PHYSICAL REVIEW A **104**, 063313 (2021)

Effective scaling approach to frictionless quantum quenches in trapped Bose gases

Tang-You Huang,[1,2] Michele Modugno,[3,4] and Xi Chen[2]

[1]International Center of Quantum Artificial Intelligence for Science and Technology (QuArtist) and Department of Physics, Shanghai University, 200444 Shanghai, China

[2]Department of Physical Chemistry, University of the Basque Country UPV/EHU, Apartado 644, 48080 Bilbao, Spain

[3]Department of Physics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain

[4]IKERBASQUE, Basque Foundation for Science, 48013 Bilbao, Spain

(Received 16 July 2021; accepted 7 December 2021; published 17 December 2021)

We work out the effective scaling approach to frictionless quantum quenches in a one-dimensional Bose gas trapped in a harmonic trap. The effective scaling approach produces an auxiliary equation for the scaling parameter interpolating between the noninteracting and the Thomas-Fermi limits. This allows us to implement a frictionless quench by inversely engineering the smooth trap frequency, as compared to the two-jump bang control. Our result is beneficial to design a shortcut-to-adiabaticity expansion of trapped Bose gases for arbitrary values of interaction, and can be directly extended to the three-dimensional case.

DOI: 10.1103/PhysRevA.104.063313

I. INTRODUCTION

Bose-Einstein condensates (BECs) and their related phenomena—such as collective excitations, collapse, and nonlinear dynamics, to mention a few—have aroused great interest since their first experimental realization [1,2]. From the theoretical point of view, weakly interacting BECs can be accurately described within the framework of the Gross-Pitaevskii (GP) theory, which provides a remarkable agreement with experimental observations. In most cases of classical hydrodynamics [3] or scaling transformations [4,5], an exact analytical solution can be found for the collective dynamics and free expansion of BECs in time-dependent harmonic traps, both in the noninteracting limit and the Thomas-Fermi (TF) regime [2]. In this vein, symmetries give birth to an intriguing property of self-similarity, which allows us to utilize the scaling approach for describing the dynamics of ultracold atomic systems, for instance, the atomic gases in the noninteracting and the hydrodynamic regimes [6,7], Tonks-Girardeau (TG) gas of impenetrable bosons [8,9], superfluid Fermi gas [7,10], and thermal clouds [11] in different geometries.

In addition, an effective scaling approach has further been proposed as an approximate solution for the evolution of both bosonic and fermionic density distributions, for describing the collective dynamics of a trapped Bose gas [12], the expansion of Fermi gas [13,14], and of quantum degenerate Bose-Fermi mixtures [15]. It consists in a self-similar evolution in the hydrodynamic regime to be satisfied on average by integrating over the spatial coordinates, reducing the complexity of the numerical treatment. Recently, the accuracy of such an effective approach in reproducing the exact solution of quasi-one-dimensional (1D) and three-dimensional (3D) GP equations for arbitrary values of the interactions has been discussed in Refs. [16,17]. Remarkably, it turns out that the space-averaged self-similarity can provide an accurate description in several situations [18].

In a slightly different but relevant topic, the concept of shortcuts to adiabaticity (STA), originally proposed for fast schemes reproducing or approaching slow adiabatic processes [19,20], has extended further the control paradigms for frictionless atomic cooling in an expanding harmonic trap [21–26]. In the context of inverse engineering, the scaling approach [21] and Lewis-Riesenfeld dynamical invariant [22] bring out the various forms of the Ermakov equation for the scaling parameter, capturing the character of the self-similar evolution. Along with it, the time-varying harmonic trap frequency is thus inversely engineered for this purpose, by choosing an interpolation function of the scaling parameter with appropriate boundary conditions. This strategy can be applicable to other ultracold atomic systems as a TG gas [27], an anisotropic gas containing quantum defects [28], and a Fermi gas [29]. However, tracking back to a quasi-1D BEC described by the GP equation in the mean-field approximation, one can realize that the original Ermakov equation obtained in the noninteracting case needs to be modified in the TF limits (where the self-similarity is exact) or in the case of a time-dependent interaction [21,30]. To remedy it, the Gaussian variational approximation [31] (which is equivalent to the moment method [32]), can be complemented by the concept of STA, for studying the dynamics of BECs [33–35], valid for weakly interacting mean-field regimes, with implications on the quantum speed limits and quantum thermodynamics [36,37]. As a matter of fact, the aforementioned variational approximation eventually breaks down, when one increases the atomic interactions, since its accuracy depends strongly on the presumed ansatz in terms of nonlinearity [34,35]. Thus, the motivation of this work is to fill the gap in more general theory, beyond the noninteracting and TF limits, on STA design for a 1D Bose gas with arbitrary interactions.

In this paper, we integrate the effective scaling approach into inverse engineering for frictionless quantum quenches in a trapped Bose gas, for arbitrary values of the atomic interaction strength. Here, we focus on the 1D GP equation, but the result can be extended to the 3D case. By assuming the scaling solution in the hydrodynamic limit, we derive the Ermakov-like equation for the scaling parameter interpolating between noninteracting and the TF limits. With this, the frictionless quench is designed, and also compared to the free expansion and two-jump trajectory of bang control. Finally, the numerical simulation is performed to check the stability of our method, and the energetic cost of STA is discussed as well.

II. EFFECTIVE SCALING APPROACH

We start by considering a quasi-1D BEC confined in a cigar-shaped trap, characterized by a longitudinal frequency ω_0 and a tight transverse frequency $\omega_\perp \gg \omega_0$. Therefore, the system can be effectively described by a wave function $\psi(x,t)$, whose dynamics is governed by the following GP equation,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}\omega^2(t)x^2\psi + g|\psi|^2\psi, \tag{1}$$

that is written here in dimensionless form, for convenience. To this end, we have used $l_0 = \sqrt{\hbar/m\omega_0}$ as the unit length (m being the particle mass), $\hbar\omega_0$ as the unit energy, and ω_0^{-1} as the unit time. The interaction strength can be written in terms of the scattering length a_s as $g = 2Na_s/\ell_0$, with N being the number of atoms and with the total density being normalized to one.

In order to elaborate the effective scaling approach, we apply the Madelung transformation $\psi = \sqrt{n(x,t)}e^{i\phi(x,t)}$, such that the Lagrangian of the system can be written as

$$\mathcal{L} = -\left[\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{8}\left(\frac{\nabla n}{n}\right)^2 + V(x,t) + gn\right]n, \tag{2}$$

where $V(x,t) = \omega^2(t)x^2/2$. The essence of the variational Lagrangian formalism is to minimize the action $S = \iint \mathcal{L}dxdt$ with respect to the parameters $q_i = \{n, \phi\}$, that is, $\delta S/\delta q_i = 0$. The latter corresponds to the Eular-Lagrangian equations $\partial\mathcal{L}/\partial q_i - d(\partial\mathcal{L}/\partial\dot{q}_i)/dx = 0$, from which the hydrodynamic equations are obtained as [2]

$$\frac{\partial n}{\partial t} + \frac{\partial(n\nabla\phi)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial(\nabla\phi)}{\partial t} + \partial_x\left(P(x,t) + \frac{1}{2}v^2 + V(x,t) + gn\right) = 0, \tag{4}$$

where $P(x,t) = -(\partial_x^2\sqrt{n})/(2\sqrt{n})$ is the so-called quantum pressure. By inserting the scaling solution $n = n_0(x/a)/a$ and $v = \dot{a}x/a$ [2] into Eq. (4), one can obtain

$$-\frac{1}{2}\frac{\ddot{a}}{a}x^2 = P(x,t) + V(x,t) + gn - P(0,t) - gn(0,0). \tag{5}$$

Multiplying the resulting equation by $n_0(\xi)$ after rescaling the spatial as $\xi = x/a$, and integrating over the coordinates, we get the following *effective* Ermakov-like equation [17]

$$\ddot{a} + \omega^2(t)a = \frac{A}{a^3} + \frac{B}{a^2}, \tag{6}$$

where A and B read

$$A = \frac{P(0,0) - E_k^0}{E_v^0}, \quad B = \frac{gn(0,0) - E_{\rm int}^0}{E_v^0}, \tag{7}$$

with $E_k^0 \equiv (1/2)\int[\partial_x\sqrt{n_0(x)}]^2dx$ being the kinetic energy, $E_v^0 \equiv \int V(x,0)n_0(x)dx$ the potential energy, and $E_{\rm int}^0 \equiv (g/2)\int n_0^2(x)dx$ the interaction energy, all evaluated at the initial time $t = 0$. Here, $n_0(x) \equiv n(x,0)$ represents the ground state of the system in the harmonic trap. As shown in Ref. [17], the parameters A and B satisfy the relation $A + B = 1$ for arbitrary (positive) interactions.

Remarkably, the Ermakov-like Eq. (6) permits us to describe the dynamics of the system in terms of an effective self-similar evolution, for arbitrary interactions. Note that this equation is more accurate than the one obtained by applying a Gaussian ansatz for arbitrary interactions [31,34],

$$\ddot{a} + \omega^2(t)a = 1/a^3 + g/(\sqrt{2\pi}a^2). \tag{8}$$

Obviously, Eq. (6) reproduces the exact scaling in the noninteracting ($g = 0$) and Thomas-Fermi limits ($g \gg 1$). Namely, for $g = 0$ the above Eq. (6) corresponds to the original Ermakov equation,

$$\ddot{a} + \omega^2(t)a = 1/a^3, \tag{9}$$

with $n_0(x) = e^{-x^2/a^2}/\sqrt{\pi a^2}$ and $A = 1, B = 0$. This is consistent with the results derived from variational control [31,34] and also from the scaling approach and Lewis-Riesenfeld dynamical invariant [21,22]. In the opposite TF regime, the ground state density is $n_0 = [\mu - \omega(0)x^2/2]/g$, with $\mu = E_{\rm int}^0 + E_v^0$ being the chemical potential (the kinetic energy can be safely neglected in this limit [2]). Then, inserting n_0 into Eq. (7) yields ($A = 0, B = 1$)

$$\ddot{a} + \omega^2(t)a = 1/a^2, \tag{10}$$

which corresponds to the exact TF result [21,30].

III. SHORTCUTS TO ADIABATICITY

In this section, we use the Ermakov-like equation (6) to construct a STA protocol for $\omega(t)$, for achieving a frictionless quench from the initial trap frequency $\omega_{\rm i} \equiv \omega(0)$ (with $\omega(0) = 1$ fixed by our notation choice) to a final value $\omega_{\rm f} \equiv \omega(t_{\rm f})$ within a short time $t_{\rm f}$, with $\gamma \equiv \sqrt{\omega_{\rm i}/\omega_{\rm f}} > 1$. That is, in a finite-time nonadiabatic expansion,the trap frequency is changed to some lower final value, while keeping the populations of the initial and final levels invariant, thus without generating friction and heating. The frictionless cooling of ultracold atoms trapped in time-dependent harmonic traps was originally investigated in two noninteracting and TF limits [21,22]. Later, the variational approximation was used to design the same process in the weakly interacting regime [34]. Here, we propose a general approach based on the formulation discussed in the previous section, for arbitrary values of the interactions, ranging from the noninteracting to the TF regime.

To this end, we recast Eq. (6) in the form of the perturbative Kepler problem in classical mechanics, in the presence of the effective potential

$$\mathcal{U}(a) = \frac{A}{2a^2} + \frac{B}{a} + \frac{1}{2}\omega^2(t)a^2, \tag{11}$$

for a fictitious particle with unit mass, satisfying the Newton equation $\ddot{a} = -\partial\mathcal{U}(a)/\partial a$, derived from Eq. (6). The total energy of the particle reads

$$E(a) = \frac{\dot{a}^2}{2} + \frac{A}{2a^2} + \frac{B}{a} + \frac{1}{2}\omega^2(t)a^2. \tag{12}$$

The conditions for an adiabatic evolution are $\dot{a} = 0$ and $\partial\mathcal{U}(a)/\partial a = 0$ [33], yielding

$$a^4\omega^2(t) - aB = A. \tag{13}$$

Then, we define the time-averaged energy

$$\mathcal{E} = \frac{1}{t_\mathrm{f}}\int_0^{t_\mathrm{f}} E(a)dt, \tag{14}$$

that will be used for quantifying the energetic cost of STA, in the discussion below. The initial boundary conditions read [33]

$$a(0) = a_\mathrm{i}, \quad \dot{a}(0) = \ddot{a}(0) = 0, \tag{15}$$

$$a(t_\mathrm{f}) = a_\mathrm{f}, \quad \dot{a}(t_\mathrm{f}) = \ddot{a}(t_\mathrm{f}) = 0, \tag{16}$$

where a_i and a_f are the unique positive real solutions of Eq. (13), at $t = 0$ and $t = t_\mathrm{f}$. These boundary conditions (15) and (16) guarantee that the initial and final states are adiabatic correspondences for designing STA protocols. Having fixed the boundary conditions, the trajectory of $a(t)$ can be interpolated, by choosing a simple polynomial ansatz of the form

$$a(t) = a_\mathrm{i} - 6(a_\mathrm{i} - a_\mathrm{f})s^5 + 15(a_\mathrm{i} - a_\mathrm{f})s^4 - 10(a_\mathrm{i} - a_\mathrm{f})s^3, \tag{17}$$

with $s = t/t_\mathrm{f}$. Consequently, the trap frequency $\omega(t)$ is determined by Eq. (6). If an imaginary trap frequency is allowed, the harmonic trap inverts to a parabolic repeller, instead of a trap, such that t_f may be formally made arbitrarily short. However, since in experimental implementations there are always imperfections and limitations related, e.g., to the trap anharmonicity and to the laser power, this poses a constraint of the amplitude of the frequency that can be physically achieved, namely $|\omega^2(t)| \leqslant \delta$, with δ being a real number [30,34,38].

In Fig. 1 we show the results for a STA protocol of a frictionless quench from $\omega_\mathrm{i} = 1$ to $\omega_\mathrm{f} = 0.1$, within a time $t_\mathrm{f} = 1$. Figures 1(a) and 1(b) illustrate the evolution of the trap frequency $\omega(t)$ and of the width $a(t)$ for different values of the interactions, $g = 0.01, 1, 100$. With the designed protocol, we use the split-operator method to propagate numerically the wave function to the final state $|\psi_\mathrm{f}\rangle$, that is then compared to the ground state of the final trap $|\tilde{\psi}_\mathrm{f}\rangle$ [see Fig. 1(c)] (the latter is computed by means of a standard by imaginary-time evolution). The corresponding fidelity, $F \equiv |\langle\tilde{\psi}_\mathrm{f}|\psi_\mathrm{f}\rangle|^2$, is shown in Fig. 2 as a function of the interaction strength g, for different values of the trap frequency. Notice that $F \equiv 1$ in both the noninteracting ($g \ll 1$) and TF limits ($g \gg 1$), where the scaling ansatz is an exact solution. Remarkably, Fig. 2(a) illustrates the fidelity always stays very close to one even in the intermediate regime, owing to the accuracy of the effective scaling approach [17]. On the contrary, the previous variational approximation [34] breaks down when the increase of the atomic interaction makes the Gaussian ansatz invalid [see Fig. 2(b)]. In Fig. 2(a), the deviations from $F = 1$ are less than 1%, and show a weak dependence on the trap frequency, i.e., the fidelity decreases as ω_f is decreased. This is due to the fact that the interaction becomes dominant when the trap frequency is negligible. The reason why the effective scaling approach yielding such excellent fidelities can be found in

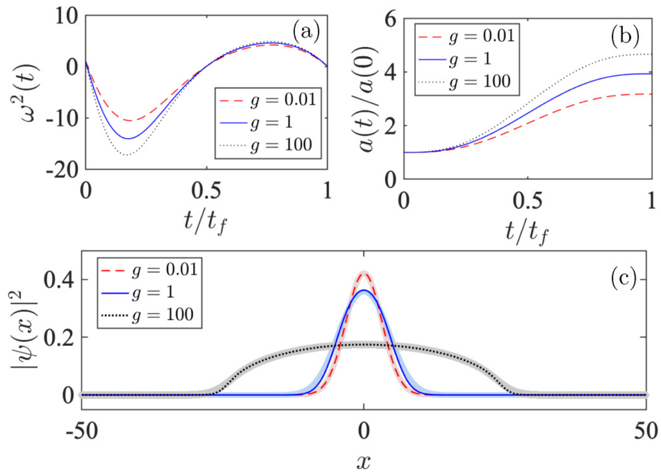

FIG. 1. (a) The designed trap frequency $\omega^2(t)$ and (b) the corresponding width $a(t)$ as a function of time, for $g = 0.01$ (red dashed line), $g = 1$ (blue solid line), and $g = 100$ (black dashed line). (c) The final states (thick lines) are compared to the corresponding stationary states (thin lines). Parameters: $\omega_\mathrm{f} = 0.1$ and $t_\mathrm{f} = 1$.

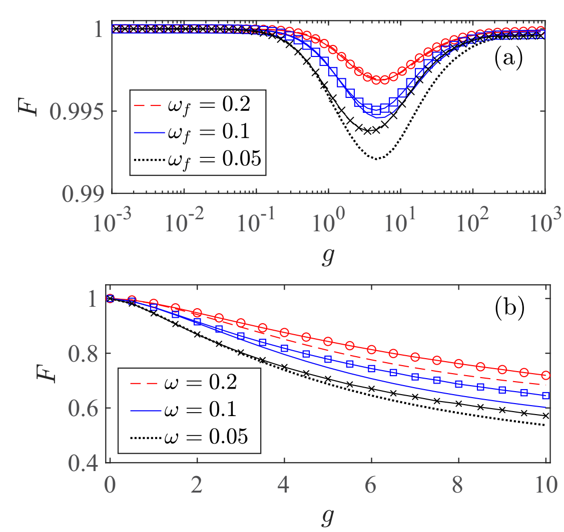

FIG. 2. (a) The fidelity F (see text) vs the interaction strength g, for a frictionless quench (symbols) and a sudden quench (lines). Three final frequencies are considered: $\omega_\mathrm{f} = 0.05, 0.1, 0.2$. Notice that, whereas in the first case the final time is fixed to $t_\mathrm{f} = 1$, in case of two-jump bang control it depends on the (final) frequency [see Eq. (23)]. (b) The fidelity obtained from the Gaussian variational approximation is compared for both frictionless and sudden quenches, to show the effective scaling approach definitely provides a better accuracy.

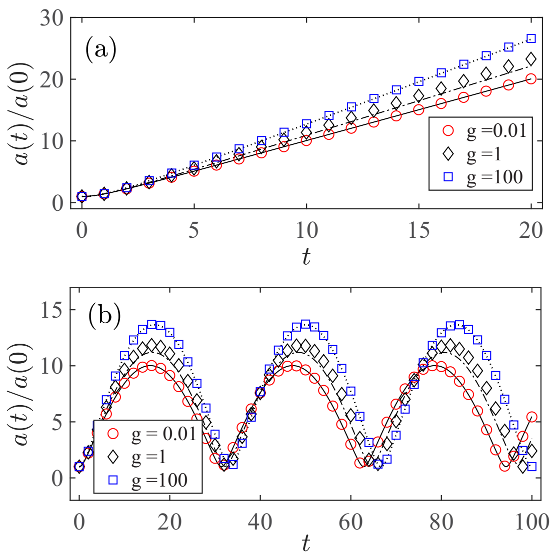

FIG. 3. (a) Dynamics of free expansion when the trap is suddenly switched off, i.e., $\omega_{\rm f} = 0$. (b) Collective oscillation modes after quenching trap frequency to $\omega_{\rm f} = 0.1$. For comparison, the center of the mass of the wave packet, $a(t) = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, calculated from numerical simulation, is denoted by symbols. Parameters: $g = 0.01$ (sold line, with red circle), $g = 1$ (dashed line, with black diamond), and $g = 100$ (dotted line, with red square), and the initial trap frequency is $\omega_0 = 1$.

Ref. [17], where it was shown that indeed the axial expansion dynamics is characterized by an approximate self-similarity. Notice that the behavior of the fidelity presented here can be also intuitively understood in terms of the stability of a particle in the presence of the effective potential [see Eq. (11)].

IV. SUDDEN QUENCH

The discontinuous piecewise-constant control, with two-jump or three-jump trajectories, also provides the alternatives of STA, relevant to the time minimization [34,39]. For completeness, here we consider the case of a sudden quench of the trap frequency:

$$\omega(t) = \begin{cases} \omega_{\rm i}, & t = 0, \\ \omega_{\rm f}, & 0 < t < t_{\rm f}. \end{cases} \tag{18}$$

In the noninteracting case, the conventional Ermakov equation (9) gives the analytical solution

$$a(t) = \sqrt{1 + \left(\omega_{\rm i}^2 - \omega_{\rm f}^2\right)\sin^2(\omega_{\rm f} t)/\omega_{\rm f}^2}, \tag{19}$$

with initial boundary conditions $a(0) = 1$, and $\dot{a}(0) = 0$, which corresponds to a collective oscillation with period $\tau = \pi/\omega_{\rm f}$. Therefore, in general the boundary conditions (15) and (16) cannot be not attained at $t = t_{\rm f}$, and this implies a heating or excitation of the system.

In the limit $\omega_{\rm f} \to 0$, the case of a free expansion Eq. (19) reduces to $a(t) = (1 + \omega_{\rm i}^2 t^2)^{1/2}$. In more general cases, analytical solutions for sudden quenches are not available, but one can still solve Eq. (6) numerically, by setting $\omega^2(t) = 0$. In Fig. 3, we demonstrate that Eq. (6) is accurate enough to describe the dynamics in sudden quenches, when the values of the nonlinear interaction are changed from zero to infinity.

Based on the above considerations, we can build a STA protocol with just two quenches, as proposed for the compression of solitons in nonlinear fibers [40]: An initial quench from $\omega_{\rm i}$ to ω_c at $t = 0$, and a second one from ω_c to $\omega_{\rm f}$, at the final time $t_{\rm f}$ such that

$$t_{\rm f} = \pi/(2\omega_c), \tag{20}$$

with $\omega_c = \sqrt{\omega_{\rm i}\omega_{\rm f}}$. Then, in the noninteracting case we have [see again Eqs. (18) and (19)]

$$a(t) = \sqrt{1 + \left(\omega_{\rm i}^2 - \omega_c^2\right)\sin^2(\omega_c t)/\omega_c^2}, \tag{21}$$

satisfying the above-mentioned conditions, $a(0) = 1$, $a(t_{\rm f}) = \gamma$, and $\dot{a}(0) = \dot{a}(t_{\rm f}) = 0$ [34,39]. Thus, Eqs. (20) and (21) provide a simple exact solution for the shortcut with just one intermediate frequency, the geometric mean of the initial and final frequencies. Importantly, the solution of two-jump bang control can be generalized for Eq. (6), without requiring a explicit form. The energy conservation in this perturbative Kepler problem implies $U(a_{\rm i}) = U(a_{\rm f})$, or, in an explicit form,

$$\omega_c = \sqrt{\frac{A}{a_{\rm i}^2 a_{\rm f}^2} + \frac{2B}{a_{\rm i} a_{\rm f}(a_{\rm i} + a_{\rm f})}}. \tag{22}$$

In this case, a simple expression for $t_{\rm f}$ is not available, but it can be written in the form of an integral,

$$t_{\rm f} = \int_{a_{\rm i}}^{a_{\rm f}} \frac{da}{\sqrt{2[U(a_{\rm in}) - U(a)]}}, \tag{23}$$

where $a_{\rm i}$ and $a_{\rm f}$ are given by Eq. (13). Accordingly, the trajectory of $a(t)$ can be obtained numerically from Eq. (6). The corresponding fidelity is plotted in Fig. 2(a) along with the results of the frictionless quench with arbitrary interactions involved. This figure shows that the two-jump bang control is less accurate than the STA by using a smooth polynomial function, when the final frequency is smaller. Another important difference between the two approaches is the fact that in the STA protocol discussed in the previous section the final time is a free parameter, whereas in the present case it is fixed by the trap frequencies and the interaction strength [see Eq. (23)]. We should emphasize that the larger repulsive interaction somehow speeds up the STA with two-jump trajectory, e.g., $t_{\rm f} = 4.9658$ for $g = 0.01$ while $t_{\rm f} = 4.5380$ for $g = 100$, when $\omega_{\rm i} = 1$ and $\omega_{\rm f} = 0.1$.

V. DISCUSSION

In the STA protocol, there is a trade-off between speed and energetic cost [41]. In principle, the transient energy excitation of the STA protocol, described by Eq. (12), is stipulated by the time-energy uncertainty, which implies an increase of the energy for shorter times. In detail, Fig. 4 illustrates the energetic cost exhibits an $\mathcal{E} \propto 1/t_{\rm f}^2$ behavior for different regimes of atomic interaction by using the STA protocols in Fig. 1, where the time-averaged energy $\mathcal{E}$ is calculated by Eq. (14). Moreover, for the same $t_{\rm f}$, the energy excitation is higher when the interaction is larger. In this sense, the given energetic cost gives the tight bound on the running time

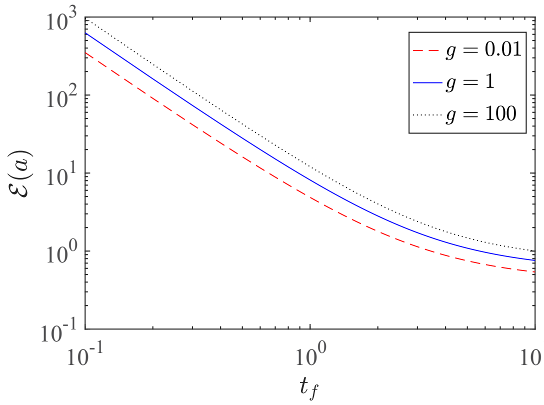

FIG. 4. Dependence of the the time-averaged energy $\mathcal{E}$ in Eq. (14), on the short time $t_{\rm f}$, where the parameters are the same as those in Fig. 1.

of STA. Additionally, we can obtain the dependence of the time-averaged energy on the final frequency $\omega_{\rm f}$ as well. The asymptotic behavior gives $t_{\rm f} \propto (\omega_{\rm i}/\omega_{\rm f})^{1/2}$ in the noninteracting limit while $t_{\rm f} \propto (\omega_{\rm i}/\omega_{\rm f})^{3/2}$ in the TF limit [34], which has fundamental implications on the quantum speed limit and the third law of thermodynamics in the request for absolute zero temperature [42]. Instead, one can further apply the Pontryagin's maximum principle in optimal control theory [30,38] to design time-optimal bang-bang control with a three-jump trajectory, when the trap frequency is bounded. We expect that the atomic interaction slows down the frictionless quenches, and the minimal time of STA in the intermediate interaction regime is between the ones in the noninteracting and TF limits (see Ref. [34]).

So far, we have considered a quasi-1D BEC confined in an elongated trap, characterized by a tight transverse confinement (with respect to the longitudinal one). In this scenario, what guarantees the effectiveness of the scaling approach is the fact that self-similarity along the transverse direction is preserved during the whole dynamics. That is to say, our approach is effective whenever the presence of a transverse trapping keeps the 3D system in a quasi-1D regime. As shown in Ref. [18], an effective self-similarity can be expected also for a full 3D expansion from almost isotropic traps, so that an analogous STA approach can be worked out in such situations.

VI. CONCLUSION

To conclude, we have employed the effective scaling approach to derive the Ermakov-like equation (6) for the scaling parameter interpolating between the noninteracting and the TF limits. By combining inverse engineering with appropriate boundary conditions, this provides a general way to design accurate STA for a 1D Bose gas, for arbitrary values of the atomic interaction strength. Here, we have considered the case of a quasi-1D condensate confined in a cigar-shaped trap, with a tight transverse frequency (that is, much larger than the longitudinal one). These results can be easily generalized to the case of almost isotropic 3D traps.

In addition, the effective scaling approach is harnessed to design a STA protocol for a trapped 1D Bose gas in an arbitrary repulsive interacting regime, where the previous methods, such as a dynamical invariant or scaling approach, cannot work successfully. We emphasize that the method is similar to, yet different from, the previous variational approximation [31,34]. In the effective scaling approach, the scaling solution in the hydrodynamic regime is used as an ansatz, but the Gaussian-shaped ground state in the noninteracting limit is replaced as a preassumed ansatz in the variational approximation. In this sense, the effective scaling approach has more reasonable accuracy, when the atomic interaction with the arbitrary value is considered. Finally, some extensions are interesting for further exploration, for instance, the soliton dynamics by quenching the interactions of the BEC from repulsive to attractive [35,43], or the expansion of a Bose gas in the crossover from the TF to TG regimes [16].

ACKNOWLEDGMENTS

This work has been supported by NSFC (Grant No. 12075145), STCSM (2019SHZDZX01-ZX042019), the Program for Eastern Scholar, Basque Government IT986-16, PGC2018-095113-B-I00, PGC2018-101355-B-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe," EU FET Open Grant Quromorphic (828826) and EPIQUS (899368), and the Ramon y Cajal program (RYC-2017-22482).

[1] E. A. Cornell and C. E. Wieman, Rev. Mod. Phys. **74**, 875 (2002).
[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[3] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[4] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
[5] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Phys. Rev. A **54**, R1753(R) (1996).
[6] V. Gritsev, P. Barmettler, and E. Demler, New J. Phys. **12**, 113005 (2010).
[7] I. L. Egusquiza, M. Modugno, and M. A. Valle Basagoiti, Phys. Rev. A **84**, 043629 (2011).
[8] P. Pedri, L. Santos, P. Öhberg, and S. Stringari, Phys. Rev. A **68**, 043601 (2003).
[9] A. Minguzzi and D. M. Gangardt, Phys. Rev. Lett. **94**, 240404 (2005).
[10] T. Schäfer, Phys. Rev. A **82**, 063629 (2010).
[11] G. M. Bruun and C. W. Clark, Phys. Rev. A **61**, 061601(R) (2000).
[12] D. Guéry-Odelin, Phys. Rev. A **66**, 033613 (2002).
[13] C. Menotti, P. Pedri, and S. Stringari, Phys. Rev. Lett. **89**, 250402 (2002).
[14] G. Diana, N. Manini, and L. Salasnich, Phys. Rev. A **73**, 065601 (2006).

[15] H. Hu, X.-J. Liu, and M. Modugno, Phys. Rev. A **67**, 063614 (2003).

[16] P. Öhberg and L. Santos, Phys. Rev. Lett. **89**, 240402 (2002).

[17] M. Modugno, G. Pagnini, and M. A. Valle-Basagoiti, Phys. Rev. A **97**, 043604 (2018).

[18] D. Viedma and M. Modugno, Phys. Rev. Research **2**, 033478 (2020).

[19] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen, and J. G. Muga, in *Advances in Atomic, Molecular, and Optical Physics*, edited by E. Arimondo, P. R. Berman, and C. C. Lin, Advances in Atomic, Molecular, and Optical Physics Vol. 62 (Academic, New York, 2013), pp. 117–169.

[20] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, Rev. Mod. Phys. **91**, 045001 (2019).

[21] J. Muga, X. Chen, A. Ruschhaupt, and D. Guéry-Odelin, J. Phys. B: At., Mol. Opt. Phys. **42**, 241001 (2009).

[22] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, Phys. Rev. Lett. **104**, 063002 (2010).

[23] J.-F. Schaff, X.-L. Song, P. Vignolo, and G. Labeyrie, Phys. Rev. A **82**, 033430 (2010).

[24] J.-F. Schaff, X.-L. Song, P. Capuzzi, P. Vignolo, and G. Labeyrie, Europhys. Lett. **93**, 23001 (2011).

[25] A. Del Campo, Europhys. Lett. **96**, 60005 (2011).

[26] W. Rohringer, D. Fischer, F. Steiner, I. E. Mazets, J. Schmiedmayer, and M. Trupke, Sci. Rep. **5**, 9820 (2015).

[27] A. del Campo, Phys. Rev. A **84**, 031606(R) (2011).

[28] D. J. Papoular and S. Stringari, Phys. Rev. Lett. **115**, 025302 (2015).

[29] S. Deng, P. Diao, Q. Yu, A. del Campo, and H. Wu, Phys. Rev. A **97**, 013628 (2018).

[30] D. Stefanatos and J.-S. Li, Phys. Rev. A **86**, 063602 (2012).

[31] V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. **77**, 5320 (1996).

[32] J. J. García-Ripoll, V. M. Pérez-García, and P. Torres, Phys. Rev. Lett. **83**, 1715 (1999).

[33] J. Li, K. Sun, and X. Chen, Sci. Rep. **6**, 38258 (2016).

[34] T.-Y. Huang, B. A. Malomed, and X. Chen, Chaos **30**, 053131 (2020).

[35] T.-Y. Huang, J. Zhang, J. Li, and X. Chen, Phys. Rev. A **102**, 053313 (2020).

[36] J. Li, T. Fogarty, S. Campbell, X. Chen, and Th. Busch, New J. Phys. **20**, 015005 (2018).

[37] T.-N. Xu, J. Li, T. Busch, X. Chen, and T. Fogarty, Phys. Rev. Research **2**, 023125 (2020).

[38] D. Stefanatos, J. Ruths, and J.-S. Li, Phys. Rev. A **82**, 063422 (2010).

[39] X. Chen and J. G. Muga, Phys. Rev. A **82**, 053403 (2010).

[40] D. Anderson, M. Lisak, B. Malomed, and M. Quiroga-Teixeiro, J. Opt. Soc. Am. B **11**, 2380 (1994).

[41] O. Abah, R. Puebla, A. Kiely, G. D. Chiara, M. Paternostro, and S. Campbell, New J. Phys. **21**, 103048 (2019).

[42] Y. Rezek, P. Salamon, K. H. Hoffmann, and R. Kosloff, Europhys. Lett. **85**, 30008 (2009).

[43] A. Di Carli, C. D. Colquhoun, G. Henderson, S. Flannigan, G.-L. Oppo, A. J. Daley, S. Kuhr, and E. Haller, Phys. Rev. Lett. **123**, 123602 (2019).