Letter

Ability of unbounded pairs of observers to achieve quantum advantage in random access codes with a single pair of qubits

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Complications in preparing and preserving quantum correlations stimulate recycling of a single quantum resource in information processing and communication tasks multiple times. Here, we consider a scenario involving multiple independent pairs of observers acting with unbiased inputs on a single pair of spatially separated qubits sequentially. In this scenario, we address whether more than one pair of observers can demonstrate quantum advantage in some specific $2 \rightarrow 1$ and $3 \rightarrow 1$ random access codes. Interestingly, we not only address these in the affirmative but also illustrate that unbounded pairs can exhibit quantum advantage. Furthermore, these results remain valid even when all observers perform suitable projective measurements and an appropriate separable state is initially shared.

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Introduction. Random access code (RAC) [1–3] is one of the fundamental communication protocols which, when assisted with quantum resources, manifests the astonishing potential of quantum systems in the context of information processing. In a $n \rightarrow m$ RAC, n is the number of bits (x_0, x_1, ..., x_{n-1}) accessed by the sender, say, Alice. On the other hand, m is the number of bits that Alice is allowed to send the receiver, say, Bob with $m < n$. In each run, Bob chooses the number y randomly (where $y \in \{0, 1, \ldots, n-1\}$) and tries to guess the bit x_y accessed by Alice, but unknown to Bob. The efficacy of RAC is limited when only classical strategies are employed. However, one can surpass the best classical strategies using quantum resources, e.g., by using either quantum communication [3] or classical bit communications assisted with a shared bipartite quantum state [4,5].

RAC assisted with quantum resources was initially introduced [1–3] in order to demonstrate the immense capabilities of quantum systems in information processing tasks. The state of a m-qubit system can be represented by a unit vector in a 2^m dimensional complex Hilbert space, which opens up the possibility of encoding and transmitting classical information with exponentially fewer qubits, for example, Alice encoding n bits into a m-qubit system (where $n \gg m$) and sending it to Bob. However, due to the Holevo bound [6], m qubits cannot transmit more than m classical bits of information faithfully. Hence, it can be inferred that exponentially many degrees of freedom of a quantum system remain inaccessible. Nevertheless, the situation becomes interesting when Bob does not need to know all the n bits of information together and chooses which bit of classical information he would like to extract out of the encoding. In order to extract different bits of information, Bob performs different measurements and these measurements are in general not commuting. Thus, by choosing a particular measurement, Bob inevitably disturbs the state and destroys some or all the information that would have been revealed by other possible measurements. This leads to the idea of RAC assisted with quantum resources. RAC has served as a powerful quantum communication task with various applications ranging from quantum finite automata [2,3,7], communication complexity [8–12], nonlocal games [13], network coding [14,15], locally decodable codes [16–18], dimension witnessing [19–22], quantum state learning [23], self-testing [24–27], quantum randomness certification [28], quantum key distribution [29], and studies of no-signaling resources [30] to characterizing quantum mechanics from information-theoretic principles [31]. Experimental demonstrations of RAC protocols have also been reported [32,33].

In the present study, we consider RAC using classical communications assisted with shared quantum correlations. In reality, it is experimentally difficult to create any quantum correlation. Moreover, environmental interactions unavoidably degrade the efficacy of any quantum correlation. To cope with these, one can recycle a single copy of any quantum resource multiple times. Furthermore, this also indicates how much quantumness in a correlation is preserved even after few cycles of local operations. Historically, this issue was first addressed by Silva *et al.* [34], where two spatially separated spin-$\frac{1}{2}$ particles were assumed to be shared between a single Alice and multiple independent Bobs. In this scenario, the maximum number of Bobs was deduced [34–39], which can

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demonstrate Bell nonlocality [40]. This idea of sharing quantum correlations by multiple sequential observers has been extended in different contexts as well [41–56]. The applications of sequential sharing of quantum correlations in different information processing tasks have also been demonstrated [27,57–63]. In all these studies, multiple observers performing sequential measurements on only one qubit have been considered, whereas the present study contemplates multiple observers performing sequential measurements on each of the two qubits. This is a more general and practical scenario for reutilizing quantum correlations in commercial quantum technologies.

In particular, we focus on recycling a single quantum resource in sequentially carrying out RAC tasks multiple times. Here, we consider the scenario where a two-qubit state is shared between two spatially separated wings. Multiple independent Alices (say, Alice^1, Alice^2, Alice^3, ...) and multiple independent Bobs (say, Bob^1, Bob^2, Bob^3, ...) act sequentially on the first and second qubits respectively with unbiased inputs. At first, Alice^1-Bob^1 executes the RAC task with the initially shared two-qubit state. Afterward, Alice^1 passes her qubit to Alice^2 and Bob^1 passes his qubit to Bob^2. Next, Alice^2-Bob^2 also passes the two qubits to Alice^3-Bob^3 after performing the RAC task and so on.

In the above scenario, we show that unbounded pairs of Alice-Bob (i.e., Alice^1-Bob^1, Alice^2-Bob^2, ...) can gain quantum advantage in executing RAC tasks. Specifically, we demonstrate that the above result holds (1) when all pairs always perform some particular $2 \to 1$ RAC, (2) when all pairs always perform some particular $3 \to 1$ RAC task, (3) when each of the pairs always performs either a $2 \to 1$ RAC or a $3 \to 1$ RAC independent of other pairs, and (4) when each pair performs a $2 \to 1$ RAC and a $3 \to 1$ RAC with different probabilities independent of other pairs. While comparing the classical and quantum strategies to demonstrate quantum advantage, we restrict the amount of shared classical bits to be equal to the amount of shared quantum bits. This constraint is quite natural in the sense that classical bits, similar to qubits, are expensive resources [5,64–66]. Since the aforementioned scenario involves two qubits, quantum strategies are compared with the classical ones assisted with two bits from a common source.

$2 \to 1$ and $3 \to 1$ RAC protocols assisted with classical communication and a two-qubit state. Let us now describe the $n \to 1$ (with $n \in \{2, 3\}$) RAC protocol using limited classical communication and shared two-qubit state. At first, Alice is given a string of n bits $x = (x_0, x_1, \dots, x_{n-1})$ chosen randomly from a uniform distribution with $x_i \in \{0, 1\}$ for all $i \in \{0, 1, \dots, n-1\}$. Next, depending on the input bit string, Alice performs one of the 2^n dichotomic measurements denoted by $A_{x_0x_1\cdots x_{n-1}}$ on her qubit. The outcome of the measurement $A_{x_0x_1\cdots x_{n-1}}$ is denoted by $a_{x_0x_1\cdots x_{n-1}} \in \{0, 1\}$. Alice then communicates the outcome of her measurement to Bob with one bit of information. Next, Bob tries to guess one of the n bits x_y (with $y \in \{0, 1, \dots, n-1\}$) given to Alice (in each run y is chosen randomly). For this purpose, Bob performs one of the n dichotomic measurements denoted by B_y on his qubit. The outcome of the measurement B_y is denoted by $b_y \in \{0, 1\}$. Finally, Bob's guess is given by $a_{x_0x_1\cdots x_{n-1}} \oplus b_y$. Hence, the RAC task will be successful, i.e., Bob's guess will be correct if and only if $a_{x_0x_1\cdots x_{n-1}} \oplus b_y = x_y$.

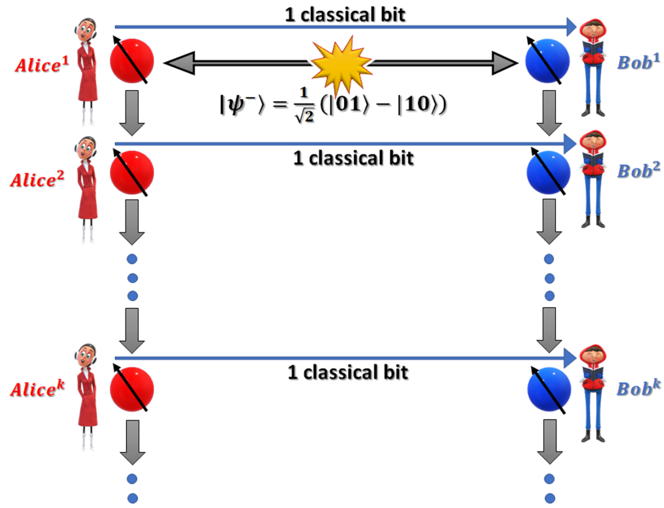

FIG. 1. Scenario for performing $n \to 1$ RAC task with multiple pairs of observers sequentially.

In the present study, we will quantify the efficacy of the RAC protocol by minimum success probability defined as

$$P^{n\to 1}_{\texttt{Min}} = \min_{x_0, x_1, \cdots, x_{n-1}, y} P(a_{x_0x_1\cdots x_{n-1}} \oplus b_y = x_y). \quad (1)$$

Results. We consider a scenario involving multiple independent Alices and multiple independent Bobs as described in Fig. 1. Alice^1-Bob^1 initially shares one pair of qubit in the singlet state, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. This first pair performs the aforementioned RAC task and then, Alice^1, Bob^1 pass their particles to Alice^2, Bob^2 respectively. Alice^2, Bob^2 also pass their particles to Alice^3, Bob^3 respectively after executing the RAC. In this way, the process continues. Note that each of the observers act with unbiased inputs. Here we want to find out how many pairs of Alice and Bob can exhibit quantum advantage. If any pair performs projective measurements, it will disturb the state maximally and the next pair may not get any quantum advantage. Hence, in order to continue the above sequential RAC task with multiple pairs of Alice-Bob, we consider weak measurements by all pairs. We should choose the weak measurement formalism in such a way that the disturbance due to this measurement is minimized for any given amount of information gain [34]. One such example is unsharp measurement (a particular class of Positive Operator-Valued Measure or POVM) [67] with generalized von Neumann–Lüders state-transformation rule [35,42].

In the present paper, we consider two particular RAC tasks. The first one is the $2 \to 1$ RAC task assisted with two (quantum or classical) bits, shared from a common source and having maximally mixed marginal at the receiver's end. In the classical strategy, a source produces two correlated bits which are shared by Alice and Bob. The two binary values 0 and 1 of Bob's bit are equiprobable. Consequently, Alice's encoding and Bob's decoding strategies are now assisted with these bits. The minimum success probability of such a classical RAC task is always less than or equal to $\frac{1}{2}$ [5]. In the case of quantum strategy, two-qubit states with maximally mixed marginal at Bob's end can only be shared in the context of this task and $P_{\tt Min}^{2\to 1} > \frac{1}{2}$ implies quantum advantage.

Another RAC task that we consider is the $3 \to 1$ RAC task assisted with two (quantum or classical) bits shared from a common source. There is no restriction on the marginals of the shared bits in this case. For classical strategies, the minimum success probability is always less than or equal to $\frac{1}{2}$ [5]. Hence, $P_{\tt Min}^{3\to 1} > \frac{1}{2}$ ensures quantum advantage.

Suppose the pair Alice^k-Bob^k for arbitrary $k \in \{1, 2, \dots\}$ performs the above RAC using the shared Bell-diagonal state,

$$\rho_{AB}^k = \frac{1}{4}\left(\mathbf{I}_4 + \sum_{i=1}^{3} t_{ii}^k \,\sigma_i \otimes \sigma_i\right), \tag{2}$$

where $(t_{uu}^k)^2 \geqslant (t_{vv}^k)^2 \geqslant (t_{ww}^k)^2$ for an arbitrary choice of $u \neq v \neq w \in \{1, 2, 3\}$; σ_i with $i = 1, 2, 3$ are the three Pauli matrices.

Next, let us present the encoding-decoding strategies adopted by the pair Alice^k-Bob^k. In the case of the $2 \to 1$ RAC, Alice^k performs one of the four POVMs denoted by $A_{x_0x_1}^k \equiv \{E_{x_0x_1}^{k,0}, E_{x_0x_1}^{k,1}\}$ with $(x_0, x_1) \in \{(00), (01), (10), (11)\}$, where

$$E_{x_0x_1}^{k,\, a_{x_0x_1}^k} = \tfrac{1}{2}\big[\mathbf{I}_2 + \lambda^k\, (-1)^{a_{x_0x_1}^k}\, \big(\hat{u}_{x_0x_1}^k \cdot \vec{\sigma}\big)\big]. \tag{3}$$

Bob^k performs one of the two POVMs denoted by $B_y^k \equiv \{E_y^{k,0}, E_y^{k,1}\}$ with $y \in \{0, 1\}$, where

$$E_y^{k,\, b_y^k} = \tfrac{1}{2}\big[\mathbf{I}_2 + \eta^k\, (-1)^{b_y^k}\, \big(\hat{v}_y^k \cdot \vec{\sigma}\big)\big]. \tag{4}$$

Here λ^k, $\eta^k \in (0, 1]$ are the sharpness parameters; $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$; $a_{x_0x_1}^k$, $b_y^k \in \{0, 1\}$ denote the outcomes of the POVMs $A_{x_0x_1}^k$ performed by Alice^k and B_y^k performed by Bob^k respectively. The unit vectors $\hat{u}_{x_0x_1}^k$ and $\hat{v}_y^k$ are given by

$$\hat{u}_{x_0x_1}^k = \left(\frac{(-1)^{x_0} t_{11}^k}{\sqrt{(t_{11}^k)^2 + (t_{22}^k)^2}}, \frac{(-1)^{x_1} t_{22}^k}{\sqrt{(t_{11}^k)^2 + (t_{22}^k)^2}}, 0\right), \tag{5}$$

$$\hat{v}_0^k = (1, 0, 0), \quad \hat{v}_1^k = (0, 1, 0). \tag{6}$$

On the other hand, for executing the aforementioned $3 \to 1$ RAC, Alice^k performs one of the eight possible POVMs denoted by $A_{x_0x_1x_2}^k \equiv \{E_{x_0x_1x_2}^{k,0}, E_{x_0x_1x_2}^{k,1}\}$ with $x_i \in \{0, 1\}$ for all $i \in \{0, 1, 2\}$, where

$$E_{x_0x_1x_2}^{k,\, a_{x_0x_1x_2}^k} = \tfrac{1}{2}\big[\mathbf{I}_2 + \lambda^k\, (-1)^{a_{x_0x_1x_2}^k}\, \big(\hat{u}_{x_0x_1x_2}^k \cdot \vec{\sigma}\big)\big]. \tag{7}$$

Bob^k performs one of the three POVMs denoted by $B_y^k \equiv \{E_y^{k,0}, E_y^{k,1}\}$ with $y \in \{0, 1, 2\}$, where

$$E_y^{k,\, b_y^k} = \tfrac{1}{2}\big[\mathbf{I}_2 + \eta^k\, (-1)^{b_y^k}\, \big(\hat{v}_y^k \cdot \vec{\sigma}\big)\big]. \tag{8}$$

We choose the unit vectors $\hat{u}_{x_0x_1x_2}^k = \dfrac{\vec{u}_{x_0x_1x_2}^k}{|\vec{u}_{x_0x_1x_2}^k|}$ and $\hat{v}_y^k$ as follows,

$$\vec{u}_{x_0x_1x_2}^k = \big((-1)^{x_0} t_{11}^k,\ (-1)^{x_1} t_{22}^k,\ (-1)^{x_2} t_{33}^k\big), \tag{9}$$

$$\hat{v}_0^k = (1, 0, 0), \quad \hat{v}_1^k = (0, 1, 0), \quad \hat{v}_2^k = (0, 0, 1). \tag{10}$$

With these, we can present the following lemma (for proof, see Appendix A [68]), which will be useful for probing the main result:

Lemma 1. Let Alice^k-Bob^k perform the $n \to 1$ RAC task (where $n = 2$ or $n = 3$) with a two-qubit Bell-diagonal state (2) using the above unsharp measurements. Then the pair achieves minimum success probability strictly greater than $\frac{1}{2}$ if $\min_{i \leqslant n}[(t_{ii}^k)^2] \neq 0$.

Next, we want to find out the postmeasurement state ρ_{AB}^{k+1} received, on average, by Alice^{k+1}-Bob^{k+1} from Alice^k-Bob^k. When Alice^k-Bob^k performs the $2 \to 1$ RAC, following the generalized von Neumann–Lüder's transformation rule, we have (see Appendix B [68])

$$\begin{aligned}\rho_{AB}^{k+1} &= \frac{1}{8} \sum_{x_0, x_1, y=0}^{1} \Bigg[\sum_{a_{x_0x_1}^k, b_y^k=0}^{1} \Big(\sqrt{E_{x_0x_1}^{k,\, a_{x_0x_1}^k}} \otimes \sqrt{E_y^{k,\, b_y^k}}\Big) \\ &\quad \times \rho_{AB}^k \Big(\sqrt{E_{x_0x_1}^{k,\, a_{x_0x_1}^k}} \otimes \sqrt{E_y^{k,\, b_y^k}}\Big)^{\dagger} \Bigg] \\ &= \frac{1}{4}\left(\mathbf{I}_4 + \sum_{i=1}^{3} t_{ii}^{k+1}\, \sigma_i \otimes \sigma_i\right).\end{aligned} \tag{11}$$

The average is taken since we have assumed that multiple Alices or multiple Bobs act independently of each other. Here, we have also used the assumption that Alice^k and Bob^k perform measurements with unbiased inputs. Similarly, when Alice^k-Bob^k performs the $3 \to 1$ RAC, it is observed that the average postmeasurement state ρ_{AB}^{k+1} received by Alice^{k+1}-Bob^{k+1} has the Bell-diagonal form (11) (see Appendix C [68] for details).

Moreover, when Alice^k-Bob^k performs the $n \to 1$ RAC task (where $n = 2$ or $n = 3$) with the state (2), it can be shown that $\min_{i \leqslant n}[(t_{ii}^{k+1})^2] \neq 0$ if $\min_{i \leqslant n}[(t_{ii}^k)^2] \neq 0$ (for details, see Appendix B [68]and Appendix C [68]).

Now, consider that the same $n \to 1$ RAC (i.e., either the $2 \to 1$ or the $3 \to 1$ RAC) is performed by each of the pairs. In such scenario, combining the above results, we can present the following: If Alice^1-Bob^1 initially shares the singlet state, then this pair achieves $P_{\tt Min}^{n\to 1} > \frac{1}{2}$ (with $n = 2$ or $n = 3$) using the aforementioned unsharp measurements. Moreover, the average postmeasurement state ρ_{AB}^2 received by Alice^2-Bob^2 is the Bell-diagonal state (2) with $k = 2$ and $\min_{i \leqslant n}[(t_{ii}^2)^2] \neq 0$. Hence, Alice^2-Bob^2 also achieves $P_{\tt Min}^{n\to 1} > \frac{1}{2}$. Subsequently, Alice^3-Bob^3 receives the Bell-diagonal state (2) with $k = 3$ and $\min_{i \leqslant n}[(t_{ii}^3)^2] \neq 0$ and exhibits $P_{\tt Min}^{n\to 1} > \frac{1}{2}$ as well. This process continues for arbitrarily many pairs. Therefore, we can present the following theorem:

Theorem 1. Unbounded pairs of Alice and Bob can demonstrate quantum advantage either in $2 \to 1$ RAC task assisted with two bits shared from a common source and having maximally mixed marginal at the receiver's end, or in $3 \to 1$ RAC task assisted with two correlated bits.

Importantly, the statements of Theorem 1 hold for all values of $\lambda^k \in (0, 1]$ and $\eta^k \in (0, 1]$ for all possible $k \in \{1, 2, \dots\}$. Moreover, for the aforementioned $n \to 1$ RAC with $n = 2$ or $n = 3$, starting with any Bell-diagonal two-qubit (entangled or separable) state given by Eq. (2) with $k = 1$ and $\min_{i\leqslant n}[(t^1_{ii})^2] \neq 0$, one gets the same result as stated in Theorem 1. Hence, the following corollary can be stated:

Corollary 1. Unbounded pairs of Alice and Bob can exhibit quantum advantage in some particular $n \to 1$ RAC task (with $n = 2$ or $n = 3$) even when each of the observers performs suitable projective measurements and the initially shared two-qubit state belongs to a particular subset of separable states.

When Alice^1-Bob^1 initially shares the singlet state and performs the aforementioned $n \to 1$ RAC (where $n = 2$ or $n = 3$) using the measurements described earlier with $\lambda^1 = \eta^1 = 1$ (i.e., projective measurements), then this pair achieves $P^{n\to 1}_{\mathtt{Min}} = \frac{1}{2}(1 + \frac{1}{\sqrt{n}})$. This is the maximum permissible value of $P^{n\to 1}_{\mathtt{Min}}$ with quantum resources [4]. In this case also, the residual quantum correlation in the average postmeasurement state is sufficient for demonstrating quantum advantage in the $n \to 1$ RAC by unbounded pairs of Alice and Bob. Hence, a single pair of qubits can be utilized indefinitely to gain quantum advantage in some particular RAC even when the optimal quantum advantage is exhibited in the first round.

Remark. We observe that when an arbitrary pair gains a large amount of quantum advantage, then only few numbers of subsequent pairs will get significant quantum advantage. On the other hand, when a pair gets a small amount of quantum advantage, a larger number of subsequent pairs can achieve significant quantum advantage. Here, "significant" quantum advantage implies that $(P^{n\to 1}_{\mathtt{Min}} - \frac{1}{2})$ is positive and large enough to be detected in a real experiment. Hence, there may exist a trade-off relation between the amount of quantum advantage gained by an arbitrary pair and the number of subsequent pairs exhibiting considerable amount of quantum advantage. Moreover, either of these two quantities can be increased at the expense of the other by suitably choosing the sharpness parameters of the measurements (see Appendix D [68]). In practical scenario, a large but finite number of sequential pairs of observers may be required to perform some communication tasks with only one pair of qubits. The number of sequential pairs required to exhibit quantum advantage depends on the particular context under consideration and that can be realized by fine-tuning the unsharpness of the measurements.

Next, we consider a more general scenario where an arbitrary pair Alice^k-Bob^k performs the aforementioned $2 \to 1$ RAC task with probability p_k and the aforementioned $3 \to 1$ RAC with probability $(1 - p_k)$, where $0 \leqslant p_k \leqslant 1$. For example, Alice^k and Bob^k can fix the task to be performed in each experimental run prior to the initiation of sequential RAC and, during the execution of sequential RAC, they perform the two different tasks accordingly. This type of scenario is particularly relevant when a sequence of RAC tasks is implemented as an intermediate step in commercial quantum computation. In such cases, different tasks may be required to be performed by the same pair of particles in different steps depending on the choices of users. In this scenario, if a singlet state or any Bell-diagonal two-qubit (entangled or separable) state given by Eq.(2) with $k = 1$ and $\min[(t^1_{11})^2, (t^1_{22})^2, (t^1_{33})^2] \neq 0$ is initially shared, then the following result is attained (see Appendix E [68] for details):

Corollary 2. Unbounded pairs of Alice and Bob can demonstrate quantum advantage when an arbitrary pair Alice^k-Bob^k performs a $2 \to 1$ RAC (assisted with two correlated bits with maximally mixed marginal at the receiver's end) with probability p_k and a $3 \to 1$ RAC (assisted with two bits shared from a common source) with probability $1 - p_k$ independent of other pairs.

When Alice^k-Bob^k performs projective measurements and $p_k = 1$ (i.e., performs $2 \to 1$ RAC with certainty), then the condition $\min[(t^x_{11})^2, (t^x_{22})^2, (t^x_{33})^2] \neq 0$ will not be satisfied for the average postmeasurement state received by all subsequent pairs (i.e., for all $x \in \{k+1, k+2, \dots\}$). Hence, all these pairs will not achieve quantum advantage in $3 \to 1$ RAC. Hence, only under unsharp measurements (with the sharpness parameters being strictly less than 1), we can state the following corollary (see Appendix E [68] for details),

Corollary 3. Unbounded pairs of Alice and Bob can demonstrate quantum advantage when an arbitrary pair Alice^k-Bob^k performs a $2 \to 1$ RAC with certainty and another arbitrary pair Alice$^{\tilde{k}}$-Bob$^{\tilde{k}}$ performs a $3 \to 1$ RAC with certainty for all choices of $k \neq \tilde{k} \in \{1, 2, \dots\}$.

Conclusions. Here we have considered a scenario involving multiple independent pairs of Alice and Bob sharing a single pair of qubits and performing some particular $2 \to 1$ and $3 \to 1$ RAC tasks with unbiased inputs sequentially. In this scenario, we have shown that unbounded pairs can gain quantum advantage even when all observers perform projective measurements. These results address the issue of recycling a single copy of a quantum resource in performing information processing tasks multiple times sequentially. This is of utmost importance since, in reality, preparing quantum correlations and preserving them against inevitable environmental interactions are difficult.

Our results point out that quantum correlations present in separable states [69] can be preserved indefinitely in spite of utilizing it in each step. Furthermore, weak measurements are not necessary for this purpose; suitable projective measurements can serve for this. Note that this is not the case for entanglement or Bell nonlocality. Hence, these results signify one fundamental difference between the quantum correlations present in entanglement and that present in separable states: The first one is destroyed only after one cycle of projective measurements while the second one is retained even after infinite cycles. The advantage of quantum information processing tasks assisted with separable states [5,66] is thus pointed out by our present study. In fact, our results open up the possibility of implementing unbounded sequence of any task, for which quantum advantage can be demonstrated even using separable states (say, for example, remote state preparation [70]), with only one pair of qubits.

There exists a complementarity between the question addressed here and the one-way communication complexity problem [71,72]. In one-way communication complexity problem, Alice and Bob are given inputs $x \in \{0, 1\}^n$ and $y \in \{0, 1\}^m$ respectively. The goal for Bob is to calculate a binary function $f(x, y)$. Alice is allowed to send limited classical communications to Bob. This game can be thought as a number of parallel RACs taking place simultaneously. The main goal of any communication complexity problem is to minimize the amount of classical communication. However, there is no restriction on the shared entanglement. On the contrary, the present study is aimed to reduce the amount of shared correlation, but does not focus on reducing the number of communicating bits.

Recently, measurement protocols have been proposed to demonstrate arbitrary many Bell-CHSH inequality [40] violations with various independent Bobs and a single Alice using unbiased inputs when a pure entangled two-qubit state is initially shared [39]. The result, however, requires arbitrarily high precision engineering for the measurement apparatus and, hence, is too strenuous to implement in a reality. On the other hand, the unsharp measurements chosen in the present study can be realized in photonic systems based on the techniques adopted in Refs. [62,63]. Moreover, our results are valid for any range of sharpness parameters and do not require any entanglement. Hence, for experimental implementation of large sequence of detecting quantum correlation with a single two-qubit state, our results are less demanding.

This study points out that there exist some communication tasks in which unbounded pairs of observers can exhibit quantum supremacy even if a single quantum resource is used. Finding out different communication tasks with the above feature merits further investigation. Next, it is worthwhile to fully characterize the set of two-qubit states for which Theorem 1 holds. It is also interesting to find out whether there exists any two-qubit state for which weak measurements are necessary for satisfying Theorem 1.

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