Fluctuation and dissipation in memoryless open quantum evolutions

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The von Neumann entropy rate for open quantum systems is, in general, written in terms of entropy production and entropy flow rates, encompassing the second law of thermodynamics. For memoryless open quantum systems, whose evolutions correspond to one-parameter quantum dynamical semigroups, we find a decomposition of the infinitesimal generator of the dynamics that unveils the fluctuation-dissipation structure of these evolutions. Thus, the fluctuation part of this decomposition allows to relate the von Neumann entropy rate with the divergence-based quantum Fisher information, at any time. Applied to continuous-in-time quantum Gaussian channels, our decomposition leads to the quantum analog of the generalized classical de Bruijn identity, thus expressing the quantum fluctuation-dissipation relation in that kind of channels. Finally, from this perspective, we analyze how stationarity arises.

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I. INTRODUCTION

The study on quantum communication channels that describe the input-output maps corresponding to quantum mechanical operations is at the core of quantum information theory (see, e.g., Ref. [1]). Formally, these maps are given, in the Schrödinger picture, by completely positive and trace-preserving operations acting on density operators, whereas, in the Heisenberg picture, they are given by identity-preserving operations acting on observables. A physically interesting property of these maps is that their composition is also a quantum channel. Accordingly, the set of quantum channels forms a semigroup. Moreover, quantum channels have an inverse only when they describe a unitary evolution. A unitary quantum channel requires for its implementation a quantum system completely isolated from its surrounding environment. In practice, the implementation of unitary channels is a great challenge. Consequently, the most common situation corresponds to nonunitary quantum channels that lead to a degradation of information during their use. To be able to assess the degradation effects, it is important to describe the evolutionary trajectory of the system that implements the channel. The description is possible through continuous maps over time. This is the point where quantum information theory meets open quantum systems theory.

We focus in this work on memoryless quantum channels that are continuous in time [2]. Therefore, the corresponding open quantum evolutions correspond to one-parameter semigroups. The density operator $\hat{\rho}_t$ of the evolved quantum system is the solution of a time-independent Markovian master equation [3]. These solutions are described by a linear map that forms a one-parameter quantum dynamical semigroup in the case of finite-dimensional Hilbert spaces [4] as well as in the case of Gaussian channels in continuous-variable systems [3,5,6], i.e., channels in infinite-dimensional separable Hilbert spaces.

In general, an open system dynamics, given by the interaction between the system and its environment, is modeled by a combination of both deterministic and random effects. One of the main approaches to the systematization of these random and deterministic effects lies in the so-called fluctuation-dissipation relations, both in the classical [7–11] and quantum domains [12–17] (see also Ref. [18]). Roughly speaking, these are relations that connect the deterministic characteristic of a system with its fluctuating aspects, both in the equilibrium and nonequilibrium regimes. Examples of these relations are the linear and nonlinear Markov fluctuation-dissipation relations [18]. In classical theory, these relations appear within Markov processes where the probability distribution of the variables of interest satisfies a Fokker-Planck equation. In particular, in the case of a linear Fokker-Planck equation the corresponding fluctuation-dissipation relations are just a set of identities that link the correlations between the variables of interest with the intensities of the fluctuation and dissipation effects in a stationary regime [18]. Also in open quantum systems the identification of diffusive (fluctuation) and dissipative terms in master equations was performed, as for example in Ref. [19].

On the other hand, the ubiquitous notion of entropy is transversal to physical and information theories, since it arises naturally as a measure of uncertainty, randomness, or lack of information about the state of a system [20]. For a quantum system described by a density operator $\hat{\rho}$, the von Neumann

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entropy is defined by $S[\hat{\rho}] = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$ where Tr denotes the trace of an operator (see, e.g., Ref. [21]). This quantity plays a key role in different contexts as in the study of entropy production in open quantum systems in general (see, e.g., Ref. [22]) and of quantum communication channels where different entropic quantities serve to characterize the information-processing performance of the channels (see, e.g., Refs. [23,24], among others). Here, we are interested in the rate of change of the von Neumann entropy $S[\hat{\rho}_t]$ in one-parameter semigroups. The analysis of the temporal behavior of this entropy is of crucial importance, for instance, in order to improve the communication rates of quantum channels, as well as for studying stationary situations.

In open quantum theory, the study of the rate of change of $S[\hat{\rho}_t]$ is usually oriented by a thermodynamic perspective [22,25]. In this framework the focus is on the decomposition of the rate of change into two parts: one corresponding to the so-called entropy production rate Π_t, and the other one to the entropy flux rate $\dot{\Phi}_t$. From this point of view, the second law of thermodynamics is nothing but the statement $\Pi_t \geqslant 0$. Considerable progress has been made in recent years from this perspective in particular in a general definition of Π_t for general quantum processes (see Ref. [22] and references therein). However, those approaches do not focus on the characterization of the fluctuation and dissipation aspects of the dynamics.

At this point a natural question arises: Is it possible to identify generically random and deterministic effects in the master equation for the density operator and for the rate of change of the von Neumann entropy in one-parameter semigroups? Here we provide a positive answer for all finite-dimensional one-parameter memoryless quantum channels and for all infinite-dimensional Gaussian one-parameter quantum channels. Both situations are described by Lindblad master equations whose solutions form quantum dynamical semigroups [3–6,26,27]. Our approach is inspired by a well-established result in classical information theory known as the de Bruijn identity [28,29], which quantifies the rate of change of the Shannon entropy of a classical random variable, the output of an additive Gaussian noise channel. More precisely, this identity quantifies how fast the channel becomes more random in terms of the nonparametric Fisher information. The fact that a de Bruijn identity can also be formulated for quantum systems was first posed in Ref. [30] for the quantum diffusion process, a useful result for formulating quantum entropy power inequalities [30–32]. However, the quantum diffusion process is a special subclass of the Gaussian one-parameter quantum channels that does not have dissipation effects. Therefore, our framework is also inspired by the recently established generalization of the de Bruijn identity for more general classical channels modeled by Langevin forces described by stochastic differential equations, called Fokker-Planck channels [33,34], capturing the trade-off between the diffusion and dissipation terms of the evolution.

First, we propose a decomposition of the nonunitary infinitesimal generator for any one-parameter quantum dynamical semigroup into two terms. The first one allows us to relate the time derivative of the von Neumann entropy with the divergence-based quantum Fisher information [35]. Accordingly, this term will be associated with the fluctuations due to the noise introduced by the open quantum dynamics. In addition, the second term will be connected with the dissipative contributions of the dynamics.

Afterwards, we focus on Gaussian channels, which are among the most important channels in information processing in both classical [28,36] and quantum [37–41] domains, as they provide a faithful model for attenuation and noise effects in most communication schemes, modeled by linear Lindblad master equations [3]. For these channels, we find the fully quantum counterpart of the generalization of the de Bruijn identity for Fokker-Planck channels [33,34]. In these cases, our proposal captures in a clear manner the diffusion and dissipation contributions to the rate of change of the von Neumann entropy given by the open quantum dynamics. Finally, we analyze stationary situations, associated with thermal states or not, within our framework.

The paper is organized as follows. In Sec. II A we review the classical de Bruijn identity for channels with additive Gaussian noise and its generalization for multidimensional classical channels corresponding to Ornstein-Uhlenbeck processes, i.e., modeled by Fokker-Planck equations with linear drift and constant diffusion. In Sec. II B we show that the diffusion term in the generalized classical de Bruijn identity can be written in terms of the Fisher information of the probability distribution under the action of Langevin forces. This provides an alternative interpretation of the diffusion term as a measure of the noise introduced by these forces. We also show that the second term in the generalized classical de Bruijn identity corresponds to the average flux of the dissipative forces, i.e., a measure of the change of the probability distribution in the directions opposite to these forces. In Sec. III we present our first main result, namely, a decomposition of the nonunitary infinitesimal generator for any quantum dynamical semigroup that splits the dynamics into two different parts: fluctuation and dissipation. Section IV A is devoted to the presentation of the notion of divergence-based quantum Fisher information and some of its properties. In Sec. IV B we present our second main contribution, namely, the application of the results of Sec. III in order to obtain a closed formula for the von Neumann entropy rate of change for quantum dynamical semigroups, where we discriminate the contributions of fluctuation and dissipation. In Sec. V A, we present the basic formalism used to describe one-parameter Gaussian channels. In Sec. V B, we apply our results of Sec. IV B to obtain the quantum counterpart of the generalized classical de Bruijn identity in the case of one-parameter Gaussian channels. We also show that the diffusion and dissipation terms of the identity obtained admit an interpretation completely analogous to that given in Sec. II B for the corresponding terms in the generalized classical de Bruijn identity. In Sec. V C we specialize the quantum de Bruijn identity for evolving Gaussian states and in Sec. V D we study the stationarity conditions in one-parameter Gaussian channels in light of our formalism. We conclude with Sec. VI where we summarize our findings. For ease of reading, some auxiliary calculations and technical proofs are presented in Appendixes A, B, and C.

II. RATE OF SHANNON ENTROPY FOR A LINEAR FOKKER-PLANCK EQUATION

A. Generalized de Bruijn identity

In classical information theory [28,36], the additive Gaussian noise is probably the most used noise model to describe many "natural" random processes viewed as a large sum of noise sources and due to the central limit theorem [36,42,43]. The Gaussian noise channel corresponds to $X_t = X_0 + \sqrt{t}\,Z$, where $Z \sim \mathcal{N}(0,1)$ is a normally distributed random variable with zero mean and unit variance, independent of the input random variable X_0 (that admits a finite variance). Here, $t > 0$ is a parameter (usually time) that controls the amount of randomness added to the system. The evolution with respect to t of the probability density function $p_t(x)$ of the output random variable X_t is ruled by the heat equation $\frac{dp_t(x)}{dt} = \frac{1}{2}\frac{\partial^2 p_t(x)}{\partial x^2}$. The de Bruijn identity plays a fundamental role in classical information theory [29,36,44], as it quantifies the rate of change of the Shannon entropy $h[p_t] = -\int_{\mathbb{R}} p_t(x)\ln p_t(x)dx$ at the output of the channel, in terms of the Fisher information $J[p_t] = \int_{\mathbb{R}} (\frac{\partial \ln p_t(x)}{\partial x})^2\, p_t(x)\,dx = -\int_{\mathbb{R}} \frac{\partial^2 \ln p_t(x)}{\partial x^2}\, p_t(x)\,dx$ [45]. More precisely,

$$\frac{dh[p_t]}{dt} = \frac{1}{2}J[p_t]. \tag{1}$$

Because $J > 0$, the fundamental interpretation of this equality is that X_t becomes more and more random as t (time) increases, with J quantifying how fast. Applications of de Bruijn identity include derivation of relevant information-theoretical inequalities [44,46–48], definition of an entropic temperature [49], and proof of monotonicity of some statistical complexity measures [50], among others [51].

The de Bruijn identity extends to the multivariate case [33,36], and its form has recently been obtained for slightly more general channels modeled by Langevin forces described by stochastic differential equations [33,34]. A similar expression for the rate of change of Shannon entropy was previously given in the thermodynamic context [52,53], but without pointing out its link with Fisher information. In particular, consider a multidimensional channel corresponding to an Ornstein-Uhlenbeck process. It is characterized by an N-dimensional random vector $\mathbf{X}_t$ that follows a stochastic differential equation with linear drift $\boldsymbol{\mu}(\mathbf{X}_t,t) = \mathbb{A}\mathbf{X}_t - \boldsymbol{\xi}$ and constant diffusion parameter $\mathbb{\Sigma}(\mathbf{X}_t,t) = \mathbb{\Sigma}$,

$$d\mathbf{X}_t = (\mathbb{A}\mathbf{X}_t - \boldsymbol{\xi})dt + \mathbb{\Sigma}\, d\mathbf{W}_t. \tag{2}$$

Here $\mathbb{A}$ and $\mathbb{\Sigma}$ are real constant matrices with dimensions $N \times N$ and $N \times M$ ($M \leqslant N$), respectively, where $\mathbb{\Sigma}$ is of full rank, $\boldsymbol{\xi}$ is an N-dimensional constant real vector, and $\mathbf{W}_t$ is an M-dimensional standard Wiener process [54–57]. For $\boldsymbol{\mu} = 0$ (null vector), the N-dimensional Gaussian additive channel is recovered. Note that the drift corresponds to the deterministic effects, whereas the diffusion term $\mathbb{\Sigma}\, d\mathbf{W}_t$ characterizes the random contributions in the dynamics.

The probability density function of the random variable $\mathbf{X}_t$ satisfies the Fokker-Planck equation [55–57]:

$$\frac{dp_t(\mathbf{x})}{dt} = -\frac{\partial}{\partial \mathbf{x}^\intercal}((\mathbb{A}\mathbf{x} - \boldsymbol{\xi})p_t(\mathbf{x})) + \frac{1}{2}\frac{\partial}{\partial \mathbf{x}^\intercal}\mathbb{D}\frac{\partial}{\partial \mathbf{x}}p_t(\mathbf{x}), \tag{3}$$

where $\frac{\partial}{\partial \mathbf{x}} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N})^\intercal$ is the gradient with respect to $\mathbf{x} = (x_1, \dots, x_N)^\intercal$ and $\mathbb{D} = \mathbb{\Sigma}\mathbb{\Sigma}^\intercal$ is the diffusion matrix, which is symmetric and positive semidefinite of rank M. For this channel, the generalized de Bruijn identity is [34]

$$\frac{dh[p_t]}{dt} = \frac{1}{2}\mathrm{tr}(\mathbb{D}\mathbb{J}[p_t]) + \mathrm{tr}(\mathbb{A}), \tag{4}$$

where $h[p_t] = -\int_{\mathbb{R}^N} p_t(\mathbf{x})\ln p_t(\mathbf{x})d^N\mathbf{x}$ ($d^N\mathbf{x} = dx_1 \cdots dx_N$) is the Shannon entropy, tr denotes trace of matrices, and the Fisher information matrix is defined as

$$\mathbb{J}[p_t] = \int_{\mathbb{R}^N} \frac{\partial \ln p_t(\mathbf{x})}{\partial \mathbf{x}}\frac{\partial \ln p_t(\mathbf{x})}{\partial \mathbf{x}^\intercal}\, p_t(\mathbf{x})\, d^N\mathbf{x}$$

which, under regularity conditions, can be written in the form [36]

$$\mathbb{J}[p_t] = -\int_{\mathbb{R}^N} p_t(\mathbf{x})\, \mathbb{H}[\ln p_t(\mathbf{x})]\, d^N\mathbf{x}, \tag{5}$$

with $\mathbb{H} = \frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}^\intercal}$ the Hessian matrix operator. The multivariate de Bruijn identity is recovered when $\mathbb{A} = 0$ (null matrix) and $\boldsymbol{\xi} = 0$ (null vector), whereas the original (scalar) de Bruijn identity is recovered when in addition $N = 1$, $\mathbb{D} = \mathbb{1}_1$. For a continuous-variable system with an underlying symplectic structure, $N = 2n$ with n the number of classical modes described by the canonically conjugate variables of position $x_i = q_i$ and momentum $x_{n+i} = p_i$, $i = 1, \dots, n$. Typical examples of this type are the description of Brownian motion and the electric field in a laser where q_i and p_i are the classical quadratures of the electric field [55,58].

In the next section we dissect the fluctuation-dissipation structure of the generalized de Bruijn identity.

B. Fluctuation-dissipation relation in the generalized de Bruijn identity

Let us rewrite the generalized de Bruijn identity (4) so as to highlight the contributions of the fluctuations and dissipation. More precisely, by expressing the Ornstein-Uhlenbeck process as $\frac{d\mathbf{X}_t}{dt} = \mathbf{F}_{dr} + \mathbf{F}_f$, where $\mathbf{F}_{dr} = \mathbb{A}\mathbf{X}_t - \boldsymbol{\xi} = \boldsymbol{\mu}(\mathbf{X}_t,t)$ are the drift forces and $\mathbf{F}_f = \mathbb{\Sigma}\frac{d\mathbf{W}_t}{dt}$ are the fluctuating or Langevin forces, the first term on the right-hand side (rhs) of Eq. (4) can be explicitly identified as induced by fluctuations due to the Langevin forces, and the second one by dissipation due to the drift forces.

To this end, let us first rewrite the diffusion matrix under the form

$$\mathbb{D} = \sum_{m=1}^{M} \mathbf{\Sigma}_m \mathbf{\Sigma}_m^\intercal, \tag{6}$$

where $\mathbf{\Sigma}_m$ are the column vectors of $\mathbb{\Sigma}$. From the linearity of the trace and the relation $\mathrm{tr}(\mathbf{\Sigma}_m\mathbf{\Sigma}_m^\intercal\mathbb{J}) = \mathbf{\Sigma}_m^\intercal\,\mathbb{J}\,\mathbf{\Sigma}_m$, the first term on the rhs of Eq. (4) can therefore be expressed as

$$\frac{1}{2}\mathrm{tr}(\mathbb{D}\mathbb{J}[p_t]) = \frac{1}{2}\sum_{m=1}^{M} J[p^{(\theta,\mathbf{\Sigma}_m)};\theta]\big|_{\theta=t}, \tag{7}$$

where $p^{(\theta,\mathbf{\Sigma}_m)} = p_t(\mathbf{x} - \delta\theta\,\mathbf{\Sigma}_m)$ is the translated probability distribution ($\delta\theta = \theta - t$), and indeed $\mathbf{\Sigma}_m^\intercal\,\mathbb{J}[p_t]\,\mathbf{\Sigma}_m = J[p^{(\theta,\mathbf{\Sigma}_m)};\theta]|_{\theta=t}$ is the Fisher information $J[p;\theta] = \int_{\mathbb{R}^N} p\,(\frac{\partial \ln p}{\partial \theta})^2 d^N\mathbf{x}$ for the translated distribution.

We remark that the rhs of Eq. (7) makes explicit the dependence on the Langevin forces, $\boldsymbol{\Sigma}_m$, and also on the rank M of the diffusion matrix $\mathbb{D}$. From the unicity of $\mathbb{D} = \mathbb{\Sigma}\mathbb{\Sigma}^\intercal$, $\mathbb{\Sigma}$ is unique up to an orthogonal transformation, i.e., $\mathbb{D} = \bar{\mathbb{\Sigma}}\bar{\mathbb{\Sigma}}^\intercal = \mathbb{\Sigma}\mathbb{\Sigma}^\intercal$ if and only if $\bar{\mathbb{\Sigma}} = \mathbb{\Sigma}\mathbb{Q}$ with $\mathbb{Q}$ an arbitrary $M \times M$ orthogonal matrix; such a transform is equivalent to use any other Langevin forces, $\bar{\boldsymbol{\Sigma}}_m$, which are the columns vectors of $\bar{\mathbb{\Sigma}}$. Furthermore, this also holds for any $M \times \bar{M}$ matrix $\mathbb{Q}$ of the Stiefel manifold ($\bar{M} \geqslant M$), i.e., such that $\mathbb{Q}\,\mathbb{Q}^\intercal$ is the $M \times M$ identity, so that the stochastic differential equation (2) with $\bar{\mathbb{\Sigma}}$ is the equivalent Langevin equation of that with the full rank $\mathbb{\Sigma}$. This shows that we can also express each particular matrix $\mathbb{D}$ with an expression like in Eq. (7) but with the $\bar{M}$ ($\bar{M} > N$) columns of a matrix $\bar{\mathbb{\Sigma}}$. However, the Langevin forces turn out to be not all independent. This kind of situation arises for the diffusion matrices $\mathbb{D}$ that come from linear Lindblad master equations [see Eq. (30)].

Moreover, from the expression $D_{KL}[p^{(\theta,\boldsymbol{\Sigma}_m)} \| p_t(\mathbf{x})] = \frac{1}{2} J[p^{(\theta,\boldsymbol{\Sigma}_m)};\theta]|_{\theta=t}\delta\theta^2 + o(\delta\theta^2)$, where D_{KL} is the Kullback-Leibler divergence, or relative entropy [36,59], the first term on the rhs of the classical de Bruijn identity (4) is essentially a measure of the noise induced by the Langevin forces $\boldsymbol{\Sigma}_m$ on the probability distribution $p_t(\mathbf{x})$.

Second, from the definition of $\mathbf{F}_{dr}(\mathbf{x}) = \mathbb{A}\mathbf{x} - \boldsymbol{\xi}$, its divergence is given by $\frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal} = \mathrm{tr}(\mathbb{A})$, which gives, together with $\langle \frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal} \rangle_{p_t} = \int_{\mathbb{R}^N} \frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal} \; p_t(\mathbf{x})\, d^N\mathbf{x}$,

$$\mathrm{tr}(\mathbb{A}) = \left\langle \frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal} \right\rangle_{p_t}. \tag{8}$$

From the fact that $p_t(\mathbf{x})\, \mathbf{F}_{dr}(\mathbf{x})$ vanishes in the boundary of $\mathbb{R}^N$ (X_t admits a mean), so that the integral of $\frac{\partial (p_t(\mathbf{x})\, \mathbf{F}_{dr}(\mathbf{x}))}{\partial \mathbf{x}^\intercal} = \frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal}\; p_t(\mathbf{x}) + \frac{\partial p_t(\mathbf{x})}{\partial \mathbf{x}^\intercal}\, \mathbf{F}_{dr}(\mathbf{x})$ vanishes, one equivalently has

$$\mathrm{tr}(\mathbb{A}) = -\int_{\mathbb{R}^N} \frac{\partial p_t(\mathbf{x})}{\partial \mathbf{x}^\intercal}\, \mathbf{F}_{dr}(\mathbf{x})\, d^N\mathbf{x}. \tag{9}$$

Note that this result can be recovered directly from the expression of $\mathbf{F}_{dr}(\mathbf{x})$, together with the identity

$$\int_{\mathbb{R}^N} \mathbf{x}\frac{\partial p_t(\mathbf{x})}{\partial \mathbf{x}^\intercal}\, d^N\mathbf{x} = -\mathbb{1}. \tag{10}$$

This is because $\mathbf{x}\, p_t(\mathbf{x})$ vanishes in the boundary of $\mathbb{R}^N$ (X_t admits a mean), so that the integral of $\frac{\partial (\mathbf{x}\, p_t(\mathbf{x}))}{\partial \mathbf{x}^\intercal} = \mathbb{1}\, p_t(\mathbf{x}) + \mathbf{x}\, \frac{\partial p_t(\mathbf{x})}{\partial \mathbf{x}^\intercal}$ vanishes. Therefore, the quantity $\mathrm{tr}(\mathbb{A})$ can be interpreted as minus the total amount of change of the probability distribution $p_t(\mathbf{x})$ in the direction given by the linear drift force $\mathbf{F}_{dr}(\mathbf{x})$, or as the average of the drift force flux [52]. In addition, by expressing $\mathbb{A} = \mathbb{A}_S + \mathbb{A}_{AS}$ with $\mathbb{A}_S = (\mathbb{A} + \mathbb{A}^\intercal)/2$ and $\mathbb{A}_{AS} = (\mathbb{A} - \mathbb{A}^\intercal)/2$, we have $\mathrm{tr}(\mathbb{A}) = \mathrm{tr}(\mathbb{A}_S)$. Accordingly, we rewrite the drift force as $\mathbf{F}_{dr}(\mathbf{x}) = \mathbf{F}_S(\mathbf{x}) + \mathbf{F}_{AS}(\mathbf{x})$ with $\mathbf{F}_S(\mathbf{x}) = \mathbb{A}_S\, \mathbf{x}$ and $\mathbf{F}_{AS}(\mathbf{x}) = \mathbb{A}_{AS}\mathbf{x} - \boldsymbol{\xi}$, in order to highlight that only the symmetric part contributes to the drift force flux. Moreover, whenever $\mathrm{tr}(\mathbb{A}) < 0$, we associate $\mathbf{F}_S$ with a dissipative force and $\mathbf{F}_{AS}$ with a nondissipative one. These names are justified because in the context where the Fokker-Planck equation (3) describes a mechanical system, $\mathbf{F}_S$ represents the force in the phase space of the system that does not conserve the energy and $\mathbf{F}_{AS}$ corresponds to the Hamiltonian phase-space force that conserves the energy.

Finally, we collect the expressions given in Eqs. (7) and (8) to obtain

$$\frac{dh[p_t]}{dt} = \frac{1}{2}\sum_{m=1}^{M} J\big[p^{(\theta,\boldsymbol{\Sigma}_m)};\theta\big]\big|_{\theta=t} + \left\langle \frac{\partial \mathbf{F}_{dr}(\mathbf{x})}{\partial \mathbf{x}^\intercal} \right\rangle_{p_t}. \tag{11}$$

Let us observe that the first term on the rhs of the de Bruijn identity (4) [or equivalently Eq. (11)] is strictly positive, because each Fisher information is positive, i.e., $J[p^{(\theta,\boldsymbol{\Sigma}_m)};\theta]|_{\theta=t} > 0$. On the other hand, the second term has no definite sign due to its dependence on the sum of the eigenvalues of $\mathbb{A}_S$, which can be negative, positive, or zero. Therefore, a necessary condition for the existence of a stationary solution of the Fokker-Planck equation (3) is the entropic balance between these quantities. This can happen only if Eq. (8) is negative, which leads to a condition on the eigenvalues of $\mathbb{A}_S$. Recall that the existence of a stationary solution also requires the matrix $\mathbb{A}$ to be asymptotically stable; that is, all its eigenvalues must strictly have negative real part. In principle, this condition does not have a direct connection with respect to the one on the eigenvalues of $\mathbb{A}_S$.

In the sequel, we will obtain an analogous generalized de Bruijn identity for the rate of change of the von Neumann entropy in Gaussian channels whose Wigner function satisfies the Fokker-Planck equation (3). Before that, in the next section we will show that such a fluctuation-dissipation relation in the quantum case has its origin in a particular decomposition of the nonunitary infinitesimal generator of the evolution in quantum dynamical semigroups.

III. FLUCTUATION AND DISSIPATION IN QUANTUM DYNAMICAL SEMIGROUPS

Here we consider memoryless quantum channels that are continuous in time, which are described by one-parameter semigroups. The density operator $\hat{\rho}_t$ of the evolved quantum system is a solution of a time-independent Markovian master equation [60]. The general form of these equations, whose solutions satisfy the completely positive condition, is known at least in finite-dimensional Hilbert spaces. In the Schrödinger picture, this corresponds to a Lindblad master equation (LME) of the form [4,61]

$$\frac{d\hat{\rho}_t}{dt} = \mathcal{L}[\hat{\rho}_t] = \mathcal{L}_{\mathrm{U}}[\hat{\rho}_t] + \mathcal{L}_{\mathrm{NU}}[\hat{\rho}_t], \tag{12}$$

where

$$\mathcal{L}_{\mathrm{U}}[\hat{\rho}_t] = \frac{1}{\imath\hbar}[\hat{H}, \hat{\rho}_t] \tag{13}$$

and

$$\mathcal{L}_{\mathrm{NU}}[\hat{\rho}_t] = \frac{1}{2\hbar}\sum_{k=1}^{K}(2\hat{L}_k\hat{\rho}_t\hat{L}_k^\dagger - \hat{L}_k^\dagger\hat{L}_k\hat{\rho}_t - \hat{\rho}_t\hat{L}_k^\dagger\hat{L}_k) \tag{14}$$

are the infinitesimal generators of the unitary and nonunitary evolution, respectively, with $\hat{H}$ the Hamiltonian of the system and $\{\hat{L}_k\}_{k=1}^K$ the Lindblad operators. When $\mathcal{L}$ is time independent, the formal solution of Eq. (12) is $\hat{\rho}_t = e^{t\mathcal{L}}\hat{\rho}_0$, where $\{\Lambda_t = e^{t\mathcal{L}}\}_{t\geqslant 0}$ is a quantum dynamical semigroup (QDS). At least for finite dimension, any QDS is precisely described by a LME [4]. Here, $\mathcal{L}$ could be bounded, or unbounded as happens for QDSs in continuous-in-time Gaussian channels [3,5,6].

In the Heisenberg picture, observables $\hat{O}_t$ evolve according to $\frac{d\hat{O}_t}{dt} = \bar{\mathcal{L}}_{\mathrm{U}}[\hat{O}_t] + \bar{\mathcal{L}}_{\mathrm{NU}}[\hat{O}_t]$, where $\bar{\mathcal{L}}$ denotes the adjoint superoperator of $\mathcal{L}$ [6,62].

QDSs are also called quantum Markov semigroups [63]. These denominations emphasize the Markov semigroup property: $\Lambda_{t+s} = \Lambda_t \Lambda_s$ [6], which is the quantum version of an analogous semigroup property in time-homogeneous classical Markov processes [64]. Accordingly, in the same way as classical Markov processes describe classical memoryless channels, QDSs are suitable for describing memoryless quantum channels.

We propose a general decomposition of $\mathcal{L}_{\mathrm{NU}}$, which, in the context of Gaussian channels, encompasses the notions of diffusion and dissipation, as follows:

$$\mathcal{L}_{\mathrm{NU}} = \mathcal{L}_1 + \mathcal{L}_{2+3}, \qquad \mathcal{L}_{2+3} = \mathcal{L}_2 + \mathcal{L}_3, \tag{15}$$

with $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ real superoperators [65] given by

$$\mathcal{L}_1[\hat{O}] = -\frac{1}{2\hbar}\sum_{k=1}^{K}([\hat{A}_k,[\hat{A}_k,\hat{O}]] + [\hat{B}_k,[\hat{B}_k,\hat{O}]]), \tag{16a}$$

$$\mathcal{L}_2[\hat{O}] = \frac{1}{2\hbar}\sum_{k=1}^{K}\frac{1}{2}\{[\hat{L}_k,\hat{L}_k^\dagger],\hat{O}\}, \tag{16b}$$

$$\mathcal{L}_3[\hat{O}] = \frac{1}{2\hbar}\sum_{k=1}^{K}(\hat{L}_k\hat{O}\hat{L}_k^\dagger - \hat{L}_k^\dagger\hat{O}\hat{L}_k). \tag{16c}$$

This follows from the Cartesian decomposition of the Lindblad operators $\hat{L}_k = \hat{A}_k + \imath\hat{B}_k$ via the Hermitian operators $\hat{A}_k$ and $\hat{B}_k$ defined by $\hat{A}_k = \frac{1}{2}(\hat{L}_k + \hat{L}_k^\dagger)$ and $\hat{B}_k = \frac{1}{2\imath}(\hat{L}_k - \hat{L}_k^\dagger)$.

Notice that $\mathcal{L}_1$ is self-adjoint, i.e., $\bar{\mathcal{L}}_1[\hat{O}] = \mathcal{L}_1[\hat{O}]$, as is $\mathcal{L}_2$, while $\mathcal{L}_3$ is antisymmetric, i.e., $\bar{\mathcal{L}}_3[\hat{O}] = -\mathcal{L}_3[\hat{O}]$. For the unit operator, $\bar{\mathcal{L}}_1[\hat{1}] = \mathcal{L}_1[\hat{1}] = 0$, $\bar{\mathcal{L}}_2[\hat{1}] = \mathcal{L}_2[\hat{1}] = \frac{1}{2\hbar}\sum_{k=1}^{K}[\hat{L}_k,\hat{L}_k^\dagger]$, and $\bar{\mathcal{L}}_3[\hat{1}] = -\mathcal{L}_3[\hat{1}] = -\mathcal{L}_2[\hat{1}]$. Consequently, $\mathcal{L}_1$ and $\mathcal{L}_{2+3}$ are infinitesimal generators of QDSs in themselves. Moreover, $\mathcal{L}_1$ is the generator of a unital QDS [66].

We recall that the infinitesimal generator $\mathcal{L}$ does not uniquely determine the form of the Hamiltonian $\hat{H}$ and Lindblad operators $\hat{L}_k$ (see, e.g., Sec. 3.2.2. of [67]). On the one hand, LME (12) is invariant under the unitary transformation of the Lindblad operators:

$$\hat{L}_k \to \hat{L}'_k = \sum_{j=1}^{K} W_{k,j}\hat{L}_j, \tag{17}$$

where $W_{k,j}$ are the entries of an arbitrary unitary matrix $\mathbb{W}$. We obtain that $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ remain invariant under this transformation (see Appendix A).

On the other hand, LME (12) is also invariant under the inhomogeneous transformations

$$\hat{L}_k \to \hat{L}_k + \alpha_k\hat{1}, \tag{18a}$$

$$\hat{H} \to \hat{H} + \hat{H}'\left(\hat{H}' = \frac{1}{2\imath\hbar}\sum_{k=1}^{K}(\alpha_k^*\hat{L}_k - \alpha_k\hat{L}_k^\dagger)\right), \tag{18b}$$

for any set of complex numbers $\{\alpha_k\}_{k=1}^{K}$. Under these transformations, it is straightforward to show that the superoperators transform as follows:

$$\mathcal{L}_1 \to \mathcal{L}_1, \tag{19a}$$

$$\mathcal{L}_2 \to \mathcal{L}_2, \tag{19b}$$

$$\mathcal{L}_3 \to \mathcal{L}_3 - \frac{1}{\imath\hbar}[\hat{H}',\cdot\,]. \tag{19c}$$

In Sec. IV B we prove that, whereas $\mathcal{L}_1$ is the infinitesimal generator of fluctuations in one-parameter quantum dynamical semigroups, $\mathcal{L}_{2+3} = \mathcal{L}_2 + \mathcal{L}_3$ is the infinitesimal generator of dissipation. The justification of this characterization lies in the contributions that these generators give to the rate of change of the von Neumann entropy of the evolved quantum states of the system.

IV. RATE OF VON NEUMANN ENTROPY FOR QUANTUM DYNAMICAL SEMIGROUPS

A. Divergence-based quantum Fisher information

As seen in Sec. II, the Fisher information is a key measure to quantify the rate of change of the Shannon entropy in the additive Gaussian channel or in the channel described by the Ornstein-Uhlenbeck process, via the classical de Bruijn identities (1) and (4), respectively [29,34]. The first attempt, to our knowledge, to find a quantum counterpart of Eq. (1) is given by König and Smith in Ref. [30], where the divergence-based quantum Fisher information (DQFI) is related to the rate of change of the von Neumann entropy for an evolution governed by the quantum diffusion semigroup. The DQFI is one of the forms of the Fisher information defined in the quantum domain, and is precisely that appearing in the rate of von Neumann entropy in QDSs, in general.

The DQFI is defined as the second derivative of the relative entropy [35], i.e.,

$$J_q[\hat{\rho}_\theta;\theta]|_{\theta=\theta_0} = \left.\frac{d^2 S[\hat{\rho}_\theta\|\hat{\rho}_{\theta_0}]}{d\theta^2}\right|_{\theta=\theta_0}, \tag{20}$$

with $S[\hat{\rho}_\theta\|\hat{\rho}_{\theta_0}] = \mathrm{Tr}(\hat{\rho}_\theta(\ln\hat{\rho}_\theta - \ln\hat{\rho}_{\theta_0}))$. As noticed in [35], this DQFI is greater than the quantum Fisher information based on the symmetric logarithmic derivative [68–70], whereas the respective classical versions coincide [36].

In the particular case of the family of density operators $\hat{\rho}_{\theta,\hat{C}_{\delta\theta}} = \hat{U}_{\theta-\theta_0}\,\hat{\rho}_{\theta_0}\,\hat{U}_{\theta-\theta_0}^\dagger$, generated by the unitary $\hat{U}_{\theta-\theta_0}$ with a generator $\hat{C}_{\delta\theta} = \imath\frac{d\hat{U}_{\theta-\theta_0}}{d\theta}\,\hat{U}_{\theta-\theta_0}$ (where $\delta\theta = \theta - \theta_0$), we have (see Appendix B)

$$J_q[\hat{\rho}_{\theta,\hat{C}_{\delta\theta}};\theta]\big|_{\theta=\theta_0} = \mathrm{Tr}(\hat{\rho}_{\theta_0}[\hat{C}_0,[\hat{C}_0,\ln\hat{\rho}_{\theta_0}]]), \tag{21}$$

where $\hat{C}_0 = \hat{C}_{\delta\theta}|_{\theta=\theta_0}$. It can be seen that when $\hat{C}_{\delta\theta}$ is independent of θ, expression (21) reduces to the one given in [30]. In what follows we consider the family $\hat{\rho}_{\theta,\hat{C}}$ generated from $\hat{\rho}_{\theta_0} = \hat{\rho}_t$ by unitaries of the form $\hat{U}_{\delta\theta} = e^{-\frac{\imath}{\hbar}\delta\theta\hat{C}}$ (with $\delta\theta = \theta - t$).

In the next section we show how the DQFI appears in the fluctuation-dissipation structure of the rate of change of the von Neumann entropy in QDSs.

B. The rate of change of von Neumann entropy

By exploiting the decomposition (15) with $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ given in Eqs. (16), we obtain that the rate of change of the von Neumann entropy, $S[\hat{\rho}_t] = -\mathrm{Tr}(\hat{\rho}_t \ln \hat{\rho}_t)$, can be written in terms of the divergence-based quantum Fisher information (21). Note first that $\frac{dS[\hat{\rho}_t]}{dt} = -\mathrm{Tr}((\frac{d\hat{\rho}_t}{dt}) \ln \hat{\rho}_t) = -\mathrm{Tr}(\mathcal{L}[\hat{\rho}_t] \ln \hat{\rho}_t) = -\mathrm{Tr}(\hat{\rho}_t \bar{\mathcal{L}}[\ln \hat{\rho}_t])$. Consequently, from the expression of $\mathcal{L} = \mathcal{L}_{\mathrm{U}} + \mathcal{L}_{\mathrm{NU}}$ given by Eqs. (13) and (15) and (16), together with the self-adjoint character of $\mathcal{L}_1$ and $\mathcal{L}_2$, and the antisymmetry of $\mathcal{L}_3$, we obtain

$$\frac{dS[\hat{\rho}_t]}{dt} = \Delta_t - \Psi_t \tag{22}$$

with $\Delta_t = -\mathrm{Tr}(\hat{\rho}_t \mathcal{L}_1[\ln \hat{\rho}_t])$ and $\Psi_t = \mathrm{Tr}(\hat{\rho}_t (\mathcal{L}_2 - \mathcal{L}_3)[\ln \hat{\rho}_t])$. Notice that we used the fact that

$$\mathrm{Tr}(\hat{\rho}_t\ [\hat{H}, \ln \hat{\rho}_t]) = 0 \tag{23}$$

(this can be established by simple algebra) so that the contribution to Eq. (22) from the unitary evolution, $\mathcal{L}_{\mathrm{U}}$, vanishes.

Now, from the expressions of $\mathcal{L}_1$ [Eq. (16a)] and of the DQFI in the form (21), we can express the first contribution on the rhs of Eq. (22) in terms of the DQFI as follows:

$$\Delta_t = \frac{1}{2} \sum_{k=1}^{K} \left(J_q\big[\hat{\rho}_{\theta,\sqrt{\hbar}\hat{A}_k}; \theta\big] + J_q\big[\hat{\rho}_{\theta,\sqrt{\hbar}\hat{B}_k}; \theta\big] \right)\Big|_{\theta=t}, \tag{24}$$

where $\hat{\rho}_{\theta,\hat{C}}$ are generated by the unitaries $\hat{U}_{\delta\theta} = e^{-\frac{i}{\hbar}\delta\theta\,\hat{C}}$ ($\delta\theta = \theta - t$), with $\hat{C} = \sqrt{\hbar}\hat{A}_k$ or $\sqrt{\hbar}\hat{B}_k$ being the Hermitian generators of the unitaries. From $S[\hat{\rho}_{\theta,\hat{C}} \| \hat{\rho}_t] = \frac{1}{2} J_q[\hat{\rho}_{\theta,\hat{C}}; \theta]|_{\theta=t} \delta\theta^2 + o(\delta\theta^2)$ (see Appendix B or Refs. [30,35]), we conclude that Δ_t is essentially a measure of the noise induced by unitaries $\hat{U}_{\delta\theta}$, with generators $\sqrt{\hbar}\hat{A}_k$ and $\sqrt{\hbar}\hat{B}_k$, on the state $\hat{\rho}_t$. Accordingly, Δ_t plays a role analogous to that given by the first term, Eq. (7), on the rhs of the classical de Bruijn identity.

In addition, let us emphasize that both quantities Δ_t and Ψ_t are invariant under the transformations given by Eqs. (17) and (18). This is due to the invariance of the superoperators, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, under the transformation in Eq. (17) and the effects (19). These effects, together with Eq. (23), justify that the energy-conserving contributions coming from $\hat{H}'$ induce no contribution in Ψ_t. As a consequence, Ψ_t characterizes the contributions of dissipative forces to the rate of change of von Neumann entropy. Accordingly, decomposition (22) reflect a fluctuation-dissipation relation in the rate of change of the von Neumann entropy for QDSs, analogous to the one given by Eq. (11) for the Shannon entropy rate in classical channels corresponding to Ornstein-Uhlenbeck processes.

We highlight that the fluctuation-dissipation relation (22) is a direct consequence of the decomposition (15) for the infinitesimal generators in the LME. In this respect, we can say that the decomposition (15) is itself a fluctuation-dissipation relation for QDSs in its own, where the infinitesimal generator $\mathcal{L}_1$ is associated with the fluctuation part of the evolution and $\mathcal{L}_{2+3}$ with the dissipation part. This decomposition determines the fluctuation-dissipation structure not only of the rate of change of the von Neumann entropy in QDSs, but also of the rate of change of the mean value of any time-independent observable, $\hat{O}$, of the quantum system being described, i.e., $\frac{d\langle \hat{O} \rangle_{\hat{\rho}_t}}{dt} = \mathrm{Tr}(\hat{\rho}_t \mathcal{L}_1[\hat{O}]) + \mathrm{Tr}(\hat{\rho}_t (\mathcal{L}_2 - \mathcal{L}_3)[\hat{O}])$, with $\langle \hat{O} \rangle_{\hat{\rho}_t} = \mathrm{Tr}(\hat{\rho}_t\ \hat{O})$.

Notice that our way to express the rate of change of the von Neumann entropy, Eq. (22), differs from the usual decomposition

$$\frac{dS[\hat{\rho}_t]}{dt} = \Pi_t - \dot{\Phi}_t \tag{25}$$

given in terms of the rate $\Pi_t = \frac{d\Sigma_t}{dt} \geqslant 0$ of the entropy production Σ_t, and the rate $\dot{\Phi}_t = \frac{d\Phi_t}{dt}$ of the entropy flux Φ_t [22]. This decomposition is one way to write the second law of thermodynamics. Although there is a general proposal for the form of the entropy production Σ_t and the entropy flux Φ_t, for a general system-environment evolution (see Ref. [22] and references herein), the entropy production, in general, depends on the evolved reduced state of the environment, which is not available in open quantum systems. However, for QDSs this problem was overcome long time ago by Spohn [25], but only in the cases when these semigroups have an invariant state $\hat{\rho}^s$, i.e., $\hat{\rho}^s = e^{t\mathcal{L}} \hat{\rho}^s$, $\forall t \geqslant 0$. In this situation the entropy production rate is $\Pi_t = -\frac{dS[\hat{\rho}_t \| \hat{\rho}^s]}{dt} \geqslant 0$.

The decomposition in Eq. (25) is useful to study stationary states, not necessarily thermal equilibrium ones, arising when $\Pi_t = \dot{\Phi}_t$. Furthermore, these states, $\hat{\rho}^s$, are thermal equilibrium states if and only if $\Pi_t = \dot{\Phi}_t = 0$. In the case of QDSs, Π_t is also a monotonic convex function of time as a consequence of $-S[\hat{\rho}_t \| \hat{\rho}^s]$ being an increasing function of time. This determines also how the approach to the stationary state is.

Our decomposition (22), that it is still valid for QDSs without a stationary state, is also useful to study stationarity in this type of open systems but on a different perspective from that given by Eq. (25). In our framework, stationarity, associated with thermal states or not, arises when Ψ_t balances Δ_t, since each DQFI in Eq. (24) is always positive. As we have already established, our decomposition (22) is a fluctuation-dissipation relation; therefore, the balance between Δ_t and Ψ_t can be interpreted as a fluctuation-dissipation equilibrium. We will confirm this point of view in Sec. V D for quantum Gaussian channels that admit a QDS description [3] that we call Gaussian dynamical semigroups (GDSs). This is the quantum counterpart of the fluctuation-dissipation equilibrium balance we found in the classical de Bruijn equation in Sec. II B.

V. RATE OF VON NEUMANN ENTROPY FOR GAUSSIAN DYNAMICAL SEMIGROUPS

A. Gaussian dynamical semigroups

In what follows, we focus on GDSs, which are the most general form of one-parameter Gaussian channels [3]. The attenuator, amplifier, and additive Gaussian noise channels are relevant examples of GDSs [3,71,72]. GDSs are also useful to describe damped collective modes in deep inelastic collisions [73]. Also, GDSs appear in all processes which can formally be described as decomposition and production of noninteracting particles or quasiparticles which can be treated at least approximately as bosons [60]. In this context GDSs are known as quasifree completely positive semigroups.

The kinematics of a quantum Gaussian channel of n bosonic modes is described by a $(2n)$-dimensional vector of canonically conjugate operators, $\hat{\mathbf{x}} = (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^\intercal$, such that $[\hat{x}_j, \hat{x}_k] = \imath\hbar \mathsf{J}_{jk}\hat{1}$, with $\mathsf{J} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$ the $2n \times 2n$ real symplectic matrix where $\mathbb{1}_n$ is the $n \times n$ identity matrix, with $\mathsf{J}^{-1} = -\mathsf{J} = \mathsf{J}^\intercal$. The dynamics of a GDS is given by LME (12) with a Hamiltonian up to quadratic order in $\hat{\mathbf{x}}$ and linear Lindblad operators,

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^\intercal \mathbb{B}\hat{\mathbf{x}} + \hat{\mathbf{x}}^\intercal \mathsf{J}\boldsymbol{\xi} \quad \text{and} \quad \hat{L}_k = \mathbf{l}_k^\intercal \mathsf{J}\hat{\mathbf{x}}, \tag{26}$$

respectively, where $\mathbb{B} \geqslant 0$ (Hessian matrix), $\boldsymbol{\xi}$ is an arbitrary $(2n)$-dimensional real vector, and the $\mathbf{l}_k$s are $(2n)$-dimensional complex vectors. Usually master equations of this type are called linear LMEs [74].

In the Weyl-Wigner representation, the symbol of $\hat{\rho}_t$ is the Wigner function $W(\mathbf{x}, t)$, where $\mathbf{x} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^\intercal$ are phase-space coordinates [75]. The time evolution of W is given by a Fokker-Planck equation of the Ornstein-Uhlenbeck form (3), with drift $\mathbb{A}$ and diffusion $\mathbb{D}$ matrices given by

$$\mathbb{A} = \mathsf{J}\mathbb{B} - \mathbb{C}\mathsf{J}, \qquad \mathbb{D} = \hbar \mathrm{Re}(\mathbb{\Gamma}), \tag{27}$$

$\mathbb{\Gamma} = \sum_{k=1}^{K} \mathbf{l}_k \mathbf{l}_k^\dagger$ is the matrix that establishes the connection with the Lindblad operators given Eq. (26), and $\mathbb{C} = \mathrm{Im}(\mathbb{\Gamma})$ is the dissipation matrix. By definition, $\mathbb{\Gamma} \geqslant 0$ so that

$$\hbar\mathbb{\Gamma} = \mathbb{D} + \imath\hbar\mathbb{C} \geqslant 0, \tag{28}$$

which can be interpreted as a generalized fluctuation-dissipation relation [74]. This matrix inequality implies that $\mathbb{D} \geqslant 0$, because $\mathbb{D}$ and $\mathbb{C}$ are symmetric and antisymmetric real matrices, respectively.

A few observations and remarks are in order here. First, defining the real vectors

$$\boldsymbol{\Sigma}_k = \sqrt{\hbar}\mathrm{Re}(\mathbf{l}_k), \qquad \bar{\boldsymbol{\Sigma}}_k = \sqrt{\hbar}\mathrm{Im}(\mathbf{l}_k), \tag{29}$$

the diffusion matrix can be expressed as

$$\mathbb{D} = \sum_{k=1}^{K} \left(\boldsymbol{\Sigma}_k \boldsymbol{\Sigma}_k^\intercal + \bar{\boldsymbol{\Sigma}}_k \bar{\boldsymbol{\Sigma}}_k^\intercal\right). \tag{30}$$

If we compare Eqs. (6) and (30), we immediately recognize in the set $\{\boldsymbol{\Sigma}_k\,,\ \bar{\boldsymbol{\Sigma}}_k\}_{k=1,\ldots,K}$ Langevin forces. However, all these forces are not necessarily linearly independent like those in Eq. (6). Because the complex vectors $\mathbf{l}_k$ are usually linearly independent (with $K \leqslant 2n$), K is the rank of the matrix $\mathbb{\Gamma}$. But the rank of the matrix $\mathbb{D}$ expressed in Eq. (30) could range from $K \leqslant \mathrm{rank}(\mathbb{D}) = M \leqslant \min\{2n, 2K\}$ being K for example when all pairs $\boldsymbol{\Sigma}_k, \bar{\boldsymbol{\Sigma}}_k$ are composed with vectors proportional to each other, and being $M = 2K \leqslant 2n$ for example when all these vectors are linearly independent. Conversely, the rank of the dissipation matrix $\mathbb{C} = \mathrm{Im}(\mathbb{\Gamma})$ could range from $0 \leqslant \mathrm{rank}(\mathbb{C}) \leqslant \min\{2n, 2K\}$. Therefore, it is not possible in GDSs to have a dynamics without diffusion while it is possible to have a dynamics without dissipation as it is the case of a quantum diffusion process. The latter is a particular case of a GDS where $\mathbb{D} = \frac{1}{2}\mathbb{1}$ and $\mathbb{C} = 0$, which is precisely the GDS considered in the context of the quantum de Bruijn identity in Ref. [30].

Recall that in GDSs the Fokker-Plank equation, Eq. (3), propagates the Wigner function $W(\mathbf{x}, t)$, that is a quasiprobability distribution that describes completely the quantum state of the system. However, in the classical mechanics context the same Fokker-Plank equation (3), with the drift $\mathbb{A}$ and diffusion $\mathbb{D}$ matrices given in Eq. (27), is also used, for example, to describe the Brownian motion in a harmonic potential. In this case the drift force in phase space can be split into $\mathbf{F}_{dr}(\mathbf{x}) = \mathbf{F}_S(\mathbf{x}) + \mathbf{F}_{AS}(\mathbf{x})$, where $\mathbf{F}_{AS}(\mathbf{x}) = \mathsf{J}\mathbb{B}\mathbf{x} - \boldsymbol{\xi}$ and $\mathbf{F}_S(\mathbf{x}) = -\mathbb{C}\mathsf{J}\mathbf{x}$ are the conservative and dissipative forces, respectively. Note that, in this case, as commented below Eq. (8), $\frac{\partial \mathbf{F}_{AS}(\mathbf{x})}{\partial \mathbf{x}^\intercal} = \mathrm{tr}(\mathbb{A}_{AS}) = \mathrm{tr}(\mathbb{J}\mathbb{B}) = 0$ because J is antisymmetric and $\mathbb{B}$ is symmetric, and $\frac{\partial \mathbf{F}_S(\mathbf{x})}{\partial \mathbf{x}^\intercal} = \mathrm{tr}(\mathbb{A}_S) = -\mathrm{tr}(\mathbb{C}\mathbb{J})$.

Finally, using successively (i) the expression for the drift matrix $\mathbb{A} = \mathsf{J}\mathbb{B} - \mathbb{C}\mathsf{J}$ in Eq. (27) so that $-\frac{\partial(\mathbb{A}\mathbf{x} - \boldsymbol{\xi})}{\partial \mathbf{x}^\intercal} = -\mathrm{tr}(\mathbb{A}) = \mathrm{tr}(\mathbb{C}\mathsf{J})$, (ii) the quadratic classical Hamiltonian $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal \mathbb{B}\mathbf{x} + \mathbf{x}^\intercal \mathsf{J}\boldsymbol{\xi}$—the Weyl symbol of the quantum Hamiltonian Eq. (26)—together with $\mathsf{J}^{-1} = -\mathsf{J} = \mathsf{J}^\intercal$ so that $(\mathbb{A}\mathbf{x} - \boldsymbol{\xi})^\intercal = -\frac{\partial H(\mathbf{x})}{\partial \mathbf{x}^\intercal}\mathsf{J} - (\mathbb{C}\mathsf{J}\mathbf{x})^\intercal$ and (judiciously) $a^\intercal b = b^\intercal a = \mathrm{tr}(b a^\intercal)$ for any vector, the Fokker-Planck equation (3) for the Wigner function $W(\mathbf{x}, t)$ of the system can be rewritten under the form

$$\begin{aligned} \frac{dW(\mathbf{x}, t)}{dt} &= [H(\mathbf{x}), W(\mathbf{x}, t)]_{cl} + \frac{1}{2}\mathrm{tr}(\mathbb{D}\mathbb{H}[W(\mathbf{x}, t)]) \\ &\quad + \mathrm{tr}(\mathbb{C}\mathbb{J})W(\mathbf{x}, t) + \mathrm{tr}\left(\mathbb{C}\mathbb{J}\, \mathbf{x}\frac{\partial W(\mathbf{x}, t)}{\partial \mathbf{x}^\intercal}\right), \end{aligned} \tag{31}$$

where the first term is the Poisson bracket, $\mathrm{tr}(\mathsf{J}\frac{\partial W(\mathbf{x},t)}{\partial \mathbf{x}}\frac{\partial H}{\partial \mathbf{x}^T})$, between the Wigner function and the quadratic classical Hamiltonian.

Let us now write the corresponding LME of a GDS in terms of the decomposition of the infinitesimal generator given Eq. (15). More precisely, in Appendix C, we show that

$$\mathcal{L}_1[\hat{O}] = \frac{1}{2}\mathrm{tr}(\mathbb{D}\hat{\mathbb{H}}[\hat{O}]), \tag{32a}$$

$$\mathcal{L}_2[\hat{O}] = \frac{1}{2}\mathrm{tr}(\mathsf{J}\mathbb{C})\hat{O}, \tag{32b}$$

$$\mathcal{L}_3[\hat{O}] = \mathrm{tr}(\mathbb{C}\mathsf{J}\hat{\mathbb{M}}[\hat{O}]) + \mathcal{L}_2[\hat{O}], \tag{32c}$$

where we define the superoperator matrices

$$\hat{\mathbb{H}}[\hat{O}] = -\frac{1}{\hbar^2}[\mathsf{J}\hat{\mathbf{x}}, [(\mathsf{J}\hat{\mathbf{x}})^\intercal, \hat{O}]] = \frac{\partial}{\partial \hat{\mathbf{x}}}\frac{\partial}{\partial \hat{\mathbf{x}}^\intercal}, \tag{33a}$$

$$\hat{\mathbb{M}}[\hat{O}] = \frac{\imath}{\hbar}\hat{\mathbf{x}}[(\mathsf{J}\hat{\mathbf{x}})^\intercal, \hat{O}] = \hat{\mathbf{x}}\,\frac{\partial}{\partial \hat{\mathbf{x}}^\intercal}, \tag{33b}$$

with the matrix notation $([\hat{\mathbf{x}}, [\hat{\mathbf{y}}^\intercal, \hat{O}]])_{ij} = [x_i, [y_j, \hat{O}]]$, and $(\hat{\mathbf{x}}\,[\hat{\mathbf{y}}^\intercal, \hat{O}])_{ij} = x_i[y_j, \hat{O}]$. The last equalities in Eqs. (33) follow from the identity $(\imath/\hbar)[(\mathsf{J}\hat{\mathbf{x}})^\intercal, \hat{O}(\hat{\mathbf{x}})] = \partial\hat{O}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}^\intercal$ given in Ref. [76]. Then, the linear LME is

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= \frac{1}{\imath\hbar}[\hat{H}, \hat{\rho}_t] + \frac{1}{2}\mathrm{tr}\big(\mathbb{D}\hat{\mathbb{H}}[\hat{\rho}_t]\big) \\ &\quad + \mathrm{tr}(\mathbb{C}\mathsf{J})\hat{\rho}_t + \mathrm{tr}(\mathbb{C}\mathsf{J}\hat{\mathbb{M}}[\hat{\rho}_t]). \end{aligned} \tag{34}$$

For one mode ($n = 1$), this expression reduces to that given in Ref. [73].

Notice that Eq. (31) is nothing but the Weyl-Wigner representation of Eq. (34). Moreover, a direct comparison of both evolution equations follows from the correspondences

$$\hat{\rho}_t \;\leftrightarrow\; W(\mathbf{x}, t), \tag{35a}$$

$$\hat{\mathbf{x}} \;\leftrightarrow\; \mathbf{x}, \tag{35b}$$

$$(\imath/\hbar)[(\mathsf{J}\hat{\mathbf{x}})^{\mathsf{T}}, \cdot\,] = \partial/\partial\hat{\mathbf{x}}^{\mathsf{T}} \;\leftrightarrow\; \partial/\partial\mathbf{x}^{\mathsf{T}} \tag{35c}$$

$$(1/\imath\hbar)[\cdot\,,\,\cdot] \;\leftrightarrow\; [\cdot\,,\,\cdot]_{cl}, \tag{35d}$$

$$\mathrm{Tr}(\cdot) \;\leftrightarrow\; \int_{\mathbb{R}^N} \cdot\, d^{2n}\mathbf{x}. \tag{35e}$$

In this way, we can go back and forth between both Eqs. (31) and (34). In the next section, we will see that these correspondences also allow to go back and forth between the generalized classical de Bruijn identity (4) and the quantum de Bruijn identity for the von Neumann entropy rate in GDS [see Eq. (36) below].

B. Quantum de Bruijn identity for Gaussian dynamical semigroups

As we have seen, decomposition (15) for a GDS splits the dynamics into diffusive and dissipative contributions, given by the corresponding superoperators $\mathcal{L}_1$ [Eq. (32a)], and $\mathcal{L}_2$ and $\mathcal{L}_3$ given in Eqs. (32b) and (32c), respectively. From this decomposition, we obtain the quantum version of the generalized de Bruijn identity for GDSs,

$$\frac{dS[\hat{\rho}_t]}{dt} = \Delta_t - \Psi_t = \frac{1}{2}\mathrm{tr}(\mathbb{D}\mathbb{J}_q[\hat{\rho}_t]) - \mathrm{tr}(\mathbb{C}\mathsf{J}\mathbb{M}[\hat{\rho}_t]), \tag{36}$$

where

$$\mathbb{J}_q[\hat{\rho}_t] = -\mathrm{Tr}(\hat{\rho}_t \hat{\mathbb{H}}[\ln \hat{\rho}_t]) \tag{37}$$

is the DQFI matrix, a quantum version of the Fisher information matrix (5), and

$$\mathbb{M}[\hat{\rho}_t] = -\mathrm{Tr}(\hat{\rho}_t \hat{\mathbb{M}}[\ln \hat{\rho}_t]). \tag{38}$$

We observe that the result (Eq. (82) of Ref. [30]) obtained for the quantum diffusion process (for which $\mathbb{D} = \frac{1}{2}\mathbb{1}$ and $\mathbb{C} = 0$) is recovered from Eq. (36) evaluated at $t = 0$ and performing the trace operation. This is the quantum analog of the classical de Bruijn identity (1). As with its classical counterpart, this result is used in the derivation of quantum entropy power inequalities [31,32].

The diffusion contribution Δ_t in Eq. (36) can also be expressed as Eq. (24) with $\hat{A}_k = \frac{1}{\sqrt{\hbar}}\boldsymbol{\Sigma}_k^{\mathsf{T}}\mathsf{J}\hat{\mathbf{x}}$ and $\hat{B}_k = \frac{1}{\sqrt{\hbar}}\bar{\boldsymbol{\Sigma}}_k^{\mathsf{T}}\mathsf{J}\hat{\mathbf{x}}$, where $\boldsymbol{\Sigma}_k = \mathrm{Re}(\mathbf{l}_k)$ and $\bar{\boldsymbol{\Sigma}}_k = \mathrm{Im}(\mathbf{l}_k)$. Therefore, the family of density operators in Eq. (24), i.e.,

$$\hat{\rho}_{\theta,\hat{C}} = \hat{U}_{\delta\theta}\,\hat{\rho}_t\,\hat{U}_{\delta\theta}^{\dagger} = \hat{T}_{\boldsymbol{\eta}}\,\hat{\rho}_t\,\hat{T}_{\boldsymbol{\eta}}^{\dagger}, \tag{39}$$

with $\hat{C} = \boldsymbol{\Sigma}_k^{\mathsf{T}}\mathsf{J}\hat{\mathbf{x}}$ or $\bar{\boldsymbol{\Sigma}}_k^{\mathsf{T}}\mathsf{J}\hat{\mathbf{x}}$, is generated by phase-space translation operators [75], $\hat{T}_{\boldsymbol{\eta}}$, with $\boldsymbol{\eta} = \boldsymbol{\Sigma}_k\delta\theta$ or $\bar{\boldsymbol{\Sigma}}_k\delta\theta$. These translations in the phase space can be directly related to the Langevin forces $\mathbf{O}_k = \boldsymbol{\Sigma}_k, \bar{\boldsymbol{\Sigma}}_k$. Indeed, if we consider the classical counterpart of the Hamiltonian generator $\hat{C}$, i.e., $C = \mathbf{O}_k^{\mathsf{T}}\mathsf{J}\mathbf{x}$, from the Hamilton equation of motion we have $\frac{d\mathbf{x}}{dt} = \mathsf{J}\frac{\partial C}{\partial \mathbf{x}} = \mathbf{O}_k$. Following the discussion below Eq. (24), we conclude that in this case the diffusion term Δ_t in Eq. (36) is a measure of the noise produced by these Langevin forces, $\boldsymbol{\Sigma}_k = \mathrm{Re}(\mathbf{l}_k)$ and $\bar{\boldsymbol{\Sigma}}_k = \mathrm{Im}(\mathbf{l}_k)$, in the quantum evolution.

Finally, from the correspondences (35) and replacing the von Neumann entropy by the Shannon entropy into the quantum de Bruijn identity (36), we can recover the classical de Bruijn equation (4), and conversely we can obtain Eq. (36) from Eq. (4). Moreover, the correspondences in Eqs. (35) imply $\mathbb{J}_q[\hat{\rho}_t] \leftrightarrow \mathbb{J}[p_t]$ for the quantum and classical information matrices given in Eqs. (37) and (5), respectively, and $\mathbb{M}[\hat{\rho}_t] \leftrightarrow -\int_{\mathbb{R}^N} W(\mathbf{x}, t)\,\mathbf{x}\,\frac{\partial \ln W(\mathbf{x},t)}{\partial \mathbf{x}^{\mathsf{T}}}\,d^{2n}\mathbf{x} = \mathbb{1}$ [see Eq. (10)], as well. Therefore, we conclude that the dissipation term Ψ_t in Eq. (36) measures the total amount of change of the density operator due to the dissipative force $\hat{\mathbf{F}}_S(\mathbf{x}) = -\mathbb{C}\mathsf{J}\hat{\mathbf{x}}$.

In the next section we focus on evolutions in GDSs starting with initial Gaussian states.

C. Gaussian states

The Fokker-Planck equation (3) associated with a linear LME propagates any initial Wigner function that describes the initial quantum state of the system [77]. Generally, these are quasiprobability distributions; i.e., they can take negative values but still $\int_{\mathbb{R}^N} W(\mathbf{x}, t)\,d^{2n}\mathbf{x} = 1$. Here, we discuss Eq. (36) specialized to Gaussian states $\hat{\sigma}_t$, which are quantum states whose Wigner functions are probability distributions. Indeed, the Wigner function of a Gaussian state $\hat{\sigma}_t$ is a multivariate Gaussian distribution with mean $\langle\hat{\mathbf{x}}\rangle_t = \mathrm{Tr}(\hat{\rho}_t\hat{\mathbf{x}})$ and covariance matrix $\mathbb{V}_t = \frac{1}{2\hbar}\mathrm{Tr}(\hat{\rho}_t(\hat{\mathbf{x}} - \langle\hat{\mathbf{x}}\rangle_t)(\hat{\mathbf{x}} - \langle\hat{\mathbf{x}}\rangle_t)^{\mathsf{T}})$,

$$W_G(\mathbf{x}, t) = e^{-\frac{1}{2\hbar}(\mathbf{x}-\langle\hat{\mathbf{x}}\rangle_t)^{\mathsf{T}}\mathbb{V}_t^{-1}(\mathbf{x}-\langle\hat{\mathbf{x}}\rangle_t)}/\mathcal{W}_t, \tag{40}$$

where $\mathcal{W}_t = (2\pi\hbar)^n\sqrt{\det(\mathbb{V}_t)}$. Moreover, the density operator of a Gaussian state can be expressed as [1,78]

$$\hat{\sigma}_t = e^{-\frac{1}{2\hbar}(\hat{\mathbf{x}}-\langle\hat{\mathbf{x}}\rangle_t)^{\mathsf{T}}\mathbb{U}_t(\hat{\mathbf{x}}-\langle\hat{\mathbf{x}}\rangle_t)}/\mathcal{Z}_t, \tag{41}$$

with $\mathcal{Z}_t = \sqrt{\det(\mathbb{V}_t + \frac{\imath}{2}\mathsf{J})}$ and $\mathbb{U}_t = 2\imath\mathsf{J}\coth^{-1}(2\imath\mathbb{V}_t\mathsf{J})$. Notice that matrices $\mathbb{V}_t$ and $\mathbb{U}_t$ satisfy

$$\mathbb{U}_t\mathbb{V}_t\mathsf{J} = \mathsf{J}\mathbb{V}_t\mathbb{U}_t. \tag{42}$$

The main reason to focus on Gaussian states is that they represent the universe of stationary states (thermal or not) in GDSs [58].

The evolution of the covariance matrix and first moments of any state in linear LME is then determined by the differential equations

$$\frac{1}{2}\frac{d\mathbb{V}_t}{dt} = \frac{\mathbb{D}}{2\hbar} + \frac{1}{2}(\mathbb{V}_t\mathbb{A}^{\mathsf{T}} + \mathbb{A}, \mathbb{V}_t), \tag{43}$$

and

$$\frac{d\langle\hat{\mathbf{x}}\rangle_t}{dt} = \mathbb{A}\langle\hat{\mathbf{x}}\rangle_t - \boldsymbol{\xi}, \tag{44}$$

respectively, with $\mathbb{A}$ and $\mathbb{D}$ given by Eq. (27) and $\boldsymbol{\xi}$ as in Eq. (26). These equations completely determine the evolution of Gaussian states in GDSs.

The Shannon entropy of a Gaussian state $\hat{\sigma}_t$ is well defined (since W_G is a probability distribution) and given by $h[W_G] = \frac{1}{2}\ln\det(\mathbb{V}_t) + n\ln(2\pi e)$. Therefore, the rate of change of the

Shannon entropy is given by

$$\frac{dh[W_G]}{dt} = \frac{1}{2}\mathrm{tr}\left(\mathbb{D}\frac{\mathbb{V}_t^{-1}}{\hbar}\right) - \mathrm{tr}(\mathsf{J}\mathbb{C}), \tag{45}$$

which is directly obtained from Eq. (4) with $\mathbb{J}[W_G] = \frac{\mathbb{V}_t^{-1}}{\hbar}$ and $\mathrm{tr}(\mathbb{A}) = -\mathrm{tr}(\mathsf{J}\mathbb{C})$. As in Eq. (36), there is no contribution in Eq. (45) of the Hamiltonian evolution.

We emphasize here that Eq. (45) is simply the classical Bruijn identity (4) for a probability distribution that corresponds to the Wigner function of a Gaussian quantum state.

In Sec. II B we showed that the first term in the classical de Bruijn identity (4) can be rewritten as in Eq. (7). However, in this case it is more interesting to rewrite the first term in Eq. (45) using the expression for $\mathbb{D}$ in Eq. (30) that contains the Langevin forces $\mathbf{O}_k = \mathbf{\Sigma}_k, \bar{\mathbf{\Sigma}}_k$, i.e.,

$$\frac{1}{2}\mathrm{tr}\left(\mathbb{D}\frac{\mathbb{V}_t^{-1}}{\hbar}\right) = \frac{1}{2}\sum_{k=1}^{K}\left(J[W_G^{(\theta,\mathbf{\Sigma}_k)};\theta] + J[W_G^{(\theta,\bar{\mathbf{\Sigma}}_k)};\theta]\right)\big|_{\theta=t}. \tag{46}$$

Here, $\mathbf{O}_k^{\mathsf{T}}\mathbb{J}[W_G]\mathbf{O}_k = J[W_G^{(\theta,\mathbf{O}_k)};\theta]|_{\theta=t}$, where $J[W_G;\theta] = \int_{\mathbb{R}^{2n}} W_G\,(\frac{\partial \ln W_G}{\partial\theta})^2 d^{2n}\mathbf{x}$ is the Fisher information of the Wigner function of the translated state given by Eq. (39), $W_G^{(\theta,\mathbf{O}_k)} = W_G(\mathbf{x} - \delta\theta\mathbf{O}_k, t)$.

Finally, for Gaussian states, we obtain that the quantum de Bruijn identity (36) reduces to

$$\frac{dS[\hat{\sigma}_t]}{dt} = \frac{1}{2}\mathrm{tr}\left(\mathbb{D}\frac{\mathbb{U}_t}{\hbar}\right) - \mathrm{tr}(\mathsf{J}\mathbb{C}\mathbb{U}_t\mathbb{V}_t), \tag{47}$$

where we used that (i) $\hat{\mathbb{H}}[\ln\hat{\sigma}_t] = -\frac{1}{\hbar}\mathbb{U}_t\,\hat{\mathbb{1}}$ and $\hat{\mathbb{M}}[\ln\hat{\sigma}_t] = \frac{1}{\hbar}\hat{\mathbf{x}}(\hat{\mathbf{x}} - \langle\hat{\mathbf{x}}\rangle_t)^{\mathsf{T}}\mathbb{U}_t$, so that $\mathbb{M}[\hat{\sigma}_t] = -\mathbb{V}_t\mathbb{U}_t - \frac{\iota}{2}\mathsf{J}\mathbb{U}_t$, and that, in general, $\mathrm{Tr}(\hat{\rho}_t\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}) - \langle\hat{\mathbf{x}}\rangle_t\langle\hat{\mathbf{x}}\rangle_t^{\mathsf{T}} = \mathbb{V}_t + \frac{\iota}{2}\mathsf{J}$, that (ii) $\mathrm{tr}(\mathbb{U}_t\mathsf{J}) = 0$, because $\mathbb{U}_t$ is symmetric and J antisymmetric, and (iii) relation (42).

In the next section we analyze the stationary situations in GDSs.

D. Stationary states in Gaussian dynamical semigroups

In GDSs if there exists a stationary state, $\hat{\sigma}^S$, this one is unique and for any initial state $\hat{\rho}_0$ we have $\hat{\rho}_t \underset{t\to+\infty}{\to} \hat{\sigma}^S$ [58,79,80]. In particular, starting with an initial Gaussian state, the evolved states remain Gaussian. Therefore, stationary states are necessarily Gaussian. A stationary Gaussian state corresponds to a thermal equilibrium situation when its density operator, $\hat{\sigma}^S = \hat{\sigma}^{th}$ in Eq. (41), is a Gibbs state of the Hamiltonian $\hat{H}$ of the QDS in Eq. (26) or of a Hamiltonian that commutes with this.

Remarkably, the quantum de Bruijn identity given by expression (36), for whatever initial state, together with the one given by Eq. (47) for an initial Gaussian state, allows a description of stationary situations. Indeed, from our approach we observe that the stationary state arises when the environment, described by $\mathbb{D}$ and $\mathbb{C}$, allows the balance between the diffusion term $\Delta_t > 0$ and the dissipative term Ψ_t i.e., $\frac{dS[\hat{\rho}_t]}{dt} \underset{t\to+\infty}{\to} 0$. While Δ_t describes the increase in noise due to Langevin forces, Ψ_t represents the contribution due to the dissipative forces. Particularly for any initial Gaussian state, the convergence $\hat{\sigma}_t \underset{t\to+\infty}{\to} \hat{\sigma}^S$ happens in such a way that $\frac{dS[\hat{\sigma}_t]}{dt} - \frac{dh[W_G]}{dt} \underset{t\to+\infty}{\to} 0$. This is a consequence of

$$\frac{dS[\hat{\sigma}_t]}{dt} = \frac{dh[W_G]}{dt} + \frac{1}{2}\mathrm{tr}\left(\mathbb{O}_t\frac{d\mathbb{V}_t}{dt}\right), \tag{48}$$

and $\frac{d\mathbb{V}_t}{dt} \underset{t\to+\infty}{\to} 0$ in Eq. (43) when $\hat{\sigma}_t \underset{t\to+\infty}{\to} \hat{\sigma}^S$. Relation (48) between von Neumann and Shannon entropy rates results from

$$\frac{dS[\hat{\sigma}_t]}{dt} = \frac{1}{2}\mathrm{tr}\left(\mathbb{U}_t\frac{d\mathbb{V}_t}{dt}\right), \tag{49a}$$

$$\frac{dh[W_G]}{dt} = \frac{1}{2}\mathrm{tr}\left(\mathbb{V}_t^{-1}\frac{d\mathbb{V}_t}{dt}\right), \tag{49b}$$

and the expansion $\mathbb{U}_t = \mathbb{V}_t^{-1} + \mathbb{O}_t$, where $\mathbb{O}_t = \sum_{m=1}^{+\infty}\frac{2\iota\mathsf{J}}{2m+1}(\frac{\iota\mathsf{J}\mathbb{V}_t^{-1}}{2})^{2m+1}$. To recover de Bruijn identity (47) from the time derivative of $S[\hat{\sigma}_t]$ given Eq. (49a), we use Eq. (43), $\mathrm{tr}(\mathbb{U}_t\mathbb{A}\mathbb{V}_t) = \mathrm{tr}(\mathbb{U}_t\mathbb{V}_t\mathbb{A}^{\mathsf{T}})$, $\mathrm{tr}(\mathsf{J}\mathbb{C}\mathbb{U}_t\mathbb{V}_t) = \mathrm{tr}(\mathbb{C}\mathsf{J}\mathbb{V}_t\mathbb{U}_t)$, and $\mathrm{tr}(\mathsf{J}\mathbb{B}\mathbb{V}_t\mathbb{U}_t) = 0$. The time derivative of $h[W_G]$ in Eq. (49b) directly follows from $\frac{d(\ln\det(\mathbb{V}_t))}{dt} = \mathrm{tr}(\mathbb{V}_t^{-1}\frac{d\mathbb{V}_t}{dt})$ (see Chaps. 1 and 9 of Ref. [81]).

The covariance matrix $\mathbb{V}^S$ of a stationary state $\hat{\sigma}^S$ is a solution of a Lyapunov equation [82], i.e., setting $\frac{d\mathbb{V}_t}{dt} = 0$ in Eq. (43), so

$$\frac{\mathbb{D}}{\hbar} + \mathbb{V}^S(-\mathbb{B}\mathsf{J} - \mathsf{J}\mathbb{C}) + (\mathsf{J}\mathbb{B} - \mathbb{C}\mathsf{J})\mathbb{V}^S = 0. \tag{50}$$

The solution of this Lyapunov equation gives $\mathbb{V}^S$ as a function of $\mathbb{B}$, $\mathbb{D}$, and $\mathbb{C}$, and this dependency determines the type of Gaussian state $\hat{\sigma}^S$ is. Notice that, according to Eq. (45), $\frac{dh[W_G]}{dt} = 0$ is equivalent to the trace of this Lyapunov equation, since $\mathrm{tr}(\mathbb{B}\mathsf{J}) = 0$. Therefore, there is a direct connection between the Lyapunov equation that determines $\mathbb{V}^S$, and hence $\hat{\sigma}^S$, and the stationarity of the Shannon entropy for the Wigner function of $\hat{\sigma}^S$, i.e., $\frac{dh[W_G]}{dt} = 0$. Besides, for a stationary state we have $\Delta_t = \Psi_t$ in Eq. (47); therefore, $\frac{dS[\hat{\sigma}_t]}{dt} = 0$. However, let us see that, for a family of thermal states, there is another way to establish the stationarity of the von Neumann entropy, i.e., $\frac{dS[\hat{\sigma}_t]}{dt} = 0$, which has a direct connection to the Lyapunov equation for $\mathbb{V}^S$. Indeed, rewriting Eq. (47) as

$$\frac{dS[\hat{\sigma}_t]}{dt} = \mathrm{tr}\left(\mathbb{U}_t\mathbb{V}_t\left(\frac{1}{2\hbar}\mathbb{V}_t^{-1}\mathbb{D} - \mathsf{J}\mathbb{C}\right)\right), \tag{51}$$

we obtain a solution of $\frac{dS[\hat{\sigma}_t]}{dt} = 0$ solving the Lyapunov equation

$$\frac{1}{2\hbar}(\mathbb{V}^S)^{-1}\mathbb{D} - \mathsf{J}\mathbb{C} = 0. \tag{52}$$

Let us see that this equation is a particular instance of Eq. (50) whose solution corresponds to the covariance matrix, as a function of $\mathbb{D}$ and $\mathbb{C}$, of a family of thermal states. Indeed, the solution is given by $\mathbb{V}^S = -\frac{1}{2\hbar}\mathbb{D}\mathbb{C}^{-1}\mathsf{J}$, with the condition $\mathbb{D}\mathsf{J}\mathbb{C} = \mathbb{C}\mathsf{J}\mathbb{D}$ for $\mathbb{V}^S$ to be symmetric. This covariance matrix $\mathbb{V}^S = \mathbb{V}^{th}$ is indeed that of the thermal state $\hat{\sigma}^{th} = \frac{e^{-\beta\hat{H}}}{\mathcal{Z}^{th}}$, where $\mathcal{Z}^{th} = \mathrm{Tr}(e^{-\beta\hat{H}})$, with β the inverse temperature, and $\hat{H}$ given by Eq. (26) where $\boldsymbol{\xi}$ must be zero [83]. This is because the Lyapunov equation (50) for this thermal state reduces to Eq. (52), using that $\mathbb{U}^{th}/\hbar = \beta\mathbb{B}$ [with $\mathbb{U}^{th}$ the matrix in Eq. (41)] together with $\mathbb{U}^{th}\mathbb{V}^{th}\mathsf{J} = \mathsf{J}\mathbb{V}^{th}\mathbb{U}^{th}$ and $\mathbb{D}\mathsf{J}\mathbb{C} = \mathbb{C}\mathsf{J}\mathbb{D}$.

An example of an equilibrium state of this type is that appearing in the quantum optical master equation, for which $\mathbb{D} = \gamma \mathbb{1}$ and $\mathbb{JC} = \alpha \mathbb{1}$, where $\gamma \geqslant 0$ and α is a real number [84].

VI. CONCLUSIONS

For memoryless continuous-in-time quantum channels, whose dynamics is described by one-parameter quantum dynamical semigroups, we have provided a decomposition given in Eq. (15) of the infinitesimal generator of the nonunitary evolution. This extends the concepts of diffusion and dissipation in these kind of open quantum systems and allows us to obtain a compact formula for the rate of change of the von Neumann entropy for quantum states, Eq. (22). We emphasize that this constitutes one of the main results of our contribution. The first term Δ_t of that formula clarifies that $\mathcal{L}_1$ in Eq. (15) is the infinitesimal generator of diffusion in the dynamics, and the second term Ψ_t indicates that $\mathcal{L}_{2+3}$ is the infinitesimal generator of dissipation. Indeed, while the positive contribution Δ_t, rewritten as in Eq. (24), corresponds to the increase of noise measured by the divergence-based quantum Fisher information, the term Ψ_t measures the contribution of dissipative forces to the rate of change of von Neumann entropy. We highlight that an analogous approach, using $\mathcal{L}_1$ and $\mathcal{L}_{2+3}$, can be done to separate the fluctuation from the dissipation contribution in the rate of change of the mean value of any time-independent observable of a quantum system that follows a one-parameter QDS dynamics.

We have then focused on channels that form Gaussian dynamical semigroups, paramount in the description of the most useful quantum channels in the data-transmission and data-processing systems in modern quantum information theory. For these channels, we have first obtained the expression (32) for the infinitesimal generators of the dynamics from our decomposition, where the dependence with the diffusion and dissipative matrices appears explicitly. Besides, from this decomposition, we have provided a series of correspondences (35) that maps the Lindblad master equation for the evolution of the density operator (34) into the Fokker-Planck equation for the evolution of the Wigner function (31), and vice versa. Finally, we obtained the rate of change of the von Neumann entropy for Gaussian dynamical semigroups (36), where the first term of our identity quantifies the contributions of diffusion due to Langevin forces, whereas the second one is the contribution due to dissipative forces. Identity (36) is nothing but the quantum counterpart of the generalized classical de Bruijn identity (4), which can also be obtained from the latter from the series of correspondences (35) (and vice versa).

In addition, we have provided an alternative perspective to study the stationarity of quantum dynamical semigroups and in particular for Gaussian dynamical semigroups. Such a study is usually addressed from the balance of the rates of the entropy production and entropy flux, whereas we considered here the balance between diffusion and dissipation in such quantum channels.

Finally, we highlight that Eq. (22) [with Δ_t given by Eq. (24)] is a quantum de Bruijn identity valid for any quantum dynamical semigroup. It is remarkable that this identity has a very similar structure to that of the classical generalized de Bruijn identity in Eq. (11), even in the case of finite-dimensional quantum systems. Thus, we believe that, like the quantum de Bruijn identity in Eq. (36), without dissipation, is a key ingredient in the derivation of entropy power inequalities in continuous variable systems, Eq. (22) with null dissipative term, i.e., $\Psi_t = 0$, could be useful to define and prove entropy power inequalities in finite-dimensional systems. Besides, the quantum de Bruijn identity (22) reveals the central role played by the divergence-based quantum Fisher information in the quantification of the amount of noise produced by the random effects into the dynamics suffered by systems that implement quantum communication channels. In this respect, our framework opens an avenue in the study of time evolution in quantum channels from a fluctuation-dissipation perspective that allows to quantify the degradation induced by noise in the information transmitted.

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APPENDIX A: UNITARY INVARIANCE IN EQ. (17)

Here we prove that the superoperators $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are invariant under the unitary transformation of the Lindblad operators in Eq. (17).

1. Invariance of $\mathcal{L}_1$

First, notice that the operators $\hat{A}_k = (\hat{L}_k + \hat{L}_k^\dagger)/2$, $\hat{B}_k = (\hat{L}_k - \hat{L}_k^\dagger)/2\imath$, and equivalently definitions to $\hat{A}'_k$ and $\hat{B}'_k$, with $\hat{L}'_k$ given in Eq. (17) and $(\hat{L}'_k)^\dagger = \sum_j W^*_{k,j} \hat{L}^\dagger_j$, transform as

$$\hat{A}_k \to \hat{A}'_k = \textstyle\sum_{j=1}^K (\mathrm{Re}(W_{k,j})\,\hat{A}_j - \mathrm{Im}(W_{k,j})\,\hat{B}_j),$$
$$\hat{B}_k \to \hat{B}'_k = \textstyle\sum_{j=1}^K (\mathrm{Im}(W_{k,j})\,\hat{A}_j + \mathrm{Re}(W_{k,j})\,\hat{B}_j).$$

Second, using the identities $\mathrm{Re}(zw^*) = \mathrm{Re}(z)\mathrm{Re}(w) + \mathrm{Im}(z)\mathrm{Im}(w)$ and $\mathrm{Im}(zw^*) = \mathrm{Im}(z)\mathrm{Re}(w) - \mathrm{Re}(z)\mathrm{Im}(w)$ and some algebra, we arrive at

$$\begin{aligned}
&[\hat{A}'_k, [\hat{A}'_k, \hat{O}]] + [\hat{B}'_k, [\hat{B}'_k, \hat{O}]] \\
&= \sum_{j,l=1}^K \mathrm{Re}(W_{k,j} W^*_{k,l})[\hat{A}_l, [\hat{A}_j, \hat{O}]] \\
&\quad + \sum_{j,l=1}^K \mathrm{Im}(W_{k,j} W^*_{k,l})[\hat{B}_l, [\hat{A}_j, \hat{O}]]
\end{aligned}$$

$$+\sum_{j,l=1}^{K}\text{Im}(W_{k,j}W_{k,l}^{*})[\hat{A}_l,[\hat{B}_j,\hat{O}]] + \text{Re}(W_{k,j}W_{k,l}^{*})[\hat{B}_l,[\hat{B}_j,\hat{O}]].$$

Finally, we obtain

$$\begin{aligned}\mathcal{L}_1'[\hat{O}] &= -\frac{1}{2\hbar}\sum_{k=1}^{K}[\hat{A}_k',[\hat{A}_k',\hat{O}]]+[\hat{B}_k',[\hat{B}_k',\hat{O}]]\\ &= -\frac{1}{2\hbar}\left(\sum_{j,l=1}^{K}\text{Re}(\delta_{j,l})[\hat{A}_l,[\hat{A}_j,\hat{O}]] + \sum_{j,l=1}^{K}\text{Im}(\delta_{j,l})[\hat{B}_l,[\hat{A}_j,\hat{O}]] + \sum_{j,l=1}^{K}\text{Im}(\delta_{j,l})[\hat{A}_l,[\hat{B}_j,\hat{O}]] + \sum_{j,l=1}^{K}\text{Re}(\delta_{j,l})[\hat{B}_l,[\hat{B}_j,\hat{O}]]\right)\\ &= -\frac{1}{2\hbar}\sum_{j=1}^{K}[\hat{A}_j,[\hat{A}_j,\hat{O}]]+[\hat{B}_j,[\hat{B}_j,\hat{O}]]\\ &=\mathcal{L}_1[\hat{O}],\end{aligned}$$

because $\sum_{k=1}^{K} W_{k,j}W_{k,l}^{*}=\delta_{j,l}$, i.e., $\mathbb{W}$ is unitary.

2. Invariance of $\mathcal{L}_2$ and $\mathcal{L}_3$

On the one hand, we have

$$\begin{aligned}\sum_{k=1}^{K}[\hat{L}_k',(\hat{L}_k')^{\dagger}] &= \sum_{j,l=1}^{K}\sum_{k=1}^{K}W_{k,j}W_{k,l}^{*}\hat{L}_j\hat{L}_l^{\dagger} - \sum_{j,l=1}^{K}\sum_{k=1}^{K}W_{k,j}W_{k,l}^{*}\hat{L}_l^{\dagger}\hat{L}_j\\ &=\sum_{j=1}^{K}\hat{L}_j\hat{L}_j^{\dagger}-\sum_{j=1}^{K}\hat{L}_j^{\dagger}\hat{L}_j\\ &=\sum_{j=1}^{K}[\hat{L}_j,(\hat{L}_j)^{\dagger}].\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{L}_2'[\hat{O}] &= \frac{1}{2\hbar}\sum_{k=1}^{K}\frac{1}{2}\{[\hat{L}_k',(\hat{L}_k')^{\dagger}],\hat{O}\}\\ &= \frac{1}{2\hbar}\sum_{k=1}^{K}\frac{1}{2}\{[\hat{L}_k,\hat{L}_k^{\dagger}],\hat{O}\}=\mathcal{L}_2[\hat{O}].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{k=1}^{K}\hat{L}_k'\hat{O}(\hat{L}_k')^{\dagger} &= \sum_{k=1}^{K}\sum_{j}W_{k,j}\hat{L}_j\hat{O}\sum_{l}W_{k,l}^{*}\hat{L}_l^{\dagger}\\ &= \sum_{j,l=1}^{K}\sum_{k=1}^{K}W_{k,j}W_{k,l}^{*}\hat{L}_j\hat{O}\hat{L}_l^{\dagger}\\ &=\sum_{j=1}^{K}\hat{L}_j\hat{O}\hat{L}_j^{\dagger}.\end{aligned}$$

A similar result is obtained for $\sum_{k=1}^{K}(\hat{L}_k')^{\dagger}\hat{O}\hat{L}_k' = \sum_{k=1}^{K}\hat{L}_k^{\dagger}\hat{O}\hat{L}_k$. Therefore, we obtain

$$\begin{aligned}\mathcal{L}_3'[\hat{O}] &= \frac{1}{2\hbar}\sum_{k=1}^{K}(\hat{L}_k'\hat{O}(\hat{L}_k')^{\dagger}-(\hat{L}_k')^{\dagger}\hat{O}\hat{L}_k')\\ &= \frac{1}{2\hbar}\sum_{k=1}^{K}(\hat{L}_k\hat{O}\hat{L}_k^{\dagger}-\hat{L}_k^{\dagger}\hat{O}\hat{L}_k)=\mathcal{L}_3[\hat{O}].\end{aligned}$$

APPENDIX B: PROOF OF EQ. (21)

Let $\hat{U}_{\delta\theta}$ be a unitary operator describing the evolution of the state of a system with respect to a parameter $\delta\theta=\theta-\theta_0$, i.e., $\hat{\rho}_\theta=\hat{U}_{\delta\theta}\,\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}$ with $\hat{U}_{\delta\theta}\,\hat{U}_{\delta\theta}^{\dagger}=\hat{U}_{\delta\theta}^{\dagger}\,\hat{U}_{\delta\theta}=\hat{1}$ the unit operator, $\hat{U}_0=\hat{1}$, and θ_0 a fixed value.

Let us consider the quantum relative entropy of the state $\hat{\rho}_\theta$ from $\hat{\rho}_{\theta_0}$ that serves as a reference,

$$S[\hat{\rho}_\theta\|\hat{\rho}_{\theta_0}]=\text{Tr}\big(\hat{\rho}_\theta\big(\ln\hat{\rho}_\theta-\ln\hat{\rho}_{\theta_0}\big)\big). \quad \text{(B1)}$$

Notice that the quantum relative entropy is positive and zero, its minimum with respect to θ, if and only if $\hat{\rho}_\theta=\hat{\rho}_{\theta_0}$. Thus, both the relative entropy and its first derivative with respect to θ vanishes at $\theta=\theta_0$.

In order to obtain expression (21), we have to calculate the second derivative of the quantum relative entropy at $\theta=\theta_0$.

The first derivative of the relative entropy is given by

$$\frac{d}{d\theta}S[\hat{\rho}_\theta\|\hat{\rho}_{\theta_0}]=\text{Tr}\left(\frac{d\hat{\rho}_\theta}{d\theta}(\ln\hat{\rho}_\theta-\ln\hat{\rho}_{\theta_0})\right) +\text{Tr}\left(\hat{\rho}_\theta\frac{d}{d\theta}\ln\hat{\rho}_\theta\right). \quad \text{(B2)}$$

Because $\hat{U}_{\delta\theta}$ is unitary, $\ln\hat{\rho}_\theta=\hat{U}_{\delta\theta}\,\ln\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}$. Thus,

$$\frac{d}{d\theta}\ln\hat{\rho}_\theta=\frac{d\hat{U}_{\delta\theta}}{d\theta}\,\ln\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}+\hat{U}_{\delta\theta}\,\ln\hat{\rho}_{\theta_0}\,\frac{d\hat{U}_{\delta\theta}^{\dagger}}{d\theta}. \quad \text{(B3)}$$

Now, tracing Eq. (B3) and using successively (i) both $\hat{\rho}_\theta=\hat{U}_{\delta\theta}\hat{\rho}_{\theta_0}\hat{U}_{\delta\theta}^{\dagger}$ and $\hat{U}_{\delta\theta}^{\dagger}\hat{U}_{\delta\theta}=\hat{1}$, (ii) $\text{Tr}(AB)=\text{Tr}(BA)$ (judiciously) together with $\hat{U}_{\delta\theta}\hat{U}_{\delta\theta}=\hat{1}$, (iii) the fact that $\hat{\rho}_{\theta_0}$ and $\ln\hat{\rho}_{\theta_0}$ commute, and (iv) $\hat{U}_{\delta\theta}\hat{U}_{\delta\theta}=\hat{1}$ so that its derivative vanishes, we obtain

$$\begin{aligned}\text{Tr}\left(\hat{\rho}_\theta\frac{d}{d\theta}\ln\hat{\rho}_\theta\right) &= \text{Tr}\left(\hat{U}_{\delta\theta}\,\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}\frac{d\hat{U}_{\delta\theta}^{\dagger}}{d\theta}\,\ln\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}\right) +\text{Tr}\left(\hat{U}_{\delta\theta}\,\hat{\rho}_{\theta_0}\,\ln\hat{\rho}_{\theta_0}\,\frac{d\hat{U}_{\delta\theta}^{\dagger}}{d\theta}\right)\\ &= \text{Tr}\left(\ln\hat{\rho}_{\theta_0}\,\hat{\rho}_{\theta_0}\,\hat{U}_{\delta\theta}^{\dagger}\frac{d\hat{U}_{\delta\theta}^{\dagger}}{d\theta}\right) +\text{Tr}\left(\hat{\rho}_{\theta_0}\,\ln\hat{\rho}_{\theta_0}\,\frac{d\hat{U}_{\delta\theta}^{\dagger}}{d\theta}\,\hat{U}_{\delta\theta}\right)\end{aligned}$$

$$= \mathrm{Tr}\left(\hat{\rho}_{\theta_0}\ \ln \hat{\rho}_{\theta_0} \frac{d(\hat{U}^\dagger_{\delta\theta}\hat{U}_{\delta\theta})}{d\theta}\right)$$
$$= 0. \tag{B4}$$

Consequently, the second term of Eq. (B2) vanishes and thus the first derivative of the relative entropy reduces to

$$\frac{d}{d\theta}S[\hat{\rho}_\theta \| \hat{\rho}_{\theta_0}] = \mathrm{Tr}\left(\frac{d\hat{\rho}_\theta}{d\theta}(\ln\hat{\rho}_\theta - \ln\hat{\rho}_{\theta_0})\right), \tag{B5}$$

which indeed vanishes at $\theta = \theta_0$.

Now differentiating expression (B5), the second derivative of the relative entropy is written

$$\frac{d^2}{d\theta^2}S[\hat{\rho}_\theta \| \hat{\rho}_{\theta_0}] = \mathrm{Tr}\left(\frac{d^2\hat{\rho}_\theta}{d\theta^2}(\ln\hat{\rho}_\theta - \ln\hat{\rho}_{\theta_0})\right) + \mathrm{Tr}\left(\frac{d\hat{\rho}_\theta}{d\theta}\frac{d}{d\theta}\ln\hat{\rho}_\theta\right),$$

where the first term on the rhs vanishes at $\theta = \theta_0$, so that

$$\left.\frac{d^2}{d\theta^2}S[\hat{\rho}_\theta \| \hat{\rho}_{\theta_0}]\right|_{\theta=\theta_0} = \left.\mathrm{Tr}\left(\frac{d\hat{\rho}_\theta}{d\theta}\frac{d}{d\theta}\ln\hat{\rho}_\theta\right)\right|_{\theta=\theta_0}. \tag{B6}$$

Let $\hat{C}_{\delta\theta}$ be the Hermitian generator of the unitary transformation $\hat{U}_{\delta\theta}$, that is,

$$\hat{C}_{\delta\theta} = -\imath\frac{d\hat{U}^\dagger_{\delta\theta}}{d\theta}\hat{U}_{\delta\theta} = \imath\,\hat{U}^\dagger_{\delta\theta}\frac{d\hat{U}_{\delta\theta}}{d\theta}. \tag{B7}$$

Then, we use expression (B3) for $\frac{d}{d\theta}\ln\hat{\rho}_\theta$, and similarly for $\frac{d}{d\theta}\hat{\rho}_\theta$ (just replace $\ln\hat{\rho}_\theta$ by $\hat{\rho}_\theta$) so that, after some algebra in the same line as previously (property of the trace, unitary property of $\hat{U}_{\delta\theta}$, commutativity of $\ln\hat{\rho}_{\theta_0}$ and $\hat{\rho}_{\theta_0}$, together with $\frac{d\hat{U}^\dagger_{\delta\theta}}{d\theta}\frac{d\hat{U}_{\delta\theta}}{d\theta} = \frac{d\hat{U}^\dagger_{\delta\theta}}{d\theta}\hat{U}_{\delta\theta}\hat{U}^\dagger_{\delta\theta}\frac{d\hat{U}_{\delta\theta}}{d\theta}$), one obtains

$$\mathrm{Tr}\left(\frac{d\hat{\rho}_\theta}{d\theta}\frac{d}{d\theta}\ln\hat{\rho}_\theta\right) = \mathrm{Tr}(\hat{\rho}_{\theta_0}[\hat{C}_{\delta\theta},[\hat{C}_{\delta\theta},\ln\hat{\rho}_{\theta_0}]]). \tag{B8}$$

Taking Eq. (B8) at $\theta = \theta_0$ where we define $\hat{C}_0 = \hat{C}_{\delta\theta}|_{\theta=\theta_0}$ and plugging the expression into (B6), we finally achieve Eq. (21).

APPENDIX C: $\mathcal{L}_1[\hat{O}]$, $\mathcal{L}_2[\hat{O}]$, AND $\mathcal{L}_3[\hat{O}]$ IN GDSS

Let us start writing $\hat{A}_k = \frac{1}{2}(\hat{L}_k + \hat{L}_k^\dagger) = \mathrm{Re}(\mathbf{l}_k^\dagger)\mathsf{J}\hat{\mathbf{x}} = \mathrm{Re}(\mathbf{a}_k)^\intercal\hat{\mathbf{p}} - \mathrm{Re}(\mathbf{b}_k)^\intercal\hat{\mathbf{q}}$ and $\hat{B} = \frac{1}{2\imath}(\hat{L}_k - \hat{L}_k^\dagger) = \mathrm{Im}(\mathbf{l}_k^\dagger)\mathbb{J}\hat{\mathbf{x}} = \mathrm{Im}(\mathbf{a}_k)^\intercal\hat{\mathbf{p}} - \mathrm{Im}(\mathbf{b}_k)^\intercal\hat{\mathbf{q}}$ with $\hat{L}_k$ in Eq. (26) and $\mathbf{l}_k^\intercal = (\mathbf{a}_k^\intercal, \mathbf{b}_k^\intercal)$. Therefore, using these expressions in Eq. (16a) we obtain

$$\begin{aligned}\mathcal{L}_1[\hat{O}] = -\frac{1}{2\hbar}\sum_{k=1}^{K}\sum_{l,j=1}^{n}(&\mathrm{Re}((\mathbf{a}_k)_l)\mathrm{Re}((\mathbf{a}_k)_j)[\hat{p}_l,[\hat{p}_j,\hat{O}]]\\ &-\mathrm{Re}((\mathbf{b}_k)_l)\mathrm{Re}((\mathbf{a}_k)_j)[\hat{q}_l,[\hat{p}_j,\hat{O}]]\\ &-\mathrm{Re}((\mathbf{a}_k)_l)\mathrm{Re}((\mathbf{b}_k)_j)[\hat{p}_l,[\hat{q}_j,\hat{O}]]\\ &+\mathrm{Re}((\mathbf{b}_k)_l)\mathrm{Re}((\mathbf{b}_k)_j)[\hat{q}_l,[\hat{q}_j,\hat{O}]]).\end{aligned}$$

Now, using the identity $\mathrm{Re}(zw^*) = \mathrm{Re}(z)\mathrm{Re}(w) + \mathrm{Im}(z)\mathrm{Im}(w)$ we get to the final result,

$$\begin{aligned}\mathcal{L}_1[\hat{O}] &= \frac{1}{2}\sum_{s=1}^{2n}\sum_{m=1}^{2n}\left(\hbar\sum_{k=1}^{K}\mathrm{Re}((\mathbf{l}_k)_s^*(\mathbf{l}_k)_m)\right)\\ &\quad\times\frac{-1}{\hbar^2}[(\mathsf{J}\hat{\mathbf{x}})_m,[(\mathsf{J}\hat{\mathbf{x}})_s,\hat{O}]]\\ &= \frac{1}{2}\sum_{s=1}^{2n}\sum_{m=1}^{2n}(\hbar\mathrm{Re}(\mathbb{\Gamma}))_{s,m}(\hat{\mathbb{H}}[\hat{O}])_{m,s}\\ &= \frac{1}{2}\mathrm{tr}(\mathbb{D}\hat{\mathbb{H}}[\hat{O}]).\end{aligned} \tag{C1}$$

Here, we used the definition in Eq. (33a), that $\mathbb{D} = \hbar\mathrm{Re}(\mathbb{\Gamma})$, and we define the matrix notation: $([\hat{\mathbf{y}},[\hat{\mathbf{y}}^\intercal,\hat{O}]])_{sm} = [\hat{y}_s,[\hat{y}_m,\hat{O}]]$.

In the following we will need the identity

$$\begin{aligned}\hat{\mathbf{x}}^\intercal\mathbb{Q}\hat{O}\hat{\mathbf{x}} &= \sum_{j=1}^{n}\sum_{l=1}^{n}\mathbb{Q}_{jl}\hat{x}_j\hat{O}\hat{x}_l = \sum_{j=1}^{n}\sum_{l=1}^{n}\mathbb{Q}_{jl}(-[\hat{x}_l,[\hat{x}_j,\hat{O}]] + \{\hat{x}_l\hat{x}_j,\hat{O}\} - \hat{x}_l\hat{O}\hat{x}_j - \imath\hbar\mathsf{J}_{lj}\hat{O})\\ &= -\hbar^2\mathrm{tr}(\mathbb{Q}\mathsf{J}\hat{\mathbb{H}}\hat{O}\mathsf{J}) + \mathrm{tr}(\mathbb{Q}\{\mathbf{x}\mathbf{x}^T,\hat{O}\}) - \mathrm{tr}(\mathbb{Q}\mathbf{x}\hat{O}\mathbf{x}^T) - \imath\hbar\mathrm{tr}(\mathbb{Q}\mathsf{J})\hat{O}\\ &= -\hbar^2\mathrm{tr}(\mathbb{Q}\mathsf{J}\hat{\mathbb{H}}[\hat{O}]\mathsf{J}) + \hbar\mathrm{tr}(\mathbb{Q}\hat{\mathbb{V}})\,\hat{O} + \hbar\,\hat{O}\,\mathrm{tr}(\mathbb{Q}\hat{\mathbb{V}}) - \mathrm{tr}(\mathbb{Q}\hat{\mathbf{x}}\hat{O}\hat{\mathbf{x}}^\intercal),\end{aligned} \tag{C2}$$

where we used the canonical commutation relation $[\hat{x}_j,\hat{x}_l] = \imath\hbar(\mathsf{J})_{j,l}\hat{\mathbb{1}}$ and the identity

$$\hat{\mathbf{x}}\hat{\mathbf{x}}^\intercal = \hbar\left(\hat{\mathbb{V}} + \frac{\imath}{2}\mathsf{J}\hat{\mathbb{1}}\right), \tag{C3}$$

with $\hat{\mathbb{V}}$ the operator matrix $\hat{\mathbb{V}} = (\hat{\mathbf{x}}\hat{\mathbf{x}}^\intercal + (\hat{\mathbf{x}}\hat{\mathbf{x}}^\intercal)^\intercal)/2$. In the case when $\hat{O} = \hat{\mathbb{1}}$ we have

$$\begin{aligned}\hat{\mathbf{x}}^\intercal\mathbb{Q}\hat{\mathbf{x}} &= 2\hbar\mathrm{tr}(\mathbb{Q}\hat{\mathbb{V}}) - \mathrm{tr}(\mathbb{Q}\hat{\mathbf{x}}\hat{\mathbf{x}}^\intercal)\\ &= \hbar\mathrm{tr}(\mathbb{Q}\hat{\mathbb{V}}) - \frac{\imath\hbar}{2}\mathrm{tr}(\mathbb{Q}\mathsf{J})\hat{\mathbb{1}},\end{aligned} \tag{C4}$$

using Eq. (C3). Identities in Eqs. (C2) and (C4) are valid for arbitrary matrices $\mathbb{Q}$, but the interesting case here is when $\mathbb{Q}$ is antisymmetric. Indeed, when $\mathbb{Q}$ is antisymmetric, because $\mathsf{J}\hat{\mathbb{H}}[\hat{O}]\mathsf{J}$ and $\hat{\mathbb{V}}$ are symmetric, we have

$$\hat{\mathbf{x}}^\intercal\mathbb{Q}\hat{O}\hat{\mathbf{x}} = -\mathrm{tr}(\mathbb{Q}\hat{\mathbf{x}}\hat{O}\hat{\mathbf{x}}^\intercal), \tag{C5}$$

$$\hat{\mathbf{x}}^\intercal\mathbb{Q}\hat{\mathbf{x}} = -\tfrac{\imath\hbar}{2}\mathrm{tr}(\mathbb{Q}\mathsf{J})\hat{\mathbb{1}}. \tag{C6}$$

Using the definition for $\mathcal{L}_2$ in Eq. (16a), we have

$$\begin{aligned}\mathcal{L}_2[\hat{O}] &= \frac{1}{2\hbar}\sum_{k=1}^{K}\frac{1}{2}\{\hat{L}_k\hat{L}_k^\dagger - \hat{L}_k^\dagger\hat{L}_k, \hat{O}\}\\ &= \frac{1}{2\hbar}\sum_{k=1}^{K}\frac{1}{2}\{-\hat{\mathbf{x}}^\mathsf{T}\mathsf{J}\mathbf{l}_k\mathbf{l}_k^\dagger\mathsf{J}\hat{\mathbf{x}} + \hat{\mathbf{x}}^\mathsf{T}\mathsf{J}\mathbf{l}_k^*\mathbf{l}_k^\mathsf{T}\mathsf{J}\hat{\mathbf{x}}, \hat{O}\}\\ &= \frac{1}{2\hbar}\left\{\hat{\mathbf{x}}\mathsf{J}\,\frac{1}{2}\left(\sum_{k=1}^{K}\mathbf{l}_k^*\mathbf{l}_k^\mathsf{T} - \sum_{k=1}^{K}\mathbf{l}_k\mathbf{l}_k^\dagger\right)\mathsf{J}\hat{\mathbf{x}}^\mathsf{T}, \hat{O}\right\}\\ &= \frac{\imath}{2\hbar}\left\{\hat{\mathbf{x}}\mathsf{J}\left(\frac{\mathbb{\Gamma}^* - \mathbb{\Gamma}}{2\imath}\right)\mathsf{J}\hat{\mathbf{x}}^\mathsf{T}, \hat{O}\right\}\\ &- \frac{\imath}{2\hbar}\{\hat{\mathbf{x}}\mathsf{J}\mathbb{C}\mathsf{J}\hat{\mathbf{x}}^\mathsf{T}, \hat{O}\}\\ &= -\frac{\imath}{2\hbar}\left\{-\imath\frac{\hbar}{2}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}^2)\hat{1}, \hat{O}\right\} = \frac{1}{2}\mathrm{tr}(\mathsf{J}\mathbb{C})\hat{O},\end{aligned}\tag{C7}$$

where we used $\mathbb{C} = (\mathbb{\Gamma}^* - \mathbb{\Gamma})/2\imath$, that $\mathsf{J}^2 = -\mathbb{1}$, and the identity in Eq. (C6).

For $\mathcal{L}_3$ in Eq. (16c) we can write

$$\begin{aligned}\mathcal{L}_3[\hat{O}] &= \frac{1}{2\hbar}\sum_{k=1}^{K}(\hat{L}_k\hat{O}\hat{L}_k^\dagger - \hat{L}_k^\dagger\hat{O}\hat{L}_k)\\ &= \frac{1}{2\hbar}\sum_{k=1}^{K}\hat{\mathbf{x}}^\mathsf{T}\mathsf{J}(-\mathbf{l}_k\mathbf{l}_k^\dagger + \mathbf{l}_k^*\mathbf{l}_k^\mathsf{T})\mathsf{J}\hat{O}\hat{\mathbf{x}}\\ &= \frac{1}{2\hbar}\hat{\mathbf{x}}^\mathsf{T}\mathsf{J}(-\mathbb{\Gamma} + \mathbb{\Gamma}^*)\mathsf{J}\hat{O}\hat{\mathbf{x}} = -\frac{\imath}{\hbar}\hat{\mathbf{x}}^\mathsf{T}\mathsf{J}\mathbb{C}\mathsf{J}\hat{O}\hat{\mathbf{x}}\\ &= \frac{\imath}{\hbar}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\mathbf{x}\hat{O}\mathbf{x}^\mathsf{T}).\end{aligned}\tag{C8}$$

But, $\hat{\mathbf{x}}\hat{O}\hat{\mathbf{x}}^T = -\hat{\mathbf{x}}[\hat{\mathbf{x}}^T, \hat{O}] + \hat{\mathbf{x}}\hat{\mathbf{x}}^T\hat{O}$ where we define the matrix notation: $(\hat{\mathbf{y}}, [\hat{\mathbf{y}}^\mathsf{T}, \hat{O}])_{sm} = \hat{y}_s, [\hat{y}_m, \hat{O}]$. Therefore,

$$\begin{aligned}\mathcal{L}_3[\hat{O}] &= \frac{\imath}{\hbar}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\mathbf{x}\hat{O}\mathbf{x}^T)\\ &= -\frac{\imath}{\hbar}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\hat{\mathbf{x}}[\hat{\mathbf{x}}^T, \hat{O}]) + \imath\,\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\hat{O}\hat{\mathbb{V}}) - \frac{1}{2}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}^2\hat{O})\\ &= -\frac{\imath}{\hbar}\mathrm{tr}(\mathbb{C}\mathsf{J}\hat{\mathbf{x}}[\hat{\mathbf{x}}^T, \hat{O}]\mathsf{J}) + \imath\hat{O}\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\hat{\mathbb{V}}) + \frac{1}{2}\mathrm{tr}(\mathsf{J}\mathbb{C})\hat{O}\\ &= \mathrm{tr}(\mathbb{C}\mathsf{J}\hat{\mathbb{M}}[\hat{O}]) + \mathcal{L}_2[\hat{O}],\end{aligned}\tag{C9}$$

where we define the superoperator matrix $\hat{\mathbb{M}}[\hat{O}] = \frac{\imath}{\hbar}(\mathbf{x})[(\mathsf{J}\mathbf{x})^\mathsf{T}, \hat{O}]$ and we used Eq. (C7), that $\mathrm{tr}(\mathsf{J}\mathbb{C}\mathsf{J}\hat{\mathbb{V}}) = 0$, $\mathsf{J}^2 = -\mathbb{1}$, and

$$\begin{aligned}(\mathsf{J}\hat{\mathbf{x}}[\hat{\mathbf{x}}^T, \hat{O}]\mathsf{J})_{lj} &= \sum_{r=1}^{2n}\sum_{s=1}^{2n}\mathsf{J}_{lr}\hat{x}_r[\hat{x}_s, \hat{O}]\mathsf{J}_{sj}\\ &= \sum_{r=1}^{2n}\sum_{s=1}^{2n}\mathsf{J}_{lr}\hat{x}_r[\mathsf{J}_{sj}\hat{x}_s, \hat{O}]\\ &= -\sum_{r=1}^{2n}\mathsf{J}_{lr}\hat{x}_r\left[\sum_{s=1}^{2n}\mathsf{J}_{js}\hat{x}_s, \hat{O}\right]\\ &= -(\mathsf{J}\hat{\mathbf{x}}[(\mathsf{J}\hat{\mathbf{x}})^T, \hat{O}])_{lj}.\end{aligned}\tag{C10}$$

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[83] In a GDS the time dependence of first moments is $\langle \hat{\mathbf{x}}\rangle_t = e^{\mathbb{A}t}(\langle \hat{\mathbf{x}}\rangle_{\hat{\rho}_0} - \mathbb{A}^{-1}\boldsymbol{\xi}) + \mathbb{A}^{-1}\boldsymbol{\xi}$. Because the GDS has a stationary state, $\mathbb{A}$ is asymptotically stable, then $\lim_{t\to+\infty}\langle \hat{\mathbf{x}}\rangle_t = \mathbb{A}^{-1}\boldsymbol{\xi} = -\mathbb{B}^{-1}\mathsf{J}\boldsymbol{\xi}$, where the last equality results for a thermal state constructed with $\hat{H}$ in Eq. (26). But $\mathbb{A}^{-1}\boldsymbol{\xi} = -\mathbb{B}^{-1}\mathsf{J}\boldsymbol{\xi}$ implies $(-\mathbb{C}\mathsf{J}\mathbb{B}^{-1}\mathsf{J})\boldsymbol{\xi} = 0$. Because $-\mathbb{C}\mathsf{J}\mathbb{B}^{-1}\mathsf{J}$ has an inverse we must have $\boldsymbol{\xi} = 0$.
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