Quantum phase transition revealed by the exceptional point in a Hopfield-Bogoliubov matrix

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We use the exceptional point in the Hopfield-Bogoliubov matrix to find the the critical point of the quantum phase transition in the bosonic system. In many previous jobs, the excitation energy vanished at the critical point. It can be stated equivalently that the critical point is obtained when the determinant of the Hopfield-Bogoliubov matrix vanishes. We analytically obtain the Hopfield-Bogoliubov matrix corresponding to the general quadratic Hamiltonian. For a single-mode system, the appearance of the exceptional point in the Hopfield-Bogoliubov matrix is equivalent to the disappearance of the determinant of the Hopfield-Bogoliubov matrix. However, in a multimode bosonic system, they are not equivalent except in some special cases. For example, in the case of perfect symmetry, that is, swapping any two subsystems and keeping the total Hamiltonian invariable, the exceptional point and the degenerate point coincide all the time when the quantum phase transition occurs. When the exceptional point and the degenerate point do not coincide, we find a significant unconventional result. With the increase in two-photon driving intensity, the normal phase changes to the superradiant phase, then the superradiant phase changes to the normal phase, and finally the normal phase changes to the superradiant phase. The discovery of the critical points will help in the design of precision measurements.

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I. INTRODUCTION

Quantum phase transitions are playing an increasingly important role in many fields, such as, quantum metrology [1–12], which involves using quantum resources to improve the measurement precision. The superradiant phase (SP) transition is one of the most important quantum phase transitions, which was proposed in the Dicke model for the first time in 1970s [13]. It described the coupling between a collection of two-level systems and a single-photon mode. In such a model, a state is recognized as the normal phase (NP) when the cavity field is in the vacuum and the atoms are in their ground states; a state is recognized as the SP when the cavity field is intensely populated with two degenerate ground states and the atoms are excited simultaneously. The NP (SP) can be revealed by the order parameter $\langle a\rangle = 0$ ($\langle a\rangle \neq 0$) [14], where $\langle a\rangle$ represents the expected value of the annihilation operator on the ground state of the cavity field.

In Refs. [15–18], the critical point, also known as the quantum phase-transition point, is revealed by the degenerate ground state. However, whether the degenerate ground state can be the general criteria for the occurrence of quantum phase transition is a question worth exploring. We will look at this question carefully and find a more rigorous criterion.

In a non-Hermitian system, purely real eigenvalues of the non-Hermitian Hamiltonian are obtained in the case of the parity-time- ($\mathcal{PT}$-) unbroken phase; the eigenvalues become imaginary in the case of the $\mathcal{PT}$-broken phase [19]. The exceptional points (EPs) separate the $\mathcal{PT}$-unbroken phase and the $\mathcal{PT}$-broken phase. Both the eigenvalues and the eigenvectors coalesce at the EPs.

In the process of the Hopfield-Bogoliubov (HB) transformation, there is a HB matrix, which is a non-Hermitian matrix generally. Like the $\mathcal{PT}$ symmetrical non-Hermitian Hamiltonian, the eigenvalues of the HB matrix can be real and imaginary. We define EPs as the separation points between the real and the imaginary eigenvalues. When one of the eigenvalues of the HB matrix is 0, the ground state of the system becomes degenerate. We simply define it as a degenerate point (DP).

In this article, we show that the EPs in the HB matrix can reveal the quantum phase transition in the bosonic system. And EPs are always associated with NP-to-SP or SP-to-NP transitions. The DP and the EP coalesce in the case of a perfectly symmetric system or a single-mode linear bosonic system. For the multimode bosonic system, the DP cannot be the quantum phase-transition point (the critical point) in many cases. Especially, in the two-mode bosonic system without counter-rotating-wave interaction, the significant unconventional results are found that with the increase in two-photon driving intensity, the NP changes to the SP, then the SP changes to the NP, and finally the NP changes to the SP. With the counter-rotating-wave interaction, the process of SP $\rightarrow$ NP $\rightarrow$ SP can be also observed.

This article is organized as follows. In Sec. II, we obtain the general HB matrix for the multimode linear bosonic system and the corresponding EP. In Sec. III, we show that the EP

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and the DP coincide all the time for the single-mode linear bosonic system. In Sec. IV, the EP and the DP do not always coincide for the two-mode linear bosonic system due to the two-photon driving and counter-rotating-wave interaction. In Sec. V, the EP in the three-mode quantum Rabi system is used to improve the estimation precision. We make a brief conclusion and outlook in Sec. VI.

II. HOPFIELD-BOGOLIUBOV MATRIX

For a general multimode linear bosonic system composed of N subsystems, the total Hamiltonian is quadratic, which can be described as

$$H=\sum_{n=1}^{N}H_n+\sum_{i=1,i<j}^{N}H_{ij}, \tag{1}$$

in which,

$$H_n=\omega_n a_n^\dagger a_n+\left(\chi_n a_n^2+\mathrm{H.c.}\right), \tag{2}$$

$$H_{ij}=g_{ij}a_i a_j+\lambda_{ij}a_i a_j^\dagger+\mathrm{Hc.}, \tag{3}$$

where H_n denotes the Hamiltonian for the nth subsystem with $n=\{1,\dots,N\}$. ω_n is the resonance frequency of the bosonic subsystem with the annihilation operator a_n and the creation operator $a_n^\dagger$, $|\chi_n|$ denotes the strength of two-photon driving, and λ_{ij} (g_{ij}) denotes the coupling strength of the rotating (counter-rotating)-wave interaction between the two subsystems.

By using an HB transformation [20,21] for the NP, the total Hamiltonian can be rewritten as a diagonal form

$$H=\sum_{n=1}^{N}\Omega_n A_n^\dagger A_n+E_g, \tag{4}$$

where the collective bosonic mode operators $A_n=\sum_{i=1}^{N}(\mu_{ni}a_i+\nu_{ni}a_i^\dagger)/\sqrt{\sum_{i=1}^{N}(|\mu_{ni}|^2-|\nu_{ni}|^2)}$ and E_g represents the ground-state energy. A_n satisfies the commutation relation: $[A_i,A_j^\dagger]=\delta_{ij}$. The coefficient vectors $(\mu_{n1},\dots,\mu_{nN},\nu_{n1},\dots,\nu_{nN})^T$ are eigenvectors of the HB matrix $\mathbf{M}$ which are derived by the commutation relation $[A_n,H]=\Omega_n A_n$,

$$\mathbf{M}=\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ -\mathbf{B}^* & -\mathbf{A}^*\end{pmatrix}, \tag{5}$$

with submatrices,

$$\mathbf{A}=\begin{pmatrix}\omega_1 & \lambda_{12} & \dots & \lambda_{1N}\\ \lambda_{12}^* & \omega_2 & \dots & \lambda_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{1N}^* & \lambda_{2N}^* & \dots & \omega_N\end{pmatrix}, \tag{6}$$

$$\mathbf{B}=\begin{pmatrix}-2\chi_1 & -g_{12} & \dots & -g_{1N}\\ -g_{12} & -2\chi_2 & \dots & -g_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ -g_{1N} & -g_{2N} & \dots & -2\chi_N\end{pmatrix}. \tag{7}$$

The spectral values of the HB matrix $\mathbf{M}$ have positive and negative symmetries due to the symmetry of the HB matrix: $\mathcal{C}\mathbf{M}\mathcal{C}^{-1}=-\mathbf{M}$, where $\mathcal{C}$ is described by

$$\mathcal{C}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & -\mathbf{0}\end{pmatrix}. \tag{8}$$

Even if all of the resonance frequency values ω_n are greater than zero, we need to emphasize that Ω_n is not necessarily a positive eigenvalue of the HB matrix $\mathbf{M}$. It can be obtained rigorously by mapping relationships,

$$\Omega_n|_{\chi_n\to 0,g_{ij}\to 0,\lambda_{ij}\to 0}\longrightarrow\omega_n. \tag{9}$$

When the determinant of the HB matrix is equal to 0 [$\mathrm{Det}(\mathbf{M})=0$], the ground state will become degenerate. In other words, DP appears at $\Omega_n=0$. In Refs. [16–18], DPs are directly treated as the critical points when the excited energy becomes 0. However, we will show that the DP is not the critical point in many cases.

There are a lot of works on $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian [22–27], which exist as the EPs separating the real and imaginary eigenenergy values. At the EPs, the eigenvalues and their corresponding eigenvectors coalesce. Similarly, due to that a general HB matrix $\mathbf{M}$ is non-Hermitian and symmetrical, there are also EPs.

For a closed system, the values of ω_n are real, $\mathbf{A}$ is the Hermitian: $\mathbf{A}=\mathbf{A}^\dagger$. Under special circumstances, there are no counter-rotating-wave interaction and two-photon driving: $\mathbf{B}=0$. In this case, the HB matrix $\mathbf{M}$ is Hermitian. If, at least, one of the eigenvalues is equal to zero, then DP exists. For a Hermitian system, there is no imaginary eigenvalue, meaning that there is no EP. In this case, the absence of quantum phase transition indicates that the DP is not the same as the critical point.

Quantum phase transition can be revealed by the EPs. When the eigenvalues of the HB matrix are real, the system is in the NP, i.e., the expectation value of the annihilation operator on the ground-state $\langle a_i\rangle=0$ [14]. When one of the eigenvalues is imaginary, the Hamiltonian of the system cannot be directly converted to Eq. (4). If not, the Hamiltonian becomes non-Hermitian. To obtain the effective Hermitian Hamiltonian with a diagonal form, one needs to first shift the bosonic operator as $a_i\to a_i+\alpha_i$ with the complex displacement α_i [28]. As a result, the expectation value of $\langle a_i\rangle=\alpha_i\neq 0$, which shows that the system is in the SP [14]. The EPs denote the transition points between the NP and the SP.

III. SINGLE-MODE BOSONIC SYSTEM AND QUANTUM RABI SYSTEM

In this section, we will show that the EP and the DP coincide all the time for the single-mode linear bosonic system and perform a corresponding comparison with the single-mode Rabi system.

For a single-mode linear bosonic system, the general Hamiltonian can be described as

$$H_1=\omega a^\dagger a+(\chi a^2+\mathrm{H.c.})/2, \tag{10}$$

where χ denotes the strength of two-photon driving. The corresponding HB matrix is obtained from Eq. (5),

$$\mathbf{M_1}=\begin{pmatrix}\omega & -\chi\\ \chi^* & -\omega\end{pmatrix}. \tag{11}$$

The eigenvalues of $\mathbf{M_1}$ are $\pm\sqrt{\omega^2 - |\chi|^2}$. By utilizing the mapping relationships in Eq. (9), we achieve that $\Omega_1 = \sqrt{\omega^2 - |\chi|^2}$. Obviously, both the EP and the DP occur when $|\omega| = |\chi|$. Namely, in the single-mode bosonic system, the DP can reveal the quantum phase transition due to that the EP and the DP appear at the same time.

The collective bosonic mode operator can be obtained by using the way in Sec. II,

$$A_1 = \frac{\chi a + (\omega - \Omega_1)a^\dagger}{\sqrt{|\chi|^2 - (\omega - \Omega_1)^2}} = \exp(i\theta)U(\xi)aU^\dagger(\xi),$$

where phase θ is given by $\exp(i\theta) = \chi/|\chi|$ and the unitary squeeze operator is $U(\xi) = \exp(\frac{\xi}{2}a^2 - \frac{\xi^*}{2}a^{\dagger 2})$ with the squeeze parameter defined as $\xi = \exp(-i\theta)\ln\frac{|\chi|+\omega-\Omega_1}{\sqrt{|\chi|^2-(\omega-\Omega_1)^2}}$. The squeeze parameter ξ diverges at the EP and the DP due to that $\sqrt{|\chi|^2 - (\omega - \Omega_1)^2} = 0$ when $|\omega| = |\chi|$. It signals the appearance of the quantum phase transition.

The ground state of the Hamiltonian H_1 is given by $|\psi_1\rangle = \exp(-\frac{\xi}{2}a^2 + \frac{\xi^*}{2}a^{\dagger 2})|0\rangle$. Performing measurements on the ground-state $|\psi_1\rangle$, we can obtain the values of $\{\theta, \omega, |\chi|\}$. The optimal estimation precision is given by the quantum Cramér-Rao (CR) bound [29–31]: $\delta^2\varphi \geqslant (\nu\mathcal{F}_\varphi)^{-1}$, where ν denotes the total number of experiments and $\mathcal{F}_\varphi$ is the quantum Fisher information about the parameter φ ($\varphi = \{\theta, \omega, |\chi|\}$). For the pure ground state, the quantum Fisher information $\mathcal{F}_\varphi$ can be achieved by $\mathcal{F}_\varphi = 4[\langle\partial_\varphi\psi|\partial_\varphi\psi\rangle - |\langle\partial_\varphi\psi|\psi\rangle|^2]$. When close to the EP, the dominant terms of the quantum Fisher information are

$$\mathcal{F}_\theta \sim (\ln\Omega_1)^2/4, \tag{12}$$

$$\mathcal{F}_\chi \sim \frac{|\chi|^2}{4\Omega_1^4}, \tag{13}$$

$$\mathcal{F}_\omega \sim \frac{|\omega|^2}{4\Omega_1^4}. \tag{14}$$

From the above equations, we can see that according to the quantum CR bound, the estimation uncertainty of parameters θ, ω, and $|\chi|$ are close to 0 ($\Omega_1 \to 0$) at the EP and the DP. It means that divergent quantum Fisher information can reveal the emergence of the critical point. In addition, θ is independent of the quantum phase transition, which is related only to the parameters ω and $|\chi|$. As a result, the scale of $\mathcal{F}_\theta$ is smaller than the scale of $\mathcal{F}_\chi$ and $\mathcal{F}_\omega$ near the critical point ($-\ln\Omega_1 \ll \frac{1}{\Omega_1^2}$).

In the quantum Rabi system, the Hamiltonian can be described as $H_R = \omega_0 a^\dagger a + \Delta/2\sigma_z + \eta(a^\dagger + a)\sigma_x$, where the cavity field frequency is ω_0, the transition frequency is Δ, the coupling strength is η, and the Pauli operator is $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. In the classical oscillator limit $\omega/\Delta \to 0$, using a Schrieffer-Wolff transformation [28] and projecting onto ground-state $|g\rangle$, we can obtain a single-mode bosonic system,

$$H_R' = \left(\omega_0 - \frac{\eta^2}{2\Delta}\right)a^\dagger a - \frac{\eta^2}{4\Delta}(a^2 + a^{\dagger 2}). \tag{15}$$

Let us redefine the parameters $\omega = \omega_0 - \frac{\eta^2}{2\Delta}$ and $\chi = -\frac{\eta^2}{2\Delta}$, and Eq. (15) becomes Eq. (10). Reference [28] showed that the NP and the SP are separated by the critical point $\eta/\sqrt{\omega_0\Delta} = 1$. The critical point $\eta/\sqrt{\omega_0\Delta} = 1$ is also the EP and the DP ($|\omega| = |\chi|$). When all eigenvalues of the HB matrix are real ($\eta/\sqrt{\omega_0\Delta} < 1$), the system is in the NP; when, at least, one of the eigenvalues is imaginary ($\eta/\sqrt{\omega_0\Delta} > 1$), the system is in the SP.

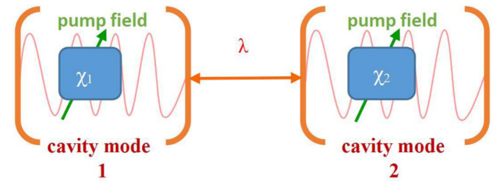

FIG. 1. The setup of two coupled cavity systems. The two cavity modes couple with each other by the rotating-wave interaction: $\lambda a_1 a_2^\dagger + \text{H.c.}$, where λ denotes the strength of resonant coupling. The cavity mode 1 (2) with angular frequency ω_{c1} (ω_{c2}) is generated by a strong pump field with angular frequency $2\omega_{d1}$ ($2\omega_{d2}$) via the second-order nonlinearity. χ_1 and χ_2 represent the strength of two-photon driving.

IV. TWO-MODE BOSONIC SYSTEM AND QUANTUM RABI SYSTEM

In this section, we will demonstrate that the EP and the DP do not always coincide. Unusual process of quantum phase transition will be revealed.

For the two-mode linear bosonic system, the general Hamiltonian can be given by

$$H_2 = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \left(\chi_1 a_1^2 + \chi_2 a_2^2 + g a_1 a_2 + \lambda a_1 a_2^\dagger + \text{H.c.}\right). \tag{16}$$

The corresponding HB matrix is obtained

$$\mathbf{M_2} = \begin{pmatrix} \omega_1 & \lambda & -2\chi_1 & -g \\ \lambda^* & \omega_2 & -g & -2\chi_2 \\ 2\chi_1^* & g^* & -\omega_1 & -\lambda^* \\ g^* & 2\chi_2^* & -\lambda & -\omega_2 \end{pmatrix}. \tag{17}$$

First, we consider that $\chi_1 = \chi_2 = g = 0$. In other words, there are only rotating-wave interactions ($\lambda \neq 0$). The eigenvalues of the HB matrix $\mathbf{M_2}$ are given by $\Omega_1 = \frac{\omega_1+\omega_2}{2} + \sqrt{(\frac{\omega_1-\omega_2}{2})^2 + |\lambda|^2}$ and $\Omega_2 = \frac{\omega_1+\omega_2}{2} - \sqrt{(\frac{\omega_1-\omega_2}{2})^2 + |\lambda|^2}$. When $|\lambda| = \sqrt{\omega_1\omega_2}$, $\Omega_2 = 0$ denotes that the DP occurs. At the DP, an eigenstate in the single-photon subspace reaches the zero energy, i.e., the eigenenergy of the vacuum state. However, the eigenvalues Ω_1 and Ω_2 are still real, meaning that the EP never appears. The ground state of the Hamiltonian H_2 is $|\psi_2\rangle = -\lambda|10\rangle + [\frac{\omega_1-\omega_2}{2} + \sqrt{|\lambda|^2 + (\frac{\omega_1-\omega_2}{2})^2}]|01\rangle$. The order parameters $\langle\psi_2|a_1|\psi_2\rangle$ and $\langle\psi_2|a_2|\psi_2\rangle$ are always equal to 0 whether $|\lambda|$ is greater than or less than $\sqrt{\omega_1\omega_2}$. It shows that no quantum phase transition occurs, which proves that the DP is not the critical point.

Then we consider that two cavity modes couple with each other by the resonant interaction as shown in Fig. 1. The two cavity modes are generated through two crystals with second-order nonlinearity (OPA) [32–34]. The Hamiltonian of two cavity systems is described as

$$H_c = \omega_{c1} a_1^\dagger a_1 + \omega_{c2} a_2^\dagger a_2 + \big[\chi_1 \exp(2i\omega_d t) a_1^2 + \chi_2 \exp(2i\omega_d t) a_2^2 + \lambda a_1 a_2^\dagger + \text{H.c.}\big], \tag{18}$$

In the rotating frame, by defining the detunings of the two cavity modes as $\omega_1 = \omega_{c1} - \omega_d$ and $\omega_2 = \omega_{c2} - \omega_d$, Eq. (16) with $g = 0$ is obtained. The eigenvalues of the HB matrix can be analytically achieved.

In the case of the perfect symmetry($\omega_1 = \omega_2,\ \chi_1 = \chi_2,\ \lambda = \pm|\lambda|$), the eigenvalues of the HB matrix $\mathbf{M_2}$ are given by $\Omega_1 = \sqrt{(\omega_1 + \lambda)^2 - 4\chi_1^2}$ and $\Omega_2 = \sqrt{(\omega_1 - \lambda)^2 - 4\chi_1^2}$. The critical point of the quantum phase transition appears at $4\chi_1^2 = \min\{(\omega_1 + \lambda)^2, (\omega_1 - \lambda)^2\}$. At the same time, one of the eigenvalues is equal to 0. This shows that the DP and the EP coincide in the two-mode bosonic system with the perfect symmetry. We verify it in an arbitrary multimode bosonic system with the perfect symmetry, i.e., swapping any two bosonic subsystems and keeping the total Hamiltonian invariable. It means that the value of the determinant of the HB matrix equal to 0 (i.e., DP) can be used to determine the occurrence of the quantum phase transition in the perfectly symmetric system.

In the case of $\omega_1 = \omega_2 = \omega > 0,\ \chi_1 > 0,\ \chi_2 = 0$, and $\lambda = |\lambda|$, the eigenvalues of $\mathbf{M_2}$ can be calculated

$$\Omega_1 = \sqrt{\omega^2 + \lambda^2 - 2\chi_1^2 - 2\sqrt{\chi_1^4 - \chi_1^2\lambda^2 + \omega^2\lambda^2}}, \tag{19}$$

$$\Omega_2 = \sqrt{\omega^2 + \lambda^2 - 2\chi_1^2 + 2\sqrt{\chi_1^4 - \chi_1^2\lambda^2 + \omega^2\lambda^2}}. \tag{20}$$

For $\lambda^2 > 5.402\,05\omega^2$, the critical two-photon driving strength is given by

$$\chi_1 = \left\{\sqrt{\frac{\lambda^2}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4\omega^2}{\lambda^2}}},\ \frac{\lambda^2 - \omega^2}{2\omega}\right\}. \tag{21}$$

When the two-photon driving strength is $\chi_1 = \chi_\pm = \sqrt{\lambda^2/2 \pm \sqrt{1 - 4\omega^2/\lambda^2}/2}$, the EPs show up and the DP does not. It shows that the DP cannot be used as a basis for judging the existence of the quantum phase transition. When the two-photon driving strength χ_1 satisfies that $\chi_1 < \chi_-$ or $\frac{\lambda^2-\omega^2}{2\omega} > \chi_1 > \chi_+$, the system is in the NP. As a conventional result [32], the system will transform from the NP to the SP when the two-photon driving strength χ_1 increases to χ_-. A very interesting and unconventional result is that the system will transform from the SP to the NP when the two-photon driving strength χ_1 increases to χ_+. When the two-photon driving strength χ_1 increases to $\frac{\lambda^2-\omega^2}{2\omega}$ (the EP coinciding with the DP), the system will transform from the NP to the SP again. The discovery of additional EPs will assist in the design of sensitive measuring instruments, which makes sense in quantum metrology.

Next, we consider the case that the counter-rotating-wave interaction cannot be negligible ($g \neq 0$), especially in the circuit quantum electrodynamics [35]. In the case of $\omega_1 = \omega_2 = \omega > 0,\ \chi_1 > 0,\ \chi_2 = 0$, and $\lambda = g = |\lambda|$, the Hamiltonian

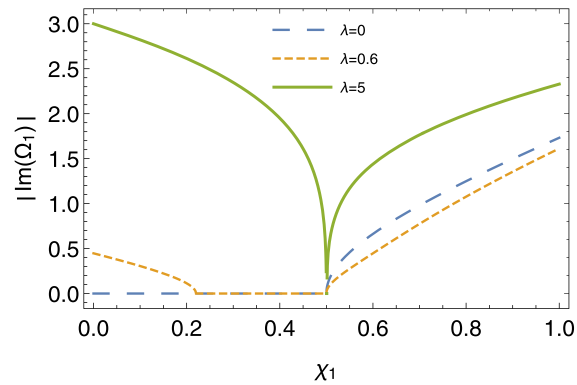

FIG. 2. The diagram of the absolute value of the imaginary part of the eigenvalue Ω_1 changing with the two-photon driving strength χ_1. Here, the value of ω is chosen to be 1 in arbitrary units.

can be described as

$$H_2' = \omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + \big(\chi_1 a_1^2 + \lambda a_1 a_2 + \lambda a_1 a_2^\dagger + \text{H.c.}\big). \tag{22}$$

By a similar calculation, the eigenvalues of the HB matrix corresponding to the Hamiltonian above can be achieved

$$\Omega_1 = \sqrt{\omega^2 - 2\chi_1^2 - 2\sqrt{\chi_1^4 - 2\chi_1\omega\lambda^2 + \omega^2\lambda^2}}, \tag{23}$$

$$\Omega_2 = \sqrt{\omega^2 - 2\chi_1^2 + 2\sqrt{\chi_1^4 - 2\chi_1\omega\lambda^2 + \omega^2\lambda^2}}. \tag{24}$$

When the minimum eigenvalue Ω_1 is imaginary, the system is in the SP. There is the ground-state bosonic coherence. As shown in Fig. 2, we calculate the imaginary part of the eigenvalue Ω_1. When $|\text{Im}(\Omega_1)|$ is nonzero, the system is in the SP; When $|\text{Im}(\Omega_1)|$ is zero, the system is in the NP. In the case of $\lambda = 0$, the system transforms from the NP to the SP with the increase in the two-photon driving strength. In the case of $\lambda = 0.6$ or $\lambda = 5$, the system is in the SP for the two-photon driving strength $\chi_1 = 0$, which is different from the previous results (without the counter-rotating wave interaction). It shows that both the counter-rotating-wave interaction and the two-photon driving can transform the system into the SP. For a proper value of coupling strength λ (such as, $\lambda = 0.6$), the system transforms from the SP to the NP and then from the NP to the SP with the increase in χ_1. Like the previous case, it is due to that the EP and the DP do not always coincide. The essential reason is that the coupling between the two bosonic systems inhibits the effect of the two-photon driving in a certain range.

Now let us consider a system composed of two quantum Rabi subsystems, which can be realized in cavity or circuit QED [36–38] as depicted in Figs. 3 and 4. The corresponding Hamiltonian is described as

$$H_{2R} = \sum_{n=1}^{2}\left[\omega_n' a_n^\dagger a_n + \frac{\Delta_n}{2}\sigma_{zn} + g_n\sigma_{xn}(a_n + a_n^\dagger)\right] + (\lambda a_1 a_2^\dagger + \text{H.c.}), \tag{25}$$

where the nth cavity field (resonator) frequency is ω_n', the nth transition frequency Δ_n, the nth cavity-atom (resonator-qubit)

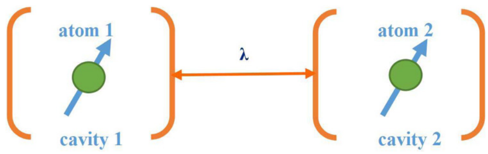

FIG. 3. The scheme of the two-mode quantum Rabi system in the cavity QED setup. The quantum Rabi subsystem composed of a two-level atom and a cavity. The two cavity modes interact with each other by the rotating-wave interaction: $\lambda a_1 a_2^\dagger + \text{H.c.}$

coupling strength g_n, and the coupling strength of the rotating-wave interaction between two cavity fields (resonators) λ. In the limit $\omega_n'/\Delta_n \to 0$ for $n = 1, 2$, using a Schrieffer-Wolff transformation [28] and projecting onto the ground state, the Hamiltonian can be rewritten as

$$H_{2R}' = \sum_{n=1}^{2}\left[\left(\omega_n' - \frac{g_n^2}{2\Delta_n}\right)a_n^\dagger a_n - \frac{g_n^2}{4\Delta_n}\left(a_n^2 + a_n^{\dagger 2}\right)\right] + (\lambda a_1 a_2^\dagger + \text{H.c.}). \tag{26}$$

By defining the parameters $\omega_n = \omega_n' - \frac{g_n^2}{2\Delta_n}$ and $\chi_n = -\frac{g_n^2}{4\Delta_n}$ with $n = \{1, 2\}$, Eq. (16) with $g = 0$ is also obtained. Different from the case of OPA, χ_n and ω_n are not independent. As a result, the system transforms from the NP to the SP with the increase in χ_1 in the case of $\chi_2 = 0$. Unlike the case of OPA, we do not get the quantum phase transition from the SP to the NP. And when $|\lambda|^2 > \omega_1'\omega_2'$, the critical point appears at the EP instead of the DP.

V. THREE-MODE QUANTUM RABI SYSTEM

In this section, we obtain the HB matrix corresponding to the three-mode quantum Rabi system and show that the quantum Fisher information will be divergent at the EP.

We consider the three-mode quantum Rabi system as shown in Ref. [18], which is described as

$$H_{3R} = \sum_{n=1}^{3}\left[\omega a_n^\dagger a_n + \frac{\Delta}{2}\sigma_{zn} + g\sigma_{xn}(a_n + a_n^\dagger)\right] + \sum_{n=1,n'=1}^{3} J(e^{i\theta} a_n a_{n'}^\dagger + e^{-i\theta} a_{n'} a_n^\dagger), \tag{27}$$

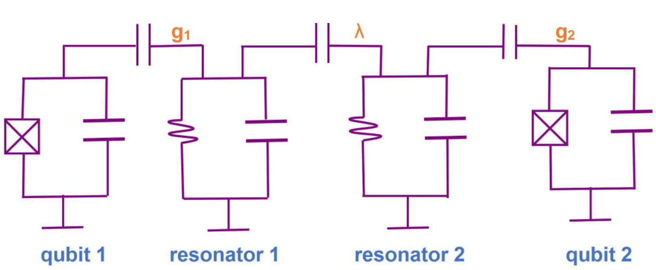

FIG. 4. The scheme of two-mode quantum Rabi systems in the circuit QED setup. The quantum Rabi subsystem composed of a transmon qubit and a LC circuit or a superconductive transmission line.

where $Je^{\pm i\theta}$ is the hopping amplitude between cavities n and n'. For the sake of a more convenient discussion, we consider $\omega > 2J > 0$. Implementing the Schrieffer-Wolff transformation in the limit of $\omega/\Delta \longrightarrow 0$ and taking a discrete Fourier transform $a_n = \frac{1}{3}\sum_q e^{-inq} a_q$ with the quasimomentum $q = \{0, \pm\frac{2\pi}{3}\}$, the reduced Hamiltonian is given by

$$H_{3R}' = \sum_q \left[\omega_q a_q^\dagger a_q - \frac{g^2}{\Delta}(a_q a_{-q} + a_q^\dagger a_{-q}^\dagger)\right] + E_0, \tag{28}$$

where E_0 is a constant and $\omega_q = \omega - \frac{2g^2}{\Delta} + 2J\cos(\theta - q)$. The eigenvalues of the corresponding HB matrix are given by

$$\left\{\Omega_q = \frac{1}{2}\left[\sqrt{(\omega_q + \omega_{-q})^2 - 16\frac{g^4}{\Delta^2}} + \omega_q - \omega_{-q}\right]\right\}_{q=\{0,\pm(2\pi/3)\}}. \tag{29}$$

In the case of $\pi/2 < \theta \leqslant \pi$, the EP and the DP appear at $g = (\sqrt{\omega + 2J\cos\theta})\Delta/2$. In the case of $0 \leqslant \theta < \pi/2$, the EP appears at $g = (\sqrt{\omega - J\cos\theta})\Delta/2$, and DP never does.

The ground state of the Hamiltonian H_{3R}' is given by

$$|\psi_3\rangle = \exp\left[\xi_0\left(a_0^{\dagger 2} - a_0^2\right) + \xi_\vartheta\left(a_\vartheta^\dagger a_{-\vartheta}^\dagger - a_\vartheta a_{-\vartheta}\right)\right]|0\rangle,$$

where the parameters are $\xi_0 = \frac{1}{8}\ln\frac{\omega+2J\cos\theta}{\omega+2J\cos\theta-4g^2/\Delta}$ and $\xi_\vartheta = \frac{1}{4}\ln\frac{\omega-J\cos\theta}{\omega-J\cos\theta-4g^2/\Delta}$ with $\vartheta = \frac{2\pi}{3}$. When close to the EP (for $\pi/2 < \theta \leqslant \pi$, $\omega + 2J\cos\theta - 4g^2/\Delta = \varepsilon \to 0$ for $0 < \theta \leqslant \pi/2$, $\omega - J\cos\theta - 4g^2/\Delta = \varepsilon \to 0$), the dominant term of the quantum Fisher information about the parameter ω is

$$\mathcal{F}_\omega \sim \frac{1}{32\varepsilon^2}, \quad \text{for } \pi/2 < \theta \leqslant \pi; \tag{30}$$

$$\mathcal{F}_\omega \sim \frac{1}{16\varepsilon^2}, \quad \text{for } 0 \leqslant \theta < \pi/2; \tag{31}$$

$$\mathcal{F}_\omega \sim \frac{3}{32\varepsilon^2}, \quad \text{for } \theta = \pi/2. \tag{32}$$

From the above equations, we can see that the quantum Fisher information will be divergent at the EP, which reveals the quantum phase transition. The optimal estimation precision of the frequency ω can be obtained with $\theta = \pi/2$. Our results show that the EPs in the HB matrix can help to find an effective way to improve the parameter estimation precision. And again it shows that the critical point is the EP instead of the DP.

VI. CONCLUSION AND OUTLOOK

We achieve the general HB matrix for linear coupled bosonic systems in arbitrary dimensions. For open systems, there are $\mathcal{PT}$-symmetrical non-Hermitian Hamiltonians, which have the EPs separating the real eigenvalues and the imaginary values. For closed systems, the HB matrix is also non-Hermitian, and the eigenvalues of the HB matrix can be real and imaginary. Therefore, we also refer to the segmentation point where the eigenvalues of HB are real and imaginary as the EP. And when one of the eigenvalues of HB matrix is 0, the ground state of the system becomes degenerate. Therefore, we simply call it the DP. In some previous works, the DP was considered as a marker for the emergence of the critical point between the NP and the SP. However, we show that the DP is not the critical point unless it coincides with the EP. The EP of the HB matrix is what really reveals the quantum phase transition between the NP and the SP.

In the single-mode bosonic or perfectly symmetric system, the DP can be the critical point due to that it coincides with the EP. In more general multimode systems, the EPs and the DPs are often not coincident. As a result, unconventional and meaningful results are obtained. With the increase in the two-photon driving strength, the phase undergoes the process of NP $\rightarrow$ SP $\rightarrow$ NP $\rightarrow$ SP in the case of neglecting the counter-rotating-wave interaction. With the counter-rotating-wave interaction, the process of SP $\rightarrow$ NP $\rightarrow$ SP can be achieved. In the conventional results, only the process of NP $\rightarrow$ SP was obtained. The fundamental reason is that the coupling between the two bosonic systems inhibits the effect of the two-photon driving to a certain extent. In addition, we apply the HB martrix into the quantum Rabi system and show that the quantum Fisher information will be divergent at EPs, which will lay the foundation for designing precision measurement.

In this article, we consider the HB matrix in the closed system, which has the Hermitian Hamiltonian. It will be interesting to further explore the dissipation phase transition with the HB matrix in the open system.

The quantum Rabi model in this article can be realized in a variety of quantum systems, such as cold atoms [39] and superconducting qubits [37]. The form of the total Hamiltonian can be obtained by a periodic modulation of the photon hopping strength between cavities [18]. And the strengths of two-photon driving can be changed by the pump field and the size of the crystal, which is feasible in experiment [30].

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