Generalized transitionless quantum driving for open quantum systems

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A general approach for transitionless quantum driving in open quantum systems is introduced. Under the assumption of adiabatic evolution for time-local master equations, we derive the generalized transitionless Lindbladian required to implement a shortcut to adiabaticity in an open-system scenario. The general counterdiabatic Lindbladian obtained accounts for a phase freedom, which translates into a set of free parameters throughout the dynamics. We discuss how our generalized approach allows us to recover the transitionless Lindbladian introduced by Vacanti *et al.* [G. Vacanti *et al.* New J. Phys. **16**, 053017 (2014)]. We show how to engineer time-independent master equations that provide the same dynamics as the time-dependent master equation provided by the standard transitionless quantum driving in open systems. We illustrate our results by applying them both to the adiabatic Deutsch algorithm under dephasing and to the Landau-Zener Hamiltonian under a bit phase flip.

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I. INTRODUCTION

Inverse quantum engineering is a useful approach to drive quantum systems through some desired path in parameter space and hence achieve a target final state [1–7]. Within a number of different approaches for inverse engineering, one can highlight the adiabatic dynamics as an important strategy, with successful applications in quantum control [8,9] and quantum computation [10,11]. However, the requirement of a sufficiently long evolution time may lead the system to undesired phenomena due to decoherence [12–16]. This has strongly motivated the investigation of methods for speeding up the adiabatic process (more precisely, to mimic the adiabatic behavior). In this scenario, transitionless quantum driving (TQD) [17–19] has been established as a widely used method for yielding shortcuts to adiabaticity, where additional fields are used to inhibit any diabatic transition between energy levels of the Hamiltonian. Transitionless quantum driving has provided numerous applications in different branches of physics [20–32], with many recent experimental realizations [33–37].

In a real physical scenario, where the quantum system is coupled with its surrounding environment, the adiabatic approximation requires a reformulation so that it is applicable to a nonunitary evolution. In that case, the closed-system adiabatic picture of a decoupled evolution of the Hamiltonian eigenspaces with distinct energy eigenvalues is replaced by a decoupled evolution of Lindblad-Jordan eigenspaces with distinct eigenvalues of the Lindbladian superoperator [13] (for alternative but similar reformulations, see Refs. [38,39]). This notion of adiabaticity has been consistently applied in different scenarios, such as quantum computation [40], eigenstate tracking of open quantum systems [14], and quantum thermodynamics [41]. As a further application, this open-system adiabatic approximation has also been used to build a theory of shortcuts to adiabaticity via transitionless evolutions, as shown by Vacanti *et al.* [42]. As an extension of closed-system TQD, the open-system TQD is established for time-local master equations by adding a counterpart to the relevant Lindbladian governing the dynamics. In turn, it requires the ability to control both fields and decohering rates so that environment engineering is taken as a tool to drive the system along an open-system adiabatic path. Recently, this protocol has been investigated experimentally in circuit quantum electrodynamics [43].

In this work we generalize the theory of TQD for open systems introduced in Ref. [42] for the case of shortcuts to adiabaticity exhibiting a phase (gauge) freedom. This brings to the realm of open systems the generalized TQD approach for closed systems theoretically proposed in Ref. [44] and experimentally realized in Ref. [34]. By considering the adiabatic dynamics in open systems and by taking the phases accompanying the evolution as free parameters, we derive a general counterdiabatic Lindbladian implementing arbitrary paths in a nonunitary evolution. We then show that, in addition to the path acceleration expected in the TQD dynamics, this set of free parameters can considerably simplify the underlying master equations allowing, for example, the derivation of time-independent Lindbladians. Moreover, as shown for closed systems, the phase freedom is potentially able to provide smooth energy requirements for local fields and interactions throughout the system dynamics [44]. This comes at the expense of convenient environment engineering, which may be achieved by employing suitable quantum

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control techniques. We then discuss the feasibility of the method through its illustration in the Deutsch quantum algorithm and the Landau-Zener Hamiltonian.

The paper is organized as follows. In Sec. II A we review some elements of the generalized TQD in closed systems, where we highlight the usefulness of the generalized phases introduced in Ref. [44]. In Sec. II B we analyze the results obtained in Ref. [42]. Building upon this previous work, we rewrite here the counterdiabatic Lindbladian introduced by Ref. [42] in terms of the right and left eigenbases of the Lindbladian superoperator. In Sec. III we discuss how the generalized phases can be employed to implement TQD for time-local master equations. In Sec. IV we illustrate our results through some applications in quantum control and quantum computation.

II. PRELIMINARY RESULTS

A. Transitionless quantum driving in closed systems

Consider a $D_{\rm S}$-dimensional Hilbert space $\mathcal{H}_{\rm S}$ describing a quantum system driven by a time-dependent Hamiltonian $H(t)$ acting on $\mathcal{H}_{\rm S}$, whose instantaneous eigenvectors are $|n(t)\rangle$ and their corresponding energies $E_n(t)$. Under sufficiently slow evolution [45–47], the system will follow an adiabatic dynamics governed by the adiabatic evolution operator

$$U_{\rm ad}(t;t_0)=\sum_{n=0}^{D_{\rm S}-1}\exp\left(-i\int_{t_0}^{t}\theta_n^{\rm ad}(\xi)d\xi\right)|n(t)\rangle\langle n(t_0)|, \quad (1)$$

where $\theta_n^{\rm ad}(t)$ is the adiabatic phase that accompanies the evolution of the nth eigenstate, which is given by [48]

$$\theta_n^{\rm ad}(t)=\frac{E_n(t)}{\hbar}+i\langle\dot{n}(t)|n(t)\rangle, \quad (2)$$

with the overdot denoting time derivative (this denotation is adopted throughout the paper). We can speed up such evolution through TQD methods [17–19]. Indeed, we can achieve the dynamics provided by the operator $U_{\rm ad}(t;t_0)$ at arbitrary finite time [24]. This occurs by letting the system evolve under the action of the standard TQD Hamiltonian

$$H_{\rm TQD}(t)=H(t)+i\hbar\sum_{n=0}^{D_{\rm S}-1}[|\dot{n}(t)\rangle\langle n(t)| + \langle\dot{n}(t)|n(t)\rangle|n(t)\rangle\langle n(t)|], \quad (3)$$

where the first term is the original Hamiltonian, whose spectral decomposition reads $H(t)=\sum_n E_n(t)|n(t)\rangle\langle n(t)|$, and the second term is the so-called counterdiabatic Hamiltonian $H_{\rm cd}(t)$, whose effect is to inhibit the typical diabatic behavior brought by fast evolutions. In particular, there are a number of situations in which the system is initially prepared in a single eigenstate $|\psi(t_0)\rangle=|k(t_0)\rangle$ of the Hamiltonian $H(t_0)$. In these cases, the adiabatic dynamics yields the evolved state

$$|\psi(t)\rangle=U_{\rm ad}(t;t_0)|\psi(t_0)\rangle=\exp\left(-i\int_{t_0}^{t}\theta_k^{\rm ad}(\xi)d\xi\right)|k(t)\rangle, \quad (4)$$

where the quantum adiabatic phase $\theta_k^{\rm ad}(t)$ works as a global phase. Therefore, it can be neglected in many applications, such as the realization of quantum gates. For these cases, it means that any phase that appears during the evolution does not contribute to the TQD evolution. It is then possible to derive an alternative TQD protocol with arbitrary quantum phases [not necessarily the adiabatic phase $\theta_n^{\rm ad}(t)$]. This introduces a phase freedom and constitutes the generalized TQD dynamics [44]. Thus, we can define the generalized TQD evolution operator

$$U_{\rm TQD}^{\rm gen}(t;t_0)=\sum_{n=0}^{D_{\rm S}-1}\exp\left(-i\int_{t_0}^{t}\theta_n(\xi)d\xi\right)|n(t)\rangle\langle n(t_0)|, \quad (5)$$

where $\{\theta_n(t)\}$ is a set of arbitrary phases to be freely adjusted according to the desired dynamics. From such an operator, we derive the generalized TQD Hamiltonian that drives the system through this path as [44]

$$H_{\rm TQD}^{\rm gen}(t)=i\hbar\sum_{n=0}^{D_{\rm S}-1}[|\dot{n}(t)\rangle\langle n(t)| - i\theta_n(t)|n(t)\rangle\langle n(t)|]. \quad (6)$$

The potential benefits of this approach have been illustrated in quantum computation and quantum control. Generalized TQD can provide feasible time-independent TQD Hamiltonians, which can be applied to implement quantum gates through controlled evolutions and to speed up the dynamics of two-level atomic systems under Landau-Zener transitions [44]. Experimentally, generalized TQD has been used to design energy-enhanced TQD microwave fields to implement shortcuts to adiabaticity in trapped ion systems [34] and nuclear magnetic resonance [49].

B. Standard transitionless quantum driving in open systems

We assume a time-local open-system dynamics described by a superoperator $\mathcal{L}[\bullet]$, which governs both the unitary (coherent) dynamics and the nonunitary contribution (due to the coupling with the reservoir). The master equation underlying the system evolution is provided by $\dot{\rho}(t)=\mathcal{L}[\rho(t)]$. In the context of the open-system adiabatic dynamics, it is convenient to rewrite the master equation in the superoperator formalism as [13] (see Appendix A)

$$|\dot{\rho}(t)\rangle\rangle=\mathbb{L}(t)|\rho(t)\rangle\rangle, \quad (7)$$

where the vector $|\rho(t)\rangle\rangle$ and the Lindbladian superoperator $\mathbb{L}(t)$ are written in a matrix basis composed of $(D_{\rm S}\times D_{\rm S})$-dimensional traceless matrices σ_n satisfying the relation $\mathrm{Tr}\{\sigma_n\sigma_m\}=D_{\rm S}\delta_{nm}$. Then we have that $|\rho(t)\rangle\rangle$ is a $D_{\rm S}^2$-dimensional coherence vector in Hilbert-Schmidt space [50], whose components are $\varrho_n(t)=\mathrm{Tr}\{\rho(t)\sigma_n^\dagger\}$ and a $(D_{\rm S}^2\times D_{\rm S}^2)$-dimensional superoperator $\mathbb{L}(t)$ with matrix elements $\mathbb{L}_{ki}(t)=(1/D_{\rm S})\mathrm{Tr}\{\sigma_k^\dagger\mathcal{L}[\sigma_i]\}$. The inner product between two coherence vectors associated with density operators ξ_1 and ξ_2 is given by $\langle\langle\xi_1|\xi_2\rangle\rangle=(1/D_S)\mathrm{Tr}\{\xi_1^\dagger\xi_2\}$, where the conjugate coherence vector $\langle\langle\xi_1|$ has components given by $\mathrm{Tr}\{\xi_1^\dagger\sigma_n\}$. In particular, for a two-level system, the Hermitian Pauli basis $\mathcal{O}_{\rm tls}=\{\mathbb{1},\sigma_x,\sigma_y,\sigma_z\}$ is a convenient choice, but we can adopt alternative bases depending on the application [51].

In the formalism of superoperators, the adiabatic dynamics is defined from the instantaneous decoupled evolution of Jordan blocks of $\mathbb{L}(t)$. As shown in Ref. [52], the adiabatic dynamics is well characterized by the open-system evolution operator $\mathcal{V}_{\mathrm{ad}}(t,t_0)=\sum_{\beta=0}^{N-1}\mathcal{V}_\beta(t,t_0)$, where each element $\mathcal{V}_\beta(t,t_0)$ reads

$$\mathcal{V}_\beta(t,t_0)=\exp\left(\int_{t_0}^{t}\lambda_\beta(\xi)d\xi\right)\sum_{n_\beta=1}^{N_\beta}\sum_{m_\beta=1}^{N_\beta}v_{n_\beta m_\beta}(t)\big|\mathcal{D}_\beta^{n_\beta}(t)\big\rangle\!\big\rangle \times\big\langle\!\big\langle\mathcal{E}_\beta^{m_\beta}(t_0)\big|, \tag{8}$$

with the elements $v_{n_\beta m_\beta}(t)$ accounting for inner transitions within a single Jordan block [52] and $\lambda_\beta(t)$ being the instantaneous eigenvalue associated with the βth block, whose right and left quasieigenvectors $|\mathcal{D}_\beta^{m_\beta}(t)\rangle\!\rangle$ and $\langle\!\langle\mathcal{E}_\beta^{m_\beta}(t)|$, respectively, obey

$$\mathbb{L}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle=\big|\mathcal{D}_\alpha^{(n_\alpha-1)}(t)\big\rangle\!\big\rangle+\lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle, \tag{9a}$$

$$\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\mathbb{L}(t)=\big\langle\!\big\langle\mathcal{E}_n^{(n_\alpha+1)}(t)\big|+\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\lambda_\alpha(t), \tag{9b}$$

with $|\mathcal{D}_\alpha^{(0)}(t)\rangle\!\rangle$ and $\langle\!\langle\mathcal{E}_\alpha^{(N_\alpha+1)}(t)|$ denoting vanishing vectors. The sets $\{|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\!\rangle\}$ and $\{\langle\!\langle\mathcal{E}_\alpha^{n_\alpha}(t)|\}$ satisfy the biorthonormalization condition $\langle\!\langle\mathcal{E}_m^{\beta}(t)|\mathcal{D}_n^{\alpha}(t)\rangle\!\rangle=\delta_{mn}\delta_{\beta\alpha}$. For one-dimensional Jordan-block decomposition of the Lindbladian $\mathbb{L}(t)$, we have block dimension $N_\alpha=1\,\forall\,\alpha$. In this case, we define $|\mathcal{D}_\alpha^{(1)}(t)\rangle\!\rangle\equiv|\mathcal{D}_\alpha(t)\rangle\!\rangle$ and $\langle\!\langle\mathcal{E}_\alpha^{(1)}(t)|\equiv\langle\!\langle\mathcal{E}_\alpha(t)|$. Then $\mathcal{V}_{\mathrm{ad}}(t,t_0)$ becomes

$$\mathcal{V}_{\mathrm{ad}}^{1\mathrm{D}}(t,t_0)=\sum_{\alpha=0}^{N-1}\exp\left(\int_{t_0}^{t}\Lambda_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t_0)|, \tag{10}$$

with $\Lambda_\alpha(t)=\lambda_\alpha(t)-\langle\!\langle\mathcal{E}_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\!\rangle$ the generalized adiabatic phase accompanying the dynamics of the nth eigenvector [52].

In the same direction as the TQD for closed systems, the TQD can be introduced here to mimic the adiabatic dynamics, but in this case the adiabatic behavior is dictated by its generalized version for open systems. In Ref. [42] Vacanti *et al.* provided an interesting and useful discussion of how we should deal with TQD in such systems. Similarly as provided for closed systems, the idea is to define a counterdiabatic term which, when added to the original Lindbladian, provides open-system TQD at finite time. To this end, the authors defined a transformation $C(t)$ which depends on the set of instantaneous right quasieigenstates of the Lindbladian. From this approach, it is then possible to find the counterdiabatic Lindbladian $\mathbb{L}_{\mathrm{cd}}(t)$ in terms of $C(t)$. By following a generalized path, we can show here how to formulate $\mathbb{L}_{\mathrm{cd}}(t)$ in notation analogous to that used for counterdiabatic Hamiltonians in closed systems [19].

First, let us briefly review the proposal for speeding up an adiabatic dynamics via TQD as originally proposed in Ref. [42]. The idea is based on achieving the perfect decoupling of Jordan-Lindblad eigenspaces, in complete agreement with the definition of adiabaticity. To this end, we consider the similarity transformation $\mathbb{L}_{\mathrm{J}}(t)=C^{-1}(t)\mathbb{L}(t)C(t)$, with $\mathbb{L}_{\mathrm{J}}(t)$ denoting the Jordan canonical form of $\mathbb{L}(t)$ and the superoperator $C(t)$ defined by

$$C(t)=\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\big|\mathcal{D}_\mu^{n_\mu}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\sigma_\mu^{n_\mu}\big|, \tag{11}$$

where $\{\langle\!\langle\sigma_\mu^{n_\mu}|\}$ is a set of time-independent vectors corresponding to a basis of traceless orthogonal matrices $\{\sigma_n\}$. From this transformation, Eq. (7) becomes

$$[\mathbb{L}_{\mathrm{J}}(t)+\dot{C}^{-1}(t)C(t)]|\rho(t)\rangle\!\rangle_{\mathrm{J}}=|\dot{\rho}(t)\rangle\!\rangle_{\mathrm{J}}, \tag{12}$$

with $|\rho(t)\rangle\!\rangle_{\mathrm{J}}=C^{-1}(t)|\rho(t)\rangle\!\rangle$. Then, following Ref. [42], we obtain

$$[\mathbb{L}_{\mathrm{J}}(t)+\mathbb{L}_{\mathrm{J}}'(t)+\mathbb{L}_{\mathrm{nd}}'(t)]|\rho(t)\rangle\!\rangle_{\mathrm{J}}=|\dot{\rho}(t)\rangle\!\rangle_{\mathrm{J}}, \tag{13}$$

where the operator $\dot{C}^{-1}(t)C(t)$ has been split into two parts, a block-diagonal operator $\mathbb{L}_{\mathrm{J}}'(t)$ and a second off-diagonal contribution $\mathbb{L}_{\mathrm{nd}}'(t)$ given, respectively, by

$$\mathbb{L}_{\mathrm{J}}'(t)=\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\sum_{l_\mu=1}^{N_\mu}C_{\mu\mu}^{n_\mu l_\mu}(t)\big|\sigma_\mu^{n_\mu}\big\rangle\!\big\rangle\big\langle\!\big\langle\sigma_\mu^{l_\mu}\big|, \tag{14}$$

$$\mathbb{L}_{\mathrm{nd}}'(t)=\sum_{\nu\neq\mu}^{N-1}\sum_{n_\nu=1}^{N_\nu}\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}C_{\nu\mu}^{n_\nu n_\mu}(t)\big|\sigma_\nu^{n_\nu}\big\rangle\!\big\rangle\big\langle\!\big\langle\sigma_\mu^{n_\mu}\big|, \tag{15}$$

with coefficients $C_{\nu\mu}^{n_\nu n_\mu}(t)=\langle\!\langle\sigma_\nu^{n_\nu}|\dot{C}^{-1}(t)C(t)|\sigma_\mu^{n_\mu}\rangle\!\rangle$. The operator $\mathbb{L}_{\mathrm{nd}}'(t)$ is associated with a transition between Jordan blocks. In order to inhibit the effects of $\mathbb{L}_{\mathrm{nd}}'(t)$, we introduce an additional operator $\mathbb{L}_{\mathrm{cd}}(t)$ given by

$$\mathbb{L}_{\mathrm{cd}}(t)=-C(t)\mathbb{L}_{\mathrm{nd}}'(t)C^{-1}(t). \tag{16}$$

Thus, the standard TQD Lindbladian reads

$$\mathbb{L}_{\mathrm{STQD}}(t)=\mathbb{L}(t)+\mathbb{L}_{\mathrm{cd}}(t). \tag{17}$$

This superoperator was obtained in Ref. [42] as the counterdiabatic Lindbladian to implement TQD in open systems. As a further step, we can go beyond Ref. [42] at this point and rewrite Eq. (16) in a more convenient way as (see Appendix B)

$$\mathbb{L}_{\mathrm{cd}}(t)=\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\left[\big|\dot{\mathcal{D}}_\mu^{n_\mu}\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\mu^{n_\mu}\big|-\sum_{k_\mu=1}^{N_\mu}\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}\big|\mathcal{D}_\mu^{n_\mu}\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\mu^{k_\mu}\big|\right], \tag{18}$$

where $\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t)=\langle\!\langle\mathcal{E}_\mu^{n_\mu}(t)|\dot{\mathcal{D}}_\mu^{k_\mu}(t)\rangle\!\rangle$. Notice the formally identical structure for $\mathbb{L}_{\mathrm{cd}}(t)$ in comparison with the standard counterdiabatic Hamiltonian $H_{\mathrm{cd}}(t)$ in Eq. (3). Equation (18) is the first contribution of our work. The dynamics induced by the Lindbladian $\mathbb{L}_{\mathrm{STQD}}(t)$ will here be referred to as the standard TQD evolution, since it allows for the exact mimicking of the adiabatic path in open systems (see Appendix B). In particular, for a one-dimensional Jordan-block decomposition, we have

$$\mathbb{L}_{\mathrm{cd}}^{1\mathrm{D}}(t)=\sum_{\alpha=0}^{N-1}|\dot{\mathcal{D}}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)|-\mathcal{G}_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)|, \tag{19}$$

with $\mathcal{G}_\alpha(t)=\langle\!\langle\mathcal{E}_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\!\rangle$, so that

$$\mathbb{L}_{\mathrm{STQD}}^{1\mathrm{D}}(t)=\mathbb{L}(t)+\mathbb{L}_{\mathrm{cd}}^{1\mathrm{D}}(t) \tag{20}$$

is the one-dimensional standard TQD Lindbladian.

III. GENERALIZED TQD IN OPEN SYSTEMS

In closed quantum systems, we can design a Hamiltonian $H(t)$ that implements a target unitary evolution by the method of inverse engineering (see, e.g., Refs. [3,5]). This can be achieved by using $H(t) = i\hbar\dot{U}(t)U^{\dagger}(t)$, with $U(t)$ denoting the evolution operator. Equivalently, we can show that we can perform inverse engineering for Lindbladian superoperators in nonunitary dynamics. To this end, we assume open systems driven by invertible dynamical maps, with the evolved density operator $|\rho(t)\rangle\rangle$ provided by a nonunitary superoperator $\mathcal{V}(t, t_0)$ as $|\rho(t)\rangle\rangle = \mathcal{V}(t, t_0)|\rho(t_0)\rangle\rangle$. We can then introduce an inversely engineered Lindbladian $\mathbb{L}_{\mathrm{IE}}(t)$ implementing the dynamics induced by $\mathcal{V}(t, t_0)$ by taking (see Appendix C)

$$\mathbb{L}_{\mathrm{IE}}(t) = \dot{\mathcal{V}}(t, t_0)\mathcal{V}^{-1}(t, t_0). \tag{21}$$

Since the evolution is encoded in the superoperator $\mathcal{V}(t, t_0)$, TQD can be directly approached by this strategy. In particular, by using an engineered Lindbladian $\mathbb{L}_{\mathrm{IE}}(t)$ from Eq. (21), we could drive the system through an adiabatic path so that the transitions among Jordan blocks are suppressed [52]. Furthermore, we can design $\mathbb{L}_{\mathrm{IE}}(t)$ capable of implementing shortcuts to the nonunitary adiabatic evolution through general TQD protocols.

A. Generalized TQD for one-dimensional Jordan blocks

Let us consider the case of one-dimensional Jordan decomposition for the original Lindbladian $\mathbb{L}(t)$. By requiring the decoupling of the Jordan blocks and by allowing arbitrary phases throughout the evolution, we write a generalized TQD evolution operator as

$$\mathcal{V}^{\mathrm{1D}}_{\mathrm{GTQD}}(t, t_0) = \sum_{\alpha=0}^{N-1} \exp\left(\int_{t_0}^{t} \Theta_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t_0)|, \tag{22}$$

where the function $\Theta_\alpha(t) \in C$ is a free generic phase. From Eq. (21) we can then show that the generalized TQD Lindbladian for one-dimensional Jordan decomposition is given by (see Appendix D)

$$\mathbb{L}^{\mathrm{1D}}_{\mathrm{GTQD}}(t) = \sum_{\alpha=0}^{N-1}[|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| + \Theta_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)|]. \tag{23}$$

As expected, we can choose $\Theta_\alpha(t)$ to recover the adiabatic phase. However, by letting $\Theta_\alpha(t)$ be a free parameter, we can achieve an infinite family of TQD evolutions mimicking the adiabatic dynamics up to a quantum phase. As we will show, this phase freedom may simplify the physical implementation. Notice also that the set of parameters $\{\Theta_n(t)\}$ cannot be considered completely arbitrary because we usually do not prepare the system in a particular eigenstate of $\mathbb{L}(t)$ at the beginning of the evolution. Such a result will be illustrated in Sec. IV. Nonetheless, we can already provide some advantages of the generalized phases concerning the feasibility of the operator $\mathbb{L}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)$. Indeed, it is possible to derive a theorem that tells us about the situations for which we could obtain a time-independent TQD Lindbladian.

Theorem 1. Let $\mathbb{L}(t)$ be a Lindblad superoperator that admits one-dimensional Jordan-block decomposition, with noncrossing eigenvalues $\lambda_\alpha(t)$ associated with right and left eigenvectors $|\mathcal{D}_\alpha(t)\rangle\rangle$ and $\langle\langle\mathcal{E}_\alpha(t)|$, respectively. If the sets $\{|\mathcal{D}_\alpha(t)\rangle\rangle\}$ and $\{\langle\langle\mathcal{E}_\alpha(t)|\}$ obey

$$\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle = \langle\langle\mathcal{E}_\eta(t_0)|\dot{\mathcal{D}}_\beta(t_0)\rangle\rangle \times \exp\left(\int_{t_0}^{t}[\mathcal{G}_\eta(\xi) - \mathcal{G}_\beta(\xi)]d\xi\right) \tag{24}$$

for every η and β, then we can derive a time-independent TQD Lindblad superoperator $\mathbb{L}^{\mathrm{1D}}_{\mathrm{GTQD}}$ by adopting generalized phases given by

$$\overline{\Theta}_\eta(t) = -\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\eta(t)\rangle\rangle \tag{25}$$

for every η.

The proof is provided in Appendix E. Such a theorem is potentially useful when we have limited ability of implementing (or simulating) reservoir effects and/or when we do not have optimal control of the time-dependent external parameters that act on the system. As a by-product, if the sets $\{|\mathcal{D}_\alpha(t)\rangle\rangle\}$ and $\{\langle\langle\mathcal{E}_\alpha(t)|\}$ obey the particular condition

$$\frac{d}{dt}[\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle] = 0 \tag{26}$$

for every η and β, then a time-independent Lindbladian can be set by choosing $\overline{\Theta}_\eta(t)$ such that

$$\overline{\Theta}_\eta = \overline{\Theta}_\beta = \mathrm{const}\ \forall\ \eta, \beta. \tag{27}$$

The proof is also provided in Appendix E. Concerning this last result, we notice that, whenever we have a time-independent eigenvector $|\mathcal{D}_\nu\rangle\rangle$, the corresponding generalized phase Θ_ν that provides a time-independent Lindbladian does not depend on the rest of the parameters $\Theta_\beta(t)$, with $\beta \neq \nu$. This means that the choice of Θ_ν can be made independently of the remaining sectors for the case of a time-independent eigenvector $|\mathcal{D}_\nu\rangle\rangle$.

B. Generalized TQD for multidimensional Jordan blocks

In order to implement the generalized TQD approach for multidimensional Jordan blocks, we start by introducing the evolution operator

$$\mathcal{V}_{\mathrm{GTQD}}(t, t_0) = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha} q_\alpha^{n_\alpha m_\alpha}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\alpha^{m_\alpha}(t_0)\big|, \tag{28}$$

where the coefficients $q_\alpha^{n_\alpha m_\alpha}(t)$ can be suitably adjusted throughout the dynamics. From Eq. (28) we can see that $\mathcal{V}_{\mathrm{GTQD}}(t, t_0)$ drives the system under a transitionless path, but it does not necessarily mimic the exact adiabatic solution. The generalized TQD Lindbladian $\mathbb{L}_{\mathrm{GTQD}}(t)$ then reads

$$\mathbb{L}_{\mathrm{GTQD}}(t) = \dot{\mathcal{V}}_{\mathrm{GTQD}}(t, t_0)\mathcal{V}^{-1}_{\mathrm{GTQD}}(t, t_0). \tag{29}$$

By using Eq. (28) in Eq. (29), we obtain (see Appendix F)

$$\mathbb{L}_{\mathrm{GTQD}}(t, t_0) = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha} \dot{q}_\alpha^{n_\alpha k_\alpha}(t)\tilde{q}_\alpha^{k_\alpha l_\alpha}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle \times \big\langle\!\big\langle\mathcal{E}_\alpha^{l_\alpha}(t)\big| + \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\big|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|, \tag{30}$$

where the coefficient $\tilde{q}_{\alpha}^{k_{\alpha}l_{\alpha}}(t)$ is associated with the operator $\mathcal{V}_{\mathrm{GTQD}}^{-1}(t,t_0)$, which satisfies $\mathcal{V}_{\mathrm{GTQD}}(t,t_0)\mathcal{V}_{\mathrm{GTQD}}^{-1}(t,t_0)=\mathbb{1}$. It is also possible to prove that the coefficient $\tilde{q}_{\alpha}^{k_{\alpha}l_{\alpha}}$ satisfies (see Appendix F)

$$\sum_{m_{\kappa}=1}^{N_{\kappa}} q_{\kappa}^{l_{\kappa}m_{\kappa}}\tilde{q}_{\kappa}^{m_{\kappa}i_{\kappa}}=\delta_{l_{\kappa}i_{\kappa}}. \tag{31}$$

The TQD Lindbladian $\mathbb{L}_{\mathrm{GTQD}}(t,t_0)$ generalizes the results presented in Sec. II A. While in the standard case the feasibility of the TQD Lindbladian depends on the spectrum of the original Lindbladian $\mathbb{L}(t,t_0)$, in the generalized approach the implementation of $\mathbb{L}_{\mathrm{GTQD}}(t,t_0)$ depends on both the spectrum of $\mathbb{L}(t,t_0)$ and the free parameters $q_{\alpha}^{n_{\alpha}m_{\alpha}}(t)$ and $\tilde{q}_{\alpha}^{k_{\alpha}l_{\alpha}}(t)$. Those parameters can be used to optimize a TQD evolution provided the experimental setup available.

Let us analyze now the particular case $q_{\alpha}^{n_{\alpha}m_{\alpha}}(t)=\exp[\int_{t_0}^{t}\lambda_{\alpha}(\xi)d\xi]v_{n_{\alpha}m_{\alpha}}(t)$, such as in Eq. (8). Then we have $\mathcal{V}_{\mathrm{GTQD}}(t,t_0)=\mathcal{V}_{\mathrm{ad}}(t,t_0)$, so the standard TQD Lindbladian in Eq. (17) is expected to be recovered. Indeed, by using the adiabatic choice for $q_{\alpha}^{n_{\alpha}m_{\alpha}}(t)$, we obtain the generalized TQD Lindbladian (see Appendix G)

$$\mathbb{L}_{\mathrm{GTQD}}^{q=v}(t)=\sum_{\alpha=0}^{N-1}\sum_{n_{\alpha}=1}^{N_{\alpha}}\lambda_{\alpha}(t)\big|\mathcal{D}_{\alpha}^{n_{\alpha}}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{n_{\alpha}}(t)\big|+\big|\dot{\mathcal{D}}_{\alpha}^{n_{\alpha}}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{n_{\alpha}}(t)\big|+\sum_{\alpha=0}^{N-1}\sum_{n_{\alpha}=1}^{N_{\alpha}}\sum_{k_{\alpha}=1}^{N_{\alpha}}\sum_{l_{\alpha}=1}^{N_{\alpha}}\dot{v}_{n_{\alpha}k_{\alpha}}(t)\tilde{v}_{k_{\alpha}l_{\alpha}}(t)\big|\mathcal{D}_{\alpha}^{n_{\alpha}}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{l_{\alpha}}(t)\big|, \tag{32}$$

where we used the existence of the inverse evolution superoperator $\mathcal{V}_{\mathrm{ad}}^{-1}(t,t_0)$ such that $\mathcal{V}_{\mathrm{ad}}(t,t_0)\mathcal{V}_{\mathrm{ad}}^{-1}(t,t_0)=\mathbb{1}$. Explicitly, we write $\mathcal{V}_{\mathrm{ad}}^{-1}(t,t_0)=\sum_{\alpha=0}^{N-1}\mathcal{V}_{\alpha}^{-1}(t,t_0)$, where [52]

$$\mathcal{V}_{\alpha}^{-1}(t,t_0)=\exp\left(-\int_{t_0}^{t}\lambda_{\alpha}(\xi)d\xi\right)\sum_{n_{\alpha}=1}^{N_{\alpha}}\sum_{m_{\alpha}=1}^{N_{\alpha}}\tilde{v}_{n_{\alpha}m_{\alpha}}(t)\big|\mathcal{D}_{\alpha}^{n_{\alpha}}(t_0)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{m_{\alpha}}(t)\big|, \tag{33}$$

where the coefficients $\tilde{v}_{n_{\alpha}m_{\alpha}}(t)$ and $v_{n_{\beta}m_{\beta}}(t)$ satisfy the relation

$$\sum_{j_{\nu}=1}^{N_{\nu}}v_{\ell_{\nu}j_{\nu}}(t)\tilde{v}_{j_{\nu}m_{\nu}}(t)=\sum_{j_{\nu}=1}^{N_{\nu}}\tilde{v}_{\ell_{\nu}j_{\nu}}(t)v_{j_{\nu}m_{\nu}}(t)=\delta_{\ell_{\nu}m_{\nu}}. \tag{34}$$

Equation (34) comes from fact that the operator $\mathcal{V}_{\mathrm{ad}}(t,t_0)$ is identified as the superoperator that block diagonalizes the Lindbladian. Although we cannot write a spectral decomposition to $\mathbb{L}_{\mathrm{GTQD}}^{q=v}(t)$, since it is in general nondiagonalizable, it is possible to write a quasispectral decomposition for an arbitrary $\mathbb{L}(t)$ as

$$\mathbb{L}(t)=\sum_{\alpha=0}^{N-1}\sum_{n_{\alpha}=1}^{N_{\alpha}}\big[\big|\mathcal{D}_{\alpha}^{(n_{\alpha}-1)}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{n_{\alpha}}(t)\big|+\lambda_{\alpha}(t)\big|\mathcal{D}_{\alpha}^{n_{\alpha}}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_{\alpha}^{n_{\alpha}}(t)\big|\big], \tag{35}$$

whence it is possible to verify the quasieigenvalue equations given by Eq. (9). Since $\mathbb{L}_{\mathrm{GTQD}}^{q=v}(t)$ has to exactly mimic the adiabatic dynamics, the functions $v_{n_{\alpha}k_{\alpha}}(t)$ and $\tilde{v}_{k_{\alpha}l_{\alpha}}(t)$ must obey (see Appendix G)

$$\sum_{k_{\alpha}=1}^{N_{\alpha}}\dot{v}_{n_{\alpha}k_{\alpha}}(t)\tilde{v}_{k_{\alpha}l_{\alpha}}(t)=\delta_{n_{\alpha}(l_{\alpha}-1)}-\big\langle\!\big\langle\mathcal{E}_{\alpha}^{n_{\alpha}}(t)\big|\dot{\mathcal{D}}_{\alpha}^{l_{\alpha}}(t)\big\rangle\!\big\rangle. \tag{36}$$

Therefore, Eq. (32) yields

$$\mathbb{L}_{\mathrm{GTQD}}^{q=v}(t)=\mathbb{L}(t)+\mathbb{L}_{\mathrm{cd}}(t)=\mathbb{L}_{\mathrm{STQD}}(t), \tag{37}$$

which shows how the generalized version of TQD can recover the standard TQD as provided in Ref. [42].

IV. APPLICATIONS

A. Deutsch algorithm under dephasing

Our first application is the Deutsch algorithm under dephasing. The problem addressed in Deutsch's algorithm [53] is to determinate whether a dichotomic real function $f: x\in\{0,1\}\to f(x)\in\{0,1\}$ is either constant [the output $f(x)$ is the same regardless of the input value x] or balanced [the output $f(x)$ assumes different values according to the input value x]. We denote by O_f the oracle operator, which is capable of computing f. The oracle O_f is given by [40]

$$O_f=(-1)^{f(0)}|0\rangle\langle 0|+(-1)^{f(1)}|1\rangle\langle 1|. \tag{38}$$

Thus, we can write the Hamiltonian that implements the adiabatic solution for the problem as

$$H^{\mathrm{DA}}(t)=U_f(t)H_0U_f^{\dagger}(t), \tag{39}$$

where $H_0=-\hbar\omega\sigma_x$ and $U_f(t)=\exp(i\frac{\pi}{2}\frac{t}{\tau}O_f)$, with $0\leqslant t\leqslant\tau$. At $t=0$, we have $H^{\mathrm{DA}}(0)=H_0$, so the initial input ground state is written as $|\psi_{\mathrm{inp}}\rangle=|+\rangle=(1/\sqrt{2})(|0\rangle+|1\rangle)$. When the evolution is slow enough, we can write the output state as the ground state of $H^{\mathrm{DA}}(t)$, which reads

$$\rho_{\mathrm{CS}}^{\mathrm{DA}}(t)=\tfrac{1}{2}[\mathbb{1}+g_{\mathrm{c}}(t)\sigma_x-g_{\mathrm{s}}(t)\sigma_y], \tag{40}$$

where $g_{\mathrm{c}}(t)=\cos(\pi Ft/2\tau)$ and $g_{\mathrm{s}}(t)=\sin(\pi Ft/2\tau)$, with $F=1-(-1)^{f(0)+f(1)}$ and the subscript CS denoting that $\rho_{\mathrm{CS}}^{\mathrm{DA}}(t)$ is obtained from the adiabatic solution for closed systems.

1. Deutsch adiabatic dynamics

Let us consider an open-system dynamics governed by Markovian phase damping, with rate $\gamma(t)$. The evolution is driven by a Lindblad master equation, which is given by

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H^{\mathrm{DA}}(t), \rho(t)] + \gamma(t)[\sigma_z \rho(t) \sigma_z - \rho(t)]. \quad (41)$$

Let us now rewrite Eq. (41) in the superoperator formalism, yielding

$$|\dot{\rho}(t)\rangle\rangle = \mathbb{L}^{\mathrm{DA}}(t)|\rho(t)\rangle\rangle, \quad (42)$$

where

$$\mathbb{L}^{\mathrm{DA}}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma(t) & 0 & 2\omega g_s(t) \\ 0 & 0 & -2\gamma(t) & 2\omega g_c(t) \\ 0 & -2\omega g_s(t) & -2\omega g_c(t) & 0 \end{bmatrix}. \quad (43)$$

The right eigenvectors of $\mathbb{L}^{\mathrm{DA}}(t)$ are (the superscript T denotes transpose)

$$|\mathcal{D}_0^{\mathrm{DA}}(t)\rangle\rangle = [1 \quad 0 \quad 0 \quad 0]^{\mathsf{T}}, \quad (44a)$$

$$|\mathcal{D}_1^{\mathrm{DA}}(t)\rangle\rangle = [0 \quad -g_c(t) \quad g_s(t) \quad 0]^{\mathsf{T}}, \quad (44b)$$

$$|\mathcal{D}_2^{\mathrm{DA}}(t)\rangle\rangle = \left[0 \quad -\tfrac{2\omega g_s(t)}{\Delta_+(t)} \quad -\tfrac{2\omega g_c(t)}{\Delta_+(t)} \quad 1\right]^{\mathsf{T}}, \quad (44c)$$

$$|\mathcal{D}_3^{\mathrm{DA}}(t)\rangle\rangle = \left[0 \quad -\tfrac{2\omega g_s(t)}{\Delta_-(t)} \quad -\tfrac{2\omega g_c(t)}{\Delta_-(t)} \quad 1\right]^{\mathsf{T}} \quad (44d)$$

and the left eigenvectors are

$$\langle\langle \mathcal{E}_0^{\mathrm{DA}}(t)| = [1 \quad 0 \quad 0 \quad 0], \quad (45a)$$

$$\langle\langle \mathcal{E}_1^{\mathrm{DA}}(t)| = [0 \quad -g_c(t) \quad g_s(t) \quad 0], \quad (45b)$$

$$\langle\langle \mathcal{E}_2^{\mathrm{DA}}(t)| = \frac{1}{\Delta(t)}\left[0 \quad \omega g_s(t) \quad \omega g_c(t) \quad \tfrac{\Delta_+(t)}{2}\right], \quad (45c)$$

$$\langle\langle \mathcal{E}_3^{\mathrm{DA}}(t)| = \frac{1}{\Delta(t)}\left[0 \quad -\omega g_s(t) \quad -\omega g_c(t) \quad -\tfrac{\Delta_-(t)}{2}\right], \quad (45d)$$

with eigenvalues $\lambda_0(t) = 0$, $\lambda_1(t) = -2\gamma(t)$, $\lambda_2(t) = \Delta_-(t)$, and $\lambda_3(t) = \Delta_+(t)$, where $\Delta_\pm(t) = -\gamma(t) \pm \Delta(t)$ and $\Delta(t) = \sqrt{\gamma^2(t) - 4\omega^2}$. The nondegenerate spectrum of $\mathbb{L}^{\mathrm{DA}}(t)$ shows that $\mathbb{L}^{\mathrm{DA}}(t)$ can be written in the one-dimensional Jordan-block form. By writing the density matrix for the initial state as $\rho^{\mathrm{DA}}(0) = |\psi_{\mathrm{inp}}\rangle\langle\psi_{\mathrm{inp}}| = |+\rangle\langle+| = \frac{1}{2}(\mathbb{1} + \sigma_x)$, we can show that the initial state is a linear combination of the vectors $|\mathcal{D}_0^{\mathrm{DA}}(0)\rangle\rangle$ and $|\mathcal{D}_1^{\mathrm{DA}}(0)\rangle\rangle$, yielding

$$|\rho^{\mathrm{DA}}(0)\rangle\rangle = [1 \quad 1 \quad 0 \quad 0]^{\mathsf{T}} = |\mathcal{D}_0^{\mathrm{DA}}(0)\rangle\rangle - |\mathcal{D}_1^{\mathrm{DA}}(0)\rangle\rangle. \quad (46)$$

Hence, if we let the system evolve adiabatically, we can write the evolved state as [52]

$$|\rho_{\mathrm{ad}}^{\mathrm{DA}}(t)\rangle\rangle = |\mathcal{D}_0^{\mathrm{DA}}(t)\rangle\rangle - \exp\left(-2\int_{t_0}^{t}\gamma(\xi)d\xi\right)|\mathcal{D}_1^{\mathrm{DA}}(t)\rangle\rangle, \quad (47)$$

where we used the adiabatic phase $\Lambda_1(t) = \lambda_1(t) = -2\gamma(t)$. Now, by rewriting Eq. (47) as a vector in the superoperator formalism, we get

$$|\rho_{\mathrm{ad}}^{\mathrm{DA}}(t)\rangle\rangle = \left[1 \quad \exp\left(-2\int_{t_0}^{t}\gamma(\xi)d\xi\right)g_c(t) \quad -\exp\left(-2\int_{t_0}^{t}\gamma(\xi)d\xi\right)g_s(t) \quad 0\right]^{\mathsf{T}}, \quad (48)$$

Then, by determining the coherence vector associated with the density matrix $\rho^{\mathrm{DA}}(t)$ and expressing the result in the Pauli basis, we obtain

$$\rho_{\mathrm{ad}}^{\mathrm{DA}}(t) = \tfrac{1}{2}\left[\mathbb{1} + \exp\left(-2\int_{t_0}^{t}\gamma(\xi)d\xi\right)g_c(t)\sigma_x - \exp\left(-2\int_{t_0}^{t}\gamma(\xi)d\xi\right)g_s(t)\sigma_y\right]. \quad (49)$$

In the limit $\gamma(t) \to 0$, we recover the density matrix for the unitary dynamics shown in Eq. (40), where the output state reads (at $t = \tau$) [40]

$$\lim_{\gamma(t)\to 0} \rho_{\mathrm{ad}}^{\mathrm{DA}}(\tau) = \rho_{\mathrm{CS}}^{\mathrm{DA}}(\tau) = \tfrac{1}{2}[\mathbb{1} + (-1)^{f(0)+f(1)}\sigma_x], \quad (50)$$

where we have used $\cos(\pi F/2) = (-1)^{f(0)+f(1)}$ for $f(x) \in \{0, 1\}$ since $F = 1 - (-1)^{f(0)+f(1)}$. This solution is the output for an optimal (decoherence-free) situation. The experimental implementation of the adiabatic Deutsch algorithm under phase damping has been implemented in trapped ions [54], where the adiabatic behavior is achieved for a long total evolution time.

2. *Deutsch standard TQD*

As a first application, let us derive a shortcut to the adiabatic Deutsch algorithm in open systems. To begin with, we observe that Eqs. (44) and (45) imply $\langle\langle \mathcal{E}_\alpha^{\mathrm{DA}}(t)|\dot{\mathcal{D}}_\alpha^{\mathrm{DA}}(t)\rangle\rangle = 0$ for $\alpha = \{0, 1\}$. Therefore, the adiabatic phases $\Lambda_\alpha(t) = \lambda_\alpha(t) - \langle\langle \mathcal{E}_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle$ for each eigenvector are simply given by its corresponding eigenvalue. Then we get

$$\Lambda_0(t) = 0, \quad \Lambda_1(t) = -2\gamma(t), \quad (51a)$$

$$\Lambda_2(t) = \lambda_2(t) - \frac{\Delta_-(t)\dot{\gamma}(t)}{2\Delta^2(t)},$$

$$\Lambda_3(t) = \lambda_3(t) + \frac{2\omega^2\dot{\gamma}(t)}{\Delta^2(t)\Delta_-(t)}. \quad (51b)$$

In particular, we can see that if $\dot{\gamma}(t) = 0$, then $\langle\langle \mathcal{E}_\alpha^{\mathrm{DA}}(t)|\dot{\mathcal{D}}_\alpha^{\mathrm{DA}}(t)\rangle\rangle = 0$ for any α. This implies a generalized parallel transport condition for the adiabatic dynamics in open systems, which is formally analogous to the parallel transport condition for unitary dynamics.

By applying the standard approach for TQD, the counterdiabatic Lindblad superoperator to be added to the adiabatic counterpart reads

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{DA}}(t) = -\sum_{\mu=0}^{3} \langle\langle \mathcal{E}_{\mu}^{\mathrm{DA}}(t) | \dot{\mathcal{D}}_{\mu}^{\mathrm{DA}}(t) \rangle\rangle | \mathcal{D}_{\mu}^{\mathrm{DA}}(t) \rangle\rangle \langle\langle \mathcal{E}_{\mu}^{\mathrm{DA}}(t) | + \sum_{\mu=0}^{3} | \dot{\mathcal{D}}_{\mu}^{\mathrm{DA}}(t) \rangle\rangle \langle\langle \mathcal{E}_{\mu}^{\mathrm{DA}}(t) |, \quad (52)$$

where $|\mathcal{D}_{\mu}^{\mathrm{DA}}(t)\rangle\rangle$ and $\langle\langle\mathcal{E}_{\mu}^{\mathrm{DA}}(t)|$ are given by Eqs. (44) and (45), respectively. By using Eqs. (44) and (45), we get a counterdiabatic Lindbladian superoperator given by

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{DA}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{F\pi}{2\tau} & h(t)g_s(t) \\ 0 & -\frac{F\pi}{2\tau} & 0 & h(t)g_c(t) \\ 0 & h(t)g_s(t) & h(t)g_c(t) & 0 \end{bmatrix}, \quad (53)$$

with $h(t) = \omega\dot{\gamma}(t)/\tau[4\omega^2 - \gamma^2(t)]$. By restricting the analysis to the simple case of the parallel transport condition $\dot{\gamma}(t) = 0$, we have $h(t) = 0$. Then Eq. (53) becomes

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{DA}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{F\pi}{2\tau} & 0 \\ 0 & -\frac{F\pi}{2\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (54)$$

Thus, the corresponding Lindbladian $\mathcal{L}_{\mathrm{cd}}^{\mathrm{DA}}[\bullet]$ is

$$\mathcal{L}_{\mathrm{cd}}^{\mathrm{DA}}[\bullet] = \frac{1}{i\hbar}[H_{\mathrm{cd}}^{\mathrm{DA}}, \bullet], \quad (55)$$

where we have the counterdiabatic Hamiltonian

$$H_{\mathrm{cd}}^{\mathrm{DA}} = -\hbar\frac{F\pi}{4\tau}\sigma_z. \quad (56)$$

Therefore, since the standard TQD Lindbladian is given by $\mathbb{L}_{\mathrm{STQD}}^{\mathrm{DA}}(t) = \mathbb{L}^{\mathrm{DA}}(t) + \mathbb{L}_{\mathrm{cd}}^{\mathrm{DA}}$, we conclude that

$$\mathcal{L}_{\mathrm{STQD}}^{\mathrm{DA}}[\bullet] = \frac{1}{i\hbar}\left[H_{\mathrm{STQD}}^{\mathrm{DA}}(t), \bullet\right] + \gamma(t)[\sigma_z \bullet \sigma_z - \bullet], \quad (57)$$

with $H_{\mathrm{STQD}}^{\mathrm{DA}}(t) = H^{\mathrm{DA}}(t) + H_{\mathrm{cd}}^{\mathrm{DA}}$. Notice that, even though the counterdiabatic contribution for the Hamiltonian is time independent, the total standard TQD Lindbladian depends on time through the Hamiltonian $H^{\mathrm{DA}}(t)$. Moreover, the additional term $H_{\mathrm{cd}}^{\mathrm{DA}}$ to be introduced in Eq. (41), which allows us to implement the transitionless evolution in an open system, does not depend on a reservoir engineering, but just on the fields that act on the system.

3. Deutsch generalized TQD

Now let us use the generalized approach of TQD in open systems to derive an alternative master equation for the Deutsch problem, which is time independent but able to provide the same results as Eq. (57). To this end, let us write the generalized TQD evolution operator as

$$\mathcal{V}_{\mathrm{GTQD}}^{\mathrm{DA}}(t) = \sum_{\alpha=0}^{N-1} \exp\left(\int_0^t \Theta_{\alpha}^{\mathrm{DA}}(\xi)d\xi\right) | \mathcal{D}_{\mu}^{\mathrm{DA}}(t) \rangle\rangle \langle\langle \mathcal{E}_{\mu}^{\mathrm{DA}}(0) |, \quad (58)$$

where $\Theta_{\alpha}^{\mathrm{DA}}(t)$ are the phases to be adjusted. However, as shown in Eq. (46), the initial state of the system is written as a superposition of $|\mathcal{D}_0^{\mathrm{DA}}(0)\rangle\rangle$ and $|\mathcal{D}_1^{\mathrm{DA}}(0)\rangle\rangle$, so the evolved state depends on the adiabatic phases $\Lambda_0(t)$ and $\Lambda_1(t)$, but it does not depend on $\Lambda_2(t)$ and $\Lambda_3(t)$. Thus, in order to derive the generalized TQD, the generalized phases $\Theta_0^{\mathrm{DA}}(t)$ and $\Theta_1^{\mathrm{DA}}(t)$ should reproduce the adiabatic phases $\Lambda_0(t)$ and $\Lambda_1(t)$, but $\Theta_2^{\mathrm{DA}}(t)$ and $\Theta_3^{\mathrm{DA}}(t)$ allow us to introduce free parameters in the evolution. In conclusion, by considering $\Theta_0^{\mathrm{DA}}(t) = \Lambda_0^{\mathrm{DA}}(t) = 0$ and $\Theta_1^{\mathrm{DA}}(t) = \Lambda_1^{\mathrm{DA}}(t) = -2\gamma(t)$, the generalized TQD Lindbladian is obtained from Eq. (23) as

$$\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{DA(1)}}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \eta_1(t) & \frac{\tau\tilde{\chi}+F\pi}{2\tau} & \frac{\omega\Theta_{23}^{-}(t)g_s(t)}{\Delta(t)} \\ 0 & \frac{\tau\tilde{\chi}-F\pi}{2\tau} & \eta_2(t) & \frac{\omega\Theta_{32}^{-}(t)g_c(t)}{\Delta(t)} \\ 0 & -\frac{\omega\Theta_{32}^{-}(t)g_s(t)}{\Delta(t)} & -\frac{\omega\Theta_{32}^{-}(t)g_c(t)}{\Delta(t)} & \chi_-(t) \end{bmatrix}, \quad (59)$$

where we have defined $\gamma(t) \equiv \gamma_0$ for a constant rate γ_0 (satisfying then $\dot{\gamma}(t) = 0$), $\tilde{\chi}(t) = \sin(F\pi t)[2\gamma_0 + \chi_+(t)]$, $\eta_1(t) = \Gamma_+(t) + g_s^2(t)\chi_+(t)$, $\eta_2(t) = \Gamma_-(t) + g_c^2(t)\chi_+(t)$, and $\Theta_{32}^{-}(t) = \Theta_3(t) - \Theta_2(t)$, with

$$\chi_{\pm}(t) = \frac{\pm\Delta_{\pm}(t)\Theta_3(t) \mp \Delta_{\mp}(t)\Theta_2(t)}{2\Delta(t)}, \quad (60a)$$

$$\Gamma_{\pm}(t) = -\gamma_0\left[1 \pm g_c^2(t) \mp g_s^2(t)\right]. \quad (60b)$$

We can fix the free parameters for convenience so that we obtain a simple Lindblad superoperator. For instance, in order to get an antisymmetric superoperator, we can adjust $\Theta_3(t)$. Indeed, from Eq. (60), if we choose

$$\Theta_3(t) = \frac{\Delta_-(t)\Theta_2(t) - 4\gamma_0\Delta(t)}{\Delta_+(t)}, \quad (61)$$

we obtain an antisymmetric Lindblad superoperator $\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{DA(2)}}(t)$ given by

$$\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{DA(2)}}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma_0 & \frac{F\pi}{2\tau} & -\tilde{\Theta}_2(t)g_s(t) \\ 0 & -\frac{F\pi}{2\tau} & -2\gamma_0 & -\tilde{\Theta}_2(t)g_c(t) \\ 0 & \tilde{\Theta}_2(t)g_s(t) & \tilde{\Theta}_2(t)g_c(t) & -\frac{2\gamma_0[\Theta_2(t)-\Delta_-(t)]}{\Delta_+(t)} \end{bmatrix}, \quad (62)$$

where

$$\tilde{\Theta}_2(t) = \frac{2\omega[2\gamma_0 + \Theta_2(t)]}{\Delta_+(t)}. \quad (63)$$

The parameter $\Theta_2(t)$ is linked to reservoir and/or Hamiltonian engineering. For example, if we are not able to perform reservoir engineering, we take $\Theta_2(t) = \Lambda_2(t)$. In this case, we can show that the resulting generalized Lindblad superoperator $\mathbb{L}^{\mathrm{DA}(2)}_{\mathrm{GTQD}}(t)$ is given by $\mathbb{L}^{\mathrm{DA}(2)}_{\mathrm{GTQD}}(t) = \mathbb{L}^{\mathrm{DA}}_{\mathrm{TTQD}}(t) = \mathbb{L}^{\mathrm{DA}}(t) + \mathbb{L}^{\mathrm{DA}}_{\mathrm{cd}}$, since $\Theta_2(t)$ and $\Theta_3(t)$ become exactly the phases which accompany the adiabatic dynamics [see Eq. (51)].

On the other hand, we can also consider others possibilities of choices for $\Theta_2(t)$ so that we can get alternative ways of driving the system. For example, from Eq. (62) we can identify some time-independent matrix elements of $\mathbb{L}^{\mathrm{DA}(2)}_{\mathrm{GTQD}}(t)$, with a time dependence on the parameter $\Theta_2(t)$ for other elements. In particular, it is worth highlighting that the right and left bases for $\mathbb{L}^{\mathrm{DA}}(t)$ satisfy Eq. (26). Then, as a consequence of Theorem 1, we can get a time-independent master equation. In particular, it is obtained if we choose $\Theta_2(t) = -2\gamma_0$ so that the Lindbladian in Eq. (62) becomes

$$\mathbb{L}^{\mathrm{DA}}_{\mathrm{ti}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma_0 & \frac{F\pi}{2\tau} & 0 \\ 0 & -\frac{F\pi}{2\tau} & -2\gamma_0 & 0 \\ 0 & 0 & 0 & -2\gamma_0 \end{bmatrix}. \tag{64}$$

For the associated master equation we obtain

$$\dot{\rho}(t) = \frac{1}{i\hbar}\big[H^{\mathrm{DA}}_{\mathrm{cd}}, \rho(t)\big] + \mathcal{R}^{\mathrm{DA}}_x[\rho(t)] + \mathcal{R}^{\mathrm{DA}}_y[\rho(t)] + \mathcal{R}^{\mathrm{DA}}_z[\rho(t)], \tag{65}$$

with $H^{\mathrm{DA}}_{\mathrm{cd}}$ given by Eq. (56) and $\mathcal{R}^{\mathrm{DA}}_k[\bullet] = \frac{\gamma_0}{2}(\sigma_k \bullet \sigma_k - \bullet)$. In order to verify whether Eqs. (57) and (65) allow us to get a shortcut to the adiabatic Deutsch algorithm in open systems, we compute the fidelity [55]

$$\mathcal{F}_{\mathrm{OS}}(\omega\tau) = \mathrm{Tr}\left\{\sqrt{\sqrt{\rho(\tau)}\rho^{\mathrm{tar}}(\gamma_0\tau)\sqrt{\rho(\tau)}}\right\}, \tag{66}$$

where $\rho(\tau)$ is a solution of Eqs. (57) and (65) at $t = \tau$ and ρ^{tar} is the target state. In our case, the target state is the adiabatic solution at the instant $t = \tau$ obtained from Eq. (49) as

$$\rho^{\mathrm{DA}}(\gamma_0\tau) = \frac{1}{2}\Big[\mathbb{1} + e^{-2\gamma_0\tau}\cos\Big(\frac{\pi F}{2}\Big)\sigma_x - e^{-2\gamma_0\tau}\sin\Big(\frac{\pi F}{2}\Big)\sigma_y\Big]. \tag{67}$$

In Fig. 1 we present the fidelity as a function of $\omega\tau$, since we set γ_0 as a multiple of ω. Both standard and generalized TQD protocols achieve a shortcut to adiabaticity in open system. It is important to highlight that a high fidelity here does not necessarily represent the optimal fidelity of the adiabatic Deutsch algorithm, because the state in Eq. (67) is not an exact solution of the problem due to decoherence. This statement can be better understood from Fig. 2, where we show the trajectory of several distinct evolutions in the Bloch sphere. Regardless of whether the system dynamics is unitary or not, as we drive the system in a time interval shorter than that required by the adiabatic conditions, the dynamics is far from the adiabatic solution, while both standard and generalized TQDs allow us to mimic the adiabatic behavior over the entire time interval $t \in [0, \tau]$. Notice also that, when the system is affected by decoherence, the state purity decreases, leading to a loss of fidelity to get the perfect output state.

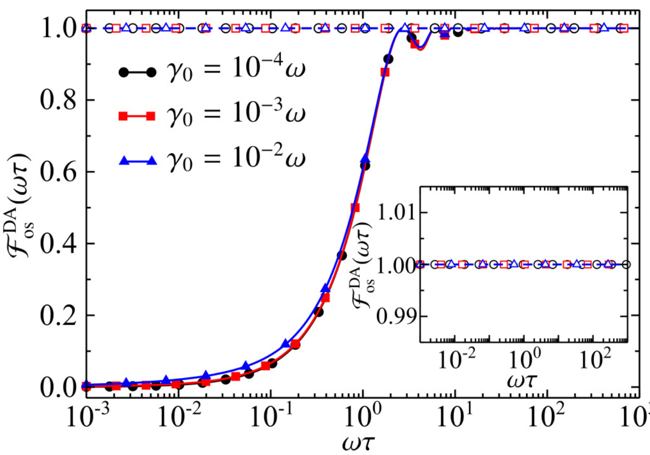

FIG. 1. Fidelity $\mathcal{F}^{\mathrm{DA}}_{\mathrm{OS}}(\omega\tau)$ to achieve the open-system adiabatic solution for the Deutsch algorithm for different values of γ_0. Solid lines with closed symbols represent $\mathcal{F}^{\mathrm{DA}}_{\mathrm{OS}}(\omega\tau)$ when the system is driven by the master equation (41), while dashed lines with open symbols represent $\mathcal{F}^{\mathrm{DA}}_{\mathrm{OS}}(\omega\tau)$ when the system is driven by the generalized TQD master equation (65). The inset shows $\mathcal{F}_{\mathrm{OS}}(\omega\tau)$ for the standard TQD evolution in Eq. (57). Here we consider the case where the function is balanced.

B. Landau-Zener model under a bit phase flip

As a second example of application of the generalized TQD approach, let us consider the Landau-Zener model, whose Hamiltonian is given by $H_{\mathrm{LZ}}(t) = (\hbar\omega_0/2)\sigma_z + [\hbar\Delta(t)/2]\sigma_x$. Here we are considering a time-independent detuning frequency ω_0 and a time-dependent field $\Delta(t)$.

1. Landau-Zener adiabatic dynamics

Let us assume that the system evolves under the bit-phase-flip decohering effect, whose Lindblad equation is

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H_{\mathrm{LZ}}(t), \rho(t)] + \gamma(t)[\sigma_y\rho(t)\sigma_y - \rho(t)], \tag{68}$$

where $\gamma(t)$ is the time-dependent bit-phase-flip decohering rate. From Eq. (68) we can write the corresponding superoperator $\mathbb{L}^{\mathrm{LZ}}(t)$ in the Pauli basis $\sigma_i = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$,

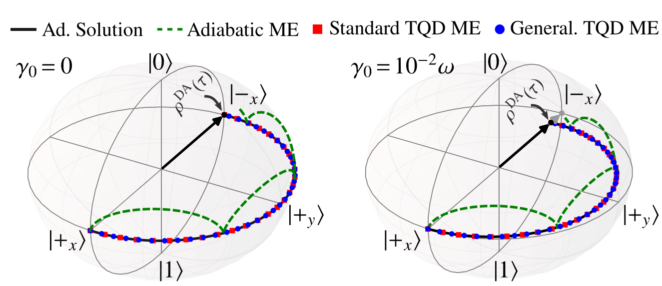

FIG. 2. Trajectories in the Bloch sphere for several quantum evolutions: adiabatic solution given by Eq. (49) (Ad. Solution), exact solution of the master equation (41) (Adiabatic ME), standard TQD approach in Eq. (57) (Standard TQD ME), and generalized TQD method in Eq. (65) (General. TQD ME). The dynamics is considered for closed (left Bloch sphere) and open (right Bloch sphere) cases, where we consider the total evolution time such that $\tau\omega = 10$.

yielding

$$\mathbb{L}^{\mathrm{LZ}}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -2\gamma(t) & -\omega_0 & 0\\ 0 & \omega_0 & 0 & -\omega_0\tan\theta(t)\\ 0 & 0 & \omega_0\tan\theta(t) & -2\gamma(t)\end{bmatrix}, \quad (69)$$

where $\theta(t)=\arctan[\Delta(t)/\omega_0]$. The right eigenvectors are given by

$$|\mathcal{D}_0^{\mathrm{LZ}}(t)\rangle\rangle=[1 \quad 0 \quad 0 \quad 0]^{\mathsf{T}}, \quad (70a)$$

$$|\mathcal{D}_1^{\mathrm{LZ}}(t)\rangle\rangle=[0 \quad \sin\theta(t) \quad 0 \quad \cos\theta(t)]^{\mathsf{T}}, \quad (70b)$$

$$|\mathcal{D}_2^{\mathrm{LZ}}(t)\rangle\rangle=\left[0 \quad -\cos\theta(t) \quad \tfrac{\gamma(t)\cos\theta(t)-\kappa(t)}{\omega_0} \quad \sin\theta(t)\right]^{\mathsf{T}}, \quad (70c)$$

$$|\mathcal{D}_3^{\mathrm{LZ}}(t)\rangle\rangle=\left[0 \quad -\cos\theta(t) \quad \tfrac{\gamma(t)\cos\theta(t)+\kappa(t)}{\omega_0} \quad \sin\theta(t)\right]^{\mathsf{T}}, \quad (70d)$$

while for the left eigenvectors we have

$$\langle\langle\mathcal{E}_0^{\mathrm{LZ}}(t)|=[1 \quad 0 \quad 0 \quad 0], \quad (71a)$$

$$\langle\langle\mathcal{E}_1^{\mathrm{LZ}}(t)|=[0 \quad \sin\theta(t) \quad 0 \quad \cos\theta(t)], \quad (71b)$$

$$\langle\langle\mathcal{E}_2^{\mathrm{LZ}}(t)|=\frac{1}{2}\left[0 \quad -\cos\theta(t)\tilde{\kappa}_+ \quad -\tfrac{\omega_0}{\kappa(t)} \quad \sin\theta(t)\tilde{\kappa}_+\right], \quad (71c)$$

$$\langle\langle\mathcal{E}_3^{\mathrm{LZ}}(t)|=\frac{1}{2}\left[0 \quad -\cos\theta(t)\tilde{\kappa}_- \quad \tfrac{\omega_0}{\kappa(t)} \quad \sin\theta(t)\tilde{\kappa}_-\right], \quad (71d)$$

where we have defined $\tilde{\kappa}_\pm=1\pm\cos\theta(t)\gamma(t)/\kappa(t)$ and $\kappa^2(t)=\gamma^2(t)\cos^2\theta(t)-\omega_0^2$, with eigenvalues $\lambda_0(t)=0$, $\lambda_1(t)=-2\gamma(t)$, and $\lambda_n(t)=-\gamma(t)-(-1)^n\kappa(t)$, where $n=\{2,3\}$.

Now, by considering the case $\Delta(0)=0$ and by preparing the system in the ground state of $H(0)$, the initial state is given by $\rho^{\mathrm{LZ}}(0)=|1\rangle\langle 1|=\frac{1}{2}(\mathbb{1}-\sigma_z)$. In the superoperator formalism we write

$$|\rho^{\mathrm{LZ}}(0)\rangle\rangle=[1 \quad 0 \quad 0 \quad -1]^{\mathsf{T}}=|\mathcal{D}_0^{\mathrm{LZ}}(0)\rangle\rangle-|\mathcal{D}_1^{\mathrm{LZ}}(0)\rangle\rangle, \quad (72)$$

where we used that $\theta(0)=0$ [since $\Delta(0)=0$] to write $|\rho(0)\rangle\rangle$ in terms of the eigenvectors of $\mathbb{L}^{\mathrm{LZ}}(0)$. If the system undergoes the adiabatic dynamics, its evolved state is given by

$$|\rho_{\mathrm{ad}}^{\mathrm{LZ}}(t)\rangle\rangle=|\mathcal{D}_0^{\mathrm{LZ}}(t)\rangle\rangle-\exp\left(\int_{t_0}^{t}\Lambda_1(\xi)d\xi\right)|\mathcal{D}_1^{\mathrm{LZ}}(0)\rangle\rangle, \quad (73)$$

where $\Lambda_1(t)=\lambda_1(t)-\langle\langle\mathcal{E}_1(t)|\dot{\mathcal{D}}_1(t)\rangle\rangle$. By using that $\langle\langle\mathcal{E}_1(t)|\dot{\mathcal{D}}_1(t)\rangle\rangle=0$, we then obtain $\Lambda_1(t)=-2\gamma(t)$. Thus, we write

$$|\rho_{\mathrm{ad}}^{\mathrm{LZ}}(t)\rangle\rangle=|\mathcal{D}_0^{\mathrm{LZ}}(t)\rangle\rangle-\exp\left(-\int_{t_0}^{t}2\gamma(\xi)d\xi\right)|\mathcal{D}_1^{\mathrm{LZ}}(t)\rangle\rangle. \quad (74)$$

The explicit vector form of the above state is

$$|\rho_{\mathrm{ad}}^{\mathrm{LZ}}(t)\rangle\rangle=\left[1 \quad -\exp\left(-\textstyle\int_{t_0}^{t}2\gamma(\xi)d\xi\right)\sin\theta(t) \quad 0 \quad -\exp\left(-\textstyle\int_{t_0}^{t}2\gamma(\xi)d\xi\right)\cos\theta(t)\right]^{\mathsf{T}}. \quad (75)$$

By returning to the density matrix for the above dynamics, we then get

$$\rho_{\mathrm{ad}}^{\mathrm{LZ}}(t)=\frac{1}{2}\left[\mathbb{1}-\exp\left(-\int_{t_0}^{t}2\gamma(\xi)d\xi\right)\sin\theta(t)\sigma_x - \exp\left(-\int_{t_0}^{t}2\gamma(\xi)d\xi\right)\cos\theta(t)\sigma_z\right]. \quad (76)$$

2. Landau-Zener standard TQD

From Eqs. (70) and (71) we find the adiabatic phases associated with each eigenvector, which are given by

$$\Lambda_0^{\mathrm{LZ}}(t)=0, \quad \Lambda_1^{\mathrm{LZ}}(t)=-2\gamma(t), \quad (77)$$

$$\Lambda_2^{\mathrm{LZ}}(t)=-\gamma(t)-\sec\theta(t)\kappa(t)-\mathcal{G}_2^{\mathrm{LZ}}(t), \quad (78)$$

$$\Lambda_3^{\mathrm{LZ}}(t)=-\gamma(t)+\sec\theta(t)\kappa(t)-\mathcal{G}_3^{\mathrm{LZ}}(t), \quad (79)$$

where $\mathcal{G}_\mu^{\mathrm{LZ}}(t)=\langle\langle\mathcal{E}_\mu^{\mathrm{LZ}}(t)|\dot{\mathcal{D}}_\mu^{\mathrm{LZ}}(t)\rangle\rangle$ are the generalized Berry phases

$$\mathcal{G}_2^{\mathrm{LZ}}(t)=\frac{\varpi(t)[\kappa(t)-\cos\theta(t)\gamma(t)]}{2\kappa^2(t)\sec\theta(t)}, \quad (80a)$$

$$\mathcal{G}_3^{\mathrm{LZ}}(t)=-\frac{\varpi(t)[\kappa(t)+\cos\theta(t)\gamma(t)]}{2\kappa^2(t)\sec\theta(t)}, \quad (80b)$$

with $\varpi(t)=\gamma(t)\dot{\theta}(t)\tan\theta(t)-\dot{\gamma}(t)$. From Eqs. (80a) and (80b) it is possible to see that, when $\varpi(t)=0$, we obtain for the Landau-Zener model the generalized parallel transport condition $\mathcal{G}_\mu^{\mathrm{LZ}}(t)=0$ for all μ. Notice that this is reached for

$$\gamma(t)=\tilde{\gamma}(t)=\gamma_0\sec\theta(t) \quad (81)$$

for some constant γ_0. From the standard TQD theory, we can then write the counterdiabatic Lindbladian as

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{LZ}}(t)=\sum_{\mu=0}^{3}\left[|\dot{\mathcal{D}}_\mu^{\mathrm{LZ}}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{\mathrm{LZ}}(t)|-\mathcal{G}_\mu^{\mathrm{LZ}}(t)|\mathcal{D}_\mu^{\mathrm{LZ}}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{\mathrm{LZ}}(t)|\right]. \quad (82)$$

By using Eqs. (70) and (71) we find

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{LZ}}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & \frac{\tilde{\varpi}(t)}{2\kappa^2(t)} & \dot{\theta}(t)\\ 0 & \frac{-3\tilde{\varpi}(t)}{2\kappa^2(t)} & 0 & \frac{3\tilde{\varpi}(t)\sin\theta(t)}{2\cos\theta(t)\kappa^2(t)}\\ 0 & -\dot{\theta}(t) & -\frac{\tilde{\varpi}(t)\sin\theta(t)}{2\cos\theta(t)\kappa^2(t)} & 0\end{bmatrix}, \quad (83)$$

where $\tilde{\varpi}=\omega_0\varpi(t)\cos^2\theta(t)$. Similarly as discussed for the Deutsch problem, we can make the Lindblad superoperator $\mathbb{L}_{\mathrm{cd}}^{\mathrm{LZ}}(t)$ antisymmetric by imposing $\varpi(t)=0$, which corresponds to the fulfillment of the generalized parallel transport condition, as provided by Eq. (81). Hence, by using this condition, we get

$$\mathbb{L}_{\mathrm{cd}}^{\mathrm{LZ}}(t)=\begin{bmatrix}0&0&0&0\\0&0&0&\dot{\theta}(t)\\0&0&0&0\\0&-\dot{\theta}(t)&0&0\end{bmatrix}. \tag{84}$$

The Lindbladian contribution $\mathcal{L}_{\mathrm{cd}}^{\mathrm{LZ}}[\bullet]$ associated with the counterdiabatic superoperator $\mathbb{L}_{\mathrm{cd}}^{\mathrm{LZ}}(t)$ is then

$$\mathcal{L}_{\mathrm{cd}}^{\mathrm{LZ}}[\bullet]=\frac{1}{i\hbar}[H_{\mathrm{cd}}^{\mathrm{LZ}}(t),\bullet], \tag{85}$$

with the counterdiabatic Hamiltonian given by

$$H_{\mathrm{cd}}^{\mathrm{LZ}}(t)=\frac{\hbar\dot{\theta}(t)}{2}\sigma_y. \tag{86}$$

Therefore, the transitionless master equation is

$$\dot{\rho}(t)=-\frac{i}{\hbar}\left[H_{\mathrm{STQD}}^{\mathrm{LZ}}(t),\rho(t)\right]+\gamma(t)[\sigma_y\rho(t)\sigma_y-\rho(t)], \tag{87}$$

where $H_{\mathrm{STQD}}^{\mathrm{LZ}}(t)=H^{\mathrm{LZ}}(t)+H_{\mathrm{cd}}^{\mathrm{LZ}}(t)$. This correction term is exactly the counterdiabatic term for the Landau-Zener Hamiltonian found in Refs. [34,44]. In particular, it allows for the implementation of the adiabatic dynamics given by

$$\tilde{\rho}_{\mathrm{ad}}^{\mathrm{LZ}}(t)=\frac{1}{2}\left[\mathbb{1}-\exp\left(-2\int_0^t\tilde{\gamma}(\xi)d\xi\right)\sin\theta(t)\sigma_x-\exp\left(-2\int_0^t\tilde{\gamma}(\xi)d\xi\right)\cos\theta(t)\sigma_z\right]. \tag{88}$$

3. Landau-Zener generalized TQD

The generalized TQD for the Landau-Zener model is obtained by implementing the Lindblad superoperator

$$\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{LZ}}(t)=\sum_{\alpha=0}^{3}\Theta_\alpha^{\mathrm{LZ}}(t)\big|\mathcal{D}_\alpha^{\mathrm{LZ}}(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha^{\mathrm{LZ}}(t)\big|+|\dot{\mathcal{D}}_\alpha^{\mathrm{LZ}}(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha^{\mathrm{LZ}}(t)\big|. \tag{89}$$

By employing the same procedure as performed for the Deutsch algorithm, we find that the generalized Lindblad superoperator can be written as

$$\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{LZ}}(t)=\begin{bmatrix}0&0&0&0\\0&-2\gamma_0\sec\theta(t)&0&\tilde{\theta}\\0&0&-2\gamma_0\sec\theta(t)&0\\0&-\tilde{\theta}&0&-2\gamma_0\sec\theta(t)\end{bmatrix}, \tag{90}$$

which is obtained by choosing $\Theta_2(t)=\Theta_3(t)=-2\gamma_0\sec\theta(t)$. The master equation then reads

$$\dot{\rho}(t)=\frac{1}{i\hbar}\left[H_{\mathrm{cd}}^{\mathrm{LZ}},\rho(t)\right]+\mathcal{R}_x^{\mathrm{LZ}}[\rho(t)]+\mathcal{R}_y^{\mathrm{LZ}}[\rho(t)]+\mathcal{R}_z^{\mathrm{LZ}}[\rho(t)], \tag{91}$$

where $H_{\mathrm{cd}}^{\mathrm{LZ}}$ is given by Eq. (86) and

$$\mathcal{R}_k^{\mathrm{LZ}}[\bullet]=\frac{\gamma_0\sec\theta(t)}{2}(\sigma_k\bullet\sigma_k-\bullet). \tag{92}$$

This illustrates the fact that, in some situations, a time-independent Lindblad superoperator cannot be obtained, but the free parameters are still useful to provide time-independent fields for the Hamiltonian driving the system. As an example, let us consider $\theta(t)=\theta_0 t/\tau$ so that Eq. (88) provides, at $t=\tau$, the density operator

$$\tilde{\rho}_{\mathrm{ad}}^{\mathrm{LZ}}(\tau)=\tfrac{1}{2}[\mathbb{1}-e^{-2\gamma_0\tau\vartheta(\theta_0)}\sin\theta_0\sigma_x-e^{-2\gamma_0\tau\vartheta(\theta_0)}\cos\theta_0\sigma_z], \tag{93}$$

where $\vartheta(\theta_0)=-\log(\cos\theta_0)/\theta_0$ for $0\leqslant\theta_0<\pi/2$. In order to achieve this state, we plot the fidelity $\mathcal{F}_{\mathrm{OS}}^{\mathrm{LZ}}(\tau\omega)$ for the adiabatic and TQD dynamics in Fig. 3, where $\mathcal{F}_{\mathrm{OS}}^{\mathrm{LZ}}(\tau\omega)$ is computed from Eq. (66), with the target state given by $\tilde{\rho}_{\mathrm{ad}}^{\mathrm{LZ}}(\tau)$. As discussed in the preceding section, the use of the TQD evolution to achieve high fidelity in open systems does not necessarily imply high fidelity for the adiabatic trajectory in closed systems. This result is supported by Fig. 4, where we illustrate several distinct trajectories in the Bloch sphere.

V. CONCLUSION

We have introduced a generalized approach for TQD in open systems, which provides a phase freedom in the Lindblad superoperator. Such phase freedom allows for an optimization of the evolution as we drive the system through a

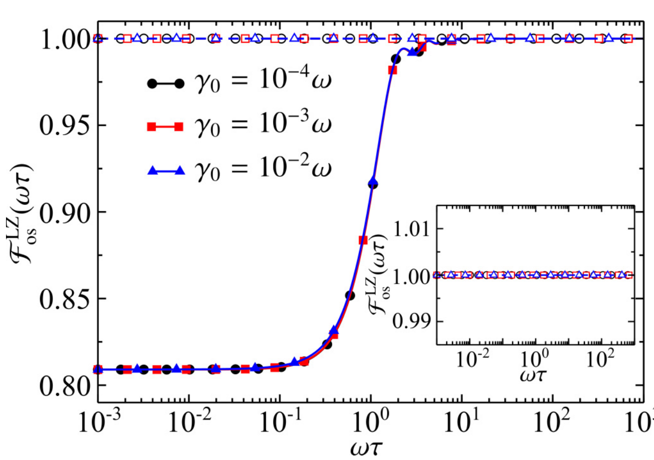

FIG. 3. Fidelity $\mathcal{F}_{\mathrm{OS}}^{\mathrm{LZ}}(\omega\tau)$ to achieve the open-system adiabatic target state of the Landau-Zener dynamics under a bit phase flip for different values of γ_0. Solid lines with closed symbols represent $\mathcal{F}_{\mathrm{OS}}^{\mathrm{LZ}}(\omega\tau)$ when the system is driven by the master equation (68), while dashed lines with open symbols represent $\mathcal{F}_{\mathrm{OS}}^{\mathrm{LZ}}(\omega\tau)$ when the system is driven by the generalized TQD master equation (91). The inset shows $\mathcal{F}_{\mathrm{OS}}(\omega\tau)$ for the standard TQD evolution in Eq. (87).

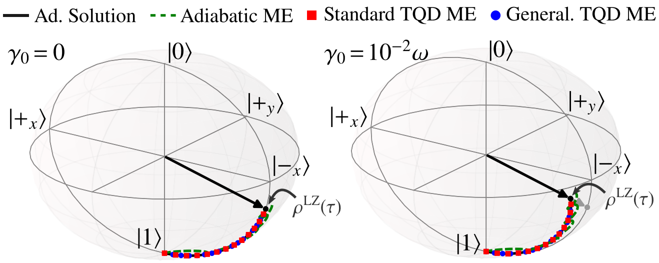

FIG. 4. Trajectories in the Bloch sphere for several quantum evolutions: adiabatic solution given by Eq. (88) (Ad. Solution), exact solution of the master equation (68) (Adiabatic ME), standard TQD approach in Eq. (87) (Standard TQD ME), and generalized TQD method in Eq. (91) (General. TQD ME). The dynamics is considered for closed (left Bloch sphere) and open (right Bloch sphere) cases, where we consider the total evolution time such that $\tau\omega = 10$

transitionless path. In particular, we discussed how to recover the standard TQD approach by suitably choosing the arbitrary phases of the generalized Lindblad superoperator. Moreover, we provided a theorem providing sufficient conditions for a time-independent generalized TQD Lindblad superoperator. As a direct consequence, we can mimic the adiabatic dynamics in open systems using time-independent control for both Hamiltonian and reservoir engineering. Our results were applied to two different situations. The first application concerns quantum computation through the adiabatic Deutsch algorithm under dephasing. For such an evolution, we determined the generalized phases in order to obtain a time-independent TQD master equation. As a second example, we considered a two-level system driven by the Landau-Zener Hamiltonian, with interest in atomic and/or molecular physics. We considered a nonunitary evolution under the bit-phase-flip channel. Such an example illustrates a time-dependent master equation, but with constant driving fields in the Hamiltonian. In both situations, we showed how reservoir engineering can be used to obtain an open-system parallel transport condition, which provides a geometric interpretation of shortcuts to adiabaticity in open systems.

The methods presented here offer the possibility of a number of applications in open-system inverse engineering, with potential impact even on universal Lindblad-like master equations [56]. For example, we can devise inverse engineering methods to design system-reservoir interactions able to track the ground state of a desired time-dependent Hamiltonian (induced adiabaticity). This is useful as an environment-based quantum control technique, which resembles the approaches proposed in Refs. [14,57].

Extending the efficiency of quantum control to open systems is an important issue, which is still more challenging for the case of many-body real systems embedded in an external environment. Our approach offers finite-time flexible control as long as the system can be described by a tractable local master equation in the adiabatic regime and the required TQD interaction can be reduced to sufficiently local couplings. This is a point worthy of further exploration in order to increase the size of quantum systems. Moreover, shortcuts to adiabaticity in open systems are also potentially fruitful in the investigation of quantum thermodynamics far from equilibrium. We have previously shown that the adiabatic behavior of open systems is compatible with the entropy variation at equilibrium [52]. We can now envisage the application of our results to study entropy production in irreversible dynamics via counterdiabatic methods. Shortcuts to quantum thermalization and out-of-equilibrium processes via inverse engineering are expected to be addressed in future research.

ACKNOWLEDGMENTS

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APPENDIX A: OPEN SYSTEMS IN THE SUPEROPERATOR FORMALISM

In the theory of open systems, the dynamics of a quantum system S, with a Hilbert space $\mathcal{H}_{\mathrm{S}}$, coupled to an environment A, with a Hilbert space $\mathcal{H}_{\mathrm{A}}$, is generically described by a master equation that takes into account the system-environment interaction. In particular, we consider here a system dynamics described by the time-local master equation [58,59]

$$\dot{\rho}(t) = \mathcal{L}_t[\rho(t)], \tag{A1}$$

where $\mathcal{L}_t[\bullet]$ is the generator of the dynamics and the subscript t makes explicit the possibility of the time dependence of the parameters associated with the environment. One way to deal with the above equation is to define an extended space where the generator $\mathcal{L}_t[\bullet]$ becomes a $(D^2 \times D^2)$-dimensional superoperator $\mathbb{L}(t)$ and the density operator becomes a supervector $|\rho(t)\rangle\rangle$. To see how it can be done, we define a set of $D^2 - 1$ operators $O = \{\sigma_n\} \in \mathcal{H}_{\mathrm{S}}$ $(n \geqslant 1)$ so that $\mathrm{Tr}\{\sigma_n\} = 0$ and $\mathrm{Tr}\{\sigma_n\sigma_m^\dagger\} = D\delta_{nm}$. The identity is introduced as a D-dimensional operator $\sigma_0 = \mathbb{1}$ so that we will be able to ensure $\mathrm{Tr}\{\rho(t)\} = 1$. In this form, we can expand $\rho(t)$ as

$$\rho = \frac{1}{D}\left[\mathbb{1} + \sum_{n=1}^{D^2-1} \varrho_n\sigma_n\right], \tag{A2}$$

with $\varrho_n = \mathrm{Tr}\{\rho\sigma_n^\dagger\}$. So, by using this expanded form of the density operator in Eq. (A1), we find the system of differential equations

$$\dot{\varrho}_k(t) = \frac{1}{D}\sum_{n=0}^{D^2-1} \varrho_i(t)\mathrm{Tr}\{\sigma_k^\dagger\mathcal{L}[\sigma_i]\}, \tag{A3}$$

where we assume that $\mathcal{L}[\bullet]$ is a linear operator. Note that if we identify the coefficient $\mathrm{Tr}\{\sigma_k^\dagger\mathcal{L}[\sigma_i]\}$ in the above equation as an element of the kth row and ith column of a $(D^2 \times D^2)$-dimensional matrix $\mathbb{L}(t)$, we can write

$$|\dot{\rho}(t)\rangle\rangle = \mathbb{L}(t)|\rho(t)\rangle\rangle, \tag{A4}$$

where $|\rho(t)\rangle\rangle$ is a D^2-dimensional vector with components $\varrho_n(t) = \mathrm{Tr}\{\rho(t)\sigma_n^\dagger\}$, $n = 0, 1, \ldots, D^2 - 1$. This proves Eq. (7).

An important remark is that, due to the non-Hermiticity of the operator $\mathcal{L}[\bullet]$, the superoperator $\mathbb{L}(t)$ may not be diagonalizable. Nevertheless, square matrices can always be written in the Jordan canonical form, where $\mathbb{L}(t)$ displays a block-diagonal structure $\mathbb{L}_J(t)$ composed of Jordan blocks $J_n(t)$. The number of Jordan blocks is the sum of the geometric multiplicities of all the eigenvalues $\lambda_\alpha(t)$ of $\mathbb{L}(t)$ [60]. Then we have the following definition.

Definition of the Jordan block. Given a $K \times K$ matrix L, the Jordan-block form of L reads

$$L_J = \begin{bmatrix} J_{k_1}[\lambda_{k_1}] & 0 & 0 & \cdots & 0 \\ 0 & J_{k_2}[\lambda_{k_2}] & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J_{k_N}[\lambda_N] \end{bmatrix}_{K\times K}, \quad \text{(A5)}$$

where each block $J_{k_1}[\lambda_{k_1}]$ is given by an upper triangular matrix of the form

$$J_k[\lambda] = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}_{k\times k}, \quad \text{(A6)}$$

with λ denoting the eigenvalues of L. Alternatively, a $K \times K$ Jordan matrix L_J can be defined as

$$L_J = J_{k_1}[\lambda_1] \oplus J_{k_2}[\lambda_2] \oplus \cdots \oplus J_{k_N}[\lambda_N] = \bigoplus_{\alpha=1}^{N} J_{k_\alpha}[\lambda_\alpha], \quad \text{(A7)}$$

where $N \leqslant K$ is the number of Jordan blocks required to write L_J in a Jordan-block form and $k_1 + k_2 + \cdots + k_N = K$.

In some cases, the coefficients λ_n may depend on other parameters (time, for example). Then, by assuming that the coefficients λ_n depend on a complete set of parameters $\xi = \{\xi_1, \ldots, \xi_M\}$, we define the Jordan matrix as

$$L_J(\xi) = J_{k_1}[\lambda_1(\xi)] \oplus J_{k_2}[\lambda_2(\xi)] \oplus \cdots \oplus J_{k_N}[\lambda_N(\xi)]. \quad \text{(A8)}$$

The notion of Jordan form is important here because, different from the generator of a unitary dynamics (the Hamiltonian), the generator $\mathbb{L}(t)$ in an open system does not always admit a diagonal form. However, every square matrix A can be diagonalized by blocks from the Jordan canonical form theorem [60].

Theorem 2 (Jordan canonical form). Let $A \in M_K$ be a $K \times K$ square matrix in the set M_K. Then there is a nonsingular matrix $S \in M_K$, positive integers $k_1, \ldots, k_N$, with $k_1 + k_2 + \cdots + k_N = K$ $(N \leqslant K)$, and scalars $\lambda_1, \ldots, \lambda_N \in C$ so that

$$A(\xi) = S(\xi) J_A(\xi) S^{-1}(\xi), \quad \text{(A9)}$$

where $J_A(\xi) = J_{k_1}[\lambda_1(\xi)] \oplus J_{k_2}[\lambda_2(\xi)] \oplus \cdots \oplus J_{k_N}[\lambda_N(\xi)]$ is the Jordan matrix associated with $A(\xi)$.

Now, by using the above discussion of the superoperator $\mathbb{L}(t)$, we can obtain its Jordan form through the matrix $S(t)$, which allows us to write

$$\mathbb{L}_J(t) = S^{-1}(t)\mathbb{L}(t)S(t) = \bigoplus_{\alpha=1}^{N} L_{N_\alpha}[\lambda_\alpha(t)], \quad \text{(A10)}$$

where N is the sum of the geometric multiplicities of all the eigenvalues $\lambda_\alpha(t)$ of $\mathbb{L}(t)$ and each block $L_{N_\alpha}[\lambda_\alpha(t)]$ is an $(N_\alpha \times N_\alpha)$-dimensional matrix given as in Eq. (A6). Since the Hilbert space of the system has dimension D, we find $N_1 + N_2 + \cdots + N_N = D^2$. In addition, as an immediate consequence of the structure of $\mathbb{L}_J(t)$, we see that $\mathbb{L}(t)$ does not always admit the existence of a complete set of eigenvectors. Instead, we define right $|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle$ and left $\langle\langle\mathcal{E}_\alpha^{n_\alpha}(t)|$ quasieigenvectors of $\mathbb{L}(t)$ associated with the eigenvalue $\lambda_\alpha(t)$, satisfying

$$\mathbb{L}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\big\rangle = \big|\mathcal{D}_\alpha^{(n_\alpha-1)}(t)\big\rangle\big\rangle + \lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\big\rangle, \quad \text{(A11a)}$$

$$\big\langle\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\mathbb{L}(t) = \big\langle\big\langle\mathcal{E}_\alpha^{(n_\alpha+1)}(t)\big| + \big\langle\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\lambda_\alpha(t). \quad \text{(A11b)}$$

The sets $\{|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\}$ and $\{\langle\langle\mathcal{E}_\alpha^{n_\alpha}(t)|\}$ constitute bases for the space associated with the operator $\mathbb{L}(t)$, satisfying the normalization condition $\langle\langle\mathcal{E}_\beta^{m_\beta}(t)|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle = \delta_{\beta\alpha}\delta_{m_\beta n_\alpha}$ and the completeness relationship

$$\sum_{\alpha=1}^{N}\sum_{n_\alpha=1}^{N_\alpha}\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\big\rangle\big\langle\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big| = \mathbb{1}_{D^2\times D^2}, \quad \text{(A12)}$$

where N is the number of Jordan blocks in Eq. (A10) and N_α is the dimension of the αth Jordan block.

APPENDIX B: STANDARD COUNTERDIABATIC LINDBLADIANS

Let us derive the standard counterdiabatic Lindblad superoperator from the similarity transformation $\mathbb{L}_J(t) = C^{-1}(t)\mathbb{L}(t)C(t)$ induced by the superoperator $C(t)$, with $\mathbb{L}_J(t)$ denoting the Jordan canonical form of $\mathbb{L}(t)$. From Eq. (12) we have

$$[\mathbb{L}_J(t) + \dot{C}^{-1}(t)C(t)]|\rho(t)\rangle\rangle_J = |\dot{\rho}(t)\rangle\rangle_J, \quad \text{(B1)}$$

with $|\rho(t)\rangle\rangle_J = C^{-1}(t)|\rho(t)\rangle\rangle_J$. We then split the term $\dot{C}^{-1}(t)C(t)$ into two contributions $\dot{C}^{-1}(t)C(t) = \mathbb{L}'_J(t) + \mathbb{L}'_{nd}(t)$ so that we can rewrite Eq. (B1) as

$$[\mathbb{L}_J(t) + \mathbb{L}'_J(t) + \mathbb{L}'_{nd}(t)]|\rho(t)\rangle\rangle_J = |\dot{\rho}(t)\rangle\rangle_J, \quad \text{(B2)}$$

where

$$\mathbb{L}'_J(t) = \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\sum_{\ell_\mu=1}^{N_\mu} C_{\mu\mu}^{n_\mu\ell_\mu}(t)\big|\sigma_\mu^{n_\mu}\big\rangle\big\rangle\big\langle\big\langle\sigma_\mu^{\ell_\mu}\big|, \quad \text{(B3)}$$

$$\mathbb{L}'_{nd}(t) = \sum_{\nu\neq\mu}^{N-1}\sum_{n_\nu=1}^{N_\nu}\sum_{\mu=0}^{N-1}\sum_{\ell_\mu=1}^{N_\mu} C_{\nu\mu}^{n_\nu\ell_\mu}(t)\big|\sigma_\nu^{n_\nu}\big\rangle\big\rangle\big\langle\big\langle\sigma_\mu^{\ell_\mu}\big|, \quad \text{(B4)}$$

with coefficients $C_{\nu\mu}^{n_\nu\ell_\mu}(t) = \langle\langle\sigma_\nu^{n_\nu}|\dot{C}^{-1}(t)C(t)|\sigma_\mu^{\ell_\mu}\rangle\rangle$. Equation (B2) leads to the counterdiabatic Lindbladian as proposed by Vacanti *et al.* in Ref. [42], which reads

$$\mathbb{L}_{cd}(t) = -C(t)\mathbb{L}'_{nd}(t)C^{-1}(t). \quad \text{(B5)}$$

Henceforth, we aim at providing $\mathbb{L}_{cd}(t)$ in terms of the right and left quasieigenbases $\{|\mathcal{D}_\beta^{m_\beta}(t)\rangle\rangle\}$ and $\{\langle\langle\mathcal{E}_\beta^{m_\beta}(t)|\}$. In this direction, we start by expressing $\mathbb{L}_{\rm cd}(t)$ as

$$\begin{aligned}\mathbb{L}_{\rm cd}(t) &= -C(t)\mathbb{L}'_{\rm nd}(t)C^{-1}(t) = -C(t)\mathbb{L}'_{\rm nd}(t)C^{-1}(t) + [C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t) - C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t)]\\ &= -C(t)[\mathbb{L}'_{\rm nd}(t) + \mathbb{L}'_{\rm J}(t)]C^{-1}(t) + C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t) = -C(t)[\dot{C}^{-1}(t)C(t)]C^{-1}(t) + C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t).\end{aligned} \tag{B6}$$

Let us now use $\dot{C}^{-1}(t)C(t) = -C^{-1}(t)\dot{C}(t)$, yielding

$$\mathbb{L}_{\rm cd}(t) = \dot{C}(t)C^{-1}(t) + C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t). \tag{B7}$$

Thus, we can compute each term on the right-hand side of Eq. (B7) as

$$\dot{C}(t)C^{-1}(t) = \left[\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}|\dot{\mathcal{D}}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\sigma_\mu^{n_\mu}|\right]\left[\sum_{\nu=0}^{N-1}\sum_{k_\nu=1}^{N_\nu}|\sigma_\nu^{k_\nu}\rangle\rangle\langle\langle\mathcal{E}_\nu^{k_\nu}(t)|\right] = \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}|\dot{\mathcal{D}}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{n_\mu}(t)| \tag{B8}$$

and

$$\begin{aligned}C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t) &= \left[\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\sigma_\mu^{n_\mu}|\right]\left[\sum_{\eta=0}^{N-1}\sum_{j_\eta=1}^{N_\eta}\sum_{\ell_\eta=1}^{N_\eta}C_{\eta\eta}^{j_\eta\ell_\eta}(t)|\sigma_\eta^{j_\eta}\rangle\rangle\langle\langle\sigma_\eta^{\ell_\eta}|\right]\left[\sum_{\nu=0}^{N-1}\sum_{k_\nu=1}^{N_\nu}|\sigma_\nu^{k_\nu}\rangle\rangle\langle\langle\mathcal{E}_\nu^{k_\nu}(t)|\right]\\ &= \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\sum_{\eta=0}^{N-1}\sum_{j_\eta=1}^{N_\eta}\sum_{\ell_\eta=1}^{N_\eta}\sum_{\nu=0}^{N-1}\sum_{k_\nu=1}^{N_\nu}\big[|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\sigma_\mu^{n_\mu}|\big]\big[C_{\eta\eta}^{j_\eta\ell_\eta}(t)|\sigma_\eta^{j_\eta}\rangle\rangle\langle\langle\sigma_\eta^{\ell_\eta}|\big]\big[|\sigma_\nu^{k_\nu}\rangle\rangle\langle\langle\mathcal{E}_\nu^{k_\nu}(t)|\big]\\ &= \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\sum_{\eta=0}^{N-1}\sum_{j_\eta=1}^{N_\eta}\sum_{\ell_\eta=1}^{N_\eta}\sum_{\nu=0}^{N-1}\sum_{k_\nu=1}^{N_\nu}\left[C_{\eta\eta}^{j_\eta\ell_\eta}(t)|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\underbrace{\langle\langle\sigma_\mu^{n_\mu}|\sigma_\eta^{j_\eta}\rangle\rangle}_{\delta_{\mu\eta}\delta_{n_\mu j_\eta}}\underbrace{\langle\langle\sigma_\eta^{\ell_\eta}|\sigma_\nu^{k_\nu}\rangle\rangle}_{\delta_{\nu\eta}\delta_{\ell_\eta k_\nu}}\langle\langle\mathcal{E}_\nu^{k_\nu}(t)|\right]\\ &= \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\sum_{k_\mu=1}^{N_\mu}\big[C_{\mu\mu}^{n_\mu k_\mu}(t)|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{k_\mu}(t)|\big] = \sum_{\mu=0}^{N-1}\sum_{n_\mu,k_\mu=1}^{N_\mu}\big[\langle\langle\sigma_\mu^{n_\mu}|\dot{C}^{-1}(t)C(t)|\sigma_\mu^{k_\mu}\rangle\rangle|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{k_\mu}(t)|\big],\end{aligned} \tag{B9}$$

where we used $C_{\mu\mu}^{n_\mu k_\mu}(t) = \langle\langle\sigma_\mu^{n_\mu}|\dot{C}^{-1}(t)C(t)|\sigma_\mu^{k_\mu}\rangle\rangle$ in the last equality. Now we use the definition of $C(t)$ in Eq. (11) to express $\dot{C}^{-1}(t)C(t)$ as

$$\dot{C}^{-1}(t)C(t) = -\sum_{\eta=0}^{N-1}\sum_{\ell_\eta=1}^{N_\eta}\sum_{\nu=0}^{N-1}\sum_{j_\nu=1}^{N_\nu}\mathcal{G}_{\eta\nu}^{\ell_\eta j_\nu}(t)|\sigma_\eta^{\ell_\eta}\rangle\rangle\langle\langle\sigma_\nu^{j_\nu}|, \tag{B10}$$

where $\mathcal{G}_{\eta\nu}^{\ell_\eta j_\nu}(t) = \langle\langle\mathcal{E}_\eta^{\ell_\eta}(t)|\dot{\mathcal{D}}_\nu^{j_\nu}(t)\rangle\rangle$, so that

$$\langle\langle\sigma_\mu^{n_\mu}|\dot{C}^{-1}(t)C(t)|\sigma_\mu^{k_\mu}\rangle\rangle = -\sum_{\eta,\nu=0}^{N-1}\sum_{\ell_\eta,j_\nu=1}^{N_\eta,N_\nu}\mathcal{G}_{\eta\nu}^{\ell_\eta j_\nu}(t)\delta_{\mu\eta}\delta_{n_\mu\ell_\eta}\delta_{\nu\mu}\delta_{j_\nu k_\mu} = -\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t). \tag{B11}$$

Therefore, by inserting Eq. (B11) in Eq. (B9), we get

$$C(t)\mathbb{L}'_{\rm J}(t)C^{-1}(t) = -\sum_{\mu=0}^{N-1}\sum_{n_\mu,k_\mu=1}^{N_\mu}\big[\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t)|\mathcal{D}_\mu^{n_\mu}(t)\rangle\rangle\langle\langle\mathcal{E}_\mu^{k_\mu}(t)|\big]. \tag{B12}$$

So from Eqs. (B7), (B8), and (B12) we conclude that

$$\mathbb{L}_{\rm cd}(t) = \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\left[|\dot{\mathcal{D}}_\mu^{n_\mu}\rangle\rangle\langle\langle\mathcal{E}_\mu^{n_\mu}| - \sum_{k_\mu=1}^{N_\mu}\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}|\mathcal{D}_\mu^{n_\mu}\rangle\rangle\langle\langle\mathcal{E}_\mu^{k_\mu}|\right]. \tag{B13}$$

Therefore, $\mathbb{L}_{\rm cd}(t)$ exhibits a structure that is formally identical to the standard counterdiabatic Hamiltonian $H_{\rm cd}(t)$ in Eq. (3). Let us now show that $\mathbb{L}_{\rm cd}(t)$ mimics the adiabatic evolution in open systems. In the superoperator formalism, the master equation is given by

$$|\dot{\rho}(t)\rangle\rangle = [\mathbb{L}(t) + \mathbb{L}_{\rm cd}(t)]|\rho(t)\rangle\rangle. \tag{B14}$$

We now use the expansion $|\rho(t)\rangle\rangle = \sum_{\alpha,n_\alpha} r_\alpha^{n_\alpha}(t)|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle$ in terms of the right quasieigenbasis of $\mathbb{L}(t)$. For each component $r_\alpha^{n_\alpha}(t)|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle$ we then get

$$\frac{d}{dt}\big[r_\alpha^{n_\alpha}(t)|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\big] = r_\alpha^{n_\alpha}(t)[\mathbb{L}(t) + \mathbb{L}_{\rm cd}(t)]|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle. \tag{B15}$$

By defining $\mathbb{L}_{\mathrm{STQD}}(t) = \mathbb{L}(t) + \mathbb{L}_{\mathrm{cd}}(t)$, we obtain

$$\begin{aligned}\mathbb{L}_{\mathrm{STQD}}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle &= [\mathbb{L}(t) + \mathbb{L}_{\mathrm{cd}}(t)]\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle \\ &= \big|\mathcal{D}_\alpha^{(n_\alpha-1)}(t)\big\rangle\!\big\rangle + \lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle + \sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\big|\dot{\mathcal{D}}_\mu^{n_\mu}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\mu^{n_\mu}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle \\ &\quad - \sum_{\mu=0}^{N-1}\sum_{n_\mu,k_\mu=1}^{N_\mu}\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t)\big|\mathcal{D}_\mu^{n_\mu}(t)\big\rangle\!\big\rangle\big\langle\!\big\langle\mathcal{E}_\mu^{k_\mu}(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle \\ &= \big|\mathcal{D}_\alpha^{(n_\alpha-1)}\big\rangle\!\big\rangle + \lambda_\alpha\big|\mathcal{D}_\alpha^{n_\alpha}\big\rangle\!\big\rangle + \big|\dot{\mathcal{D}}_\alpha^{n_\alpha}\big\rangle\!\big\rangle - \sum_{n_\mu=1}^{N_\mu}\mathcal{G}_{\alpha\alpha}^{n_\mu n_\alpha}\big|\mathcal{D}_\alpha^{n_\mu}\big\rangle\!\big\rangle, \end{aligned} \tag{B16}$$

so that, after projecting Eq. (B15) over $\langle\!\langle\mathcal{E}_\beta^{j_\beta}(t)|$, the dynamics for the coefficient $r_\beta^{j_\beta}(t)$ reads

$$\dot{r}_\beta^{j_\beta}(t) = \lambda_\beta(t)r_\beta^{j_\beta}(t) - \sum_{n_\alpha=1}^{N_\alpha}\mathcal{G}_{\beta\beta}^{j_\beta n_\alpha}(t)r_\beta^{n_\alpha}(t) + r_\beta^{(j_\beta+1)}(t). \tag{B17}$$

Equation (B17) implies the adiabatic behavior, with the Jordan blocks decoupled from each other and the standard open-system adiabatic phase fixed [13,52]. Therefore, the Lindbladian $\mathbb{L}_{\mathrm{STQD}}(t) = \mathbb{L}'(t) = \mathbb{L}(t) + \mathbb{L}_{\mathrm{cd}}(t)$ exactly mimics the adiabatic dynamics in open systems.

APPENDIX C: INVERSE ENGINEERING IN OPEN SYSTEMS

Let us start from the time-local master equation provided by Eq. (7) and assume that the evolved density operator $|\rho(t)\rangle\!\rangle$ can be obtained by a nonunitary superoperator $\mathcal{V}(t,t_0)$ as $|\rho(t)\rangle\!\rangle = \mathcal{V}(t,t_0)|\rho(t_0)\rangle\!\rangle$. Then

$$\dot{\mathcal{V}}(t,t_0)|\rho(t_0)\rangle\!\rangle = \mathbb{L}(t)\mathcal{V}(t,t_0)|\rho(t_0)\rangle\!\rangle, \tag{C1}$$

which holds for any initial state $|\rho(t_0)\rangle\!\rangle$. Therefore,

$$\dot{\mathcal{V}}(t,t_0) = \mathbb{L}(t)\mathcal{V}(t,t_0). \tag{C2}$$

We assume an open-system evolution driven by an invertible dynamical map. Then we consider a superoperator $\mathcal{V}^{-1}(t,t_0)$ so that $\mathcal{V}^{-1}(t,t_0)\mathcal{V}(t,t_0) = \mathbb{1}$. By multiplying Eq. (C2) by $\mathcal{V}^{-1}(t,t_0)$, we get

$$\mathbb{L}(t) = \dot{\mathcal{V}}(t,t_0)\mathcal{V}^{-1}(t,t_0). \tag{C3}$$

Equation (C3) generalizes the inverse engineering approach to the realm of open quantum systems.

APPENDIX D: GENERALIZED TQD FOR 1D JORDAN DECOMPOSITION

Let $\mathbb{L}(t)$ be a Lindbladian superoperator that admits one-dimensional Jordan-block decomposition. Then the phase-free generalized TQD evolution operator reads

$$\mathcal{V}_{\mathrm{GTQD}}^{\mathrm{1D}}(t,t_0) = \sum_{\alpha=0}^{N-1}\exp\left(\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t_0)|, \tag{D1}$$

so that the associated Lindbladian is

$$\begin{aligned}\mathbb{L}_{\mathrm{GTQD}}^{\mathrm{1D}}(t) &= \sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1}\frac{d}{dt}\left[\exp\left(\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\!\rangle\right]\langle\!\langle\mathcal{E}_\beta(t)|\exp\left(-\int_{t_0}^{t}\Theta_\beta(\xi)d\xi\right)\langle\!\langle\mathcal{E}_\alpha(t_0)|\mathcal{D}_\beta(t_0)\rangle\!\rangle \\ &= \sum_{\alpha=0}^{N-1}\frac{d}{dt}\left[\exp\left(\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\!\rangle\right]\langle\!\langle\mathcal{E}_\alpha(t)|\exp\left(-\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right) \\ &= \sum_{\alpha=0}^{N-1}\Theta_\alpha(t)\exp\left(\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right)|\mathcal{D}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)|\exp\left(-\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right) \\ &\quad + \exp\left(\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right)|\dot{\mathcal{D}}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)|\exp\left(-\int_{t_0}^{t}\Theta_\alpha(\xi)d\xi\right) \\ &= \sum_{\alpha=0}^{N-1}\Theta_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)| + |\dot{\mathcal{D}}_\alpha(t)\rangle\!\rangle\langle\!\langle\mathcal{E}_\alpha(t)|. \end{aligned} \tag{D2}$$

The standard TQD Lindbladian $\mathbb{L}_{\mathrm{STQD}}^{1\mathrm{D}}(t)$ can be recovered by imposing that $\Theta_\alpha(t)$ is equal to the open-system adiabatic phase, i.e., $\Theta_\alpha(t) = \lambda_\alpha(t) - \langle\langle \mathcal{E}_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle$. In that case, we have

$$\begin{aligned}
\mathbb{L}_{\mathrm{GTQD}}^{1\mathrm{D}}(t) &= \overbrace{\sum_{\alpha=0}^{N-1} \lambda_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)|}^{\mathbb{L}(t)} \\
&\quad + \underbrace{\sum_{\alpha=0}^{N-1} |\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| - \langle\langle\mathcal{E}_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)|}_{\mathbb{L}_{\mathrm{cd}}^{1\mathrm{D}}(t)} \\
&= \mathbb{L}(t) + \mathbb{L}_{\mathrm{cd}}^{1\mathrm{D}}(t) = \mathbb{L}_{\mathrm{STQD}}^{1\mathrm{D}}(t).
\end{aligned} \tag{D3}$$

APPENDIX E: PROOF OF THEOREM 1

We now demonstrate Theorem 1. To this end, we need to consider the time derivative of $\mathbb{L}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)$, which yields

$$\begin{aligned}
\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t) &= \frac{d}{dt}\left[\sum_{\alpha=0}^{N-1} \Theta_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| + |\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)|\right] \\
&= \sum_{\alpha=0}^{N-1}[\dot{\Theta}_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| + \Theta_\alpha(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| + \Theta_\alpha(t)|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\dot{\mathcal{E}}_\alpha(t)|] \\
&\quad + \sum_{\alpha=0}^{N-1}[|\ddot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| + |\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\dot{\mathcal{E}}_\alpha(t)|].
\end{aligned} \tag{E1}$$

By computing the matrix elements $\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)|_{\eta\beta} = \langle\langle\mathcal{E}_\eta(t)|\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)|\mathcal{D}_\beta(t)\rangle\rangle$, we obtain

$$\begin{aligned}
\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)|_{\eta\beta} &= \sum_{\alpha=0}^{N-1}[\dot{\Theta}_\alpha(t)\delta_{\eta\alpha}\delta_{\alpha\beta} + \Theta_\alpha(t)\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\delta_{\alpha\beta} + \Theta_\alpha(t)\delta_{\eta\alpha}\langle\langle\dot{\mathcal{E}}_\alpha(t)|\mathcal{D}_\beta(t)\rangle\rangle] \\
&\quad + \sum_{\alpha=0}^{N-1}[\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle\langle\langle\dot{\mathcal{E}}_\alpha(t)|\mathcal{D}_\beta(t)\rangle\rangle + \langle\langle\mathcal{E}_\eta(t)|\ddot{\mathcal{D}}_\alpha(t)\rangle\rangle\delta_{\alpha\beta}] \\
&= \dot{\Theta}_\eta(t)\delta_{\eta\beta} + [\Theta_\beta(t) - \Theta_\eta(t)]\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle + \langle\langle\mathcal{E}_\eta(t)|\ddot{\mathcal{D}}_\beta(t)\rangle\rangle + \langle\langle\dot{\mathcal{E}}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle,
\end{aligned} \tag{E2}$$

where we have used the identity $\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\alpha(t)\rangle\rangle = -\langle\langle\dot{\mathcal{E}}_\eta(t)|\mathcal{D}_\alpha(t)\rangle\rangle$ (due to the left-right eigenvector orthonormalization) and the completeness relation $\sum_{\alpha=0}^{N-1}|\mathcal{D}_\alpha(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)| = \mathbb{1}$. Thus, in order to get a time-independent Lindbladian superoperator, we need to find parameters $\Theta_\beta(t)$ so that $\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t) = 0$. This occurs if and only if we require

$$\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)|_{\eta\eta} = \dot{\Theta}_\eta(t) + \frac{d}{dt}[\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\eta(t)\rangle\rangle] = 0, \tag{E3}$$

$$\dot{\mathbb{L}}_{\mathrm{GTQD}}^{1\mathrm{D}}(t)|_{\eta,\beta\neq\eta} = [\Theta_\beta(t) - \Theta_\eta(t)]\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle + \frac{d}{dt}[\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle] = 0. \tag{E4}$$

Let us assume that we have

$$\frac{d}{dt}[\langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\beta(t)\rangle\rangle] = 0\,\forall\,\eta,\beta. \tag{E5}$$

Under this condition, it is possible to solve Eqs. (E3) and (E4) by simply imposing

$$\Theta_\beta = \Theta_\eta = \mathrm{const}\,\forall\,\eta,\beta. \tag{E6}$$

On the other hand, even if Eq. (E5) is not satisfied, we may find an analytical solution. By explicitly solving Eq. (E3) for $\Theta_\eta(t)$ and by denoting the solution by $\overline{\Theta}_\eta(t)$, we obtain

$$\begin{aligned}
\overline{\Theta}_\eta(t) - \overline{\Theta}_\eta(t_0) &= -\int_{t_0}^{t}\frac{d}{d\xi}[\langle\langle\mathcal{E}_\eta(\xi)|\dot{\mathcal{D}}_\eta(\xi)\rangle\rangle]d\xi \\
&= \langle\langle\mathcal{E}_\eta(t_0)|\dot{\mathcal{D}}_\eta(t_0)\rangle\rangle - \langle\langle\mathcal{E}_\eta(t)|\dot{\mathcal{D}}_\eta(t)\rangle\rangle
\end{aligned} \tag{E7}$$

so that

$$\overline{\Theta}_{\eta}(t) = -\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\eta}(t)\rangle\rangle \equiv \mathcal{G}_{\eta}(t). \tag{E8}$$

However, we still have to further impose Eq. (E4) in order to guarantee that $\overline{\Theta}_{\eta}(t)$ provides a time-independent Lindbladian. By using Eq. (E8) in Eq. (E4), we get

$$\begin{aligned}\mathbb{L}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)|_{\eta,\beta\neq\eta} &= [\overline{\Theta}_{\beta}(t) - \overline{\Theta}_{\eta}(t)]\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle + \frac{d}{dt}[\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle]\\ &= [\mathcal{G}_{\beta}(t) - \mathcal{G}_{\eta}(t)]\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle + \frac{d}{dt}[\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle] = 0.\end{aligned} \tag{E9}$$

Therefore, we have a differential equation for $\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle$, which can be rewritten as

$$\frac{\frac{d}{dt}[\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle]}{\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle} = \mathcal{G}_{\eta}(t) - \mathcal{G}_{\beta}(t). \tag{E10}$$

Equation (E10) then yields

$$\frac{d}{dt}\ln[\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle] = \mathcal{G}_{\eta}(t) - \mathcal{G}_{\beta}(t), \tag{E11}$$

where, by integrating, we find

$$\ln\left[\frac{\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle}{\langle\langle\mathcal{E}_{\eta}(t_0)|\dot{\mathcal{D}}_{\beta}(t_0)\rangle\rangle}\right] = \int_{t_0}^{t}[\mathcal{G}_{\eta}(\xi) - \mathcal{G}_{\beta}(\xi)]d\xi. \tag{E12}$$

Therefore, we conclude that the condition to be satisfied is given by

$$\langle\langle\mathcal{E}_{\eta}(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle = \langle\langle\mathcal{E}_{\eta}(t_0)|\dot{\mathcal{D}}_{\beta}(t_0)\rangle\rangle \exp\left(\int_{t_0}^{t}[\mathcal{G}_{\eta}(\xi) - \mathcal{G}_{\beta}(\xi)]d\xi\right) \forall\, \eta, \beta. \tag{E13}$$

We can now analyze a particular case. As discussed in Ref. [54], from the set of right eigenvectors of the Lindbladian, it is always possible to get a time-independent eigenvector $|\mathcal{D}_0(t)\rangle\rangle = |\mathcal{D}_0\rangle\rangle$ associated with an eigenvalue $\lambda_0 = 0$. Let us consider the matrix elements of the Lindbladian for this constant eigenvector. From Eqs. (E3) and (E4) we write

$$\langle\langle\mathcal{E}_0(t)|\dot{\mathbb{L}}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)|\mathcal{D}_0(t)\rangle\rangle = \dot{\Theta}_0(t) + \frac{d}{dt}[\langle\langle\mathcal{E}_0(t)|\dot{\mathcal{D}}_0(t)\rangle\rangle] = 0, \tag{E14}$$

$$\langle\langle\mathcal{E}_0(t)|\dot{\mathbb{L}}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)|\mathcal{D}_{\beta\neq0}(t)\rangle\rangle = [\Theta_{\beta}(t) - \Theta_0(t)]\langle\langle\mathcal{E}_0(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle + \frac{d}{dt}[\langle\langle\mathcal{E}_0(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle] = 0. \tag{E15}$$

Therefore, since $|\mathcal{D}_0(t)\rangle\rangle = |\mathcal{D}_0\rangle\rangle$, we can always choose a corresponding time-independent left eigenvector $\langle\langle\mathcal{E}_0(t)| = \langle\langle\mathcal{E}_0|$ so that $\langle\langle\mathcal{E}_0(t)|\dot{\mathcal{D}}_{\beta}(t)\rangle\rangle = 0\,\forall\,\beta$ (due to the orthonormalization condition). Hence

$$\langle\langle\mathcal{E}_0(t)|\dot{\mathbb{L}}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)|\mathcal{D}_0(t)\rangle\rangle = \dot{\Theta}_0(t) = 0, \tag{E16}$$

$$\langle\langle\mathcal{E}_0(t)|\dot{\mathbb{L}}^{\mathrm{1D}}_{\mathrm{GTQD}}(t)|\mathcal{D}_{\beta\neq0}(t)\rangle\rangle = 0. \tag{E17}$$

This shows that $\Theta_0(t)$ can be chosen to be constant independently of the remaining parameters $\Theta_{\beta\neq0}(t)$, i.e., the definition of the phase specifically associated with the vanishing eigenvalue $\lambda_0 = 0$ is decoupled from the other sectors $\beta \neq 0$.

APPENDIX F: GENERALIZED TQD LINDBLAD SUPEROPERATOR FOR MULTIDIMENSIONAL JORDAN BLOCKS

Let us obtain the generalized TQD Lindbladian for multidimensional Jordan blocks, which is provided by Eq. (30). First, we write the evolution superoperator, which reads

$$\mathcal{V}_{\mathrm{GTQD}}(t, t_0) = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha} q_{\alpha}^{n_\alpha m_\alpha}\big|\mathcal{D}_{\alpha}^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_{\alpha}^{m_\alpha}(t_0)\big|, \tag{F1}$$

where the time dependence of the coefficients q is omitted. Then we can write the inverse evolution superoperator $\mathcal{V}^{\mathrm{gen}}_{\mathrm{TQD}-1}(t, t_0)$ as

$$\mathcal{V}^{-1}_{\mathrm{GTQD}}(t, t_0) = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha} \tilde{q}_{\alpha}^{n_\alpha m_\alpha}\big|\mathcal{D}_{\alpha}^{n_\alpha}(t_0)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_{\alpha}^{m_\alpha}(t)\big|, \tag{F2}$$

where the coefficients $\tilde{q}_\alpha^{n_\alpha m_\alpha}$ are such that $\mathcal{V}_{\rm GTQD}(t,t_0)\mathcal{V}_{\rm GTQD}^{-1}(t,t_0)=\mathbb{1}$. In order to determine the conditions to be obeyed by $q_\alpha^{n_\alpha m_\alpha}$ and $\tilde{q}_\alpha^{n_\alpha m_\alpha}$, let us explicitly consider the expression

$$\begin{aligned}\mathcal{V}_{\rm GTQD}(t,t_0)\mathcal{V}_{\rm GTQD}^{-1}(t,t_0) &= \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha}\sum_{\beta=0}^{N-1}\sum_{k_\beta=1}^{N_\beta}\sum_{\ell_\beta=1}^{N_\beta} q_\alpha^{n_\alpha m_\alpha}\tilde{q}_\beta^{k_\beta \ell_\beta}\langle\langle\mathcal{E}_\alpha^{m_\alpha}(t_0)|\mathcal{D}_\beta^{k_\beta}(t_0)\rangle\rangle|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\beta^{\ell_\beta}(t)| \\ &= \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha}\sum_{\ell_\alpha=1}^{N_\alpha} q_\alpha^{n_\alpha m_\alpha}\tilde{q}_\alpha^{m_\alpha \ell_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{\ell_\alpha}(t)|,\end{aligned} \tag{F3}$$

so that we write

$$\langle\langle\mathcal{E}_\eta^{l_\eta}(t)|\mathcal{V}_{\rm GTQD}(t,t_0)\mathcal{V}_{\rm GTQD}^{-1}(t,t_0)|\mathcal{D}_\kappa^{i_\kappa}(t)\rangle\rangle = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{m_\alpha=1}^{N_\alpha}\sum_{\ell_\alpha=1}^{N_\alpha} q_\alpha^{n_\alpha m_\alpha}\tilde{q}_\alpha^{m_\alpha \ell_\alpha}\langle\langle\mathcal{E}_\eta^{l_\eta}(t)|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{\ell_\alpha}(t)|\mathcal{D}_\kappa^{i_\kappa}(t)\rangle\rangle = \delta_{\eta\kappa}\sum_{m_\kappa=1}^{N_\kappa} q_\kappa^{l_\kappa m_\kappa}\tilde{q}_\kappa^{m_\kappa i_\kappa}. \tag{F4}$$

Thus, we can get $\mathcal{V}_{\rm GTQD}(t,t_0)\mathcal{V}_{\rm GTQD}^{-1}(t,t_0)=\mathbb{1}$ by imposing

$$\sum_{m_\kappa=1}^{N_\kappa} q_\kappa^{l_\kappa m_\kappa}\tilde{q}_\kappa^{m_\kappa i_\kappa} = \delta_{l_\kappa i_\kappa}. \tag{F5}$$

Now we can derive the generalized TQD Lindbladian $\mathbb{L}_{\rm GTQD}(t,t_0)$, which implements the dynamics governed by the evolution superoperator $\mathcal{V}_{\rm GTQD}(t,t_0)$. From Eq. (29) we obtain

$$\begin{aligned}\mathbb{L}_{\rm GTQD}(t,t_0) &= \dot{\mathcal{V}}_{\rm GTQD}(t,t_0)\mathcal{V}_{\rm GTQD}^{-1}(t,t_0) \\ &= \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\left[\dot{q}_\alpha^{n_\alpha k_\alpha}\tilde{q}_\alpha^{k_\alpha l_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)| + q_\alpha^{n_\alpha k_\alpha}\tilde{q}_\alpha^{k_\alpha l_\alpha}|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)|\right].\end{aligned} \tag{F6}$$

Let us then apply the normalization condition for the parameters $q_\alpha^{n_\alpha m_\alpha}$ and $\tilde{q}_\alpha^{n_\alpha m_\alpha}$ in Eq. (F6). By using Eq. (F5) in the second term of Eq. (F6), we then get

$$\mathbb{L}_{\rm GTQD}(t,t_0) = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\dot{q}_\alpha^{n_\alpha k_\alpha}\tilde{q}_\alpha^{k_\alpha l_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)| + \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}(t)|. \tag{F7}$$

Equation (F7) then reproduces the generalized TQD Lindbladian for multidimensional Jordan blocks provided by Eq. (30). Naturally, further conditions can be imposed over $q_\alpha^{n_\alpha m_\alpha}$ and $\tilde{q}_\alpha^{n_\alpha m_\alpha}$ for specific transitionless evolutions, such as in the dynamics induced by the standard TQD Lindbladian.

APPENDIX G: RECOVERING THE STANDARD TQD LINDBLAD SUPEROPERATOR FOR A GENERAL JORDAN DECOMPOSITION

Let us consider $q_\alpha^{n_\alpha m_\alpha}(t)=\exp[\int_{t_0}^t\lambda_\alpha(\xi)d\xi]v_{n_\alpha m_\alpha}(t)$, which means that the adiabatic evolution is mimicked. In order to satisfy Eq. (F5), we then have $\tilde{q}_\alpha^{n_\alpha m_\alpha}(t)=\exp[-\int_{t_0}^t\lambda_\alpha(\xi)d\xi]\tilde{v}_{n_\alpha m_\alpha}(t)$. Our aim here is to determine the conditions over the parameters $v_{k_\alpha l_\alpha}$ and $\tilde{v}_{k_\alpha l_\alpha}$ such that the generalized Lindblad superoperator $\mathbb{L}_{\rm GTQD}(t)$ reduces to the standard Lindblad superoperator $\mathbb{L}_{\rm STQD}(t)$. Thus, by computing $\mathbb{L}_{\rm GTQD}(t,t_0)$ according with Eq. (F7), we get

$$\begin{aligned}\mathbb{L}_{\rm GTQD}(t,t_0) &= \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}(t)| + \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\frac{d}{dt}\left[\exp\left(\int_{t_0}^t\lambda_\alpha(\xi)d\xi\right)v_{n_\alpha k_\alpha}\right] \\ &\quad\times\left(-\int_{t_0}^t\lambda_\alpha(\xi)d\xi\right)\tilde{v}_{k_\alpha l_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)|,\end{aligned} \tag{G1}$$

which yields

$$\begin{aligned}\mathbb{L}_{\rm GTQD}(t,t_0) &= \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\lambda_\alpha(t)v_{n_\alpha k_\alpha}\tilde{v}_{k_\alpha l_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)| + \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\dot{v}_{n_\alpha k_\alpha}\tilde{v}_{k_\alpha l_\alpha}|\mathcal{D}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{l_\alpha}(t)| \\ &\quad+\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}(t)|.\end{aligned} \tag{G2}$$

Now we use the normalization condition between $v_{k_\alpha l_\alpha}$ and $\tilde{v}_{k_\alpha l_\alpha}$, provided by $\mathcal{V}_{\mathrm{ad}}(t,t_0)\mathcal{V}_{\mathrm{ad}}^{-1}(t,t_0)=\mathbb{1}$. Then Eq. (G2) can be rewritten as

$$\mathbb{L}_{\mathrm{GTQD}}(t)=\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|+\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\big|\dot{\mathcal{D}}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|+\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\sum_{k_\alpha=1}^{N_\alpha}\sum_{l_\alpha=1}^{N_\alpha}\dot{v}_{n_\alpha k_\alpha}\tilde{v}_{k_\alpha l_\alpha}\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{l_\alpha}(t)\big|. \quad \text{(G3)}$$

Now, let us work on the last term of Eq. (G3). First, let us consider the original Lindblad superoperator $\mathbb{L}(t)$, as provided by Eq. (7). Its quasispectral decomposition reads

$$\mathbb{L}(t)=\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\big|\mathcal{D}_\alpha^{(n_\alpha-1)}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|+\lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|. \quad \text{(G4)}$$

We can then rewrite the standard TQD Lindblad superoperator $\mathbb{L}_{\mathrm{STQD}}(t)=\mathbb{L}(t)+\mathbb{L}_{\mathrm{cd}}(t)$ by using Eqs. (G4) and (18), yielding

$$\begin{aligned}\mathbb{L}_{\mathrm{STQD}}(t)=&\sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}\big|\mathcal{D}_\alpha^{(n_\alpha-1)}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|+\lambda_\alpha(t)\big|\mathcal{D}_\alpha^{n_\alpha}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\\&+\sum_{\mu=0}^{N-1}\sum_{n_\mu=1}^{N_\mu}\left[\big|\dot{\mathcal{D}}_\mu^{n_\mu}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\mu^{n_\mu}(t)\big|-\sum_{k_\mu=1}^{N_\mu}\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t)\big|\mathcal{D}_\mu^{n_\mu}(t)\big\rangle\!\big\rangle\!\big\langle\!\big\langle\mathcal{E}_\mu^{k_\mu}(t)\big|\right],\end{aligned} \quad \text{(G5)}$$

where $\mathcal{G}_{\mu\mu}^{n_\mu k_\mu}(t)=\langle\!\langle\mathcal{E}_\mu^{n_\mu}(t)|\dot{\mathcal{D}}_\mu^{k_\mu}(t)\rangle\!\rangle$. Thus, we can recover the standard Lindblad superoperator $\mathbb{L}_{\mathrm{STQD}}(t)$ from the generalized Lindblad superoperator $\mathbb{L}_{\mathrm{GTQD}}(t)$ by imposing $\mathbb{L}_{\mathrm{GTQD}}(t)=\mathbb{L}_{\mathrm{STQD}}(t)$. From Eqs. (G3) and (G5), this is achieved by requiring

$$\sum_{k_\alpha=1}^{N_\alpha}\dot{v}_{n_\alpha k_\alpha}(t)\tilde{v}_{k_\alpha l_\alpha}(t)=\delta_{n_\alpha(l_\alpha-1)}-\big\langle\!\big\langle\mathcal{E}_\alpha^{n_\alpha}(t)\big|\dot{\mathcal{D}}_\alpha^{l_\alpha}(t)\big\rangle\!\big\rangle. \quad \text{(G6)}$$

Hence, Eq. (G6) connects the generalized TQD formalism with the standard counterdiabatic approach.

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