Quantum non-Markovian "casual bystander" environments

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Quantum memory effects can be induced even when the (time-dependent) dynamical degrees of freedom associated to the environment are not affected at all by the open system during their joint coupled evolution. In this paper, based on a completely positive bipartite representation of the system-environment dynamics, we found the more general interactions that lead to this class of quantum non-Markovian "casual bystander" environments. General properties of the resulting dynamics are studied with a focus on the system-environment correlations, a collisional measurement-based representation, and the quantum regression hypothesis. Memory effects are also characterized through an operational approach, which in turn allows one to detect when the studied properties apply. Single and multipartite qubits dynamics support and exemplify the developed results.

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I. INTRODUCTION

Both in classical and quantum realms, memory effects emerge whenever a set of dynamical degrees of freedom is not considered as part of the system of interest [1–4]. In classical systems (or incoherent ones), the presence of memory effects can be related to departures from a Markov property defined in a probabilistic frame. In contrast, the definition of memory effects and non-Markovianity is much more subtle in a quantum regime.

Given that any quantum system is affected by a measurement process, a reasonable approach to defining quantum non-Markovianity is to study the properties of the (unperturbed) system density-matrix propagator. In fact, the theory of quantum semigroups [5] is usually taken as a landmark of quantum Markovianity. Thus, any departure of the propagator properties with respect to that of a Lindblad evolution can be proposed as a signature of quantum non-Markovianity [6,7]. This approach has shown to be a very fruitful tool to study memory effects in quantum systems, leading to the formulation of many, in general inequivalent, memory witnesses (see, for example, Refs. [8–14]). Independently of the chosen memory witness, one interesting perspective that these studies provide is the understanding of memory effects through an "environment-to-system backflow of information" [8–11]. Information stored in the environment degrees of freedom may influence the system at later times, giving a solid and clear understanding of memory effects.

In spite of the simplicity and efficacy of the previous theoretical perspective, it has been shown that memory effects may emerge even when the degrees of freedom associated to the environment are not affected at all during the system evolution, which implies the absence of any "physical" environment-to-system backflow of information. A clear situation where this occurs is in quantum systems coupled to incoherent degrees of freedom that have a fixed classical stochastic dynamics [15,16], that is, their underlying stochastic evolution is completely independent of the system degrees of freedom.

While the previous drawback in the definition of information flows remains under debate [15–22], alternative operational approaches to quantum memory effects [23–32] furnish a possible solution. For example, by subjecting a system to three successive measurements, a conditional past-future (CPF) correlation [27] provides a memory witness that is consistent with the usual approach to non-Markovianity in terms of conditional probabilities [1]. This object is defined by the correlation between the first and last outcomes conditioned to a given intermediate value. It vanishes in a Markovian regime (defined in terms of conditional probabilities). In addition, by randomizing the intermediate postmeasurement system state, even in presence of memory effects, the conditional past-future correlation vanishes when the environment is not affected by the system evolution [32]. Thus, it detects the presence or absence of "bidirectional system-environment information flows." This last property provides a solution to the previous drawback. In fact, a solid experimental procedure for distinguishing between memory effects induced by environments that are affected or not by their interaction with the system is established.

The previous advancements left open an interesting issue the formulation of which is *independent* of any memory witness definition. Besides environments consisting of incoherent degrees of freedom with a fixed classical stochastic dynamics, in which other situations may it occur that the environment is not affected at all by its interaction with the system? More specifically, we ask about the most general system-environment interactions that guarantee this property. The resolution of this problem is relevant for achieving a clear understanding of intrinsically different memory effects, that is, those where the environment is or is not altered by its interaction with the system.

The main goal of this paper is to answer the previous question. We characterize the most general system-environment interactions that, even when the system develops memory effects, guarantee that the environment self-dynamics and state remain independent of the system degrees of freedom. For these "casual bystander" environments (CBEs), special interest is paid to the resulting system dynamics, the system-environment correlations, a collisional measurement-based representation [33–40], and the validity of the quantum regression hypothesis when analyzing system operator correlations [41–44]. General expressions for the CPF correlation [27,32] are also provided. The main results are exemplified by studying these properties and indicators for a class of qubit dynamics with single and multipartite dephasing channels.

The paper is outlined as follows. In Sec. II we derive the most general interaction consistent with a CBE. General properties of the resulting system dynamics are analyzed in Sec. III. In Sec. IV we study single and multipartite examples. The conclusions are provided in Sec. V. Calculus details are presented in the Appendix.

II. QUANTUM CASUAL BYSTANDER ENVIRONMENTS

We consider a system (s) interacting with uncontrollable quantum degrees of freedom that constitute the environment (e). Correspondingly, in the total Hilbert space $\mathcal{H}_s\otimes\mathcal{H}_e$, the bipartite density matrix is denoted as ρ_t^{se}. Its evolution is written as

$$\frac{d}{dt}\rho_t^{se}=(\mathcal{L}_s+\mathcal{L}_e+\mathcal{L}_{se})[\rho_t^{se}],\tag{1}$$

where $\mathcal{L}_s$ and $\mathcal{L}_e$ define the system and environment isolated dynamics, respectively, while $\mathcal{L}_{se}$ introduces their mutual interaction. As usual, the marginal system and environment states follow by tracing out the complementary degrees of freedom:

$$\rho_t^s=\mathrm{Tr}_e[\rho_t^{se}],\qquad \rho_t^e=\mathrm{Tr}_s[\rho_t^{se}],\tag{2}$$

where $\mathrm{Tr}[\cdots]$ is the trace operation. By *definition*, a CBE is characterized by a density matrix ρ_t^e that is completely independent of the system state and dynamics. Consequently, its time evolution $(d\rho_t^e/dt)$ must also fulfill the same property. From Eq. (1) we get $(d\rho_t^e/dt)=\mathcal{L}_e[\rho_t^e]+\mathrm{Tr}_s(\mathcal{L}_{se}[\rho_t^{se}])$, leading to the condition

$$\mathrm{Tr}_s(\mathcal{L}_{se}[\rho_t^{se}])=\mathcal{A}[\rho_t^e].\tag{3}$$

Here, $\mathcal{A}$ is an arbitrary superoperator (acting on ρ_t^e) that does not depend on any degree of freedom (or operator) associated to the system Hilbert space. Furthermore, this equality must be valid at all times. The criterion (3) allows us to find which kind of system-environment couplings fulfill the proposed definition.

A. Unitary coupling

A unitary coupling is set by a bipartite Hamiltonian H_{se} such that

$$\mathcal{L}_{se}[\bullet]=-i[H_{se},\bullet].\tag{4}$$

In order to check condition (3), we introduce a complete orthogonal basis $\{|s\rangle\}$ of the system Hilbert space such that $\sum_s|s\rangle\langle s|=I_s$, where I_s is the system identity matrix. We get

$$\mathrm{Tr}_s(\mathcal{L}_{se}[\rho_t^{se}])=-i\sum_{s,s'}\big[\langle s|H_{se}|s'\rangle,\langle s'|\rho_t^{se}|s\rangle\big].\tag{5}$$

Thus, independence of the environment state of the system degrees of freedom requires $\langle s|H_{se}|s'\rangle=\delta_{s,s'}Q_e$, which in turn implies $H_{se}=I_s\otimes Q_e$, where $Q_e=Q_e^\dagger$ is an arbitrary operator acting on $\mathcal{H}_e$. Nevertheless, this solution implies that system and environment do not interact. Consequently, it is impossible to obtain a *non-Markovian* CBE if it interacts unitarily with the system of interest.

The previous result is not valid when a *Born-Markov approximation* applies [2], where it is possible to approximate $\rho_t^{se}\simeq\rho_t^s\otimes\rho_0^e$. Thus, when this approximation is valid, the environment can be considered as a casual bystander one, being characterized by a stationary state, $\rho_t^e=\mathrm{Tr}_s[\rho_t^{se}]\simeq\rho_0^e$. Nevertheless, in this situation memory effects do not develop. In fact, the evolution of ρ_t^s can be well approximated by a Lindblad equation.

B. Dissipative coupling

After discarding the unitary property, now we consider dissipative system-environment couplings. Thus, the environment is defined by a set of quantum degrees of freedom the interaction of which with the system is approximated by an arbitrary (nondiagonal) Lindblad superoperator:

$$\mathcal{L}_{se}[\bullet]=\sum_{i,j}\gamma_{i,j}\bigg(T_i\bullet T_j^\dagger-\frac{1}{2}\{T_j^\dagger T_i,\bullet\}_+\bigg),\tag{6}$$

where $\{a,b\}_+\equiv ab+ba$. We notice that the extra degrees of freedom ("superenvironment") necessary to induce this coupling are irrelevant here because their influence can always be related to an underlying Born-Markov approximation.

In Eq. (6) the complex parameters $\{\gamma_{i,j}\}$ define the (diagonal and nondiagonal) rate coefficients of the dissipative channels corresponding to the bipartite operators $\{T_i\}$. Thus, assuming that the evolution induced by $\mathcal{L}_{se}$ is completely positive, they constitute an Hermitian positive definite matrix [2,5].

In order to check condition (3), we introduce the replacements $T_i\to V_k\otimes B_\alpha$, $T_j\to V_l\otimes B_\beta$, and $\gamma_{i,j}\to\gamma_{k\alpha,l\beta}$, where $\{V_k\}$ and $\{B_\alpha\}$ are arbitrary operators acting on $\mathcal{H}_s$ and $\mathcal{H}_e$, respectively. Hence, $\mathcal{L}_{se}$ can be rewritten as

$$\mathcal{L}_{se}[\bullet]=\sum_{k\alpha,l\beta}\gamma_{k\alpha,l\beta}\bigg(V_kB_\alpha\bullet B_\beta^\dagger V_l^\dagger-\frac{1}{2}\{V_l^\dagger V_kB_\beta^\dagger B_\alpha,\bullet\}_+\bigg).\tag{7}$$

Here, the indices run in the intervals $k,l=1,\cdots,(\dim\mathcal{H}_s)^2-1$ and $\alpha,\beta=1,\cdots,(\dim\mathcal{H}_e)^2-1$.

By taking the trace to the interaction superoperator $\mathcal{L}_{se}[\rho_t^{se}]$, we get

$$\mathrm{Tr}_s(\mathcal{L}_{se}[\rho_t^{se}])=\sum_{\alpha,\beta,s}\bigg(B_\alpha\langle s|D_{\alpha,\beta}\rho_t^{se}|s\rangle B_\beta^\dagger-\frac{1}{2}B_\beta^\dagger B_\alpha\times\langle s|D_{\alpha,\beta}\rho_t^{se}|s\rangle-\frac{1}{2}\langle s|\rho_t^{se}D_{\alpha,\beta}|s\rangle B_\beta^\dagger B_\alpha\bigg),$$

where as before $\{|s\rangle\}$ is a complete base in $\mathcal{H}_s$. Furthermore, we introduced the system operators $\{D_{\alpha,\beta}\}$:

$$D_{\alpha,\beta} \equiv \sum_{k,l} \gamma_{k\alpha,l\beta} V_l^\dagger V_k, \qquad D_{\alpha,\beta}^\dagger = D_{\beta,\alpha}. \tag{8}$$

The symmetry property $D_{\alpha,\beta}^\dagger = D_{\beta,\alpha}$ is inherited from condition $\rho_t^{se} = (\rho_t^{se})^\dagger$ in Eq. (1). It is simple to realize that the constraint (3) is fulfilled if

$$\langle s|D_{\alpha,\beta} = \Gamma_{\alpha,\beta}\langle s|, \qquad D_{\alpha,\beta}|s\rangle = \Gamma_{\alpha,\beta}|s\rangle, \tag{9}$$

where $\Gamma_{\alpha,\beta}$ are, in general, complex coefficients. Applying $\sum_{s'}|s'\rangle$ and $\sum_{s'}\langle s'|$ to the left and right equalities, using that $D_{\alpha,\beta}^\dagger = D_{\beta,\alpha}$, we get the equivalent conditions

$$D_{\alpha,\beta} = \Gamma_{\alpha,\beta} I_s, \qquad \Gamma_{\alpha,\beta}^* = \Gamma_{\beta,\alpha}. \tag{10}$$

Introducing the final constraints (10) into Eq. (7) and by using that $\mathcal{L}_{se}$ must have the properties of a Lindblad equation [2,5] (Hermiticity, trace preservation, and completely positivity) we can write the bipartite interaction generator as

$$\mathcal{L}_{se}[\bullet] = \sum_{\alpha,\beta} \Gamma_{\alpha,\beta}\left(B_\alpha \mathbb{S}_{\alpha,\beta}[\bullet]B_\beta^\dagger - \frac{1}{2}\{B_\beta^\dagger B_\alpha, \bullet\}_+\right). \tag{11}$$

Here, the coefficients $\{\Gamma_{\alpha,\beta}\}$ define an (arbitrary) Hermitian positive definite matrix. Furthermore, $\{\mathbb{S}_{\alpha,\beta}\}$ is a set of (arbitrary) system superoperators that fulfill the symmetry $\mathbb{S}_{\alpha,\beta}^\dagger = \mathbb{S}_{\beta,\alpha}$ and are trace preserving, $\mathrm{Tr}_s(\mathbb{S}_{\alpha,\beta}[\rho^s]) = \mathrm{Tr}_s[\rho^s]$. Hence, they can be written in a Kraus-like representation [2] as

$$\mathbb{S}_{\alpha,\beta}[\bullet] = \sum_k W_k^{\alpha\beta} \bullet W_k^{\dagger\beta\alpha}, \qquad \sum_k W_k^{\dagger\beta\alpha} W_k^{\alpha\beta} = I_s, \tag{12}$$

where $\{W_k^{\alpha\beta}\}$ are (arbitrary) system operators that for each pair of indices (α,β) guarantee trace preservation. In Eq. (11), it is always possible to choose a new base of environment operators, unitarily related to $\{B_\alpha\}$, where the matrix of rate coefficients becomes diagonal [2,5], $\Gamma_{\alpha,\beta} \to \delta_{\alpha,\beta}\Gamma_\alpha$. This property implies that the system superoperators $\{\mathbb{S}_{\alpha,\beta}\}$ can always be related, via the same unitary transformation, to a set of (arbitrary but standard) completely positive system transformations (superoperators).

From Eqs. (1) and (11), the bipartite system-environment evolution can finally be written as

$$\frac{d}{dt}\rho_t^{se} = (\mathcal{L}_s + \mathcal{L}_e)[\rho_t^{se}] + \sum_{\alpha,\beta} \Gamma_{\alpha,\beta}\, B_\alpha \mathbb{S}_{\alpha,\beta}[\rho_t^{se}]B_\beta^\dagger - \frac{1}{2}\sum_{\alpha,\beta}\Gamma_{\alpha,\beta}\{B_\beta^\dagger B_\alpha, \rho_t^{se}\}_+, \tag{13}$$

where $\mathcal{L}_s$ and $\mathcal{L}_e$ are arbitrary. This equation is the main result of this section. It defines the most general dissipative system-environment coupling that is consistent with a quantum non-Markovian CBE. In fact, after applying the (system) trace operation to Eq. (13), and using the property (12), the density matrix ρ_t^e of the environment [Eq. (2)] evolves as

$$\frac{d}{dt}\rho_t^e = \mathcal{L}_e[\rho_t^e] + \sum_{\alpha,\beta}\Gamma_{\alpha,\beta}\left(B_\alpha \rho_t^e B_\beta^\dagger - \frac{1}{2}\{B_\beta^\dagger B_\alpha, \rho_t^e\}_+\right). \tag{14}$$

As expected, this Lindblad equation does not depend on the system degrees of freedom. In contrast, the time evolution of the system state, $\rho_t^s = \mathrm{Tr}_e[\rho_t^{se}]$, assuming uncorrelated initial conditions $\rho_0^{se} = \rho_0^s \otimes \rho_0^e$, from Eq. (13) can formally be written as a time-convoluted equation,

$$\frac{d}{dt}\rho_t^s = \mathcal{L}_s[\rho_t^s] + \int_0^t dt' \mathcal{K}_s(t-t')[\rho_{t'}^s], \tag{15}$$

which anticipates the presence of system memory effects. From standard calculation steps, the superoperator $\mathcal{K}_s(t)$ can be defined in a Laplace domain $[f(z) = \int_0^\infty dt e^{-zt} f(t)]$ from the relation $\mathrm{Tr}_e[\mathcal{G}_z^{se}(\mathcal{L}_e + \mathcal{L}_{se})\rho_0^e][\bullet] = \mathrm{Tr}_e[\mathcal{G}_z^{se}\rho_0^e]\mathcal{K}_s(z)[\bullet]$, where $\mathcal{G}_z^{se} = [z - (\mathcal{L}_s + \mathcal{L}_e + \mathcal{L}_{se})]^{-1}$ is the bipartite system-environment propagator.

Both Eqs. (13) and (14) can always be reduced to a standard diagonal form [2,5], which can be read by taking $\Gamma_{\alpha,\beta} = \delta_{\alpha,\beta}\Gamma_\alpha$, where $\{\Gamma_\alpha\}$ are positive rate coefficients. Furthermore, Eq. (15) can always be transformed into a convolutionless form [45].

III. GENERAL PROPERTIES

On the basis of Eq. (13), it is possible to establish general properties that characterize the system-environment dynamics.

A. System-environment correlations

Even for uncorrelated initial conditions, the evolution (13) induces correlations between the system and the degrees of freedom associated to the environment. These correlations can be characterized from the bipartite state ρ_t^{se}. We show that, for uncorrelated initial conditions, $\rho_0^{se} = \rho_0^s \otimes \rho_0^e$, there always exists an environment basis where ρ_t^{se} can be written with the structure

$$\rho_t^{se} = \sum_c \rho_c(t) \otimes |c_t\rangle\langle c_t|. \tag{16}$$

Here, $\{\rho_c(t)\}$ are un-normalized states (matrices) in $\mathcal{H}_s$ and $\{|c_t\rangle\langle c_t|\}$ are orthogonal time-dependent projectors in $\mathcal{H}_e$, that is, $\langle c_t|c_t'\rangle = \delta_{cc'}$. Consistently, the system and environment states read

$$\rho_t^s = \sum_c \rho_c(t), \qquad \rho_t^e = \sum_c p_c(t)|c_t\rangle\langle c_t|, \tag{17}$$

where $p_c(t) \equiv \mathrm{Tr}_s[\rho_c(t)]$. From these expressions, it follows that $\{|c_t\rangle\}$ is the basis in which ρ_t^e becomes diagonal at time t. Furthermore, $\rho_c(t)$ is the conditional state of the system given that the environment is in the state $|c_t\rangle\langle c_t|$, while its trace define the probabilities $p_c(t)$.

The formal solution (16) implies that ρ_t^{se} is a *separable state* [46] with a null system-environment discord [47]. Hence, no quantum entanglement (between the system and the environment) is produced during the evolution.

The validity of Eq. (16) can be established from the bipartite evolution (13). From these equations, for the system conditional states $\{\rho_c(t)\}$, using that $\mathrm{Tr}_e(\mathcal{L}_e[\bullet]) = 0$, we get the evolution

$$\frac{D}{Dt}\rho_c(t) = \mathcal{L}_s[\rho_c(t)] + \sum_{\tilde{c} \atop \tilde{c}\neq c}\phi_{c\tilde{c}}(t)\rho_{\tilde{c}}(t) - \sum_{\tilde{c} \atop \tilde{c}\neq c}\phi_{\tilde{c}c}(t)\rho_c(t) + \sum_{\tilde{c}}\gamma_{c\tilde{c}}(t)\mathbb{S}_{c\tilde{c}}(t)[\rho_{\tilde{c}}(t)] - \sum_{\tilde{c}}\gamma_{\tilde{c}c}(t)\rho_c(t). \quad (18)$$

Here, a "total-time-derivative" was introduced:

$$\frac{D}{Dt}\rho_c(t) \equiv \frac{d}{dt}\rho_c(t) + \sum_{\tilde{c}}\langle c_t|\frac{d}{dt}[\Pi_t^{\tilde{c}}]|c_t\rangle\rho_{\tilde{c}}(t), \quad (19)$$

where $\Pi_t^c \equiv |c_t\rangle\langle c_t|$. Furthermore, the time-dependent rates are

$$\phi_{c\tilde{c}}(t) = \langle c_t|\mathcal{L}_e[\Pi_t^{\tilde{c}}]|c_t\rangle \geqslant 0, \quad (20)$$

and similarly

$$\gamma_{\tilde{c}c}(t) = \sum_{\alpha,\beta}\Gamma_{\alpha,\beta}\ \langle\tilde{c}_t|B_\alpha|c_t\rangle\langle c_t|B_\beta^\dagger|\tilde{c}_t\rangle \geqslant 0, \quad (21)$$

where the inequalities follow straightforwardly from the diagonal rate representation $\Gamma_{\alpha,\beta}\to\delta_{\alpha,\beta}\Gamma_\alpha$. Finally, in Eq. (18) the system superoperators $\mathbb{S}_{c\tilde{c}}(t)[\bullet]$ read

$$\mathbb{S}_{c\tilde{c}}(t)[\bullet] = \frac{\sum_{\alpha,\beta}\Gamma_{\alpha,\beta}\ \langle c_t|B_\alpha|\tilde{c}_t\rangle\langle\tilde{c}_t|B_\beta^\dagger|c_t\rangle\mathbb{S}_{\alpha,\beta}[\bullet]}{\sum_{\alpha,\beta}\Gamma_{\alpha,\beta}\ \langle c_t|B_\alpha|\tilde{c}_t\rangle\langle\tilde{c}_t|B_\beta^\dagger|c_t\rangle}, \quad (22)$$

where $\{\mathbb{S}_{\alpha,\beta}\}$ are defined by Eq. (12). For each pair of indices $(c,\tilde{c})$ these superoperators are trace preserving, $\mathrm{Tr}_s(\mathbb{S}_{c\tilde{c}}(t)[\rho]) = \mathrm{Tr}_s[\rho]$, and completely positive. This last property follows straightforwardly from the diagonal rate representation. On the other hand, the evolution of the environment probabilities $\{p_c(t)\}$ [Eq. (17)] follows by taking the trace of Eq. (18), which by using that $\{\mathbb{S}_{c\tilde{c}}(t)\}$ are trace preserving yields

$$\frac{D}{Dt}p_c(t) = +\sum_{\tilde{c} \atop \tilde{c}\neq c}\phi_{c\tilde{c}}(t)p_{\tilde{c}}(t) - \sum_{\tilde{c} \atop \tilde{c}\neq c}\phi_{\tilde{c}c}(t)p_c(t) + \sum_{\tilde{c} \atop \tilde{c}\neq c}\gamma_{c\tilde{c}}(t)p_{\tilde{c}}(t) - \sum_{\tilde{c} \atop \tilde{c}\neq c}\gamma_{\tilde{c}c}(t)p_c(t). \quad (23)$$

Here, $(D/Dt)p_c(t)$ follows from Eq. (19) under the replacement $\rho_c(t)\to p_c(t)$.

Consistently with the definition of a CBE, the evolution of the probabilities $\{p_c(t)\}$ [Eq. (23)] does not depend on the system degrees of freedom. On the other hand, at a given time, it has the structure of a classical master equation [1], where the gain and loss terms have a clear stochastic interpretation in term of transitions between the bath states. We notice that the evolution of the system states $\{\rho_c(t)\}$ [Eq. (18)] involves the same incoherent coupling with rates $\phi_{c\tilde{c}}(t)$, while the coupling with rates $\gamma_{c\tilde{c}}(t)$ is endowed by the application of the superoperators $\mathbb{S}_{c\tilde{c}}(t)$ in each environment transition $c\leftarrow\tilde{c}$. This (unidirectional) dependence of the system evolution on the underlying (stochastic) environment dynamics *explains* the correlation structure defined by Eq. (16). The physical meaning of this association is also supported by analyzing an incoherent environment case.

B. Incoherent environment

If the degrees of freedom of the environment do not develop any (quantum) coherence, at any time its density matrix ρ_t^e is diagonal in a fixed base $\{|c\rangle\}$. Thus, the previous results must be read under the replacement

$$|c_t\rangle\langle c_t| \to |c\rangle\langle c|. \quad (24)$$

In consequence, the total time derivative [Eq. (19)] becomes an usual time derivative, $(D/Dt)\to(d/dt)$, and the rates $\phi_{c\tilde{c}}(t)$ and $\gamma_{\tilde{c}c}(t)$, as well as the superoperators $\mathbb{S}_{c\tilde{c}}(t)$, do not depend on time. In this situation, the probabilities $\{p_c(t)\}$ obey a standard *time-independent* classical master equation [of the form of Eq. (23)], a property that is shared by the states $\{\rho_c(t)\}$ [Eq. (18)]. Consistently, under these changes these equations recover the description corresponding to a system driven by incoherent degrees of freedom that follow their own stochastic dynamics [see Eq. (27) in Ref. [16] and examples in Ref. [15]].

The interpretation of Eqs. (18) and (23) remains the same as in the previous coherent general case. Nevertheless, in that case the eigenbase $\{|c_t\rangle\langle c_t|\}$, where the underlying stochastic environment dynamics develops, is time dependent due to the intrinsic quantum nature of the environmental degrees of freedom. This effect is taken into account through the total time derivative D/Dt [Eq. (19)].

C. Measurement based stochastic representation

A clear understanding of the system-environment coupling can also be achieved by representing their dynamics with a measurement-based bipartite state $\rho_{\mathrm{st}}^{se}(t)$ the time evolution of which, in contrast to Lindblad equations, is a stochastic one. As is well known, this last feature (denoted with the subscript "st ") allows one to relate $\rho_{\mathrm{st}}^{se}(t)$ with a continuous-in-time measurement process [48,49]. Averaging over (measurement) realizations (denoted with an overbar symbol) it follows that

$$\rho_t^{se} = \overline{\rho_{\mathrm{st}}^{se}(t)}. \quad (25)$$

In principle, the state $\rho_{\mathrm{st}}^{se}(t)$ can be formulated from the incoherentlike representation (18). A deeper understanding is achieved by assuming that (solely) the degrees of freedom of the environment are subjected to a measurement process that resolves the transitions induced by the (bath) operators $\{B_\alpha\}$. Thus, $\rho_{\mathrm{st}}^{se}(t)$ follows from the standard quantum jump approach [48,49] [for simplicity we consider in Eqs. (13) and (14) the diagonal case $\Gamma_{\alpha,\beta}=\delta_{\alpha,\beta}\Gamma_\alpha$, denoting $\mathbb{S}_{\alpha,\alpha}\leftrightarrow\mathbb{S}_\alpha$]. The environment state is recovered as $\rho_t^e = \overline{\rho_{\mathrm{st}}^e(t)}$, where $\rho_{\mathrm{st}}^e(t) = \mathrm{Tr}_s[\rho_{\mathrm{st}}^{se}(t)]$. For a CBE, the evolution of $\rho_{\mathrm{st}}^e(t)$ is independent of the system dynamics, being defined by the transitions associated to $\{B_\alpha\}$. Similarly, from the evolution (13) it follows that the bipartite state (assuming uncorrelated initial conditions) must take the form

$$\rho_{\mathrm{st}}^{se}(t) = \rho_{\mathrm{st}}^s(t)\otimes\rho_{\mathrm{st}}^e(t). \quad (26)$$

It is simple to realize that the dynamics of the system state $\rho_{\mathrm{st}}^s(t)$ must include the action of the superoperator $\mathbb{S}_\alpha$ whenever the environment suffers a transition (jump) corresponding to the operator B_α. In this way, the bipartite system-environment correlations [Eq. (16)] are built up in average. On the other hand, between environment transitions, the state $\rho_{\rm st}^{s}(t)$ evolves under the action of $\mathcal{L}_s$. Thus, the system dynamics can be seen as a *collisional* one [33,34], where the occurrence of the sudden (collisional) changes $\rho_{\rm st}^{s} \rightarrow \mathbb{S}_\alpha[\rho_{\rm st}^{s}]$ is dictated by the environment transitions associated to the operator B_α. We notice that this representation recovers the results of Ref. [34], which can be read as a particular case of the general dynamics (13).

D. Quantum regression hypothesis

The underlying system-environment dynamics is Markovian [Eq. (13)]. Thus, the quantum regression theorem (QRT) [48] is valid in the bipartite space $\mathcal{H}_s\otimes\mathcal{H}_e$. Introducing a vector of system operators $\mathbf{A}\leftrightarrow\mathbf{A}\otimes I_e$, where $\mathbf{A}=(A_1,A_2,\cdots,A_{\dim(\mathcal{H}_s)^2})$, their expectation value at a time τ (for simplicity, also denoted with an overbar symbol) can be written as

$$\overline{\mathbf{A}(\tau)}=\mathrm{Tr}_{se}\big(\mathcal{G}_{\tau,0}^{se}\big[\rho_0^{se}\big]\mathbf{A}\big). \tag{27}$$

The bipartite propagator is $\mathcal{G}_{\tau,\tau_0}^{se}\equiv\exp[(\tau-\tau_0)\mathcal{L}_T]$, with $\mathcal{L}_T\equiv(\mathcal{L}_s+\mathcal{L}_e+\mathcal{L}_{se})$, where $\mathcal{L}_{se}$ follows from Eq. (13). Given an extra system operator $O\leftrightarrow O\otimes I_e$, the correlations $\overline{O(t)\mathbf{A}(t+\tau)}$ follow from the QRT, which implies [48]

$$\overline{O(t)\mathbf{A}(t+\tau)}=\mathrm{Tr}_{se}\big(\mathbf{A}\mathcal{G}_{t+\tau,t}^{se}\big[\rho_t^{se}O\big]\big). \tag{28}$$

Now, we search conditions under which the QRT is valid on $\mathcal{H}_s$. Thus, we explore if the previous (system) operator correlations can be written only in terms of the system propagator. Assuming uncorrelated initial conditions, $\rho_0^{se}=\rho_0^s\otimes\rho_0^e$, the system propagator $\mathcal{G}_{\tau,0}^{s}$ can be written as

$$\mathcal{G}_{\tau,0}^{s}[\bullet]\equiv\mathrm{Tr}_e\big(\mathcal{G}_{\tau,0}^{se}[\bullet]\otimes\rho_0^e\big). \tag{29}$$

Thus, from Eq. (27), the operator expectation values can be rewritten as

$$\overline{\mathbf{A}(\tau)}=\mathrm{Tr}_s\big(\mathcal{G}_{\tau,0}^{s}\big[\rho_0^{s}\big]\mathbf{A}\big). \tag{30}$$

On the other hand, it is simple to realize that the correlations (28) cannot be written only in terms of the system propagator $\mathcal{G}_{t+\tau,t}^{s}$, which implies that the QRT is not valid in general on $\mathcal{H}_s$. Nevertheless, *assuming* that the environment begins in its stationary state (ρ_∞^e), and that the stationary bipartite state ($\lim_{t\rightarrow\infty}\rho_t^{se}$) does not involve system-environment correlations,

$$\rho_0^{se}=\rho_0^s\otimes\rho_\infty^e,\quad \lim_{t\rightarrow\infty}\rho_t^{se}=\rho_\infty^s\otimes\rho_\infty^e, \tag{31}$$

in the *long time limit* the operator correlations become

$$\lim_{t\rightarrow\infty}\overline{O(t)\mathbf{A}(t+\tau)}=\mathrm{Tr}_s\big(\mathbf{A}\mathcal{G}_{\tau,0}^{s}\big[\rho_\infty^{s}O\big]\big). \tag{32}$$

In deriving this equality the (stationary) time-translation symmetry $\lim_{t\rightarrow\infty}\mathcal{G}_{t+\tau,t}^{se}=\mathcal{G}_{\tau,0}^{se}$ was used. Consequently, if the conditions (31) are fulfilled the QRT is valid in the stationary regime *even* when the system dynamics is non-Markovian. The explicit meaning of the restricted validity of the QRT becomes clear by writing $\overline{\mathbf{A}(\tau)}=\hat{\boldsymbol{T}}(\tau)\overline{\mathbf{A}(0)}$, and the stationary correlations as $\lim_{t\rightarrow\infty}\overline{O(t)\mathbf{A}(t+\tau)}=\hat{\boldsymbol{T}}(\tau)\lim_{t\rightarrow\infty}\overline{O(t)\mathbf{A}(t)}$, where $\hat{\boldsymbol{T}}(\tau)$ is a matrix in the space corresponding to the vector of operators $\overline{\mathbf{A}(0)}$ [48].

Interestingly, the same condition for the validity of the stationary QRT [Eq. (31)] arises in quantum systems coupled to arbitrary incoherent degrees of freedom [42]. In fact, given that no condition on $\mathcal{G}_{\tau,0}^{se}$ [related to the constraint (3)] was demanded in the previous derivation, this result is valid in general whenever the bipartite (system-environment) dynamics is a Markovian (Lindblad) one.

E. Operational memory witness

It is not possible in general to infer if the environment is (or is not) a casual bystander one from the time-convoluted system master equation [Eq. (15)], or from its convolutionless form, or from the validity (or not) of the QRT [Eqs. (31) and (32)]. This is one of the limitations of nonoperational memory witnesses, which is surpassed by operational measurement based approaches.

We consider a CPF correlation [27]. It is defined by a set of three successive measurements performed over the system of interest. Here, they are taken as projective ones, corresponding to Hermitian operators $O_{\underline{\mathbf{m}}}$, denoted in successive order with $\underline{\mathbf{m}}=\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}}$. Their eigenvectors and eigenvalues read $O_{\underline{\mathbf{m}}}|m\rangle=m|m\rangle$, where correspondingly $\{m\}=\{x\},\ \{y\},\ \{z\}$. The measurements are performed at the initial time $t=0$ (past), at time t (present), and $t+\tau$ (future), respectively. After the intermediate measurement at time t, the system postmeasurement state is externally modified [32] as $\rho_y=|y\rangle\langle y|\rightarrow\rho_{\breve{y}}=|\breve{y}\rangle\langle\breve{y}|$. A *deterministic scheme* (d) is defined by the condition $\breve{y}=y$. Thus, no change is introduced. A *random scheme* (r) is defined by a random election of $\breve{y}$ (over the set $\{y\}$) with an *arbitrary* conditional probability $\wp(\breve{y}|x)$, which may (or not) depend on the outcomes $\{x\}$ of the first measurement performed at time $t=0$. The CPF correlation depends on the chosen scheme. In both cases, it reads

$$C_{pf}(t,\tau)|_{\breve{y}}\overset{d/r}{=}\sum_{z,x}zx[P(z,x|\breve{y})-P(z|\breve{y})P(x|\breve{y})], \tag{33}$$

where $\{z\}$ and $\{x\}$ are the eigenvalues of $O_{\underline{\mathbf{z}}}$ and $O_{\underline{\mathbf{x}}}$, respectively. With $P(a|b)$ we denote the conditional probability of a given b.

In the deterministic scheme, the CPF correlation vanishes in a Markovian regime, where past and future outcomes are conditionally independent: $P(z,x|\breve{y})=P(z|\breve{y})P(x|\breve{y})$ [27]. Thus, $C_{pf}(t,\tau)|_{\breve{y}}\overset{d}{\neq}0$ detects memory effects independently of their underling mechanism. This last property can be understood from the change of the bipartite state ρ_{se} after the intermediate (projective) measurement:

$$\rho_{se}\rightarrow|y\rangle\langle y|\otimes\frac{\langle y|\rho_{se}|y\rangle}{\mathrm{Tr}_{se}[|y\rangle\langle y|\rho_{se}]}. \tag{34}$$

In this expression, everything except for the postmeasurement system state $|y\rangle\langle y|$ depends in general on the initial measurement outcome x. In fact, the dependence of the environment postmeasurement state $\langle y|\rho_{se}|y\rangle/\mathrm{Tr}_{se}[|y\rangle\langle y|\rho_{se}]$ on the previous system history allows one to detect memory effects [take, for example, ρ_{se} from Eq. (16)].

In the random scheme, the CPF correlation also detects non-Markovian effects but in addition it allows one to classify the corresponding memory effects. In fact, if the environment is not affected during the system evolution it vanishes identically [32]. Consequently, the presence of *a (non-Markovian) CBE implies that* $C_{pf}(t,\tau)|_{\breve{y}}\overset{r}{=}0$. Alternatively, in the random scheme, the condition $C_{pf}(t,\tau)|_{\check{y}} \overset{r}{\neq} 0$ indicates departures with respect to a (non-Markovian) CBE, here defined by the condition Eq. (3).

The previous features of the random scheme can be understood from Eq. (34) by introducing the (random) system transformation $|y\rangle\langle y| \to |\check{y}\rangle\langle\check{y}|$ and averaging (marginalizing) the environment postmeasurement states (associated to the outcomes $\{y\}$) with their probabilities $\{\mathrm{Tr}_{se}[|y\rangle\langle y|\rho_{se}]\}$. This last ingredient is introduced because the CPF correlation is defined with the renewed conditional outcome $\check{y}$. Hence, after the intermediate measurement the bipartite state transforms as

$$\rho_{se} \to |\check{y}\rangle\langle\check{y}| \otimes \sum_y \langle y|\rho_{se}|y\rangle = |\check{y}\rangle\langle\check{y}| \otimes \mathrm{Tr}_s[\rho_{se}]. \qquad (35)$$

In contrast to Eq. (34), here the final postmeasurement bath state is $\mathrm{Tr}_s[\rho_{se}]$. As the transformation (35) is valid independently of the environment nature, in general this state depends on the previous system history (x outcomes). Consequently, in this scheme the CPF correlation also detects memory effects. Nevertheless (by definition), for a CBE the (bath) state $\mathrm{Tr}_s[\rho_{se}]$ is independent of the present or past system states [take ρ_{se} from Eq. (16); see also Eq. (3)], which implies that a Markovian property characterizes the conditional outcome statistics. Consequently, the CPF correlation vanishes identically for CBEs.

The conditional probabilities appearing in Eq. (33) can explicitly be calculated from the joint outcome probability $P(z,\check{y},x) \leftrightarrow P(z,t+\tau;\check{y},t;x,0)$. As shown in Refs. [27,32], this object can be calculated after knowing the system-environment propagator. In order to maintain a description as simple as possible, we consider Eq. (13) in the diagonal case $\Gamma_{\alpha,\beta} = \delta_{\alpha,\beta}\Gamma_\alpha$ (denoting $\mathbb{S}_{\alpha,\alpha} \leftrightarrow \mathbb{S}_\alpha$) and assuming that the composition of two arbitrary superoperators can be written as a superoperator included in the master equation, that is, $\mathbb{S}_\alpha\mathbb{S}_{\alpha'} = \mathbb{S}_{\alpha''}$. Under these two conditions, the bipartite propagator, $\rho_t^{se} = \mathcal{G}_{t,0}^{se}[\rho_0^s \otimes \rho_0^e]$, can be written in a general way as

$$\rho_t^{se} = \rho_0^s \otimes \mathcal{F}_0(t)\big[\rho_0^e\big] + \sum_{\substack{\alpha\\ \alpha\neq 0}} \mathbb{S}_\alpha\big[\rho_0^s\big] \otimes \mathcal{F}_\alpha(t)\big[\rho_0^e\big]. \qquad (36)$$

Here, the environment superoperators $\{\mathcal{F}_\alpha(t)\}$, which act on the initial environment state ρ_0^e, depend on each specific problem. With this expression at hand, below it is shown how the CPF correlation can be explicitly calculated from the expressions for $P(z,\check{y},x)$ found in Ref. [32].

1. Deterministic scheme

In the deterministic scheme, the joint outcome probability can be written as [32]

$$\frac{P(z,\check{y},x)}{P(x)} \overset{d}{=} \mathrm{Tr}_{se}(E_z\mathcal{G}_{t+\tau,t}^{se}\{\rho_{\check{y}} \otimes \mathrm{Tr}_s(E_{\check{y}}\mathcal{G}_{t,0}^{se}[\rho_x^{se}])\}). \qquad (37)$$

Here, $E_m \equiv |m\rangle\langle m|$ and $\rho_m \equiv |m\rangle\langle m|$ $[m = z, \check{y}, x]$ represent the (positive) effect measurement operators and postmeasurement states, respectively [27]. They coincide because here the measurements are defined by one-dimensional projectors. Furthermore, $\rho_x^{se} \equiv \rho_x \otimes \rho_0^e$ and $P(x) = \langle x|\rho_0^s|x\rangle$. From the previous expression, using the bipartite propagator (36), we get

$$\frac{P(z,\check{y},x)}{P(x)} \overset{d}{=} \sum_{\alpha,\beta} \langle z|\mathbb{S}_\alpha[\rho_{\check{y}}]|z\rangle\langle\check{y}|\mathbb{S}_\beta[\rho_x]|\check{y}\rangle \times \mathrm{Tr}_e\big[\mathcal{F}_\alpha(\tau)\mathcal{F}_\beta(t)\rho_0^e\big]. \qquad (38)$$

Using that $P(z,x|\check{y}) = P(z,\check{y},x)/P(\check{y})$, where $P(\check{y}) = \sum_{z,x} P(z,\check{y},x)$, the CPF correlation (33) reads

$$C_{pf}(t,\tau)|_{\check{y}} \overset{d}{=} \frac{1}{P(\check{y})^2} \sum_{\alpha,\beta,\mu} \Theta^{\alpha\beta\mu}|_{\check{y}} \Lambda_{\alpha\beta\mu}(t,\tau). \qquad (39)$$

The time-independent coefficients $\Theta^{\alpha\beta\mu}|_y$ only depend on the chosen observables:

$$\Theta^{\alpha\beta\mu}|_{\check{y}} = \langle\check{y}|\mathbb{S}_\alpha^\#[O_{\underline{z}}]|\check{y}\rangle\langle\check{y}|\mathbb{S}_\beta[O_{\underline{x}}\rho_{\underline{x}}]|\check{y}\rangle\langle\check{y}|\mathbb{S}_\mu[\rho_{\underline{x}}]|\check{y}\rangle, \qquad (40)$$

where the dual superoperator $\mathbb{S}_\alpha^\#$ is defined from $\mathrm{Tr}_s(O\mathbb{S}_\alpha[\rho]) = \mathrm{Tr}_s(\rho\mathbb{S}_\alpha^\#[O])$. Furthermore, we defined the state $\rho_{\underline{x}} \equiv \sum_x P(x)|x\rangle\langle x| = \sum_x \langle x|\rho_0^s|x\rangle|x\rangle\langle x|$. The time dependence in Eq. (39) follows from

$$\Lambda_{\alpha\beta\mu}(t,\tau) = +\mathrm{Tr}_e\big[\mathcal{F}_\alpha(\tau)\mathcal{F}_\beta(t)\rho_0^e\big]\,\mathrm{Tr}_e\big[\mathcal{F}_\mu(t)\rho_0^e\big] - \mathrm{Tr}_e\big[\mathcal{F}_\alpha(\tau)\mathcal{F}_\mu(t)\rho_0^e\big]\,\mathrm{Tr}_e\big[\mathcal{F}_\beta(t)\rho_0^e\big], \qquad (41)$$

which only depends on the initial environment state ρ_0^e and environment superoperators $\{\mathcal{F}_\alpha(t)\}$. The probability $P(\check{y})$ is

$$P(\check{y}) = \sum_\alpha \langle\check{y}|\mathbb{S}_\alpha[\rho_{\underline{x}}]|\check{y}\rangle\mathrm{Tr}_e\big[\mathcal{F}_\alpha(t)\rho_0^e\big]. \qquad (42)$$

The previous formula gives an exact analytical expression for the CPF correlation that is valid for a broad class of problems (see Sec. IV below).

2. Random scheme

In the random scheme, $P(z,\check{y},x)$ reads [32]

$$\frac{P(z,\check{y},x)}{P(x)} \overset{r}{=} \mathrm{Tr}_{se}\big(E_z\mathcal{G}_{t+\tau,t}^{se}\big\{\rho_{\check{y}} \otimes \mathrm{Tr}_s\big(\mathcal{G}_{t,0}^{se}\big[\rho_x^{se}\big]\big)\big\}\big)\wp(\check{y}|x), \qquad (43)$$

where as before $E_m \equiv |m\rangle\langle m|$, $\rho_m \equiv |m\rangle\langle m|$ $[m = z, \check{y}, x]$, and $\rho_x^{se} \equiv \rho_x \otimes \rho_0^e$, while $P(x) = \langle x|\rho_0^s|x\rangle$. Furthermore, the conditional probability $\wp(\check{y}|x)$ can be freely chosen. Using the propagator expression (36), it follows that

$$\frac{P(z,\check{y},x)}{P(x)} \overset{r}{=} \sum_\alpha \langle z|\mathbb{S}_\alpha[\rho_{\check{y}}]|z\rangle\mathrm{Tr}_e\big[\mathcal{F}_\alpha(\tau)\rho_t^e\big]\wp(\check{y}|x). \qquad (44)$$

Consequently, independently of the chosen measurement observables, it is confirmed that the CPF correlation [Eq. (33)] vanishes identically in this scheme:

$$C_{pf}(t,\tau)|_{\check{y}} \overset{r}{=} 0. \qquad (45)$$

In fact, in contrast to Eq. (38), the sum term $[\sum_\alpha \cdots]$ in Eq. (44) can be read as $P(z|\check{y})$, leading to the Markovian structure $P(z,\check{y},x) = P(z|\check{y})\wp(\check{y}|x)P(x) \to P(z,x|\check{y}) = P(z|\check{y})P(x|\check{y})$.

IV. EXAMPLES

In order to exemplify the developed results, we consider different single and multipartite system dynamics interacting with a quantum non-Markovian CBE.

A. Single qubit system

The system is a qubit, while the quantum degrees of freedom of the environment correspond to a two-level fluorescent system with decay rate γ and Rabi frequency Ω [48]. Their mutual evolution [Eq. (13)] is written as

$$\frac{d}{dt}\rho_t^{se} = -i\frac{\Omega}{2}[\sigma_x, \rho_t^{se}] + \gamma\Big(\sigma\mathbb{S}[\rho_t^{se}]\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho_t^{se}\}_+\Big). \tag{46}$$

The operators σ_x, $\sigma^\dagger$, and σ are, respectively, the x-Pauli matrix and the raising and lowering operators in the two-dimensional environment Hilbert space $\mathcal{H}_e$. The unique system contribution is the superoperator [see Eq. (12)]

$$\mathbb{S}[\bullet] = \sigma_z[\bullet]\sigma_z, \tag{47}$$

where σ_z is the z-Pauli matrix in the system Hilbert space $\mathcal{H}_s$. Thus, the system is subjected to a dephasing process driven by the transitions of the fluorescent system.

1. System-environment propagator

Considering (bipartite) uncorrelated initial conditions, $\rho_0^{se} = \rho_0^s \otimes \rho_0^e$, the propagator of Eq. (46) can be written with the structure (36):

$$\rho_t^{se} = \rho_0^s \otimes \mathcal{F}_+(t)\big[\rho_0^e\big] + \mathbb{S}\big[\rho_0^s\big] \otimes \mathcal{F}_-(t)\big[\rho_0^e\big]. \tag{48}$$

At any time, this bipartite state is a separable one [Eq. (16)]. Consistently, a positive partial transpose criterion [46] is fulfilled. On the other hand, the environment superoperators $\mathcal{F}_\pm$ can be written as

$$\mathcal{F}_\pm(t)[\bullet] = \tfrac{1}{2}(\mathcal{G}_{t,0}^+[\bullet] \pm \mathcal{G}_{t,0}^-[\bullet]), \tag{49}$$

where the auxiliary superoperators $\mathcal{G}_t^\pm[\bullet]$ are defined by the evolutions,

$$\frac{d}{dt}\mathcal{G}_{t,t_0}^\pm = -i\frac{\Omega}{2}[\sigma_x, \mathcal{G}_{t,t_0}^\pm] + \gamma\Big(\pm\sigma\mathcal{G}_{t,t_0}^\pm\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \mathcal{G}_{t,t_0}^\pm\}_+\Big), \tag{50}$$

with the initial conditions $\mathcal{G}_{t_0,t_0}^\pm = I_e$. These equations can be solved in an analytical way.

Both superoperators $\mathcal{G}_{t,t_0}^\pm$ define the system and environment dynamics:

$$\rho_t^e = \mathcal{G}_{t,0}^+\big[\rho_0^e\big], \quad f(\tau|t) \equiv \mathrm{Tr}_e\big(\mathcal{G}_{t+\tau,t}^-\big[\rho_t^e\big]\big). \tag{51}$$

In fact, from Eqs. (46) and (50) it is simple to realize that $\mathcal{G}_{t,t_0}^+$ is the propagator of the environment degrees of freedom. Furthermore, from Eq. (48) it is simple to show that the function $f(\tau|t)$ sets the system coherence decay:

$$\rho_t^s = \begin{pmatrix} p_{\mathrm{up}} & f(t|0)c_{\mathrm{up}} \\ f(t|0)c_{\mathrm{dn}} & p_{\mathrm{dn}} \end{pmatrix}, \tag{52}$$

where p_{up} and p_{dn} are the initial upper and lower system populations while the nondiagonal contributions c_{up} and c_{dn} are the initial system coherences. Furthermore,

$$f(\tau|t) = e^{-\gamma\tau}a_t + e^{-\gamma\tau/4}\bigg[b_t\cosh(\Delta\tau) + c_t\frac{\sinh(\Delta\tau)}{\Delta}\bigg], \tag{53}$$

where for shortening the expression we introduced the coefficient $\Delta \equiv \sqrt{(\gamma/4)^2 - \Omega^2}$. Explicit expressions for the time-dependent coefficients a_t, b_t, and c_t can be found in the Appendix.

2. Operator correlations

From Eq. (48), using that $\lim_{t\to\infty}\mathcal{G}_{t,t_0}^+[\rho_0^e] = \rho_\infty^e$ and $\lim_{t\to\infty}\mathcal{G}_{t,t_0}^-[\rho_0^e] = 0$, it follows that

$$\lim_{t\to\infty}\rho_t^{se} = \rho_\infty^s \otimes \rho_\infty^e, \qquad \rho_\infty^s = \tfrac{1}{2}\big(\rho_0^s + \mathbb{S}\big[\rho_0^s\big]\big), \tag{54}$$

where ρ_∞^e is the stationary state of a two-level fluorescent system (see the Appendix). Thus, when the environment at the initial time begins in its stationary state, the conditions (31) are fulfilled, indicating the validity of the QRT in the stationary regime. This conclusion is corroborated by the following explicit calculation of operator expectation values and correlations. Nevertheless, we also found that for some operator correlations the QRT is valid at all times.

Introducing the vector of system operators $\sigma \equiv \{\sigma_x, \sigma_y, \sigma_z\}$, their expectation values, from Eq. (27), read

$$\overline{\sigma(\tau)} = \hat{\boldsymbol{T}}(\tau|0)\,\overline{\sigma(0)}, \tag{55}$$

where $\overline{\sigma(0)} = \mathrm{Tr}_s[\sigma(0)\rho_0^s]$. Similarly, for the Pauli operators there are nine possible correlations $\overline{\sigma_i(t)\sigma_j(t+\tau)}$. For six of them, from Eq. (28) we get

$$\overline{\sigma_x(t)\sigma(t+\tau)} = \hat{\boldsymbol{T}}(\tau|t)\,\overline{\sigma_x(t)\sigma(t)}, \tag{56a}$$

$$\overline{\sigma_y(t)\sigma(t+\tau)} = \hat{\boldsymbol{T}}(\tau|t)\,\overline{\sigma_y(t)\sigma(t)}. \tag{56b}$$

In these expressions, the matrix $\hat{\boldsymbol{T}}(\tau|t)$ reads

$$\hat{\boldsymbol{T}}(\tau|t) = \mathrm{diag}\{f(\tau|t), f(\tau|t), 1\}. \tag{57}$$

When the environment begins in its stationary state $\rho_0^e = \rho_\infty^e = \lim_{t\to}\rho_t^e$, it follows that $f(\tau|t) = f(\tau|0)$ [see Eq. (51)], and consequently $\hat{\boldsymbol{T}}(\tau|t) = \hat{\boldsymbol{T}}(\tau|0)$. Thus, the six correlations (56) *at any time* (t and τ) evolve as the expectation values [Eq. (55)] indicating the absence of any departure with respect to the QRT.

The unique correlations [Eq. (28)] that depart (at any finite time) from the predictions of the QRT are

$$\overline{\sigma_z(t)\sigma(t+\tau)} = \tilde{\boldsymbol{T}}(\tau,t)\overline{\sigma_z(t)\sigma(t)}, \tag{58}$$

where the matrix $\tilde{\boldsymbol{T}}(\tau,t)$ is

$$\tilde{\boldsymbol{T}}(\tau,t) = \mathrm{diag}\bigg\{\frac{f(t+\tau|0)}{f(t|0)}, \frac{f(t+\tau|0)}{f(t|0)}, 1\bigg\}. \tag{59}$$

If the semigroup property $f(t+\tau|0) = f(\tau|t)f(t|0)$ becomes valid, the QRT is recovered. This happens when the system coherence can be approximated by an exponential decay behavior. On the other hand, using that $\overline{\sigma_z(t)\sigma(t)} = \mathrm{diag}\{f(t|0), f(t|0), 1\}\overline{\sigma_z(0)\sigma(0)}$ we get $\lim_{t\to\infty}\overline{\sigma_z(t)\sigma(t)} = \{0,0,1\}$. Thus, consistently with Eq. (31), in the long-time limit the correlations (58) obey the same evolution as the operator expectation values [Eq. (55)], $\lim_{t\to\infty}\overline{\sigma_z(t)\sigma(t+\tau)} = \hat{\boldsymbol{T}}(\tau|0)\{0,0,1\}$, indicating the validity of the QRT in the stationary regime.

3. Memory witnesses

A deeper understanding of the memory effects developed in the studied model can be achieved by comparing operational and nonoperational memory witnesses.

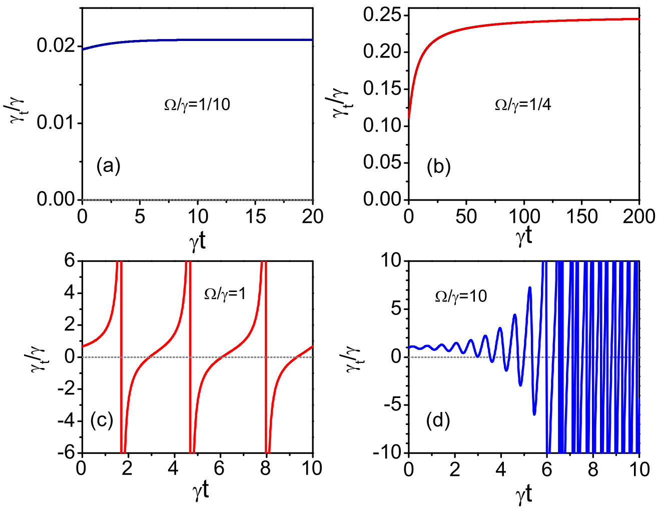

FIG. 1. Time dependence of the decoherence rate Eq. (60) corresponding to the underlying dynamics (46). The environment begins in its stationary state. In each plot, the parameters are (a) $\Omega/\gamma = 1/10$, (b) $\Omega/\gamma = 1/4$, (c) $\Omega/\gamma = 1$, and (d) $\Omega/\gamma = 10$.

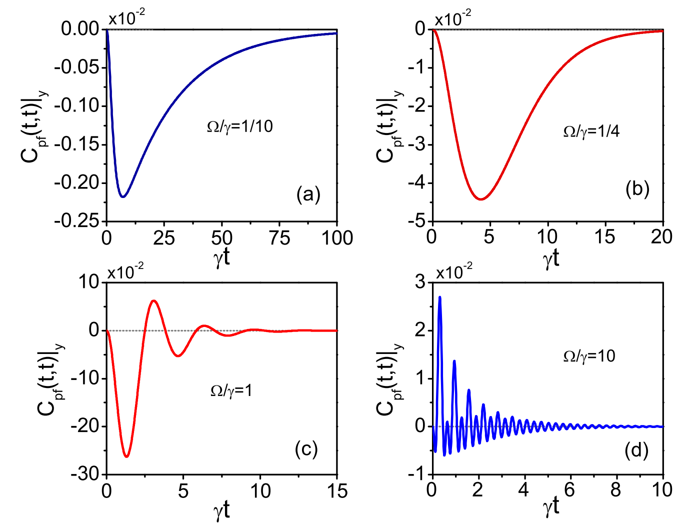

FIG. 2. CPF correlation [Eq. (62)] with equal time intervals $\tau = t$ corresponding to the system-environment model (46). The parameters Ω/γ are the same as in Fig. 1. Similarly, the environment begins in its stationary state. In all cases, the system initial condition is such that $\langle x\rangle = 0$.

For *nonoperational approaches*, the central ingredient to analyze is the system density-matrix evolution. From Eq. (52), straightforwardly we get

$$\frac{d}{dt}\rho_t^s = \frac{1}{2}\gamma_t\left(\mathbb{S}[\rho_t^s] - \rho_t^s\right), \qquad \gamma_t = -\frac{d}{dt}\ln[f(t|0)], \tag{60}$$

where $f(t|0)$ follows from Eq. (53).

The negativity of γ_t can be used as an indicator of memory effects [14]. Under this criterion the dynamics is non-Markovian if at some time interval $\gamma_t < 0$ and Markovian if $\gamma_t \geqslant 0$ at all times. We assume that the environment begins in its stationary state, $\rho_0^e = \rho_\infty^e$. Thus, $f(\tau|t) = f(\tau|0)$ [see Eq. (51)]. In Fig. 1 we plot γ_t for different values of the quotient Ω/γ. For $\Omega/\gamma \leqslant 1/4$ the rate is always positive, while for $\Omega/\gamma > 1/4$ it develops periodical divergences. Thus, the dynamics (under the negative rate criteria) is non-Markovian in this last regime. The same conclusion follows from the trace distance between two initial states [8]. On the other hand, for $\Omega/\gamma \gg 1/4$, the rate γ_t approaches a constant value (initial stage), implying that a Markovian regime is (asymptotically) reached again.

The previous rate behaviors can be understood from the underlying environment dynamics. For $\Omega/\gamma \ll 1/4$, the probability distribution of the elapsed time between environment (fluorescent) transitions approaches an exponential function with average time $[2\Omega^2/\gamma]^{-1}$ [48]. Consequently, the coherence decay function (induced by the application of the superoperator $\mathbb{S}$) can be approximated as $f(\tau|0) \approx \exp[-t(2\Omega^2/\gamma)]$, which implies $\gamma_t \approx 2\Omega^2/\gamma$ [Fig. 1(a)]. This regime changes drastically when $\Omega/\gamma = 1/4$ [Fig. 1(b)], where the environment starts to develop Rabi oscillations. Around $\Omega/\gamma \approx 1$, the system coherence $f(t|0)$ decays and vanishes in an oscillatory way. Consequently, γ_t [Eq. (60)] develops periodic divergences [Fig. 1(c)]. For $\Omega/\gamma \gg 1$, the effect of the (fast) environment Rabi oscillations over the system cancels out on average, leading to the approximate coherence decay $f(\tau|0) \approx \exp(-t\gamma)$. Thus, $\gamma_t \approx \gamma$, indicating that a Markovian regime is approached. This tendency is clearly seen in Fig. 1(d) at the initial stage ($\gamma t \lesssim 2$). The posterior ($\gamma t > 5$) divergent oscillations of γ_t emerge because $f(\tau|0)$ oscillates around zero. Nevertheless, in this regime $|f(\tau|0)| \ll 1$. Thus, when $\Omega/\gamma \gg 1$ the divergent oscillations do not contradict that a Markovian regime is being approached.

The previous dynamical regimes can alternatively be analyzed from the *operational approach*. For the dynamics (46), using the explicit propagator Eq. (48), all statistical objects that define the CPF correlation can be explicitly evaluated. We consider that the three consecutive measurements are performed in the $\hat{x}$ direction of the system Bloch sphere. Thus, in all cases, the possible measurement outcomes (eigenvalues) are $m = \pm 1$ $(m = x, y, z)$, while the corresponding eigenvectors are $|m\rangle = (1/\sqrt{2})(|up\rangle + m|dn\rangle)$, where $|up\rangle$ and $|dn\rangle$ are, respectively, the upper and lower states of the system [see Eq. (52)].

In the deterministic scheme, the joint outcome probability Eq. (38) becomes

$$P(z, \breve{y}, x) \overset{d}{=} \tfrac{1}{4}[1 + z\breve{y}f(\tau|t) + zxf(t+\tau|0) + \breve{y}xf(t|0)]P(x). \tag{61}$$

The CPF correlation Eq. (39) reads

$$C_{pf}(t,\tau)|_{\breve{y}} \overset{d}{=} \frac{1-\langle x\rangle^2}{4[P(\breve{y})]^2}[f(t+\tau|0) - f(\tau|t)\,f(t|0)], \tag{62}$$

with $P(\breve{y}) = [1 + \breve{y}\langle x\rangle f(t|0)]/2$, jointly with $\langle x\rangle = \sum_{x=\pm1} xP(x)$ and $P(x) = \langle x|\rho_0^s|x\rangle$ where $\{|x\rangle\}$ are the eigenvectors of the $\hat{x}$-Pauli matrix.

In Fig. 2, for the same parameter regimes shown in Fig. 1, we plot the CPF correlation at equal time intervals, $\tau = t$. The environment also begins in its stationary state, $\rho_0^e = \rho_\infty^e$.

As is well known, operational and nonoperational memory witnesses do not coincide in general [27]. In fact, here the

CPF correlation indicates *the presence of memory effects for all parameter regimes*, even when the rate γ_t is positive at all times. Consistently, for $\Omega/\gamma \ll 1/4$, the maximal absolute value of the CPF correlation diminishes [Fig. 2(a)], indicating the proximity of a Markovian regime. When $\Omega/\gamma = 1/4$ [Fig. 2(b)], the CPF is negative at all times and does not develop oscillations. For $\Omega/\gamma \approx 1$, it develops oscillations and its absolute value is maximal [Fig. 2(c)], indicating strong memory effects. Consistently, when $\Omega/\gamma \gg 1$, the CPF correlation oscillates but with a smaller amplitude [Fig. 2(d)], indicating again the approaching of a Markov regime.

In contrast to memory witnesses based only on the unperturbed system dynamics, the CPF correlation indicates a Markovian regime only in the limits $\Omega/\gamma \to 0$ and ∞. In addition, the operational approach gives a much deeper characterization when considering the random scheme. From Eq. (44), for the joint probabilities we get

$$P(z, \breve{y}, x) \overset{r}{=} \frac{1}{2}[1 + z\breve{y} f(\tau|t)]\wp(\breve{y}|x)P(x). \tag{63}$$

As expected, a Markovian property is fulfilled, leading consistently to $C_{pf}(t,\tau)|_{\breve{y}} \overset{r}{=} 0$ [Eq. (45)] for arbitrary initial environment states. This result indicates the presence of a CBE, *a property that cannot be resolved with nonoperational approaches.*

B. Multipartite qubit systems

The developed formalism also allows one to study the coupling of multipartite systems with a CBE. In contrast to Eq. (46), here we consider a set of N qubits. For simplicity, we assume the system-environment evolution

$$\frac{d}{dt}\rho_t^{se} = -i\frac{\Omega}{2}[\sigma_x, \rho_t^{se}] + \gamma\left(\sigma\mathbb{S}_{\underline{\mathbf{a}}}[\rho_t^{se}]\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho_t^{se}\}_+\right) + \varphi\left(\sigma^\dagger\mathbb{S}_{\underline{\mathbf{b}}}[\rho_t^{se}]\sigma - \frac{1}{2}\{\sigma\sigma^\dagger, \rho_t^{se}\}_+\right). \tag{64}$$

As before, σ_x, $\sigma^\dagger$, and σ are, respectively, the x-Pauli matrix and the raising and lowering operators in the two-dimensional environment Hilbert space $\mathcal{H}_e$. Thus, the environment corresponds to a two-level fluorescentlike system with Rabi frequency Ω and decay rate γ, while the rate φ scales the presence of thermally induced excitations [2].

Each of the system superoperators $\mathbb{S}_\alpha[\bullet] \equiv \sigma_\alpha \bullet \sigma_\alpha$ ($\alpha = \underline{\mathbf{a}}, \underline{\mathbf{b}}$) are defined by an arbitrary (multipartite) Pauli string $\sigma_\alpha = \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}$, which consists in the external product of N arbitrary Pauli operators acting on each qubit. These superoperators are applied over the system whenever the environment suffers a transition between its (two) states. From Eq. (64), it follows that $\mathbb{S}_{\underline{\mathbf{a}}}$ is applied when an environmental transition between the upper and lower states occurs, while $\mathbb{S}_{\underline{\mathbf{b}}}$ is applied for the inverse (thermally induced) transition.

The bipartite propagator associated to Eq. (64) can be written with the structure Eq. (36). The label of the superoperators $\{\mathcal{F}_\alpha(t)\}$ runs over the values $\alpha = \mathbf{0}, \underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}$, where $\mathbb{S}_{\underline{\mathbf{c}}} = \mathbb{S}_{\underline{\mathbf{b}}}\mathbb{S}_{\underline{\mathbf{a}}}$. The calculations that lead to explicit solutions for the environment superoperators $\{\mathcal{F}_\alpha(t)\}$ are presented in the Appendix. From these expressions and Eq. (36), it follows that

$$\lim_{t\to\infty} \rho_t^{se} = \rho_\infty^s \otimes \rho_\infty^e, \qquad \rho_\infty^s = \frac{1}{4}\left(\rho_0^s + \sum_\alpha \mathbb{S}_\alpha[\rho_0^s]\right), \tag{65}$$

where ρ_∞^s is the multipartite system stationary state while the (two-level) state ρ_∞^e follows by tracing out the system degrees of freedom in Eq. (64). In consequence, as in the previous example, the QRT is valid for stationary correlations of the system.

From Eq. (36) the system state at any time, $\rho_t^s = \mathrm{Tr}_e(\rho_t^{se})$, can straightforwardly be written as a statistical superposition of Kraus maps:

$$\rho_t^s = p_t^{\mathbf{0}}\rho_0 + \sum_{\alpha=\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},} p_t^\alpha \mathbb{S}_\alpha[\rho_0], \tag{66}$$

where the weights are $p_t^\alpha \equiv \mathrm{Tr}_e(\mathcal{F}_\alpha(t)[\rho_0^e])$. Similarly, the density-matrix evolution can be written as

$$\frac{d\rho_t^s}{dt} = \sum_{\alpha=\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},} \gamma_t^\alpha\left(\mathbb{S}_\alpha[\rho_t^s] - \rho_t^s\right). \tag{67}$$

Simple expressions for the probabilities $\{p_t^\alpha\}$ and rates $\{\gamma_t^\alpha\}$ are obtained when $\varphi = \gamma$ in Eq. (64). We get

$$p_t^{\mathbf{0}} = \frac{1}{2}e^{-\gamma t}\left[\cosh(\gamma t) + \frac{\gamma^2}{\chi^2}\cos(\chi t) + \frac{\Omega^2}{\chi^2}\right], \tag{68a}$$

$$p_t^{\underline{\mathbf{a}}} = p_t^{\underline{\mathbf{b}}} = \frac{1}{4}[1 - e^{-2\gamma t}], \tag{68b}$$

$$p_t^{\underline{\mathbf{c}}} = \frac{1}{2}e^{-\gamma t}\left[\cosh(\gamma t) - \frac{\gamma^2}{\chi^2}\cos(\chi t) - \frac{\Omega^2}{\chi^2}\right], \tag{68c}$$

where $\chi \equiv \sqrt{\gamma^2 + \Omega^2}$. From these expressions, the rates in Eq. (67) are

$$\gamma_t^{\underline{\mathbf{a}}} = \gamma_t^{\underline{\mathbf{b}}} = \frac{1}{2}\gamma, \quad \gamma_t^{\underline{\mathbf{c}}} = \frac{1}{2}\frac{\chi\sin(\chi t)}{(\Omega/\gamma)^2 + \cos(\chi t)}. \tag{69}$$

Notice that $\gamma_t^{\underline{\mathbf{c}}}$ presents an oscillatory behavior at any time, which develops divergences only when $(\Omega/\gamma)^2 < 1$. When $\Omega = 0$, it reduces to $\gamma_t^{\underline{\mathbf{c}}} = (1/2)\gamma\tan(\gamma t)$, recovering the rates of the "trigonometric eternal non-Markovian" dynamics introduced in Ref. [50], where the environment dynamics is an incoherent one. Thus, we can read Eq. (64) as a *quantum (coherent) generalization* ($\Omega \neq 0$) of the incoherent environment studied in Ref. [50].

The CPF correlation can also be obtained in the present case. Assuming that the three measurements correspond to the observable σ_α ($\alpha = \underline{\mathbf{a}}$ or $\underline{\mathbf{b}}$), from Eq. (39) we get (with $\varphi = \gamma$)

$$C_{pf}(t,\tau)|_{\breve{y}} \overset{d}{=} -\frac{(1-\langle x\rangle^2)}{[2^N P(\breve{y})]^2}e^{-(t+\tau)\gamma/2}\left[\frac{\gamma^2}{\chi^2}\sin(t\chi)\sin(\tau\chi) - \frac{4\gamma^2\Omega^2}{\chi^4}\sin^2(t\chi/2)\sin^2(\tau\chi/2)\right], \tag{70}$$

where as before $\chi = \sqrt{\gamma^2 + \Omega^2}$ and $P(\breve{y}) = 2^{-N}\{1 + \breve{y}\langle x\rangle e^{-\gamma t}[\Omega^2 + \gamma^2\cos(t\chi)]/\chi^2\}$. Furthermore, $\langle x\rangle = \sum_x x\langle x|\rho_0^s|x\rangle$, where $\{x\}$ and $\{|x\rangle\}$ are, respectively, the eigenvalues and eigenvectors associated to the first measurement observable. When the three measurements correspond to the observable $\sigma_{\underline{c}}$, we get $C_{pf}(t,\tau)|_y \overset{d}{=} 0$. This property does not imply Markovianity because it is only valid for these particular measurement observables. In contrast, in the random scheme Eq. (45) guarantees that $C_{pf}(t,\tau)|_y \overset{r}{=} 0$ for *any* system observables and initial environment states, a property consistent with a CBE.

V. SUMMARY AND CONCLUSIONS

Quantum memory effects can be induced by environments the state and dynamical behavior of which are not affected at all by their interaction with the system of interest. Based on a bipartite completely positive representation of the system-environment dynamics, in this paper we have explored the most general interaction structures that are consistent with this class of non-Markovian CBEs.

While unitary interactions must be discarded, we have found the most general dissipative coupling structures [Eq. (13)] that are consistent with the demanded constraint [Eq. (3)]. The degrees of freedom associated to the environment are governed by a Lindblad evolution. The corresponding system dynamic turns out to be defined by a set of arbitrary completely positive transformations the action of which is conditioned to the environment dynamics.

The bipartite system-environment state can always be written as a separable one [Eq. (16)], indicating the absence of quantum entanglement between both parts. Nevertheless, in contrast to a purely incoherent case, the environment may develop quantum coherent behaviors. Consistently, by subjecting the degrees of freedom of the environment to a continuous-in-time measurement process, a product state characterizes the bipartite stochastic dynamics [Eq. (26)], where a collisional dynamics defines the stochastic system evolution.

Similarly to incoherent environments, here the QRT is not valid in general. Nevertheless, *stationary* (system) operator correlations evolve in the same way as expectation values when the bipartite system-environment stationary state is an uncorrelated one [Eq. (31)]. Consequently, outside the stationary regime operator correlations can be used as a witness of memory effects. Nevertheless, given that the absence of stationary system-environment correlations may emerge in different models, a deeper characterization of non-Markovianity can be achieved through an operational approach.

The CPF correlation is an operational memory witness that relies on performing three consecutive measurement processes over the system of interest. This object was explicitly calculated in terms of a bipartite propagator [Eq. (36)] associated to the studied system-environment coupling. In a deterministic scheme, where the system state is not modified after the intermediate measurement, the CPF correlation [Eq. (39)] detects departures with respect to a Markovian regime (defined in terms of conditional probabilities). In a random scheme, where the intermediate postmeasurement state is selected in a random way, the CPF correlation vanishes when the environment is a casual bystander one [Eq. (45)]. This feature provides an explicit experimental procedure for detecting when the studied properties apply.

All previous conclusions were supported by the explicit study of single and multipartite qubits dynamics. The developed approach furnishes a solid basis for constructing alternative underlying mechanisms that lead to quantum memory effects. On the other hand, added to incoherent environments with a classical self-fluctuating dynamics, the studied dynamics define the most general situation where quantum memory effects are not endowed with a physical environment-to-system backflow of information. While unitary system-environment interactions were discarded, the present results motivate us to ask about different dynamical regimes where an effective non-Markovian casual bystander environmental action could be recovered.

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APPENDIX: AUXILIARY EXPRESSIONS AND CALCULUS DETAILS

Auxiliary expressions and calculation details are provided.

1. Coefficients of the coherence decay function

The coherence decay function $f(\tau|t) \equiv \mathrm{Tr}_e(\mathcal{G}^-_{t+\tau,t}[\rho^e_t])$, where $\mathcal{G}^-_{t+\tau,t}$ is defined by the evolution (50), can be written as in Eq. (53), where the time-dependent coefficients are

$$a_t = \left[1 - \frac{2\gamma\Omega\overline{\sigma_y(t)} - \overline{\sigma_z(t)}\gamma^2}{\gamma^2+2\Omega^2}\right], \tag{A1a}$$

$$b_t = \frac{2\gamma\Omega\overline{\sigma_y(t)} - \overline{\sigma_z(t)}\gamma^2}{\gamma^2+2\Omega^2}, \tag{A1b}$$

$$c_t = -\frac{6\gamma\Omega\overline{\sigma_y(t)} + \overline{\sigma_z(t)}(\gamma^2+8\Omega^2)}{4(\gamma^2+2\Omega^2)}. \tag{A1c}$$

The overbar symbol denotes the expectation values $\overline{\sigma_i(t)} = \mathrm{Tr}_e[\rho^e_t\sigma_i]$, where σ_i are Pauli operators in $\mathcal{H}_e$. The environment state follows from $\rho^e_t = \mathcal{G}^+_{t,0}[\rho^e_0]$, where $\mathcal{G}^+_{t,0}$ is also defined by the evolution (50). The stationary environment state $\rho^e_\infty = \lim_{t\to\infty}\rho^e_t$ reads

$$\rho^e_\infty = \frac{1}{\gamma^2+2\Omega^2}\begin{pmatrix} \Omega^2 & -i\gamma\Omega \\ +i\gamma\Omega & \gamma^2+\Omega^2 \end{pmatrix}. \tag{A2}$$

Assuming that the environment begins in this state, the previous expectations value $\overline{\sigma_i(\infty)} = \lim_{t\to\infty}\overline{\sigma_i(t)} = \mathrm{Tr}_e[\rho^e_\infty\sigma_i]$ follows straightforwardly:

$$\overline{\sigma_y(\infty)} = \frac{2\gamma\Omega}{\gamma^2+2\Omega^2}, \qquad \overline{\sigma_z(\infty)} = -\frac{\gamma^2}{\gamma^2+2\Omega^2}. \tag{A3}$$

2. Multipartite system-environment propagator

The bipartite propagator corresponding to the model (64) can be written as in Eq. (36). The solution for the set of environment superoperators $\{\mathcal{F}_\alpha(t)\}$ can be obtained by defining the vector

$$\mathcal{F} \equiv \{\mathcal{F}_{\mathbf{0}}(t), \mathcal{F}_{\underline{\mathbf{a}}}(t), \mathcal{F}_{\underline{\mathbf{b}}}(t), \mathcal{F}_{\underline{\mathbf{c}}}(t)\}. \tag{A4}$$

It is written as $\mathcal{F} = (1/4)H \cdot \mathcal{G}$, where H is a four-dimensional Hadamard matrix, $H = \{\{1, 1, 1, 1\}, \{1, 1, -1, -1\}, \{1, -1, 1, -1\}, \{1, -1-, 1, 1\}\}$. The components of the vector $\mathcal{G}$ are denoted as

$$\mathcal{G} \equiv \{\mathcal{G}_{t,0}^{++}, \mathcal{G}_{t,0}^{+-}, \mathcal{G}_{t,0}^{-+}, \mathcal{G}_{t,0}^{--}\}, \tag{A5}$$

which in turn can be written as $\mathcal{G} = H \cdot \mathcal{F}$. With these definitions, the underlying model (64) implies the time evolutions

$$\frac{d\mathcal{G}^{uv}}{dt} = -i\frac{\Omega}{2}[\sigma_x, \mathcal{G}^{uv}] - \frac{\gamma}{2}\{\sigma^\dagger\sigma, \mathcal{G}^{uv}\}_+ - \frac{\varphi}{2}\{\sigma\sigma^\dagger, \mathcal{G}^{uv}\}_+ + u(\gamma\sigma\mathcal{G}^{uv}\sigma^\dagger) + v(\varphi\sigma^\dagger\mathcal{G}^{uv}\sigma), \tag{A6}$$

with initial conditions $\mathcal{G}_{t_0,t_0}^{uv} = I_e$. For shortening the expressions, we denoted $\mathcal{G}_{t,t_0}^{uv} \leftrightarrow \mathcal{G}^{uv}$. The supra-indices are $u = \pm$ and $v = \pm$. The explicit analytical expressions for the four superoperators $\mathcal{G}^{uv}$ can be obtained by solving their evolution via Laplace transform techniques.

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