper, we confirm that full tripartite steering inseparability has been experimentally generated for three optical modes. Armstrong *et al.* [60] produced CV EPR states and tested the van Loock–Furusawa type inequalities, measuring  $B_I$  and  $B_{II}$  as given by Eq. (51) of our paper. Using Criterion 4b, the experimental values show violation of both  $B_I \geq 2$  and  $B_{II} \geq 2$ , implying full tripartite two-way steering inseparability. In a second experiment, Armstrong *et al.* [38] measured the steering along each bipartition of the three modes  $A$ ,  $B$ , and  $C$  by violating the steering inequalities  $S_{A|BC} \geq 1$ ,  $S_{B|AC} \geq 1$ , and  $S_{C|AB} \geq 1$  [Eqs. (17) and (123)]. This confirms full tripartite steering inseparability, by our definition. The experimental results of [60] also indicate that genuine tripartite steering has been generated for a CV cluster state. This is based on the data reported for the van Loock–Furusawa inequalities. In Sec. VD, we show that Criteria 6c and 7 (with an adjusted phase choice) certify both forms of genuine tripartite steering for this data, by Definitions 1 and 3. Deng *et al.* [43] generated four-mode cluster states and examined full steering inseparability with the focus on monogamy relations between these modes, but within the Gaussian-state assumption.

For tripartite states, the distributions of bipartite steering and of bipartite entanglement among the three subsystems will be constrained. In Sec. VI, we present results for the monogamy of steering and for the monogamy of entanglement, as certified with respect to a particular witness. We study the monogamy properties for the tripartite CV GHZ, CV EPR, and CV SS states. We find that a limited amount of bipartite entanglement may exist between the nodes of the tripartite network, for each of the three types of states, but

that bipartite steering as measured by the CV EPR steering criterion cannot. The monogamy relations we derive do not require the assumption of Gaussian systems and may therefore be useful for CV secret sharing protocols.

### ACKNOWLEDGMENTS

This research has been supported by the Australian Research Council Discovery Project Grants schemes under Grant No. DP180102470. This work was funded by LabEx

ENS-ICFP Grants No. ANR-10-LABX-0010 and No. ANR-10-IDEX-0001-02 PSL\*. This work received funding from MCIN and AEI under Project No. PID2020-115761RJ-I00, from a fellowship from the la Caixa Foundation (Grant No. 100010434), and from the European Union's Horizon 2020 research and innovation program under Marie Skłodowska-Curie Grant No. 847648 (Fellowship Code No. LCF/BQ/PI21/11830025). The authors wish to thank Nippon Telegraph and Telephone (NTT) Research for their financial and technical support.

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- [1] P. van Loock and S. L. Braunstein, *Phys. Rev. Lett.* **84**, 3482 (2000).
- [2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [3] J. Jing, J. Zhang, Y. Yan, F. Zhao, C. Xie, and K. Peng, *Phys. Rev. Lett.* **90**, 167903 (2003).
- [4] B. Jungnitsch, T. Moroder, and O. Gühne, *Phys. Rev. Lett.* **106**, 190502 (2011).
- [5] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, *Phys. Rev. Lett.* **106**, 250404 (2011).
- [6] M. Hillery, V. Bužek, and A. Berthiaume, *Phys. Rev. A* **59**, 1829 (1999).
- [7] S. Bose, V. Vedral, and P. L. Knight, *Phys. Rev. A* **57**, 822 (1998).
- [8] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, *Nat. Phys.* **5**, 19 (2009).
- [9] W. Dür, G. Vidal, and J. I. Cirac, *Phys. Rev. A* **62**, 062314 (2000).
- [10] P. van Loock and A. Furusawa, *Phys. Rev. A* **67**, 052315 (2003).
- [11] G. Svetlichny, *Phys. Rev. D* **35**, 3066 (1987).
- [12] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Going beyond Bell's theorem, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Springer, New York, 1989), pp. 69–72.
- [13] N. D. Mermin, *Phys. Rev. Lett.* **65**, 1838 (1990).
- [14] D. Mayers and A. Yao, in *Proceedings of the 39th Annual Symposium on Foundations of Computer Science*, Conf. Proc. No. 98CB36280 (IEEE, New York, 1998), pp. 503–509.
- [15] D. Mayers and A. Yao, *Quantum Info. Comput.* **4**, 273 (2004).
- [16] J. Barrett, L. Hardy, and A. Kent, *Phys. Rev. Lett.* **95**, 010503 (2005).
- [17] A. Acín, N. Gisin, and L. Masanes, *Phys. Rev. Lett.* **97**, 120405 (2006).
- [18] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [19] D. Rosset, C. Branciard, T. J. Barnea, G. Pütz, N. Brunner, and N. Gisin, *Phys. Rev. Lett.* **116**, 010403 (2016).
- [20] M. G. M. Moreno, S. Brito, R. V. Nery, and R. Chaves, *Phys. Rev. A* **101**, 052339 (2020).
- [21] T. Holz, H. Kampermann, and D. Bruß, *Phys. Rev. Research* **2**, 023251 (2020).
- [22] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- [23] S. J. Jones, H. M. Wiseman, and A. C. Doherty, *Phys. Rev. A* **76**, 052116 (2007).
- [24] H. M. Wiseman, S. J. Jones, and A. C. Doherty, *Phys. Rev. Lett.* **98**, 140402 (2007).
- [25] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, *Phys. Rev. A* **80**, 032112 (2009).
- [26] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, *Rev. Mod. Phys.* **92**, 015001 (2020).
- [27] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, *Phys. Rev. A* **85**, 010301(R) (2012).
- [28] Q. Y. He and M. D. Reid, *Phys. Rev. Lett.* **111**, 250403 (2013).
- [29] C.-Y. Huang, N. Lambert, C.-M. Li, Y.-T. Lu, and F. Nori, *Phys. Rev. A* **99**, 012302 (2019).
- [30] B. Opanchuk, L. Arnaud, and M. D. Reid, *Phys. Rev. A* **89**, 062101 (2014).
- [31] N. Walk, S. Hosseini, J. Geng, O. Thearle, J. Y. Haw, S. Armstrong, S. M. Assad, J. Janousek, T. C. Ralph, T. Symul, H. M. Wiseman, and P. K. Lam, *Optica* **3**, 634 (2016).
- [32] S. Kocsis, M. J. W. Hall, A. J. Bennet, D. J. Saunders, and G. J. Pryde, *Nat. Commun.* **6**, 5886 (2015).
- [33] Q. He, L. Rosales-Zárata, G. Adesso, and M. D. Reid, *Phys. Rev. Lett.* **115**, 180502 (2015).
- [34] B. Yadin, M. Fadel, and M. Gessner, *Nat. Commun.* **12**, 2410 (2021).
- [35] C. Wilkinson, M. Thornton, and N. Korolkova, Quantum steering is the resource for secure tripartite quantum state sharing, [arXiv:2106.06337](https://arxiv.org/abs/2106.06337) (2021).
- [36] M. D. Reid, *Phys. Rev. A* **40**, 913 (1989).
- [37] I. Kogias, A. R. Lee, S. Ragy, and G. Adesso, *Phys. Rev. Lett.* **114**, 060403 (2015).
- [38] S. Armstrong, M. Wang, R. Y. Teh, Q. Gong, Q. He, J. Janousek, H.-A. Bachor, M. D. Reid, and P. K. Lam, *Nat. Phys.* **11**, 167 (2015).
- [39] D. Cavalcanti, P. Skrzypczyk, G. H. Aguilar, R. V. Nery, P. H. S. Ribeiro, and S. P. Walborn, *Nat. Commun.* **6**, 7941 (2015).
- [40] M. Fadel, T. Zibold, B. Décamps, and P. Treutlein, *Science* **360**, 409 (2018).
- [41] P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärtner, and M. K. Oberthaler, *Science* **360**, 413 (2018).
- [42] B. Opanchuk, L. Rosales-Zárata, R. Y. Teh, B. J. Dalton, A. Sidorov, P. D. Drummond, and M. D. Reid, *Phys. Rev. A* **100**, 060102(R) (2019).
- [43] X. Deng, Y. Xiang, C. Tian, G. Adesso, Q. He, Q. Gong, X. Su, C. Xie, and K. Peng, *Phys. Rev. Lett.* **118**, 230501 (2017).
- [44] M. D. Reid, *Phys. Rev. A* **88**, 062108 (2013).

- [45] S. Cheng, A. Milne, M. J. W. Hall, and H. M. Wiseman, *Phys. Rev. A* **94**, 042105 (2016).
- [46] P. Marian and T. A. Marian, *Phys. Rev. A* **103**, 062224 (2021).
- [47] O. Gühne and G. Tóth, *Phys. Rep.* **474**, 1 (2009).
- [48] L. K. Shalm, D. R. Hamel, Z. Yan, C. Simon, K. J. Resch, and T. Jennewein, *Nat. Phys.* **9**, 19 (2012).
- [49] R. Y. Teh and M. D. Reid, *Phys. Rev. A* **90**, 062337 (2014).
- [50] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, *Phys. Rev. Lett.* **109**, 070401 (2012).
- [51] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, *Phys. Rev. A* **88**, 014102 (2013).
- [52] S. Szalay, *Quantum* **3**, 204 (2019).
- [53] Z. Ren, W. Li, A. Smerzi, and M. Gessner, *Phys. Rev. Lett.* **126**, 080502 (2021).
- [54] M. Gessner, M. Fadel, R. Y. Teh, and M. D. Reid (unpublished).
- [55] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, *Phys. Rev. Lett.* **88**, 170405 (2002).
- [56] S. Dutta, A. Mukherjee, and M. Banik, *Phys. Rev. A* **102**, 052218 (2020).
- [57] T. Aoki, N. Takei, H. Yonezawa, K. Wakui, T. Hiraoka, A. Furusawa, and P. van Loock, *Phys. Rev. Lett.* **91**, 080404 (2003).
- [58] A. S. Coelho, F. A. S. Barbosa, K. N. Cassemiro, A. S. Villar, M. Martinelli, and P. Nussenzveig, *Science* **326**, 823 (2009).
- [59] M. Pysher, Y. Miwa, R. Shahrokhshahi, R. Bloomer, and O. Pfister, *Phys. Rev. Lett.* **107**, 030505 (2011).
- [60] S. Armstrong, J.-F. Morizur, J. Janousek, B. Hage, N. Treps, P. K. Lam, and H.-A. Bachor, *Nat. Commun.* **3**, 1026 (2012).
- [61] M. Chen, N. C. Menicucci, and O. Pfister, *Phys. Rev. Lett.* **112**, 120505 (2014).
- [62] S. Gerke, J. Sperling, W. Vogel, Y. Cai, J. Roslund, N. Treps, and C. Fabre, *Phys. Rev. Lett.* **114**, 050501 (2015).
- [63] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, P. Hu, X.-Y. Yang, W.-J. Zhang, H. Li, Y. Li, X. Jiang, L. Gan, G. Yang, L. You, Z. Wang *et al.*, *Science* **370**, 1460 (2020).
- [64] R. Y. Teh, L. Rosales-Zárate, B. Opanchuk, and M. D. Reid, *Phys. Rev. A* **94**, 042119 (2016).
- [65] E. G. Cavalcanti and M. D. Reid, *Phys. Rev. A* **77**, 062108 (2008).
- [66] L. Rosales-Zárate, R. Y. Teh, S. Kiesewetter, A. Brolis, K. Ng, and M. D. Reid, *J. Opt. Soc. Am. B* **32**, A82 (2015).
- [67] B. Julsgaard, A. Kozhekin, and E. S. Polzik, *Nature (London)* **413**, 400 (2001).
- [68] N. Korolkova, G. Leuchs, R. Loudon, T. C. Ralph, and C. Silberhorn, *Phys. Rev. A* **65**, 052306 (2002).
- [69] M. G. Raymer, A. C. Funk, B. C. Sanders, and H. de Guise, *Phys. Rev. A* **67**, 052104 (2003).
- [70] R. Y. Teh and M. D. Reid, *Phys. Rev. A* **100**, 022126 (2019).
- [71] R. Y. Teh and M. D. Reid, *Phys. Rev. A* **102**, 049902(E) (2020).
- [72] E. G. Cavalcanti, P. D. Drummond, H. A. Bachor, and M. D. Reid, *Opt. Express* **17**, 18693 (2009).
- [73] A. Sørensen, L. M. Duan, J. I. Cirac, and P. Zoller, *Nature (London)* **409**, 63 (2001).
- [74] A. S. Sørensen and K. Mølmer, *Phys. Rev. Lett.* **86**, 4431 (2001).
- [75] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, *Nature (London)* **464**, 1165 (2010).
- [76] L. Rosales-Zárate, B. J. Dalton, and M. D. Reid, *Phys. Rev. A* **98**, 022120 (2018).
- [77] M. Fadel and M. Gessner, *Phys. Rev. A* **102**, 012412 (2020).
- [78] R. Schmied, J.-D. Bancal, B. Allard, M. Fadel, V. Scarani, P. Treutlein, and N. Sangouard, *Science* **352**, 441 (2016).
- [79] F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, *Rev. Mod. Phys.* **90**, 025004 (2018).
- [80] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, *Rev. Mod. Phys.* **90**, 035005 (2018).
- [81] Z. Qin, M. Gessner, Z. Ren, X. Deng, D. Han, W. Li, X. Su, A. Smerzi, and K. Peng, *npj Quantum Inf.* **5**, 3 (2019).
- [82] B. J. Dalton, J. Goold, B. M. Garraway, and M. D. Reid, *Phys. Scr.* **92**, 023005 (2017).
- [83] B. J. Dalton, B. M. Garraway, and M. D. Reid, *Phys. Rev. A* **101**, 012117 (2020).
- [84] M. Fadel, L. Ares, A. Luis, and Q. He, *Phys. Rev. A* **101**, 052117 (2020).
- [85] B. Opanchuk, Q. Y. He, M. D. Reid, and P. D. Drummond, *Phys. Rev. A* **86**, 023625 (2012).
- [86] Q. Y. He, T. G. Vaughan, P. D. Drummond, and M. D. Reid, *New J. Phys.* **14**, 093012 (2012).
- [87] C. Gross, H. Strobel, E. Nicklas, T. Zibold, N. Bar-Gill, G. Kurizki, and M. K. Oberthaler, *Nature (London)* **480**, 219 (2011).
- [88] Q. Y. He, P. D. Drummond, M. K. Olsen, and M. D. Reid, *Phys. Rev. A* **86**, 023626 (2012).
- [89] J. Peise, I. Kruse, K. Lange, B. Lücke, L. Pezzè, J. Arlt, W. Ertmer, K. Hammerer, L. Santos, A. Smerzi, and C. Klempt, *Nat. Commun.* **6**, 8984 (2015).
- [90] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, *Phys. Rev. Lett.* **119**, 010402 (2017).
- [91] D. K. Shin, B. M. Henson, S. S. Hodgman, T. Wasak, J. Chwedeńczuk, and A. G. Truscott, *Nat. Commun.* **10**, 4447 (2019).
- [92] M. Bonneau, W. J. Munro, K. Nemoto, and J. Schmiedmayer, *Phys. Rev. A* **98**, 033608 (2018).
- [93] A. Bergschneider, V. M. Klinkhamer, J. H. Becher, R. Klemt, L. Palm, G. Zürn, S. Jochim, and P. M. Preiss, *Nat. Phys.* **15**, 640 (2019).
- [94] A. S. Villar, M. Martinelli, C. Fabre, and P. Nussenzveig, *Phys. Rev. Lett.* **97**, 140504 (2006).
- [95] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989).
- [96] H. F. Hofmann and S. Takeuchi, *Phys. Rev. A* **68**, 032103 (2003).
- [97] S. L. W. Midgley, A. J. Ferris, and M. K. Olsen, *Phys. Rev. A* **81**, 022101 (2010).
- [98] V. Händchen, T. Eberle, S. Steinlechner, A. Samblowski, T. Franz, R. F. Werner, and R. Schnabel, *Nat. Photonics* **6**, 596 (2012).
- [99] W. Tittel, H. Zbinden, and N. Gisin, *Phys. Rev. A* **63**, 042301 (2001).
- [100] A. M. Lance, T. Symul, W. P. Bowen, T. Tyc, B. C. Sanders, and P. K. Lam, *New J. Phys.* **5**, 4 (2003).
- [101] A. M. Lance, T. Symul, W. P. Bowen, B. C. Sanders, and P. K. Lam, *Phys. Rev. Lett.* **92**, 177903 (2004).
- [102] D. Mondal, C. Datta, J. Singh, and D. Kaszlikowski, *Phys. Rev. A* **99**, 012312 (2019).
- [103] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevA.105.012202> for four-partite



- steering in CV EPR state and the derivation of generalized monogamy relation.
- [104] V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, *Phys. Rev. A* **67**, 022320 (2003).
- [105] P. van Loock and S. L. Braunstein, *Phys. Rev. A* **63**, 022106 (2001).
- [106] M. Wang, X. Deng, Z. Qin, and X. Su, *Phys. Rev. A* **100**, 022328 (2019).
- [107] V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A* **61**, 052306 (2000).
- [108] T. J. Osborne and F. Verstraete, *Phys. Rev. Lett.* **96**, 220503 (2006).
- [109] M. Koashi and A. Winter, *Phys. Rev. A* **69**, 022309 (2004).
- [110] B. Toner, *Proc. R. Soc. A* **465**, 59 (2009).
- [111] A. Streltsov, G. Adesso, M. Piani, and D. Bruß, *Phys. Rev. Lett.* **109**, 050503 (2012).
- [112] Y.-K. Bai, Y.-F. Xu, and Z. D. Wang, *Phys. Rev. Lett.* **113**, 100503 (2014).
- [113] S.-W. Ji, M. S. Kim, and H. Nha, *J. Phys. A* **48**, 135301 (2015).
- [114] L. Lami, C. Hirche, G. Adesso, and A. Winter, *Phys. Rev. Lett.* **117**, 220502 (2016).
- [115] Y. Xiang, I. Kogias, G. Adesso, and Q. He, *Phys. Rev. A* **95**, 010101(R) (2017).
- [116] G. Adesso and F. Illuminati, *New J. Phys.* **8**, 15 (2006).
- [117] T. Hiroshima, G. Adesso, and F. Illuminati, *Phys. Rev. Lett.* **98**, 050503 (2007).
- [118] L. Rosales-Zárate, R. Y. Teh, B. Opanchuk, and M. D. Reid, *Phys. Rev. A* **96**, 022313 (2017).
- [119] W. P. Bowen, R. Schnabel, P. K. Lam, and T. C. Ralph, *Phys. Rev. Lett.* **90**, 043601 (2003).
- [120] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).
- [121] Q. Y. He, Q. H. Gong, and M. D. Reid, *Phys. Rev. Lett.* **114**, 060402 (2015).
- [122] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).
- [123] P. Marian and T. A. Marian, *J. Phys. A* **51**, 065301 (2018).