

Quantifying multipartite quantum entanglement in a semi-device-independent manner

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We propose two semi-device-independent approaches that are able to quantify unknown multipartite quantum entanglement experimentally, where the only information that has to be known beforehand is quantum dimension, and the concept that plays a key role is nondegenerate Bell inequalities. Specifically, using the nondegeneracy of multipartite Bell inequalities, we obtain useful information on the purity of target quantum state. Combined with an estimate of the maximal overlap between the target state and pure product states, and a continuous property of the geometric measure of entanglement, we shall prove the information on purity allows us to give a lower bound for this entanglement measure. In addition, we show that a different combination of the above results also converts to a lower bound for the relative entropy of entanglement. As a demonstration, we apply our approach on 5-partite qubit systems with the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality, and show that useful lower bounds for the geometric measure of entanglement can be obtained if the Bell expression value is larger than 3.60, and those for the relative entropy of entanglement can be given if the Bell expression value is larger than 3.80, where the Tsirelson bound is 4.

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I. INTRODUCTION

Quantum entanglement plays a fundamental role in quantum physics and quantum information, where it often serves as the key factor in physical effects or key resource in information processing tasks [1,2]. Therefore, how to certify the existence of quantum entanglement and even quantify it in physical experiments are two important problems. However, due to the imperfection of quantum operations and inevitable quantum noise, fulfilling these two tasks *reliably* is extremely challenging. As a result, though some methods like entanglement witnesses have been applied widely in quantum laboratories [3], they usually depend heavily on accurate knowledge on involved quantum systems, and possibly give incorrect results when it is not fully available [4]. Meanwhile, some other methods, like quantum tomography, consume too many resources, making it hard to apply them on large systems [5,6].

To overcome these difficulties, a promising idea is to design protocols for these tasks in such a way that beforehand assumption needed on involved quantum systems, particularly on the precisions of quantum devices or quantum operations, is as little as possible, which allows us to draw reliable conclusions on quantum entanglement that we are interested in. Following this idea, various *device-independent* approaches have been proposed to tackle the problem of characterizing unknown quantum entanglement [7–11]. The key idea of these approaches is that the judgements are only based on quantum nonlocality that we can observe in quantum laboratories reliably, where one has to build nontrivial relations between

quantum nonlocality and the aspects of quantum entanglement that we want to know. Indeed, a lot of interesting results of this kind have been reported or even demonstrated to certify the existence of genuine multipartite entanglement [12–18].

If only focusing on the issue of *quantifying* unknown quantum entanglement experimentally, a lot of results have also been reported under the idea of device independence [19–23]. For example, inspired by the Navascues-Pironio-Acin (NPA) method [24], in Ref. [21] a device-independent approach to quantify the negativity, a measure of entanglement [25], was provided. In Ref. [20], based on the idea of semiquantum nonlocal games [26], an approach that quantifies negative-partial-transposition entanglement was reported, where one does not have to put any trust onto measurement devices. In Ref. [22], a new method with excellent performance was proposed to characterize the quantitative relation between entanglement measure and Clauser-Horne-Shimony-Holt inequality violations.

Particularly, in Ref. [23] another general approach that is able to provide analytic results on entanglement measures, like the entanglement of distillation and the entanglement of formation, was proposed. Basically, this is a semi-device-independent approach, where the only assumption that we have to make beforehand is quantum dimension, and the key idea of this approach is introducing the concept of nondegenerate Bell inequalities, which plays a crucial role in providing nontrivial information on the purities of target quantum states. As a result, the purity information allows us to quantify the target entanglement by lower bounding coherent information, which is known to be a lower bound for the entanglement

measures that we are interested in [27]. However, an apparent drawback of the approach in Ref. [23] is that it only works for bipartite entanglement.

For multipartite quantum entanglement, it has been known that its mathematical characterization, especially quantification, is a notoriously hard problem. However, it turns out that the geometric measure of entanglement (GME) and the relative entropy of entanglement (REE) are two quite successful measures for multipartite entanglement [28–31]. In this paper we propose two theoretical approaches to quantify the two above measures of unknown multipartite quantum states in a semi-device-independent manner. The concept of nondegenerate Bell inequalities is essential to these approaches. Indeed, combined with the purity information provided by applying nondegenerate Bell inequality onto experimental statistics data, we manage to lower bound the GME by proving a continuous property of this entanglement measure. Furthermore, with the help of the purity information, we show that the REE can also be quantified by estimating the maximal overlap between the target state and pure product states. To achieve these tasks, we need to certify the nondegeneracy of multipartite Bell inequalities.

As a demonstration of our approaches, we show that the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality [32–34] is nondegenerate for qubit systems, and then we demonstrate that nontrivial lower bounds on the GME and the REE of multipartite quantum entanglement can be obtained when the violation of the MABK inequality is sufficient, where it can be seen that the approaches have decent performance.

II. NONDEGENERATE BELL INEQUALITIES

Bell inequalities are crucial tools in the current paper, and in history they played a key role in the development of quantum mechanics [35]. In a so-called n -partite Bell settings, n space-separated parties share a physical system. Each party, say $i \in [n] \equiv \{1, 2, \dots, n\}$, has a set of measurement devices labeled by a finite set X_i , and the corresponding set of possible measurement outcomes are labeled by a finite set A_i . Without communications, all parties choose random measurement devices from their own X_i to measure their subsystems, respectively, and record the outcomes. By repeating the whole process for sufficient times, they find out the joint probability distribution of outcomes for any given choices of measurement devices, denoted $p(a_1 a_2 \dots a_n | x_1 x_2 \dots x_n)$, where $x_i \in X_i$ and $a_i \in A_i$.

For simplicity we call the above joint probability distribution a *quantum correlation*, and write it as $p(\vec{a}|\vec{x})$ (or just p if the context is clear), where $\vec{a} = (a_1 a_2 \dots a_n)$ and $\vec{x} = (x_1 x_2 \dots x_n)$. Then a (linear) Bell inequality is a relation that $p(\vec{a}|\vec{x})$ must obey if the system is classical, and it can be expressed as

$$I(p) = \sum_{\vec{a}, \vec{x}} c_{\vec{x}}^{\vec{a}} p(\vec{a}|\vec{x}) \leq C_l, \quad (1)$$

where for any \vec{x} and \vec{a} , $c_{\vec{x}}^{\vec{a}}$ is a real number.

However, a remarkable fact on quantum mechanics is that, if the shared physical system is quantum, Bell inequalities can be violated. Suppose the shared quantum state is ρ , then

according to quantum mechanics $p(\vec{a}|\vec{x})$ can be written as

$$p(\vec{a}|\vec{x}) = \text{Tr} \left[\left(\bigotimes_{i=1}^n M_{x_i}^{a_i} \right) \rho \right], \quad (2)$$

where for any i and x_i , $M_{x_i}^{a_i}$ is the measurement operators with outcome a_i for the measurement with label x_i performed by the i th party. For convenience of later discussions, we let $I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n})$ be the Bell expression value achieved by ρ and $M_{x_i}^{a_i}$. Then, as mentioned above, if we let

$$C_q \equiv \max I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}), \quad (3)$$

where the maximum is taken over all possibilities of ρ and $M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}$, then it is possible that $C_q > C_l$, indicating that quantum systems are able to produce stronger correlations than classical ones.

In the joint quantum system, suppose the dimensions of the subsystems are d_1, d_2, \dots, d_n , respectively, then we call the vector $\vec{d} \equiv (d_1 d_2 \dots d_n)$ the *dimension vector* of the joint system. In this paper we are interested in the maximal value of $I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n})$ for fixed dimension vector \vec{d} . Similar with C_q , we denote it as $C_q(\vec{d})$.

The concept of nondegenerate Bell inequalities was proposed when studying bipartite quantum systems [23]. As we will see later, it can also be applied in the multipartite case and plays a key role in entanglement measure quantification.

Suppose $I \leq C_l$ is a Bell inequality for an n -partite quantum system with dimension $d_1 \times d_2 \times \dots \times d_n$. We say it is *nondegenerate* on dimension vector $\vec{d} = (d_1 \dots d_n)$, if there exist two real number $0 \leq \epsilon_1 < \epsilon_2 \leq C_q(\vec{d})$, such that for any two quantum states of this system, $|\alpha\rangle$ and $|\beta\rangle$ with $\langle \alpha | \beta \rangle = 0$, and any quantum measurement sets $M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}$, the relation that

$$I(|\alpha\rangle\langle\alpha|, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \geq C_q(\vec{d}) - \epsilon_1$$

always implies that

$$I(|\beta\rangle\langle\beta|, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \leq C_q(\vec{d}) - \epsilon_2.$$

Roughly speaking, if $I \leq C_l$ is a nondegenerate Bell inequality on dimension vector \vec{d} , then for any two orthogonal quantum states, at most one of them is able to achieve near maximum violation using the same measurements.

We further let $M = \sum_{\vec{a}, \vec{x}} c_{\vec{x}}^{\vec{a}} \left(\bigotimes_{i=1}^n M_{x_i}^{a_i} \right)$, then it can be seen that M is a Hermitian operator. And for any ρ with dimension vector \vec{d} , it holds that $I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) = \text{Tr}(\rho M)$. Suppose $\lambda_1(M) \geq \dots \geq \lambda_r(M)$ are the eigenvalues of M , where $r = d_1 \times \dots \times d_n$. For any integer t with $1 \leq t \leq r$, let

$$C(I, \vec{d}, t) \equiv \max \sum_{k=1}^t \lambda_k(M),$$

where the maximum is taken over all possible local quantum measurements. Then we immediately have that $C_q(\vec{d}) = C(I, \vec{d}, 1)$. Furthermore, an important fact that allows us to certify the nondegeneracy of Bell inequalities is that, for any multipartite Bell expression I and any dimension vector \vec{d} , $I \leq C_l$ is nondegenerate if and only if $C(I, \vec{d}, 2) < 2C(I, \vec{d}, 1)$,

and when $I \leq C_l$ is nondegenerate, the parameters can be chosen by the relations $\epsilon_1 < C(I, \vec{d}, 1) - \frac{1}{2}C(I, \vec{d}, 2)$ and $\epsilon_1 + \epsilon_2 = 2C(I, \vec{d}, 1) - C(I, \vec{d}, 2)$ [23].

To illustrate the existence of nondegenerate multipartite Bell expressions, we consider the MABK expression over qubits [32–34]. In fact, the nondegeneracy property of this inequality has been observed in Ref. [36], where it was proved that the first two eigenvalues of the Bell operator satisfy $\lambda_1^2(M) + \lambda_2^2(M) \leq 2^{n-1}$. This implies that if $\lambda_1(M) > 2^{n/2-1}$, we have

$$\lambda_2(M) \leq \sqrt{2^{n-1} - \lambda_1^2(M)} < \lambda_1(M),$$

which indicates $C(I, \vec{d}, 2) < 2C(I, \vec{d}, 1)$. Meanwhile, it is known that the maximal value that $\lambda_1(M)$ can achieve is $2^{n/2-1/2}$, where the corresponding state can be the n -qubit Greenberger-Horne-Zeilinger (GHZ) state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ [32–34]. Therefore, the MABK expression is nondegenerate over qubits.

III. QUANTIFYING THE GEOMETRIC MEASURE OF ENTANGLEMENT

The geometric measure of entanglement is a well-known measure for multipartite quantum entanglement [28,29]. Suppose $|\psi\rangle$ is a pure state of a joint system composed by n subsystems. Define $G(|\psi\rangle)$ to be the maximal overlap between $|\psi\rangle$ and a product pure state, that is to say,

$$G(|\psi\rangle) = \sup_{|\phi\rangle \in \text{sep}_n} |\langle \psi | \phi \rangle|, \tag{4}$$

where sep_n is the set of n -partite product pure states. Then for $|\psi\rangle$, its geometric measure of entanglement is defined to be

$$E_G(|\psi\rangle) \equiv 1 - G(|\psi\rangle)^2. \tag{5}$$

For a mixed state ρ of this joint system, the geometric measure can be defined by convex roof construction, which is

$$E_G(\rho) \equiv \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E_G(|\psi_i\rangle). \tag{6}$$

The GME has many nontrivial applications in quantum physics and quantum information, for example quantifying the difficulty of multipartite state discrimination under local operations and classical communications (LOCC) [37], constructing entanglement witness [29,38], characterizing ground states of condensed matter systems and detecting phase transitions [39,40], and so on. Therefore, it will be very nice if we can quantify the GME reliably in quantum laboratories. We now show how the concept of nondegenerate Bell inequalities allows us to achieve this, and the approach is composed by three steps as below.

Step 1. Suppose ρ is the global state that produces the quantum correlation $p(\vec{a}|\vec{x})$. Let the underlying measurements be $M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}$; that is, $p(\vec{a}|\vec{x}) = \text{tr}[(\bigotimes_{i=1}^n M_{x_i}^{a_i})\rho]$. Now, since a crucial component in the definition of GME is the maximum overlap

$$G(|\psi\rangle) = \max_{|\phi\rangle \in \text{sep}_n} |\langle \phi | \psi \rangle| = \max_{|\phi\rangle \in \text{sep}_n} F(|\phi\rangle, |\psi\rangle),$$

where F is the fidelity, we wish to quantify the related fidelity

$$\max_{|\phi\rangle \in \text{sep}_n} F(|\phi\rangle\langle\phi|, \rho) = \max_{|\phi\rangle \in \text{sep}_n} \sqrt{\langle \phi | \rho | \phi \rangle}$$

in a fully device-independent manner, where sep_n is the set of product pure states.

Suppose $|\phi\rangle \in \text{sep}_n$ is the state that maximizes $F(|\phi\rangle\langle\phi|, \rho)$. Let $q^*(\vec{a}|\vec{x})$ be the correlation produced by $|\phi\rangle$ upon measurements $M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}$. Since $|\phi\rangle$ is a product pure state, the correlation q^* is a product correlation; that is, there exists probability distributions $q_i^*(a_i|x_i)$ such that $q^*(\vec{a}|\vec{x}) = \prod_{i=1}^n q_i^*(a_i|x_i)$. When ρ and $|\phi\rangle$ are measured, the fidelity between them should increase [2]; that is, for any \vec{x} the resulting probability distribution $p_{\vec{x}} \equiv p(\cdot|\vec{x})$ and $q_{\vec{x}}^* \equiv q^*(\cdot|\vec{x})$ satisfy

$$\sum_{\vec{a}} \sqrt{q^*(\vec{a}|\vec{x})p(\vec{a}|\vec{x})} = F(q_{\vec{x}}^*, p_{\vec{x}}) \geq F(|\phi\rangle\langle\phi|, \rho),$$

hence it holds that $\min_{\vec{x}} F(q_{\vec{x}}^*, p_{\vec{x}}) \geq F(|\phi\rangle\langle\phi|, \rho)$. Since q^* is a product correlation, we have

$$\max_q \min_{\vec{x}} F(q_{\vec{x}}, p_{\vec{x}}) \geq F(|\phi\rangle\langle\phi|, \rho),$$

where the outmost maximization is over product correlations q and $q_{\vec{x}} \equiv q(\cdot|\vec{x})$. By the max-min inequality, it holds that

$$\min_{\vec{x}} \max_q F(q_{\vec{x}}, p_{\vec{x}}) \geq \max_q \min_{\vec{x}} F(q_{\vec{x}}, p_{\vec{x}}),$$

then we have

$$\min_{\vec{x}} \max_q F(q_{\vec{x}}, p_{\vec{x}}) \geq F(|\phi\rangle\langle\phi|, \rho).$$

Then by numerical calculations on the correlation data, we can get an upper bound on the fidelity between the target state and a pure product state, denoted as \hat{F} . For example, once \vec{x} is fixed, the inner maximization can be computed using symmetric embedding [41] and the shifted higher-order power method (SHOPM) algorithm [42], yielding a correct answer up to numerical precision with very high probability (see also Ref. [43]).

Step 2. Since computing GME for a mixed state requires complicated optimization over ensembles, it would be ideal for the quantification of GME if ρ is a pure state. Therefore, we wish to bound the purity of ρ , defined as $\text{Tr}(\rho^2)$, from below, which is accomplished by the nondegeneracy property of Bell inequalities [23].

Let $\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of ρ . Suppose $I \leq C_l$ is a nondegenerate Bell inequality with parameters ϵ_1 and ϵ_2 satisfying $0 \leq \epsilon_1 < \epsilon_2$. If $I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \geq C(I, \vec{d}, 1) - \epsilon_1$, then there is i such that $I(|\psi_i\rangle\langle\psi_i|, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \geq C(I, \vec{d}, 1) - \epsilon_1$. Thus, by nondegeneracy of $I \leq C_l$, we have

$$\begin{aligned} C(I, \vec{d}, 1) - \epsilon_1 &\leq I(\rho, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \\ &= \sum_j a_j I(|\psi_j\rangle\langle\psi_j|, M_{x_1}^{a_1}, \dots, M_{x_n}^{a_n}) \\ &\leq a_i C(I, \vec{d}, 1) + (1 - a_i)[C(I, \vec{d}, 1) - \epsilon_2]. \end{aligned}$$

This implies that $a_i \geq 1 - \epsilon_1/\epsilon_2$. Since the order of eigenstates in the spectral decomposition is arbitrary, for convenience we now relabel the index i found above to 1, then it

holds that $a_1 \geq 1 - \epsilon_1/\epsilon_2$. This allows us to lower bound the purity.

Step 3. In the previous two steps we obtained a lower bound for a_1 in the spectral decomposition of ρ and an upper bound \hat{F} for $F(|\phi\rangle\langle\phi|, \rho) = \sqrt{\langle\phi|\rho|\phi\rangle}$ among all product

pure states $|\phi\rangle$. The following theorem shows that, if $\hat{F} \leq a_1$, then we can derive a lower bound for $E_G(\rho)$ by proving a continuous property of GME.

Theorem. Suppose $\hat{F} \leq a_1$, then it holds that

$$E_G(\rho) \geq \max_{c \in [\frac{\hat{F}}{\sqrt{a_1}}, \sqrt{a_1}]} \frac{a_1 - c^2}{1 - c^2} \left[1 - \left(\frac{\hat{F}}{\sqrt{a_1}} c + \sqrt{1 - \frac{\hat{F}^2}{a_1}} \sqrt{1 - c^2} \right)^2 \right].$$

Proof Suppose $\rho = \sum_j \tilde{a}_j |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|$ is an ensemble of ρ that obtains the GME of ρ . Let c be a real number in the interval $[\hat{F}/\sqrt{a_1}, \sqrt{a_1}]$. Consider the sets of indices

$$J_1 = \{j : |\langle\psi_1|\tilde{\psi}_j\rangle| \geq c\},$$

$$J_2 = \{j : |\langle\psi_1|\tilde{\psi}_j\rangle| < c\},$$

which form a partition of the set of all indices j . Intuitively, the set J_1 consists of components with high fidelity with $|\psi_1\rangle$. Let $\mu = \sum_{j \in J_1} \tilde{a}_j$. We have

$$\begin{aligned} a_1 &= \langle\psi_1|\rho|\psi_1\rangle \\ &= \sum_j \tilde{a}_j |\langle\psi_1|\tilde{\psi}_j\rangle|^2 \\ &= \sum_{j \in J_1} \tilde{a}_j |\langle\psi_1|\tilde{\psi}_j\rangle|^2 + \sum_{j \in J_2} \tilde{a}_j |\langle\psi_1|\tilde{\psi}_j\rangle|^2 \\ &\leq \mu + (1 - \mu)c^2, \end{aligned}$$

thus

$$\mu \geq \frac{a_1 - c^2}{1 - c^2} \geq 0,$$

which is lower bound for the sum of weights of components whose indices belong to J_1 . Note that $\mu \rightarrow 1$ when $a_1 \rightarrow 1$ if $c < \sqrt{a_1}$, and $\mu = 1$ if $a_1 = c = 1$. By the definition of \hat{F} , for any product pure state $|\phi\rangle$, we have

$$\hat{F}^2 \geq \langle\phi|\rho|\phi\rangle = \sum_i a_i |\langle\phi|\psi_i\rangle|^2 \geq a_1 |\langle\phi|\psi_1\rangle|^2,$$

thus

$$|\langle\phi|\psi_1\rangle| \leq \frac{\hat{F}}{\sqrt{a_1}}.$$

On the other hand, there are states $\{|\phi_j\rangle\}$ such that

$$E_G(\rho) = 1 - \sum_j \tilde{a}_j |\langle\phi_j|\tilde{\psi}_j\rangle|^2.$$

By the triangle inequality of fidelity, for every $j \in J_1$, we have

$$\begin{aligned} \arccos |\langle\phi_j|\tilde{\psi}_j\rangle| &\geq \arccos |\langle\phi_j|\psi_1\rangle| - \arccos |\langle\psi_1|\tilde{\psi}_j\rangle| \\ &\geq \arccos \left(\frac{\hat{F}}{\sqrt{a_1}} \right) - \arccos(c). \end{aligned}$$

As $\hat{F}/\sqrt{a_1} \leq c$, the inequality above implies

$$|\langle\phi_j|\tilde{\psi}_j\rangle| \leq \frac{\hat{F}}{\sqrt{a_1}} c + \sqrt{1 - \frac{\hat{F}^2}{a_1}} \sqrt{1 - c^2}.$$

For $j \in J_2$, we upper bound the overlap via $|\langle\phi_j|\tilde{\psi}_j\rangle| \leq 1$, thereby obtaining a lower bound for the GME of ρ as

$$\begin{aligned} E_G(\rho) &\geq 1 - \mu \left(\frac{\hat{F}}{\sqrt{a_1}} c + \sqrt{1 - \frac{\hat{F}^2}{a_1}} \sqrt{1 - c^2} \right)^2 - (1 - \mu) \\ &= \mu \left[1 - \left(\frac{\hat{F}}{\sqrt{a_1}} c + \sqrt{1 - \frac{\hat{F}^2}{a_1}} \sqrt{1 - c^2} \right)^2 \right] \\ &\geq \frac{a_1 - c^2}{1 - c^2} \left[1 - \left(\frac{\hat{F}}{\sqrt{a_1}} c + \sqrt{1 - \frac{\hat{F}^2}{a_1}} \sqrt{1 - c^2} \right)^2 \right]. \end{aligned}$$

Note that the above relation holds for any $c \in [\hat{F}/\sqrt{a_1}, \sqrt{a_1}]$, which concludes the proof.

In particular, if ρ is a pure state, then it holds that $a_1 = 1$. In that case, the lower bound in Proposition 1 reads

$$E_G(|\psi_1\rangle\langle\psi_1|) \geq 1 - \hat{F}^2, \quad (7)$$

which agrees with the definition of GME on pure states, indicating that our lower bound is tight in this case.

Therefore, combining all the above three steps together, we obtain a semi-device-independent approach to quantify the GME for unknown multipartite entanglement. We now demonstrate that this approach indeed works well by quantifying the GME of an n -partite quantum system with the MABK inequality ($n = 3, 5$). Recall that we have known that this inequality is nondegenerate. At the same time we would like to stress that in principle the approach can be applied on any multipartite quantum systems with known dimensions.

There exist many configurations that achieve the maximum violation to the MABK inequality, and it turns out that they are essentially equivalent [44]. For example, one can let the state be

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes n} + e^{\frac{2\pi i}{8}(n-1)} |1\rangle^{\otimes n}),$$

then measure the observables σ_x and σ_y on each qubit. That is, for each site, we select

$$\begin{aligned} M_0^0 &= |+\rangle\langle+|, & M_0^1 &= |-\rangle\langle-|, \\ M_1^0 &= |+i\rangle\langle+i|, & M_1^1 &= |-i\rangle\langle-i|, \end{aligned}$$

where $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$ and $|\pm i\rangle = 1/\sqrt{2}(|0\rangle \pm i|1\rangle)$.

To obtain physical statistic data of the Bell experiments, we perturb the state $|\Phi\rangle$ and the above optimal measurements,

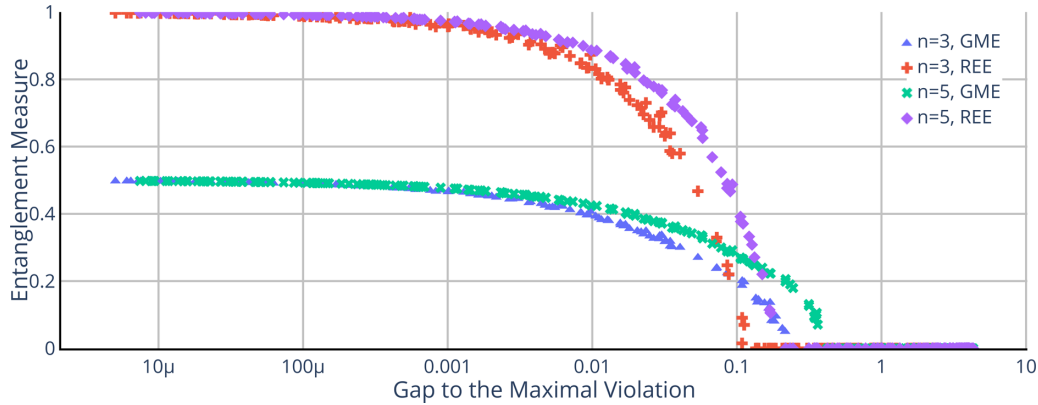


FIG. 1. Lower bounds for the GME and the REE, where $n = 3, 5$. Note that the maximal Bell expression value is 2 ($n = 3$) and 4 ($n = 5$), respectively, and we focus on the gaps to the maximal Bell expression values.

which produces a series of legitimate quantum correlations. We then apply our approach to each correlation. The result is shown in Fig. 1.

It can be seen from the blue (triangle) scatter plot in Fig. 1 that, when $n = 3$, if the gap between the Bell expression value I and the Tsirelson bound 2 is smaller than 0.20 (that is, when $I \geq 1.80$), our approach is able to provide a nontrivial result on the GME. Furthermore, when the violation approaches the maximum, our approach gives a tight result 0.5, considering that the maximal violation is achieved by $|\Phi\rangle$.

Similarly, the green scatter (X) plot in Fig. 1 also illustrates the result of our approach on 5-partite qubit systems. The same patterns as in the case $n = 3$ can be observed here: nontrivial GME lower bounds can be obtained when the gap between the Bell expression value I and the Tsirelson bound 4 is smaller than 0.40 (that is, when $I \geq 3.60$), and it approaches the tight result of 0.5 when the gap tends to zero.

IV. QUANTIFYING THE RELATIVE ENTROPY OF ENTANGLEMENT

Interestingly, steps 1 and 2 introduced above are already sufficient for us to lower bound the relative entropy of entanglement (REE) in a semi-device-independent manner.

The REE of ρ is defined to be the minimal relative entropy of ρ to the set of separable states, that is,

$$E_R(\rho) \equiv \min_{\sigma \in \mathcal{D}} S(\rho \| \sigma) = \min_{\sigma \in \mathcal{D}} \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma), \quad (8)$$

where \mathcal{D} is the set of all separable states [30,31]. It turns out that the REE has many profound applications in quantum information theory. For example, for bipartite quantum states, REE can lower bound the entanglement of formation and upper bound the entanglement of distillation [31]. Therefore, quantifying the REE reliably in experiments is naturally very important and rewarding.

We now show that $E_R(\rho)$ has a close relation with the quantity \hat{F} introduced above. In fact, it has been known that [45]

$$E_R(\rho) \geq G(\rho) - S(\rho), \quad (9)$$

where $S(\rho)$ is the Von Neumann entropy and

$$G(\rho) \equiv -\log_2 \left\{ \max_{|\phi\rangle \in \text{sep}_n} \langle \phi | \rho | \phi \rangle \right\}. \quad (10)$$

Since $\max_{|\phi\rangle \in \text{sep}_n} \langle \phi | \rho | \phi \rangle \leq \hat{F}^2$, it holds that

$$E_R(\rho) \geq -2 \log_2(\hat{F}) - S(\rho). \quad (11)$$

In the meantime, in step 2 we get a lower bound for the purity of ρ (in terms of a_1). Combining this fact and the approach introduced in Ref. [46], we can derive an upper bound for $S(\rho)$ (see [23] for a complete demonstration). According to Eq. (11), this implies that we are able to lower bound the REE and any other multipartite entanglement measures that are lower bounded by the REE (for example, the generalized robustness of entanglement [45,47,48]).

Still using the MABK inequality and the samples of quantum correlations generated above, we test the performance of the second approach, and the result can also be seen in Fig. 1. Particularly, when $n = 3$, our approach can give positive lower bound for the REE when the Bell expression value is larger than 1.88; when $n = 5$, it can provide nontrivial results when the Bell expression value is larger than 3.80.

V. DISCUSSION

In this paper, based on the concept of nondegenerate Bell inequalities, we show that multipartite quantum entanglement can be quantified experimentally in a semi-device-independent way. The key information provided by this concept is on the purity of the target quantum systems. Based on this, by studying the mathematical properties of the geometric measure of entanglement and the relative entropy of entanglement, we can provide nontrivial lower bounds for these two well-known entanglement measures. Our approaches do not need any trust on the precision of the involved quantum devices except the information on their dimensions and have decent performance, hence might be valuable for characterizing unknown multipartite states in future quantum experiments.

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