

Noise limits on two-photon interferometric sensingHannah Scott¹, Dominic Branford¹, Niclas Westerberg², Jonathan Leach¹ and Erik M. Gauger¹¹*SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*²*School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, United Kingdom*

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When a photon interferes with itself while traversing a Mach-Zehnder interferometer, the output port where it emerges is influenced by the phase difference between the interferometer arms. This allows for highly precise estimation of the path length difference (delay) but is extremely sensitive to phase noise. By contrast, a delay between the arms of the two-photon Hong-Ou-Mandel interferometer directly affects the relative indistinguishability of the photon pair, affecting the rate of recorded coincidences. This likewise allows for delay estimation, notably less precise but with the advantage of being less sensitive to perturbations of the photons' phase. Focusing on two-photon input states, we here investigate to what degree of noise Mach-Zehnder interferometry retains its edge over Hong-Ou-Mandel interferometry. We also explore the competing benefits of different two-photon inputs for a Mach-Zehnder interferometer and under what parameter regimes each input performs best.

DOI: [10.1103/PhysRevA.104.053704](https://doi.org/10.1103/PhysRevA.104.053704)**I. INTRODUCTION**

Interference lies at the heart of optical metrology: Observing changes in the recorded interference patterns allows for precise measurements of sample and environmental parameters. This is commonly realized with Michelson, Mach-Zehnder, Fabry-Pérot, Sagnac, and Hong-Ou-Mandel interferometers [1–4]. The quintessential task for interferometry is estimating optical delays which manifest as phase shifts in the traditional Mach-Zehnder (MZ) interferometer or distinguishability in Hong-Ou-Mandel (HOM) interferometry. For example, laser-interferometric gravitational wave detectors use sensitivity to physical displacements to achieve extraordinarily precise measurements of mechanical displacements across a broad range of frequencies [5,6].

The operating principle of the MZ interferometer derives from interference fringes with a period determined by the optical frequency [2]; these fringes shift according to the relative phase in the interferometer. By contrast, HOM [7] interferometers only possess a single interference dip with a width determined by the spectral distribution of the input photons; this dip is displaced according to the relative delay between the two input paths.

In both cases, path delay can be estimated from the readout of detectors placed at the two interferometer output ports. For the single-photon MZ interferometer this is through the ratio of clicks between detectors [3,8,9]; in the two-photon HOM case the delay influences the rate at which photons bunch [10–16].

The MZ analyses can be extended to include multiphoton inputs with nonmonochromatic [17–19] and monochromatic light [3,20,21]. This is expected to give rise to significant benefits through the application of nonclassical light which can obtain more favorable scalings in the number of particles used [3,20,21]. However, reaching the high-photon regime

where such nonclassical light becomes beneficial compared to classical light is experimentally demanding [21]. Instead, one of the foremost concerns is in the probing of so-called delicate samples which are sensitive to high-photon numbers and can genuinely benefit from the application of optimum few-photon probe states [22–26].

The narrow wavelength-order fringes of the Mach-Zehnder interferometry may intuitively be expected to result in higher sensitivity than the much broader HOM dip, which varies on the order of the inverse spectral width rather than inverse frequency. In principle, MZ interferometry allows for arbitrarily high precision with sufficiently high frequency; however, it suffers from the phase-wrapping problem [27–29] where multiple phases can produce the same output signal. In practice, identifying the true phase requires prior information or the use of adaptive techniques. By comparison, the HOM effect remains technically distinct across half the dip, affording a much larger dynamic range at the cost of generally reduced estimation precision.

This gap between phase-insensitive and phase-sensitive interferometry has recently been narrowed with experiments that have reported much improved sensitivities to HOM-based sensing [11,12]. This raises the question of whether there are settings where HOM interferometry, which may already be practically desirable due to the wider dynamic range and relatively simpler optics, can compete with or even surpass phase-sensitive interferometry.

In this work we thus focus on interferometry with photon pairs as input states to Mach-Zehnder and Hong-Ou-Mandel interferometers as the archetypal phase-sensitive and phase-insensitive interferometers. Rather than limiting our study to monochromatic inputs, we consider Gaussian two-photon spectral distributions such as those readily produced by generic down-conversion sources. We will be interested in the comparative performance between those different approaches,

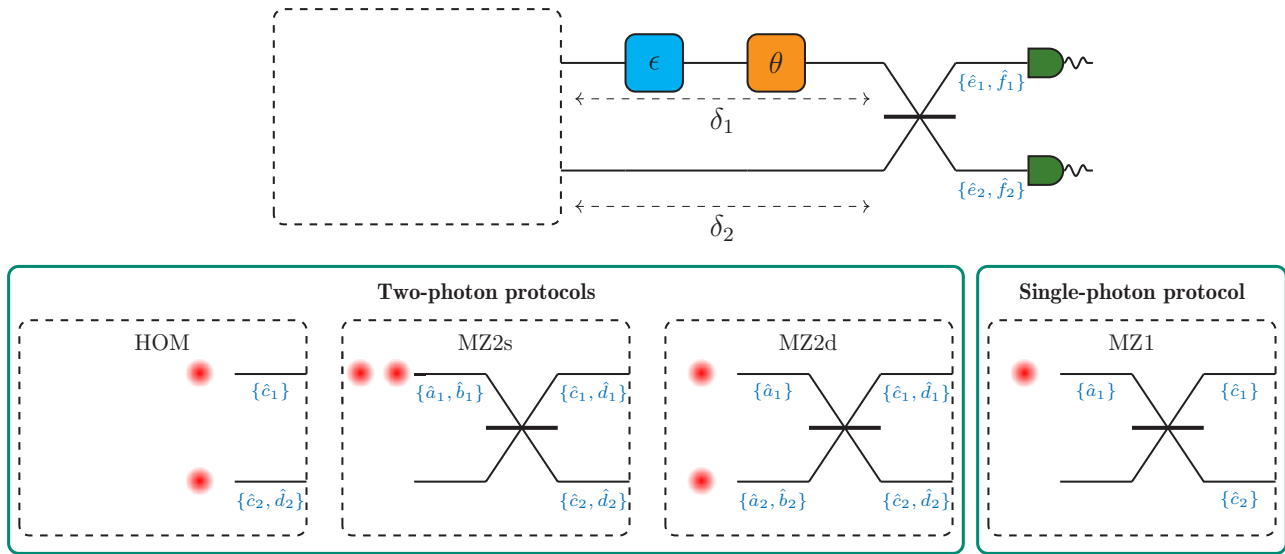


FIG. 1. Generalized schematic for our four protocol configurations. The upper arm contains two phase shifts: A frequency-dependent shift ϵ and a frequency-independent shift θ . Noise is subsequently modeled as some variation in these shifts. Photon modes are labeled at the different stages of the schematic; at each stage we work with a combination of two orthogonal modes to model the initial indistinguishability within the photon pair.

including a comparison of different two-photon MZ input states. We later additionally compare to the results for a single-photon MZ protocol.

With our focus on photon pair inputs, we compare the performance of different protocols at varying degrees of phase noise, including the high-noise limit. Our noise model is similar to that of Refs. [30–32], which explored the ultimate quantum limit of phase estimation¹ in the presence of phase diffusion. Much as the best probe states for Mach-Zehnder interferometry are undermined by noise [31,33–35], we may anticipate that the HOM interferometer is capable of delivering valuable precision (relative to a two-photon Mach-Zehnder interferometer) in high-noise regimes.

This article is organized as follows. In Sec. II we describe our protocols with fixed frequency-dependent and frequency-independent phase shifts. We then move to our noisy model in Sec. III and derive the relevant probabilities and expressions for the Fisher information. In Sec. IV we outline how our model can be tweaked to account for frequency-independent input photons and noise that is uncorrelated between photon modes. Section V presents our results, where we compare the resilience of our various protocols and models to increased noise and also identify and discuss a number of interesting emerging features. We summarize these results and present some concluding comments in Sec. VI.

II. PROTOCOLS

We consider and compare four different protocols, illustrated in Fig. 1 with the common parameter encoding and measurements along the different state preparations: First a standard Hong-Ou-Mandel protocol (HOM), wherein two

photons interfere at a beam splitter and detectors are placed at the two output ports, and then three protocols using a Mach-Zehnder interferometer (MZI), consisting of two beam splitters such that the output ports of the first beam splitter are directed towards the input ports of the second beam splitter, with detectors at the outputs of the second beam splitter. Between the two beam splitters, the upper and lower interferometer arms have path lengths δ_1 and δ_2 , respectively. We consider a pair of two-photon MZ protocols, one in which both photons enter via the same initial input port (MZ2s) and another in which both enter via different input ports (MZ2d), and in addition a conventional single-photon MZ protocol (MZ1).

The predominant type of noise affecting the photons can be expressed as an unknown fluctuating phase shift in one or both of the arms. We can write this as $e^{-i\phi(\omega)}$, where $\phi(\omega)$ has some unspecified frequency dependence. Taylor expanding around ω_0 , we then write $e^{-i\phi(\omega)} \approx \exp\{-i[\phi(\omega_0) + \frac{\partial\phi(\omega_0)}{\partial\omega}(\omega - \omega_0) + \dots]\}$. Truncating terms beyond linear order in $\omega - \omega_0$, we can write the total phase shift from noise as

$$\exp\left[-i\left\{\overbrace{\phi(\omega_0) - \frac{\partial\phi(\omega_0)}{\partial\omega}\omega_0}^{\theta} + \overbrace{\frac{\partial\phi(\omega_0)}{\partial\omega}\omega}^{\epsilon}\right\}\right]. \quad (1)$$

Here we identify a frequency-independent component θ and a frequency-dependent component ϵ to the phase shift. The latter can be thought of as representing fluctuations in the path length, such as those that might arise from vibrations or heating in the system. By contrast, θ is a “pure” phase shift that leaves the path length unaffected. The effects of these shifts are illustrated for different frequency modes in Fig. 2. For the remainder of these sections we will work with unknown but fixed ϵ and θ ; we will later average over them in Sec. III to capture fluctuations in time.

¹This is equivalent to delay estimation with monochromatic photons.

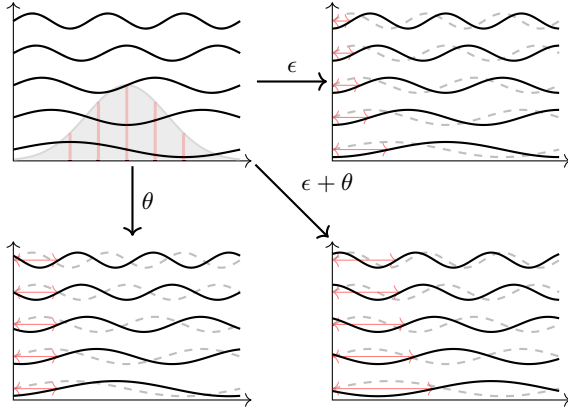


FIG. 2. For a Gaussian wave packet, the effect of our fixed shifts ϵ and θ on the different frequency modes is shown. The ϵ shift is frequency dependent and can be thought of as some modification to the actual delay δ . The top right shows the differing intensity of an ϵ shift at differing frequencies. The θ shift is frequency independent; the bottom left shows that all modes experience the same effect from a θ shift. The bottom right shows a combined ϵ and θ shift. Our noise model is obtained in Sec. III by averaging over these shifts.

For simplicity, we limit these shifts to the upper arm of the interferometer, essentially assuming noise to be localized noise entirely within the upper arm. Though in reality we expect noise to be present in both arms, this localization gives rise to equivalent detection probabilities with phase-insensitive measurements. For justification and further discussion, see Appendix B.

A. Optical modes

A biphoton state generated by spontaneous parametric down-conversion (SPDC) will, in practice, exhibit some degree of nonspatial distinguishability (such as a mismatch of polarizations) between the two photons. Thus we write one photon in the initial superposition $\sqrt{\alpha}\hat{a}^\dagger(\omega) + \sqrt{1-\alpha}\hat{b}^\dagger(\omega)$, with \hat{b} some orthogonal photonic mode. The visibility $\alpha \in [0, 1]$ therefore encodes the relative indistinguishability of the photon pair upon generation. The modes labeled in Fig. 1 (\hat{a}_j, \hat{b}_j ; \hat{c}_j, \hat{d}_j ; and \hat{e}_j, \hat{f}_j) are then the pairs of orthogonal photonic modes in a given arm at each stage of the protocol. The subscripts $j = \{1, 2\}$ denote distinct spatial modes, corresponding to the two arms of the interferometer.

B. Common optics

We describe the initial preparation illustrated by the lower part of Fig. 1 individually in the following subsections, alongside the resulting detection probabilities. For the MZ protocols the initial states are given in terms of the $\{\hat{a}_{1,2}, \hat{b}_{1,2}\}$ modes which pass through a beam splitter with transforms

$$\begin{aligned}\hat{a}_1^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[i\hat{c}_1^\dagger(\omega) + \hat{c}_2^\dagger(\omega)], \\ \hat{a}_2^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[\hat{c}_1^\dagger(\omega) + i\hat{c}_2^\dagger(\omega)],\end{aligned}$$

$$\begin{aligned}\hat{b}_1^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[i\hat{d}_1^\dagger(\omega) + \hat{d}_2^\dagger(\omega)], \\ \hat{b}_2^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[\hat{d}_1^\dagger(\omega) + i\hat{d}_2^\dagger(\omega)].\end{aligned}\quad (2)$$

The HOM protocol inputs photons directly into the delay stage and so the input state is written in terms of the modes $\{\hat{c}_{1,2}, \hat{d}_2\}$.

After the initial preparation stage all photons pass through a common set of linear optics to reach the detectors where the mode transformations are described by

$$\begin{aligned}\hat{c}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon)}e^{-i\theta}\hat{c}_1^\dagger(\omega), \\ \hat{c}_2^\dagger(\omega) &\rightarrow e^{-i\omega\delta_2}\hat{c}_2^\dagger(\omega), \\ \hat{d}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon)}e^{-i\theta}\hat{d}_1^\dagger(\omega), \\ \hat{d}_2^\dagger(\omega) &\rightarrow e^{-i\omega\delta_2}\hat{d}_2^\dagger(\omega),\end{aligned}\quad (3)$$

which encode the local delays alongside the phase shifts ϵ and θ . The detection probabilities depend only on the path length difference $\delta = \delta_2 - \delta_1$. The photons then interfere at the (final) beam splitter,

$$\begin{aligned}\hat{c}_1^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[i\hat{e}_1^\dagger(\omega) + \hat{e}_2^\dagger(\omega)], \\ \hat{c}_2^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[\hat{e}_1^\dagger(\omega) + i\hat{e}_2^\dagger(\omega)], \\ \hat{d}_1^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[i\hat{f}_1^\dagger(\omega) + \hat{f}_2^\dagger(\omega)], \\ \hat{d}_2^\dagger(\omega) &\rightarrow \frac{1}{\sqrt{2}}[\hat{f}_1^\dagger(\omega) + i\hat{f}_2^\dagger(\omega)],\end{aligned}\quad (4)$$

after which detectors measure whether a photon is found in one of the arms. In Appendix A we derive the general detection probabilities in terms of the form of the output state.

C. HOM protocol

For the HOM protocol, we take the biphoton state [7,36]

$$\begin{aligned}|\psi_{\text{HOM}}^{\text{in}}\rangle &= \int d\omega \phi(\omega)\hat{c}_1^\dagger(\omega_p - \omega) \\ &\times [\sqrt{\alpha}\hat{c}_2^\dagger(\omega) + \sqrt{1-\alpha}\hat{d}_2^\dagger(\omega)]|0\rangle\end{aligned}\quad (5)$$

as input, with

$$\phi(\omega) = (2\pi\sigma^2)^{-1/4}e^{-(\omega-\omega_p/2)^2/4\sigma^2}.\quad (6)$$

Here ω_p is the pump frequency and σ the spectral width [36]. Evolving $|\psi_{\text{HOM}}^{\text{in}}\rangle$ according to Eqs. (3) and (4), we obtain the output state

$$\begin{aligned}|\psi_{\text{HOM}}^{\text{out}}\rangle &= \frac{1}{2} \int d\omega \phi(\omega)e^{-i[(\delta_1+\epsilon)(\omega_p-\omega)+\delta_2\omega+\theta]} \\ &\times [i\hat{e}_1^\dagger(\omega_p - \omega) + \hat{e}_2^\dagger(\omega_p - \omega)] \\ &\times [\sqrt{\alpha}\{\hat{e}_1^\dagger(\omega) + i\hat{e}_2^\dagger(\omega)\} \\ &+ \sqrt{1-\alpha}\{\hat{f}_1^\dagger(\omega) + i\hat{f}_2^\dagger(\omega)\}]|0\rangle.\end{aligned}\quad (7)$$

Equations (A4) and (A5) together with Eq. (7) then give the probabilities of detection at one or both detectors. We find

$$P_{1,\text{HOM}} = P_{2,\text{HOM}} = \frac{1}{4}(1 + \alpha e^{-2\sigma^2(\delta-\epsilon)^2}), \quad (8)$$

the probability of detection at detector 1 only and detector 2 only, and

$$P_{c,\text{HOM}} = \frac{1}{2}(1 - \alpha e^{-2\sigma^2(\delta-\epsilon)^2}), \quad (9)$$

the probability of coincidence at both detectors. These probabilities concur with the probabilities seen in standard HOM analyses [36].

One thing we immediately notice is that θ drops out: The HOM protocol is not affected by arbitrary frequency-independent phase shifts and therefore is immune to frequency-independent noise.² In traditional HOM analyses one often treats $P_{1,\text{HOM}} + P_{2,\text{HOM}}$ as a unified bunching probability, but we here keep them separate for consistency with the MZ analysis.

D. Two-photon MZ protocol: Same input port

We now consider our MZ2s protocol, where both photons enter via the same port (specifically, we choose the upper left port). Our initial state is thus a modified version of our biphoton state from Eq. (5):

$$|\psi_{\text{MZ2s}}^{\text{in}}\rangle = \frac{1}{\sqrt{1+\alpha}} \int d\omega \phi(\omega) [\sqrt{\alpha} \hat{a}_1^\dagger(\omega) + \sqrt{1-\alpha} \hat{b}_1^\dagger(\omega)] \times \hat{a}_1^\dagger(\omega_p - \omega) |0\rangle. \quad (10)$$

The normalization of $\frac{1}{\sqrt{1+\alpha}}$ is required as both photons entering via the same port results in a visibility-dependent overlap of their initial modes.

Up to an irrelevant global phase, we apply Eqs. (2)–(4) to obtain the output state

$$|\psi_{\text{MZ2s}}^{\text{out}}\rangle = \frac{1}{\sqrt{1+\alpha}} \int d\omega \phi(\omega) \times [\sin(\Omega^-) \hat{e}_1^\dagger(\omega_p - \omega) + \cos(\Omega^-) \hat{e}_2^\dagger(\omega_p - \omega)] \times [\sqrt{\alpha} \{\sin(\Omega^+) \hat{e}_1^\dagger(\omega) + \cos(\Omega^+) \hat{e}_2^\dagger(\omega)\} + \sqrt{1-\alpha} \{\sin(\Omega^+) \hat{f}_1^\dagger(\omega) + \cos(\Omega^+) \hat{f}_2^\dagger(\omega)\}] |0\rangle, \quad (11)$$

with

$$\Omega^- = \frac{1}{2}\{\theta - (\delta - \epsilon)(\omega_p - \omega)\}, \quad (12)$$

$$\Omega^+ = \frac{1}{2}\{\theta - (\delta - \epsilon)\omega\}. \quad (13)$$

Taking this output state with Eqs. (A4) and (A5), we obtain the detection probabilities

$$P_{1,\text{MZ2s}} = \frac{1}{8} \left[2 + e^{-2\sigma^2(\delta-\epsilon)^2} + \cos[2\theta - \omega_p(\delta - \epsilon)] - 4e^{-\frac{1}{2}\sigma^2(\delta-\epsilon)^2} \cos\left(\theta - \frac{\omega_p}{2}(\delta - \epsilon)\right) \right], \quad (14)$$

$$P_{2,\text{MZ2s}} = \frac{1}{8} \left[2 + e^{-2\sigma^2(\delta-\epsilon)^2} + \cos[2\theta - \omega_p(\delta - \epsilon)] + 4e^{-\frac{1}{2}\sigma^2(\delta-\epsilon)^2} \cos\left(\theta - \frac{\omega_p}{2}(\delta - \epsilon)\right) \right], \quad (15)$$

$$P_{c,\text{MZ2s}} = \frac{1}{4} \{2 - \cos[2\theta - \omega_p(\delta - \epsilon)] - e^{-2\sigma^2(\delta-\epsilon)^2}\}. \quad (16)$$

As both photons enter the same port, the state at every point has the form $\hat{a}_\zeta(\omega)[\sqrt{1-\alpha}\hat{a}_\zeta(\omega') + \sqrt{\alpha}\hat{b}_\zeta(\omega')]$, where \hat{a}_ζ and \hat{b}_ζ are orthogonal photonic modes and \hat{a}_ζ (\hat{b}_ζ) can be a superposition of the spatial modes \hat{a}_1 and \hat{a}_2 (\hat{b}_1 and \hat{b}_2). This prohibits any HOM-like interference which only occurs when the photons enter a beam splitter in nonidentical spatial (superposition) modes. Moreover, as the photonic modes experience identical optical transforms, the nonzero distinguishability ($\alpha < 1$) does not affect the measured outputs, and hence the visibility dependence vanishes from the MZ2s output probabilities.

E. Two-photon MZ protocol: Different input ports

We now consider the MZ2d protocol, where both photons enter the MZI via different ports. This is similar to the two-photon case of Ref. [37], though our protocol is generalized to nonmonochromatic inputs.

Our initial state is the same as in Eq. (5), but with \hat{c} and \hat{d} relabeled as \hat{a} and \hat{b} to reflect that we now have an additional beam splitter. Again, up to an irrelevant global phase, we apply Eqs. (2)–(4) to obtain the output state

$$|\psi_{\text{MZ2d}}^{\text{out}}\rangle = \int d\omega \phi(\omega) \times [\sin(\Omega^-) \hat{e}_1^\dagger(\omega_p - \omega) + \cos(\Omega^-) \hat{e}_2^\dagger(\omega_p - \omega)] \times [\sqrt{\alpha} \{\cos(\Omega^+) \hat{e}_1^\dagger(\omega) + \sin(\Omega^+) \hat{e}_2^\dagger(\omega)\} + \sqrt{1-\alpha} \{\cos(\Omega^+) \hat{f}_1^\dagger(\omega) + \sin(\Omega^+) \hat{f}_2^\dagger(\omega)\}] |0\rangle, \quad (17)$$

with Ω^- and Ω^+ as defined in Eqs. (12) and (13).

Together with Eqs. (A4) and (A5), we obtain the detection probabilities

$$P_{1,\text{MZ2d}} = P_{2,\text{MZ2d}} = \frac{1}{8} \{2 - (1 - \alpha)e^{-2\sigma^2(\delta-\epsilon)^2} - (1 + \alpha) \cos[\omega_p(\epsilon - \delta) + 2\theta]\}, \quad (18)$$

$$P_{c,\text{MZ2d}} = \frac{1}{4} \{2 + (1 - \alpha)e^{-2\sigma^2(\delta-\epsilon)^2} + (1 + \alpha) \cos[\omega_p(\epsilon - \delta) + 2\theta]\}. \quad (19)$$

Visibility dependence is seen for the MZ2d protocol, while it was absent from the MZ2s protocol. We also note that, like the HOM case, $P_1 = P_2$. This initially seems a major deviation from the conventional MZ1 scenario, where varying the phase can bias the photon towards one detector or the other, a feature that remains present in the MZ2s protocol. The crucial difference in this configuration is that after the initial beam splitter, both photons have opposite phase. Setting a phase in the upper arm that biases one photon towards a certain detector must equally bias the other photon towards the other detector: No combination of delay and fixed shifts

²Because the HOM state is still separable at the stage where phase shifts are applied, the frequency-independent shift θ can be thought of as a global phase for the HOM case.

can result in P_1 or P_2 more likely than the other, and hence $P_1 = P_2$.

In the limiting case of $\delta = \epsilon = 0$, our MZ2d coincidence probability in Eq. (19) concurs with the coincidence probability in Ref. [38], which examines two photons entering opposite input ports of an MZI with equal path lengths in both arms but a fixed frequency-independent phase shift in one arm.

F. Single-photon MZ protocol

For the MZ1 protocol we have the input state

$$|\psi_{\text{MZ1}}^{\text{in}}\rangle = \int d\omega \phi(\omega) \hat{a}^\dagger(\omega) |0\rangle, \quad (20)$$

with $\phi(\omega)$ as defined in Eq. (6). Applying the MZ mode evolutions described by Eqs. (2) to (4), we obtain the output state

$$|\psi_{\text{MZ1}}^{\text{out}}\rangle = \frac{1}{2} \int d\omega \phi(\omega) [(e^{-i\delta_2\omega} - e^{-i[\theta+(\delta_1+\epsilon)\omega]}) \hat{c}_1^\dagger(\omega) + (e^{-i[\theta+(\delta_1+\epsilon)\omega]} + ie^{-i\delta_2\omega}) \hat{c}_2^\dagger(\omega)] |0\rangle. \quad (21)$$

The resulting detection probabilities are then

$$\begin{aligned} P_{1,\text{MZ1}} &= \frac{1}{4} \int d\omega |\phi(\omega)|^2 |e^{-i\delta_2\omega} - e^{-i[\theta+(\delta_1+\epsilon)\omega]}|^2 \\ &= \frac{1}{2} - \frac{1}{2} e^{-\frac{1}{2}\sigma^2(\delta-\epsilon)^2} \cos\left(\theta - \frac{\omega_p}{2}(\delta - \epsilon)\right), \end{aligned} \quad (22)$$

the probability of detection at detector 1, and

$$\begin{aligned} P_{2,\text{MZ1}} &= \frac{1}{4} \int d\omega |\phi(\omega)|^2 |ie^{-i[\theta+(\delta_1+\epsilon)\omega]} + ie^{-i\delta_2\omega}|^2 \\ &= \frac{1}{2} + \frac{1}{2} e^{-\frac{1}{2}\sigma^2(\delta-\epsilon)^2} \cos\left(\theta - \frac{\omega_p}{2}(\delta - \epsilon)\right), \end{aligned} \quad (23)$$

the probability of detection at detector 2.

III. MODELING NOISE

To move from our probabilities in Sec. II to those that model noise, we now assume some uncertainty in our previously fixed phase shifts. Assuming that both ϵ and θ fluctuate around zero, we average over them with the Gaussian weighting factors

$$J_\epsilon(\epsilon) = \frac{e^{-\epsilon^2/2\eta_\epsilon^2}}{\sqrt{2\pi}\eta_\epsilon}, \quad J_\theta(\vartheta) = \frac{e^{-\vartheta^2/2\eta_\theta^2}}{\sqrt{2\pi}\eta_\theta}. \quad (24)$$

Here η_ϵ and η_θ control the width of the Gaussian weighting factors, the strength of these noise processes. These terms give rise to shot-to-shot variations on top of the path length δ on the order of η_θ/ω_p and η_ϵ .

The probabilities for such noise distributions are given by

$$P_j^\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\vartheta d\epsilon J_\epsilon(\epsilon) J_\theta(\vartheta) P_j. \quad (25)$$

For simplicity, we have chosen to use a Gaussian noise distribution with support $(-\infty, \infty)$ for frequency-independent (θ) noise. Note, however, that this is, for the probabilities under consideration, equivalent to a more conventional wrapped $[-\pi, \pi]$ Gaussian distribution. This equivalence is shown

in Appendix E, and our core results are contrasted against results where a von Mises distribution is chosen for frequency-independent noise.³

It is important to emphasize that although η_ϵ and η_θ represent the degree of uncertainty in the pair of phase shifts and these shifts may fluctuate over time as multiple photon pairs traverse the interferometer, in a single run of the protocol both photons will experience the same constant (but unknown) ϵ and θ shifts simultaneously. This fact leads to some interesting consequences that we will observe in Sec. V and explore in more detail in Appendix C.

From these probabilities we calculate the Fisher information, which quantifies the information obtained about a parameter of interest (the path length difference δ) from an average measurement, given a set of measurement outcomes and their associated probabilities. For a parameter δ and measurement outcomes $m \in \mathcal{M}$, with $P(m|\delta)$ the probability of outcome m given δ , the Fisher information can be written [39]

$$F(\delta) = \sum_{m \in \mathcal{M}} \frac{1}{P(m|\delta)} \left(\frac{\partial}{\partial \delta} P(m|\delta) \right)^2. \quad (26)$$

The single-parameter Fisher information can then be used to bound the variance of an unbiased estimator for that parameter. With $\tilde{\delta}$ an unbiased estimator for δ and N the number of independent measurements, the Cramér-Rao bound is given by

$$\text{var}(\tilde{\delta}) \geq \frac{1}{NF(\delta)} \quad (27)$$

and represents the ultimate limit on the precision of an unbiased estimator $\tilde{\delta}$.

A. The HOM protocol

For our HOM protocol the probabilities are

$$P_{1,\text{HOM}}^\eta = P_{2,\text{HOM}}^\eta = \frac{1}{4} \left(1 + \frac{\alpha e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} \right), \quad (28)$$

$$P_{c,\text{HOM}}^\eta = \frac{1}{2} \left(1 - \frac{\alpha e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} \right). \quad (29)$$

If we let $\eta_\epsilon = 0$, the scenario with no noise, this reduces to $P_{j,\text{HOM}}$ with $\epsilon = 0$, as we would expect.

From these probabilities, we can calculate the Fisher information

$$F_{\text{HOM}}^\eta = \frac{16\alpha^2\delta^2\sigma^4}{(4\eta_\epsilon^2\sigma^2+1)^2 [(4\eta_\epsilon^2\sigma^2+1)e^{4\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)} - \alpha^2]}. \quad (30)$$

As we would expect, large frequency-dependent noise washes out all information from the protocol: $F_{\text{HOM}}^\eta \rightarrow 0$ as $\eta_\epsilon \rightarrow \infty$.

³The frequency-dependent (ϵ) noise, like δ , does not have such a symmetry as the different frequency modes accumulate a phase due to ϵ noise (or change in δ).

B. The MZ2s protocol

For the MZ2s protocol, our noisy probabilities can be calculated as in the HOM case but with an additional averaging over θ . The resulting probabilities are

$$P_{1,MZ2s}^\eta = \frac{1}{8} \left(2 + \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} + \cos(\delta\omega_p) e^{-2\eta_\theta^2 - \eta_\epsilon^2\omega_p^2/2} - \frac{4 \cos\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right) e^{-\kappa}}{\sqrt{\eta_\epsilon^2\sigma^2+1}} \right), \quad (31)$$

$$P_{2,MZ2s}^\eta = \frac{1}{8} \left(2 + \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} + \cos(\delta\omega_p) e^{-2\eta_\theta^2 - \eta_\epsilon^2\omega_p^2/2} + \frac{4 \cos\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right) e^{-\kappa}}{\sqrt{\eta_\epsilon^2\sigma^2+1}} \right), \quad (32)$$

$$P_{c,MZ2s}^\eta = \frac{1}{4} \left(2 - \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} - \cos(\delta\omega_p) e^{-2\eta_\theta^2 - \eta_\epsilon^2\omega_p^2/2} \right), \quad (33)$$

with

$$\kappa = \frac{4[\sigma^2(\delta^2 + \eta_\theta^2\eta_\epsilon^2) + \eta_\theta^2] + \eta_\epsilon^2\omega_p^2}{8(1 + \eta_\epsilon^2\sigma^2)}. \quad (34)$$

The Fisher information can then be calculated from the above probabilities, though the resulting expression is not particularly illuminating so we here omit it; see the Supplemental Material [40] for the full expression. Once again, information decays to zero as η_ϵ is increased. However, intriguingly, the same is not true when increasing η_θ : In this case the information instead decays until it resembles a HOM-like Fisher information curve, converging at around $\eta_\theta \approx 3.5$. This curious result will be discussed in more detail in Sec. V and Appendix C.

C. The MZ2d protocol

We follow the same method to obtain the noisy probabilities for the MZ2d protocol:

$$P_{1,MZ2d}^\eta = P_{2,MZ2d}^\eta = \frac{1}{8} \left(2 - \frac{(1-\alpha)e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} - (1+\alpha)\cos(\delta\omega_p) e^{-2\eta_\theta^2 - \eta_\epsilon^2\omega_p^2/2} \right), \quad (35)$$

$$P_{c,MZ2d}^\eta = \frac{1}{4} \left(2 + \frac{(1-\alpha)e^{-2\delta^2\sigma^2/(4\eta_\epsilon^2\sigma^2+1)}}{\sqrt{4\eta_\epsilon^2\sigma^2+1}} + (1+\alpha)\cos(\delta\omega_p) e^{-2\eta_\theta^2 - \eta_\epsilon^2\omega_p^2/2} \right). \quad (36)$$

Once again, we omit the unwieldy Fisher information; the full expression is given in the Supplemental Material [40]. Like the MZ2s protocol, we see a full decay to zero at high η_ϵ but decay to a fixed HOM-like curve at high η_θ .

D. The MZ1 protocol

Finally, following the same method, we derive the noisy probabilities for the MZ1 protocol,

$$P_{1,MZ1}^\eta = \frac{1}{2} \left(1 - \frac{\cos\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right) e^{-\kappa}}{\sqrt{\eta_\epsilon^2\sigma^2+1}} \right), \quad (37)$$

$$P_{2,MZ1}^\eta = \frac{1}{2} \left(1 + \frac{\cos\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right) e^{-\kappa}}{\sqrt{\eta_\epsilon^2\sigma^2+1}} \right), \quad (38)$$

with κ as defined in Eq. (34). The Fisher information is therefore

$$F_{MZ1}^\eta = \frac{[\omega_p \sin\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right) + 2\delta\sigma^2 \cos\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right)]^2}{4(\eta_\epsilon^2\sigma^2+1)^2[(\eta_\epsilon^2\sigma^2+1)e^{2\kappa} - \cos^2\left(\frac{\delta\omega_p}{2\eta_\epsilon^2\sigma^2+2}\right)]}. \quad (39)$$

Again, as η_ϵ increases the Fisher information tends to zero. In this case, the same is true for frequency-independent noise via its dependence on κ [Eq. (34)], i.e., $F_{MZ1}^\eta \rightarrow 0$ as $\eta_\theta \rightarrow \infty$.

IV. TWO-PHOTON MODEL VARIATIONS

For the purpose of probing and better understanding the origin of some of the subtleties we will discuss in the next section, we introduce some slight variations for the noise model, which primarily affect the two-photon MZ case. These are explained in the following; the full expression for the probabilities and Fisher information, for every combination of protocol and model, are given in the Supplemental Material [40].

A. Mode-uncorrelated noise

So far we have applied a common phase shift to both the \hat{c} and \hat{d} photonic modes and so the noise affects the indistinguishable and distinguishable components in a correlated fashion. We now consider the case where the two-photon modes experience distinct phase shifts such that the propagation transformations for the MZ2s and MZ2d protocols, previously given in Eq. (3), now take the form

$$\begin{aligned} \hat{c}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon_1)} e^{-i\theta_1} \hat{c}_1^\dagger(\omega), \\ \hat{c}_2^\dagger(\omega) &\rightarrow e^{-i\omega\delta_2} \hat{c}_2^\dagger(\omega), \\ \hat{d}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon_2)} e^{-i\theta_2} \hat{d}_1^\dagger(\omega), \\ \hat{d}_2^\dagger(\omega) &\rightarrow e^{-i\omega\delta_2} \hat{d}_2^\dagger(\omega). \end{aligned} \quad (40)$$

Now ϵ_1 and θ_1 solely shift the \hat{c}_1 mode, with ϵ_2 and θ_2 being shifts for the \hat{d}_1 mode. The initial state and other transformations remain unchanged. This leads to a subtly different output state, from which we can derive the probabilities as before.

For the MZ1 protocol, this obviously need not be considered as only one photon, and thus one single mode, is considered. For the HOM protocol, the orthogonal mode does not interfere at the beam splitter, and any phase shifts applied drop out. Hence, this only has a tangible effect on the MZ2s and MZ2d protocols, where a photon in a superposition of modes interferes *with itself*.

A minor effect of working with mode-uncorrelated noise is that the resulting probabilities for the MZ2s protocol are now

TABLE I. Comparison of the Fisher information behavior of our different protocols and mode-correlated noise (MC), mode-uncorrelated noise (MU), frequency-entangled photon (FE), and independent photon (Ind.) variants. Here α is the visibility of our input photon pair. The dagger entry means that for the MZ2s protocol with mode-uncorrelated noise, the visibility dependence is only present at nonzero noise. If $\eta_\epsilon = \eta_\theta = 0$, the information remains constant as visibility is varied.

Fisher information behavior	HOM		MZ2s				MZ2d					
	MC		MU		MC		MU		MC		MU	
	FE	Ind.	FE	Ind.	FE	Ind.	FE	Ind.	FE	Ind.	FE	Ind.
Residual oscillating information as $\delta \rightarrow \infty$ (Fig. 5)					✓		✓		✓		✓	
Visibility dependence	✓	✓	✓	✓	✓		†	†	✓	✓	✓	✓
HOM-like residual at high η_θ independent of α [Fig. 4(a)]					✓	✓						
HOM-like residual at high η_θ <i>inversely</i> proportional to α [Fig. 4(b)]									✓	✓		
HOM-like residual at high η_θ proportional to α	✓	✓	✓	✓			✓	✓				
No HOM-like residual at high η_θ											✓	✓

visibility dependent as the uncorrelated noise means the two photonic modes see different noise and so only experience the same optics on average. This dependence naturally drops out at zero noise.

B. Independent photons

We have previously assumed our photons to be frequency entangled, such as a photon pair generated by SPDC. We can also compare them to the case where the input photons are independent (but with frequencies peaked around the same value).

For the HOM and MZ2d protocols, the initial state then looks like

$$\begin{aligned}
 |\psi_{\text{HOM}}^{\text{in}}\rangle &= |\psi_{\text{MZ2d}}^{\text{in}}\rangle \\
 &= \int d\omega_1 d\omega_2 \phi(\omega_1) \phi(\omega_2) [\sqrt{\alpha} \hat{a}_2^\dagger(\omega_1) \\
 &\quad + \sqrt{1-\alpha} \hat{b}_2^\dagger(\omega_1)] \hat{a}_1^\dagger(\omega_2) |0\rangle, \quad (41)
 \end{aligned}$$

where \hat{a} and \hat{b} can be relabeled \hat{c} and \hat{d} for the HOM case as we omit the first beam splitter. For the MZ2s protocol we get

$$\begin{aligned}
 |\psi_{\text{MZ2s}}^{\text{in}}\rangle &= \frac{1}{\sqrt{1+\alpha}} \int d\omega_1 d\omega_2 \phi(\omega_1) \phi(\omega_2) [\sqrt{\alpha} \hat{a}_1^\dagger(\omega_1) \\
 &\quad + \sqrt{1-\alpha} \hat{b}_1^\dagger(\omega_1)] \hat{a}_1^\dagger(\omega_2) |0\rangle. \quad (42)
 \end{aligned}$$

The MZ1 protocol features only a single photon, so the input is as in Eq. (20).

In deriving the output states the same transformations as detailed in Sec. II can be followed, optionally replacing the transformations in Eq. (3) with those in Eq. (40) if we wish to model mode-uncorrelated noise. These output states are of a common form, and the general detection probabilities for states of this form are given in Eqs. (A7) and (A8). From here, obtaining the noisy probabilities and the Fisher information for each protocol follows the same identical steps as for frequency-entangled photons.

V. RESULTS

The three two-photon protocols (HOM, MZ2s, and MZ2d), for frequency-entangled and separable photons, with

photonic-mode-correlated and -uncorrelated noise are tabulated in Table I along with a characterization of their behavior in different regimes. In the following subsections we discuss in more depth some of the more interesting observations.

For simplicity, we choose not to specify a specific value for the pump frequency and instead scale other quantities relative to an arbitrary ω_p . The spectral width is then written as some fraction of the pump frequency; we fix $\sigma = \frac{\omega_p}{100}$ in the following, which is within the experimentally viable range of the ratios of Refs. [10,11]. While both η_θ and η_ϵ are unbounded in principle, their action within the Mach-Zehnder is limited to the 2π periodicity, with $\eta_\theta \sim 2\pi$ or $\eta_\epsilon \sim 4\pi/\omega_p$ approaching the regime where the effective phase is uniform in 2π .

A. Frequency-dependent noise

Figure 3 illustrates how increasing frequency-dependent noise affects our protocols. Unsurprisingly, when noise is near zero, the HOM protocol is far outclassed by the two MZ protocols. The MZ2d protocol performs best in this low-noise regime, but it drops rapidly as noise increases. The MZ2s protocol is more resilient, suffering a sizable performance hit as noise is increased but retaining its relative advantage for longer. The HOM protocol performs worst but experiences only comparatively minor information loss. For all protocols information decays as η_ϵ is increased, as the phase shifts due to noise grow larger and thus sensitivity to the true delay we wish to measure, δ , is reduced. In the limit $\eta_\epsilon \rightarrow \infty$ the phase shifts from noise become effectively wholly random and detection probabilities are thus constant at $P_1 = P_2 = \frac{P_c}{2}$. The Fisher information is therefore everywhere equal to zero.

Similar plots, for a frequency-independent photon input, are given in Fig. 8 in Appendix D. Though the exact values differ, the same trends follow with increasing noise. This is also true for our mode-uncorrelated noise model.

B. Frequency-independent noise

Initially, increasing frequency-independent noise seems to affect the MZ protocols analogously to frequency-dependent noise, while the HOM protocol is the exception, being entirely unaffected. However, there is an interesting high-noise limit, which we depict in Fig. 4(a). Rather than information tending

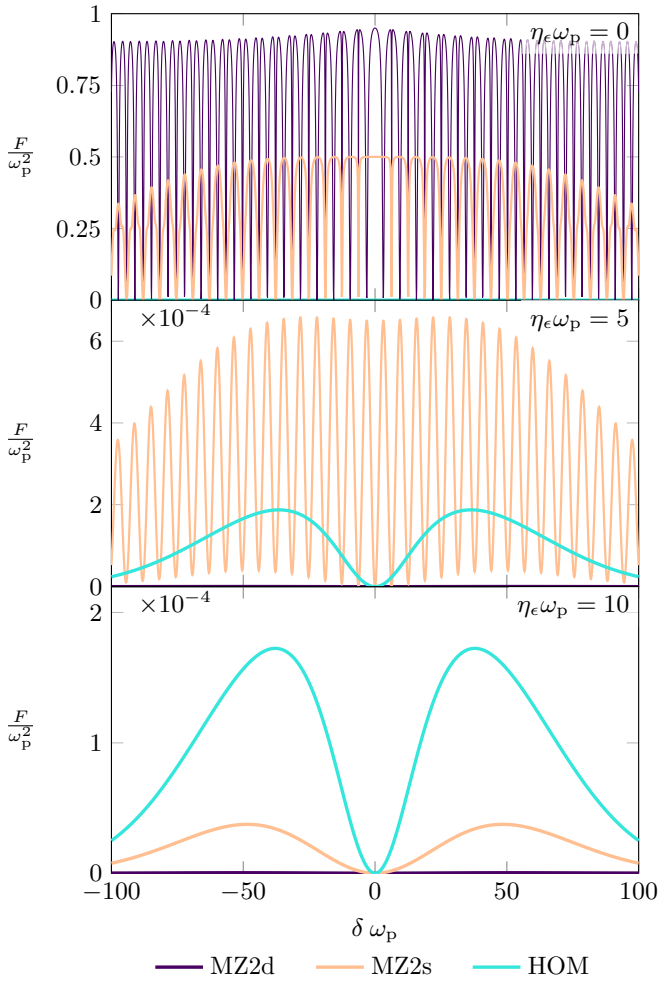


FIG. 3. Comparison of the two-photon protocols, in the frequency-entangled mode-correlated noise configuration with $\sigma = \frac{\omega_p}{100}$ and $\alpha = 0.9$. As frequency noise increases, all protocols see a marked reduction in the Fisher information. The MZ2d protocol performs best at low noise, but is most sensitive to noise. The HOM protocol initially performs by far the worst, but is significantly more resilient to noise. The MZ2s protocol lies in between in regard to both initial performance and resilience to noise.

towards zero everywhere, increased η_θ merely washes out the fringes while on either side of $\delta = 0$ two peaks remain, much like the familiar HOM information curve. Indeed, for the MZ2s protocol this limit exactly matches a HOM curve with 50% visibility. The MZ2d protocol remains visibility dependent even in this limit. Curiously, this result suggests that for a low-visibility ($\alpha < 0.5$) photon pair the MZ2s protocol is always preferable to the HOM protocol for maximizing information.

In low-noise scenarios ($\eta_\theta \lesssim 1.5$), the MZ2d protocol has an intuitive dependence on visibility: Information is highest at $\alpha = 1$ and lowest (though nonzero) at $\alpha = 0$. However, when noise is larger ($\eta_\theta \gtrsim 2.3$) and particularly in the high- η_θ limit, information is inversely proportional to visibility, as seen in Fig. 4(b). At $\alpha = 1$, information vanishes, whereas at $\alpha = 0$ the curve again matches the HOM curve for 50% visibility. In the transition region $1.5 \lesssim \eta_\theta \lesssim 2.3$ the visibility dependence

is more complicated, with some information peaks still proportional to visibility while others are inversely proportional. The visibility-proportional peaks gradually decay with larger η_θ until they become negligible at $\eta_\theta \approx 2.3$.

Considering our mode-uncorrelated noise model, this limit remains for the MZ2s protocol, though it now has a direct visibility dependence (no information at $\alpha = 0$, matches 50% visibility HOM at $\alpha = 1$). However, the MZ2d information now tends to zero everywhere at high η_θ .

If we instead take our initial photons to be frequency independent we see the same general behavior (though the precise value of the Fisher information varies) for both mode-correlated noise and mode-uncorrelated noise. This initially perplexing behavior, which, given the $\alpha \rightarrow 0$ limit, may at first glance appear to suggest interference between wholly distinguishable photons, arises in fact from classical correlations between the paths of the two photons: At large η_θ uncertainty in the value of the frequency-independent phase shift increases, and in the limit $\eta_\theta \rightarrow \infty$ the shift becomes wholly random. However, though this θ shift is random, the same size shift is experienced by both photons. This allows some δ dependence to remain even with distinguishable photons. In the $\eta_\theta \rightarrow \infty$ limit δ -dependent interference at the second beam splitter only occurs when one photon is in each mode (when the photons are bunched they each exit the second beam splitter stochastically). In the MZ2d protocol this component is suppressed, according to visibility, by HOM interference at the first beam splitter giving rise to the increasing Fisher information in spite of decreasing visibility. A deeper analysis of these classical correlations, as well as a discussion of the extra subtleties in the mode-uncorrelated noise model, is given in Appendix C.

C. Oscillatory information

For the MZ protocols, information peaks at $\delta = 0$, which decays as the frequency components lose a common phase. However, when our input photons are frequency entangled F does not decay to zero, but rather we see a constant regular oscillatory information. This is not the case when the input photons are independent: There is still a central peak and some oscillation in the decay, but information ultimately drops to zero at large delays. Both cases are shown in Fig. 5, in the scenario with zero noise.⁴ When noise is introduced (not shown), these fringes decay faster than the central peaks and no fringes remain in the high frequency-independent noise limit.

Such oscillations arise because the frequencies of down-converted photons, though themselves unknown, will always sum to the constant pump frequency ω_p . A quick inspection

⁴The central peaks (at $\delta \rightarrow 0$) coincide at $\frac{1}{2}(4\sigma^2 + \omega_p^2)$ for the MZ2s protocol regardless of whether the photons are frequency-entangled or independent (full expressions are given in Supplemental Material [40]). For the MZ2d protocol, however, the frequency-entangled peak is $2\sigma^2 + \frac{1}{2}(1 + \alpha)\omega_p^2 - 2\alpha\sigma^2$, whereas the frequency-independent peak drops the final term and is simply $2\sigma^2 + \frac{1}{2}(1 + \alpha)\omega_p^2$. This makes separable photons marginally favorable around $\delta \sim 0$.

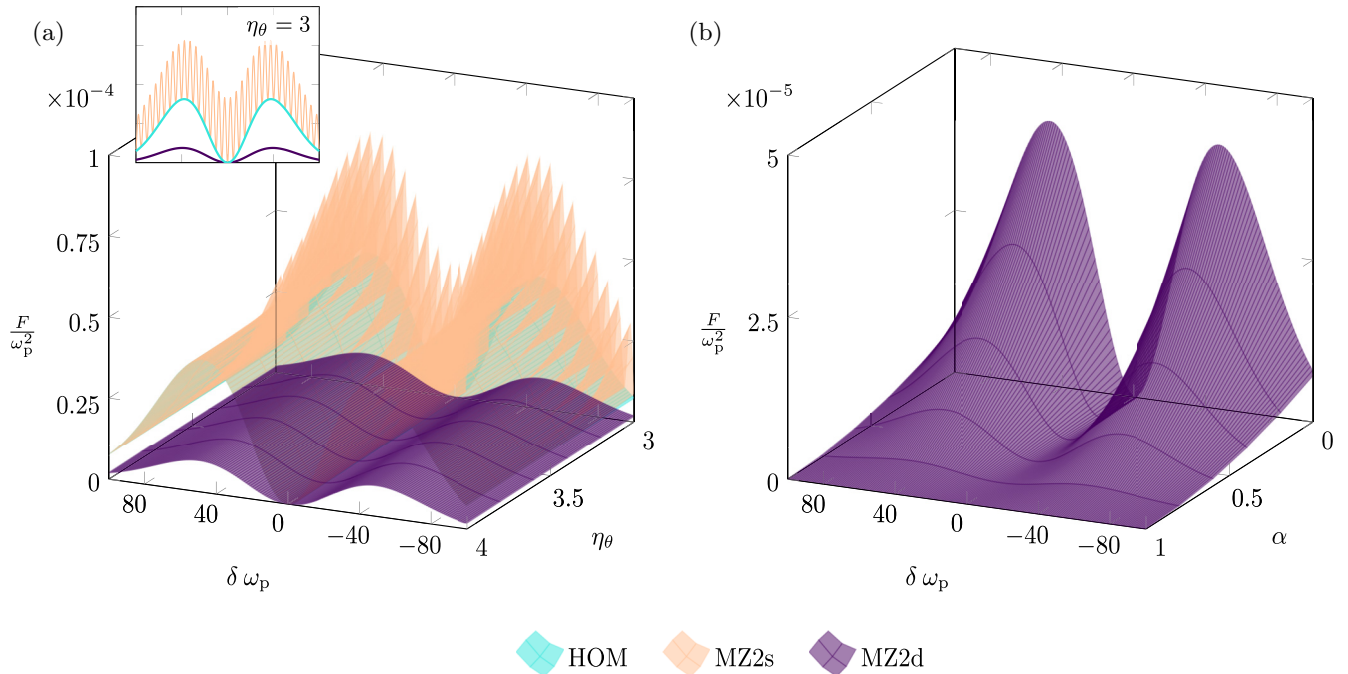


FIG. 4. We examine the behavior at and approaching the high frequency-independent noise limit. (a) Comparison of the two-photon protocols, in the frequency-entangled mode-correlated noise configuration, at high frequency-independent noise with $\sigma = \frac{\omega_p}{100}$ and $\alpha = 0.5$. Frequency-independent noise does not affect the HOM protocol. Due to classical correlations (see Appendix C) between the two photons, for our MZ protocols information is not entirely washed out. Instead, as η_θ increases, both protocols tend towards a HOM-like information curve. For the MZ2d protocol, this curve is visibility dependent. For the MZ2s protocol, which is always visibility independent, it tends towards a curve that exactly matches the HOM curve at 50% visibility. The inset shows the three curves for the far face ($\eta_\theta = 3$). (b) Visibility dependence of the MZ2d protocol, in the high frequency-independent noise limit ($\eta_\theta = 4$) with $\sigma = \frac{\omega_p}{100}$. The information has an inverse dependence on the visibility, and at $\alpha = 0$ the curve matches that of our HOM protocol with $\alpha = 0.5$. See Appendix C for further discussion.

of the frequency-entangled coincidence probabilities for the MZ2s [Eq. (33)] and MZ2d protocols [Eq. (36)] reveal these are indeed oscillations at the pump frequency. These oscillations concur with results from previous two-photon MZ experiments where down-converted photons enter via the same port [17] and via different ports [18], and similar oscillatory behavior is observed in the two-photon Franson interferometer [41].

By contrast, when the photons are independent, their frequencies no longer sum to some constant ω_p . As a result, all probabilities tend to constants at high delays and information decays entirely. Appendix F shows the relation between the detected signal (probabilities) and the resulting Fisher information in the oscillating region.

D. Zero visibility and comparison with the MZ1 protocol

We now want to compare our two-photon MZ protocols to the MZ1 protocol. The MZ1 protocol has no fringes, so it is natural to compare it more directly to the MZ2s and MZ2d protocols with independent photons. We also choose $\alpha = 0$ for the MZ2d protocol so as to prevent interference at the initial beam splitter, which yields additional information.

We might suppose that both the MZ2s and MZ2d protocols are now equivalent to the MZ1 protocol run twice ($2 \times \text{MZ1}$).

However, Fig. 6(b) shows this is only true for the MZ2s protocol, which does indeed have double the information of the MZ1 protocol. Though the MZ2d protocol shares the same central peak, its overall shape is different from the MZ2s protocol (and thus different from the $2 \times \text{MZ1}$ protocol). This is a limitation of the measurements which do not distinguish between the two photonic modes. In the MZ1s and MZ2s protocols we can recognize whether the photon(s) are detected in the same spatial mode they started or the opposite, as the photon(s) always start in a common mode. In the MZ2d protocol the same coincidence event is observed when each photon is detected in the original spatial mode as when they are detected in the opposite spatial mode due to this nondistinguishing measurement. This gives rise to an apparent loss in precision, albeit one which could be alleviated with a distinguishing measurement.

Returning to frequency-entangled photons, Fig. 6(a) shows that we see similarly different Fisher information for both protocols despite $\alpha = 0$, though at large delays the information coincides at the fringes.

E. Maximal information

As a final comparison, Fig. 7 explores the resilience of the MZ1 protocol to noise in comparison to its frequency-entangled two-photon counterparts. Plotting only the maximum information, as both types of noise increase, we again

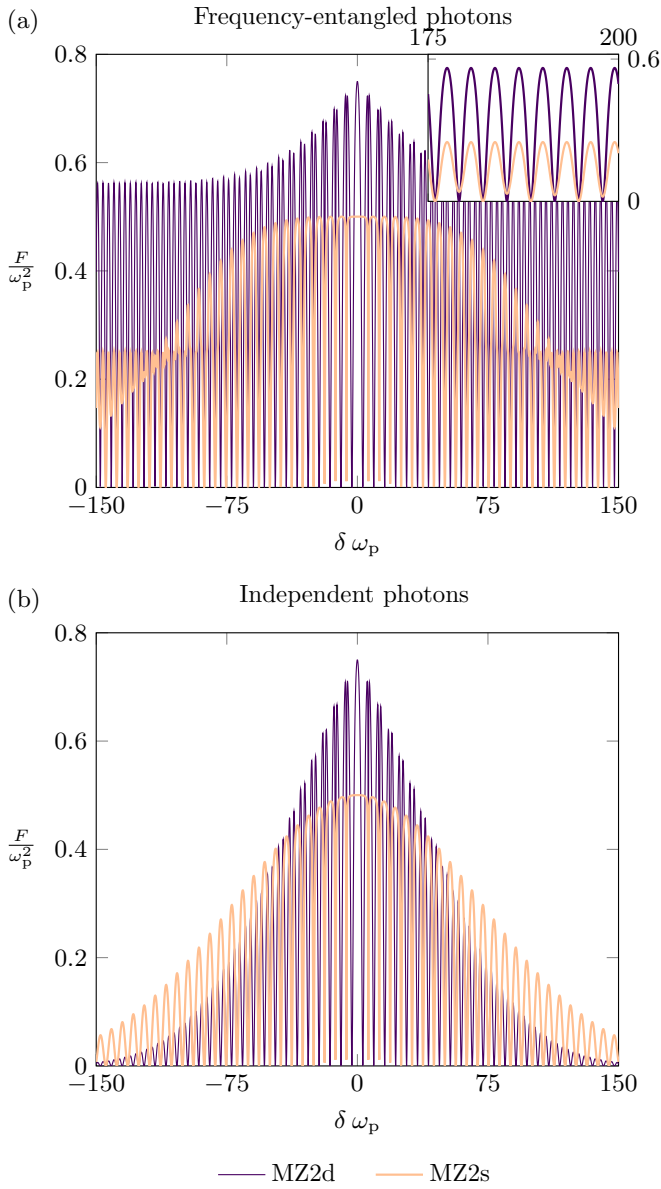


FIG. 5. Fisher information for (a) frequency-entangled vs (b) independent photons with $\sigma = \frac{\omega_p}{100}$, $\alpha = 0.5$, and $\eta_\epsilon = \eta_\theta = 0$. When the photons are frequency entangled, information plateaus to some regular oscillatory information at high $|\delta|$, either side of the central peak [a closeup is shown in the inset in (a)]. These enduring fringes are not present when the input photons are frequency independent, with information gradually decaying to zero at high $|\delta|$.

see the superior performance of the MZ2d protocol at low noise (provided a sufficiently high visibility). We also see, however, that the MZ1 protocol proves slightly more resilient to noise than the MZ2s protocol. Thus, in scenarios where noise is sufficiently high so that the MZ2d protocol is no longer preferable, but before reaching such high values that the HOM protocol is preferable to all the MZ protocols, the best choice is not the MZ2s protocol but rather two individual runs of the MZ1 protocol.

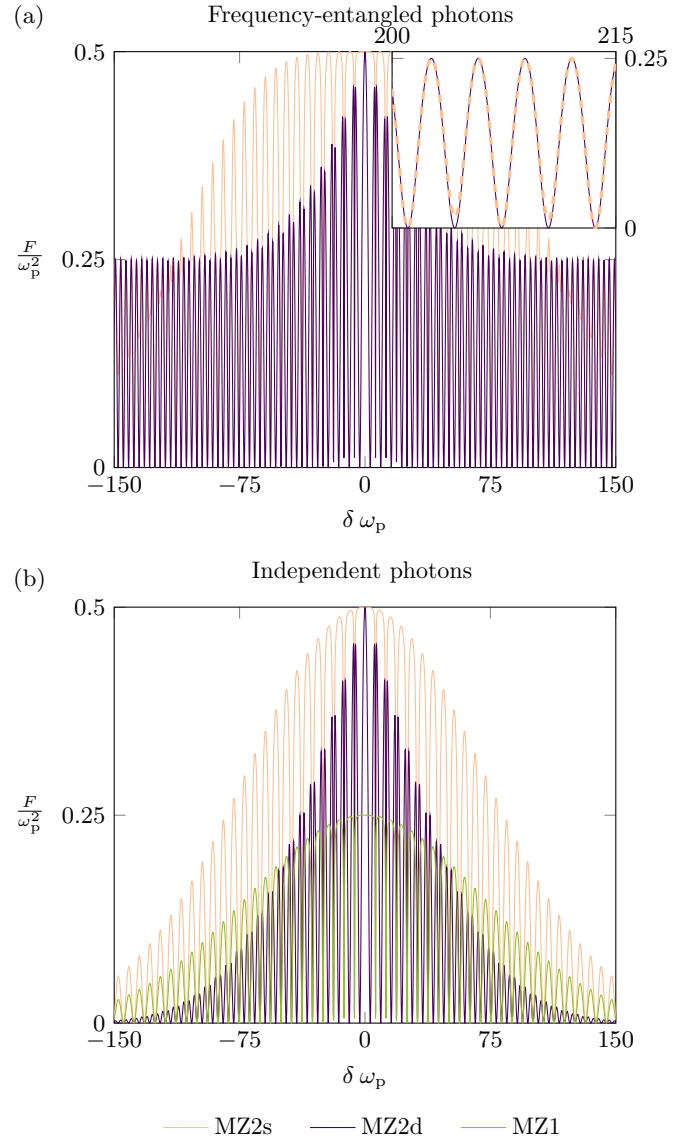


FIG. 6. Fisher information for (a) frequency-entangled vs (b) independent photons, with additional comparison to the single-photon MZ1 protocol in (b). We choose zero visibility to minimize any two-photon interference effects and plot $\sigma = \frac{\omega_p}{100}$ and $\eta_\epsilon = \eta_\theta = 0$. Though the central peaks of the MZ2d and MZ2s protocols coincide, the MZ2d protocol generally performs worse at zero visibility. This is true everywhere when the input photons are independent, but for the frequency-entangled case the two curves converge at the fringes [shown in the inset in (a)]. When the photons are independent, the MZ2s protocol has exactly double the information compared to the MZ1 protocol.

We also see that we require higher values of $\eta_\epsilon \omega_p$ to obtain a similar information decrease compared to lower η_θ values. The disruptive effects of frequency-dependent noise naturally scale with the frequency. Specifically, recalling that the central frequency is $\frac{\omega_p}{2}$, we would expect the performance with frequency-dependent noise η_ϵ to be roughly equivalent to the performance with frequency-independent noise

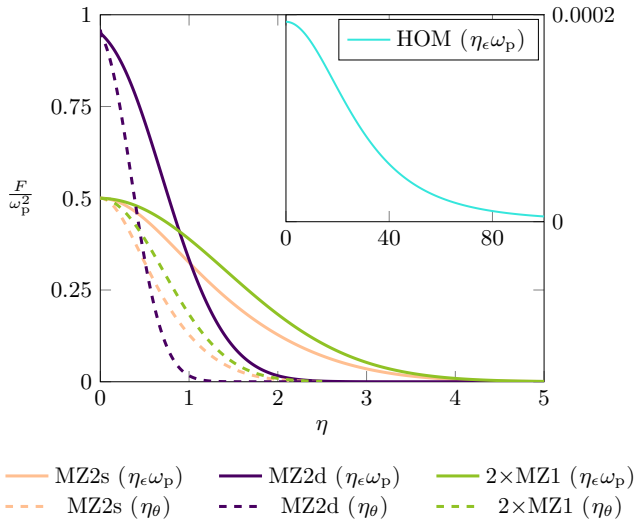


FIG. 7. Maximum Fisher information for the frequency-entangled two-photon Mach-Zehnder protocols against noise with $\sigma = \frac{\omega_p}{100}$ and $\alpha = 0.9$. For comparison, we also plot twice the MZ1 protocol maximum. The HOM protocol performance is shown in the inset. The solid lines show the information decrease as frequency-dependent noise (scaled with ω_p) grows and the dashed lines show the decrease with increasing frequency-independent noise. We saw previously that in the absence of noise two runs of the MZ1 protocol performs comparably to the MZ2s protocol, and the MZ2d protocol beats both provided $\alpha > 0$. We now note that the MZ1 protocol is in fact slightly more resilient to noise: While the MZ2s protocol emerges as superior to the MZ2d protocol as noise grows, it now dips below the performance of two MZ1 protocol runs, except in the high- η_θ limit where the MZ2s protocol retains some residual information [see Fig. 4(a)] while the MZ1 protocol information decays fully to zero.

$\eta_\theta = \eta\epsilon \frac{\omega_p}{2}$.⁵ Thus, normalizing as we do with respect to the pump frequency ω_p , we see both sets of maximal Fisher information curves in Fig. 7 separated by a factor of 2.

In perfect conditions (no noise, $\alpha = 1$) the MZ2d protocol performs twice as well as the MZ2s protocol and four times as well as the MZ1 protocol. This can be associated with the monochromatic limits where the MZ2d probe state equates to a Heisenberg-scaling two-photon NOON state, while the MZ1 and MZ2s probes are shot-noise-limited Fock states [3]. Additionally, Table II gives the requisite noise values that reduce the peak information by 50% compared to the peak information in the absence of noise.

VI. CONCLUSION

Our results confirm that under ideal, i.e., noiseless, conditions Mach-Zehnder interferometry offers superior performance over Hong-Ou-Mandel interferometry for the purposes of delay estimation on a per photon basis. Once noise is introduced, we show that, in keeping with expectations, the HOM

⁵For an immediate justification for why they cannot be exactly equal, recall that the Fisher information tends to different high noise limits for each type of noise.

TABLE II. Noise values where peak information is halved compared to the zero-noise peak. The HOM protocol is impervious to frequency-independent noise.

Noise	MZ2d	MZ2s	MZ1	HOM
$\eta\epsilon\omega_p$	0.8	1.32	1.66	27.36
η_θ	0.4	0.66	0.83	

protocol proves remarkably resilient to frequency-dependent noise (equivalent to some unknown jitter in the actual delay) and is in fact wholly unaffected by frequency-independent noise (representing some random phase shift in one or both of the arms). In the following, we discuss the performance of MZ interferometry as the level of noise increases, with a focus on the difference between our three two-photon MZ protocols.

Generally, we have found that two independent runs of a conventional single-photon MZ protocol is preferable to a simultaneous two-photon run where both enter via the *same* input port of the MZI, as in our MZ2s protocol, for *any* finite amount of noise. While both are matched at vanishing noise, the superiority of two independent input photons increases with noise severity. The ranking between the (*different* input port) MZ2d protocol and the $2 \times$ MZ1 protocol depends on the level of noise. The best protocol choice for a given noise regime is summarized in Table III.

Specifically, in low-noise scenarios, the MZ2d protocol offers optimal performance provided the visibility of the initial photon pair is sufficiently high. However, beyond a certain noise threshold (the exact point varies depending on other parameters, but for $\alpha = 0.9$ and $\sigma = \frac{\omega_p}{100}$ the threshold is $\eta\epsilon\omega_p \approx 0.9$ or $\eta_\theta \approx 0.45$) the performance of the MZ2d protocol dips below both the $2 \times$ MZ1 protocol and the MZ2s protocol, with the $2 \times$ MZ1 protocol marginally preferable. The experimental setup of Ref. [11] falls within this moderate noise regime: The 2-fs average delay drift and a spectral width $\sigma \approx 4.6 \text{ ps}^{-1}$ correspond to frequency-dependent noise of roughly $\eta\epsilon\omega_p \approx 1.15$.

In high-noise scenarios ($\eta\epsilon\omega_p > 5.6$ or $\eta_\theta > 2.8$ for $\alpha = 0.9$ and $\sigma = \frac{\omega_p}{100}$) the performance of all MZ protocols drops substantially and the best choice is generally to drop the initial beam splitter and perform the delay estimation based on the HOM approach. The exception to this is if visibility

TABLE III. Optimal protocol choice for varying noise regimes. Threshold values are calculated for $\alpha = 0.9$. At higher visibility both the MZ2d and HOM protocols perform better, tightening the range of noise values where the $2 \times$ MZ1 protocol (which is unaffected by visibility) is optimal. At lower visibility the inverse is true: The $2 \times$ MZ1 protocol becomes optimal over a wider range of noise values.

Noise regime		
Low	Moderate	High
$\eta\epsilon\omega_p < 0.9$	$0.9 < \eta\epsilon\omega_p < 5.6$	$\eta\epsilon\omega_p > 5.6$
$\eta_\theta < 0.45$	$0.45 < \eta_\theta < 2.8$	$\eta_\theta > 2.8$
MZ2d	$2 \times$ MZ1	HOM

is low ($\alpha < 0.5$) and noise is largely frequency independent; in this case the MZ2s protocol may remain the superior protocol owing to the classical correlations discussed in Appendix C. Throughout, we see broadly the same qualitative results whether our photons are frequency entangled (as in an SPDC pair) or frequency independent. The only remarkable difference is that the former produces regular oscillatory information at delays larger than the single-photon coherence time, whereas information decays rapidly at large delays if the photons are independent. The results for mode-correlated vs mode-uncorrelated noise are qualitatively similar, though notable differences arise in the interesting limit of high frequency-independent noise, where classical correlation between photon paths results in some residual information remaining when all other interference has been washed out.

In this paper we have not accounted for the possibility of photon loss, instead treating our detectors as 100% efficient. In practical scenarios, photon loss can result in some ambiguity as to the location of the second photon when only a single detector clicks. With conventional bucket detectors, which lack photon-number-resolving capabilities, it is unclear whether this single click represents a true bunching event or is simply a consequence of one photon being lost. This ambiguity can be overcome with the introduction of number-resolving detectors [15]. In accounting for loss, the practical detection probabilities will differ slightly depending on the number-resolving capabilities of the detectors. For either detector type, the Supplemental Material [40] contains the ability to generate loss-dependent expressions for any of the two-photon protocol configurations we have considered.

In summary, our results provide a rigorous investigation of the effects of noise in MZ and HOM interferometers traversed by photon pairs. We have confirmed that HOM interferometry is indeed largely impervious to phase noise and thus remains the favored choice for noisy scenarios. By exploring three different two-photon MZ protocols, we have quantitatively established their considerable resilience to phase noise and uncovered interesting subtleties and differences in performance depending on how the photons are fed in. Notably, the existence of the HOM-like feature in the Fisher information arising from classical correlations and the fact that the MZ2d protocol performs up to twice as well in low-noise scenarios compared to the MZ2s protocol and the $2\times$ MZ1 protocol might be interesting for further exploration and may become increasingly relevant as more phase-stable Mach-Zehnder setups, e.g., on integrated photonic chips, become more readily available.

ACKNOWLEDGMENTS

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APPENDIX A: GENERAL DETECTION PROBABILITIES

1. Frequency-entangled photons

Our initial two-photon protocols involve a frequency-entangled photon pair (such as that generated by SPDC).

Looking at the output states for our HOM [Eq. (7)], MZ2s [Eq. (11)], and MZ2d [Eq. (17)] protocols, these can be written in the general form

$$|\psi^{\text{out}}\rangle = \int d\omega \sum_{i=1}^2 \sum_{j=1}^2 [C_{e_i, e_j}(\omega) \hat{e}_i^\dagger(\omega_p - \omega) \hat{e}_j^\dagger(\omega) + C_{e_i, f_j}(\omega) \hat{e}_i^\dagger(\omega_p - \omega) \hat{f}_j^\dagger(\omega)], \quad (\text{A1})$$

where the C coefficient functions are now all that differ between protocols.

We can now define the positive-operator-valued measure (POVM) elements associated with our detection events. For the case where both photons arrive at a single detector we have

$$\Pi_j = \int d\omega_1 d\omega_2 \left[\frac{1}{2} \hat{e}_j^\dagger(\omega_1) \hat{e}_j^\dagger(\omega_2) |0\rangle \langle 0| \hat{e}_j(\omega_2) \hat{e}_j(\omega_1) + \hat{e}_j^\dagger(\omega_1) \hat{f}_j^\dagger(\omega_2) |0\rangle \langle 0| \hat{f}_j(\omega_2) \hat{e}_j(\omega_1) \right], \quad (\text{A2})$$

where $j \in \{1, 2\}$ indicates which detector we are considering. The factor of $\frac{1}{2}$ in the first term is included to account for double counting. For the case of a coincidence at both detectors we have

$$\Pi_c = \int d\omega_1 d\omega_2 [\hat{e}_1^\dagger(\omega_1) \hat{e}_2^\dagger(\omega_2) |0\rangle \langle 0| \hat{e}_2(\omega_2) \hat{e}_1(\omega_1) + \hat{e}_1^\dagger(\omega_1) \hat{f}_2^\dagger(\omega_2) |0\rangle \langle 0| \hat{f}_2(\omega_2) \hat{e}_1(\omega_1) + \hat{f}_1^\dagger(\omega_1) \hat{e}_2^\dagger(\omega_2) |0\rangle \langle 0| \hat{e}_2(\omega_2) \hat{f}_1(\omega_1)]. \quad (\text{A3})$$

These elements are straightforward sums of projectors onto orthogonal states; therefore their positivity is apparent. We require one further element $\mathbb{1} - (\Pi_1 + \Pi_2 + \Pi_c)$ to strictly complete the POVM; however, as we are only considering states of the form of Eq. (A1), the events associated with that element all occur with probability zero and our sets of probabilities for each protocol sum to one.

We can now calculate the detection probabilities for our general output state. We have

$$P_j = \langle \psi^{\text{out}} | \Pi_j | \psi^{\text{out}} \rangle = \int d\omega \left[|C_{e_j, f_j}(\omega)|^2 + \frac{1}{2} |C_{e_j, e_j}(\omega) + C_{e_j, e_j}(\omega_p - \omega)|^2 \right] \quad (\text{A4})$$

for the probability of detection at detector j and

$$P_c = \langle \psi^{\text{out}} | \Pi_c | \psi^{\text{out}} \rangle = \int d\omega [|C_{e_1, f_2}(\omega)|^2 + |C_{e_2, f_1}(\omega)|^2 + |C_{e_1, e_2}(\omega) + C_{e_2, e_1}(\omega_p - \omega)|^2] \quad (\text{A5})$$

for the probability of coincidence at both detectors. We can now insert the relevant coefficient functions for each output state to obtain the fixed shift probabilities presented in Sec. II.

2. Independent photons

For the alternative model discussed in Sec. IV, we now assume both input photons to have independent frequencies.

The general output state is now of the form

$$|\psi^{\text{out}}\rangle = \int d\omega_1 d\omega_2 \sum_{i=1}^2 \sum_{j=1}^2 [C_{e_i, e_j}(\omega_1, \omega_2) \hat{e}_i^\dagger(\omega_2) \hat{e}_j^\dagger(\omega_1) + C_{e_i, f_j}(\omega_1, \omega_2) \hat{e}_i^\dagger(\omega_2) \hat{f}_j^\dagger(\omega_1)]. \quad (\text{A6})$$

Our POVM elements are the same as before [Eqs. (A2) and (A3)] and we can calculate the detection probabilities as

$$P_j = \langle \psi^{\text{out}} | \Pi_j | \psi^{\text{out}} \rangle = \int d\omega_1 d\omega_2 [|C_{e_j, f_j}(\omega_1, \omega_2)|^2 + \frac{1}{2} |C_{e_j, e_j}(\omega_1, \omega_2) + C_{e_j, e_j}(\omega_2, \omega_1)|^2] \quad (\text{A7})$$

for the probability of detection at detector j and

$$P_c = \langle \psi^{\text{out}} | \Pi_c | \psi^{\text{out}} \rangle = \int d\omega_1 d\omega_2 [|C_{e_1, f_2}(\omega_2, \omega_1)|^2 + |C_{e_2, f_1}(\omega_1, \omega_2)|^2 + |C_{e_1, e_2}(\omega_2, \omega_1) + C_{e_2, e_1}(\omega_1, \omega_2)|^2] \quad (\text{A8})$$

for the probability of coincidence at both detectors.

Once again, inserting the relevant coefficients for the output state of a specific protocol yields the (fixed δ and phase shift) detection probabilities for that protocol.

APPENDIX B: NOISE IN TWO ARMS

Throughout this paper we have opted to localize noise entirely within one arm of the interferometer. To motivate this choice and show that this neat simplification is sufficient (even if physically noise occurs in both arms), we will explicitly model noise split between two arms for our MZ2s protocol in the following. This requires us to first introduce an additional pair of fixed phase shifts for the lower arm; we therefore modify the propagation transformations given in Eq. (3) to now read

$$\begin{aligned} \hat{c}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon_1)} e^{-i\theta_1} \hat{c}_1^\dagger(\omega), \\ \hat{c}_2^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_2+\epsilon_2)} e^{-i\theta_2} \hat{c}_2^\dagger(\omega), \\ \hat{d}_1^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_1+\epsilon_1)} e^{-i\theta_1} \hat{d}_1^\dagger(\omega), \\ \hat{d}_2^\dagger(\omega) &\rightarrow e^{-i\omega(\delta_2+\epsilon_2)} e^{-i\theta_2} \hat{d}_2^\dagger(\omega). \end{aligned} \quad (\text{B1})$$

The exact same method to derive the noisy probabilities now follows: Obtain the output state, calculate the fixed shift probabilities, and then average over the fixed shifts. The only additional requirement is the introduction of a second pair of integrals over ϵ_2 and θ_2 . The resulting noisy probabilities are

$$P_{1, \text{MZ2s}}^\eta = \frac{1}{8} \left(2 + \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon\sigma^2+1)}}{\sqrt{4\eta_\epsilon\sigma^2+1}} + \cos(\delta\omega_p) e^{-2\eta_{\bar{\theta}} - \eta_\epsilon\omega_p^2/2} - \frac{4 \cos\left(\frac{\delta\omega_p}{2\eta_\epsilon\sigma^2+2}\right) \exp\left(-\frac{4\delta^2\sigma^2+4\eta_{\bar{\theta}}(\eta_\epsilon\sigma^2+1)+\eta_\epsilon\omega_p^2}{8\eta_\epsilon\sigma^2+8}\right)}{\sqrt{\eta_\epsilon\sigma^2+1}} \right), \quad (\text{B2})$$

$$P_{2, \text{MZ2s}}^\eta = \frac{1}{8} \left(2 + \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon\sigma^2+1)}}{\sqrt{4\eta_\epsilon\sigma^2+1}} + \cos(\delta\omega_p) e^{-2\eta_{\bar{\theta}} - \eta_\epsilon\omega_p^2/2} + \frac{4 \cos\left(\frac{\delta\omega_p}{2\eta_\epsilon\sigma^2+2}\right) \exp\left(-\frac{4\delta^2\sigma^2+4\eta_{\bar{\theta}}(\eta_\epsilon\sigma^2+1)+\eta_\epsilon\omega_p^2}{8\eta_\epsilon\sigma^2+8}\right)}{\sqrt{\eta_\epsilon\sigma^2+1}} \right), \quad (\text{B3})$$

$$P_{c, \text{MZ2s}}^\eta = \frac{1}{4} \left(2 - \frac{e^{-2\delta^2\sigma^2/(4\eta_\epsilon\sigma^2+1)}}{\sqrt{4\eta_\epsilon\sigma^2+1}} - \cos(\delta\omega_p) e^{-2\eta_{\bar{\theta}} - \eta_\epsilon\omega_p^2/2} \right). \quad (\text{B4})$$

From the above we see that the frequency-dependent (frequency-independent) noise terms only enter through an effective total frequency-dependent (frequency-independent) noise term $\eta_\epsilon = \sqrt{\eta_{\epsilon_1}^2 + \eta_{\epsilon_2}^2}$ ($\eta_{\bar{\theta}} = \sqrt{\eta_{\bar{\theta}_1}^2 + \eta_{\bar{\theta}_2}^2}$). The form matches Eqs. (31)–(33), which can be recovered by taking $\eta_\epsilon = \eta_\epsilon$ and $\eta_{\bar{\theta}} = \eta_{\bar{\theta}}$. The same equivalence holds for the HOM, MZ1, and MZ2d protocols and also in the case of independent photons for each protocol.

APPENDIX C: CLASSICAL CORRELATIONS AND THE HIGH FREQUENCY-INDEPENDENT NOISE LIMIT

In Sec. V we noted that at high frequency-independent noise some residual information remains unscathed for most of our model variants. To justify our claim that this is a result of classical correlations, let us return to our single-photon MZ probabilities with fixed ϵ and θ shifts. These are $P_{1, \text{MZ1}}$, the probability that the photon is detected at detector 1, and $P_{2, \text{MZ1}}$, the probability that the photon is detected at detector 2, and are given in Eqs. (22) and (23), respectively.

We want to now consider what happens if we run MZ1 twice, but with the same fixed ϵ and θ shifts in both runs. This leads to three possible outcomes

$$P_{1, \text{CC}} = (P_{1, \text{MZ1}})^2, \quad (\text{C1})$$

$$P_{2, \text{CC}} = (P_{2, \text{MZ1}})^2, \quad (\text{C2})$$

$$P_{c, \text{CC}} = 2 \times P_{1, \text{MZ1}} \times P_{2, \text{MZ1}}, \quad (\text{C3})$$

the probabilities that both photons arrive at detector 1, both at detector 2, and a coincidence at both detectors.

We can now perform the same procedure in Sec. III, averaging over ϵ and θ with appropriate weighting to obtain noisy probabilities. The full expressions for these are given in the Supplemental Material [40].

If we now take the limit $\eta_\epsilon \rightarrow \infty$, our new probabilities tend to constants, just as in a single MZ1 run. However, while the same holds for a single MZ1 run in the limit $\eta_\theta \rightarrow \infty$, what we instead see is that for two MZ1 runs, correlated with

the same (albeit unknown) θ , the probabilities do still vary with delay. Specifically, we see

$$\lim_{\eta_\theta \rightarrow \infty} P_{1,CC}^\eta = \frac{1}{8} \left(2 + \frac{e^{-\delta^2 \sigma^2 / (2\eta_\epsilon^2 \sigma^2 + 1)}}{\sqrt{2\eta_\epsilon^2 \sigma^2 + 1}} \right), \quad (\text{C4})$$

$$\lim_{\eta_\theta \rightarrow \infty} P_{2,CC}^\eta = \frac{1}{8} \left(2 + \frac{e^{-\delta^2 \sigma^2 / (2\eta_\epsilon^2 \sigma^2 + 1)}}{\sqrt{2\eta_\epsilon^2 \sigma^2 + 1}} \right), \quad (\text{C5})$$

$$\lim_{\eta_\theta \rightarrow \infty} P_{c,CC}^\eta = \frac{1}{4} \left(2 - \frac{e^{-\delta^2 \sigma^2 / (2\eta_\epsilon^2 \sigma^2 + 1)}}{\sqrt{2\eta_\epsilon^2 \sigma^2 + 1}} \right). \quad (\text{C6})$$

Because these probabilities were derived from two independent runs of the MZ1 protocol, which could be taken some arbitrary time apart, with only the requirement that ϵ and θ remain constant (but unknown) between each run, these probabilities must be the result of classical correlations.

We can then note these are the same probabilities we get for the HOM protocol with independent photons and $\alpha = 0.5$, which is the same $\eta_\theta \rightarrow \infty$ limit we see for the independent photon MZ2s protocol. While the equivalent frequency-entangled probabilities differ slightly, this nevertheless suggests that the residual Fisher information seen in this limit is exactly the information that remains from these classical correlations.

This is sufficient to explain the MZ2s case with mode-correlated noise. However, the MZ2d protocol with mode-correlated noise sees a peculiar inverse visibility dependence. At $\alpha = 0$, the photon pair will not interfere at the initial beam splitter. The same logic now holds in terms of classical correlations: The two photons behave independently but experience the same θ shift. As visibility increases, however, so does the degree of interference at the first beam splitter. When the two photons bunch the analogy to two correlated MZ1 protocol runs breaks down. Rather than each photon independently interfering with itself, the photon pair now acts as one. At $\alpha = 1$ it becomes wholly impossible to exploit the classical correlations as the two photons will be completely entangled after the first beam splitter; hence the Fisher information drops to zero.

It is perhaps even more straightforward to understand the behavior with mode-uncorrelated noise. The MZ2s protocol retains the same residual information in the high- η_θ limit, but it now decays at low visibility. This naturally follows as low visibility means the second photon has a larger contribution from the orthogonal \hat{d} mode, which experiences a different θ shift. Because the shifts now differ between modes, classical correlations only exist between the paths of the \hat{c} modes. Letting α drop all the way to zero puts both photons in wholly distinguishable modes and thus the Fisher information vanishes as there are no correlations between the two.

The MZ2d protocol with mode-uncorrelated noise has no residual information at all in this limit. This odd one out can be explained with a combination of the two previous cases. At high visibility the photons are more likely to bunch, which provides no classical correlations to exploit. At low visibility the photons will behave independently, but both modes experience different θ shifts so no correlations exist.

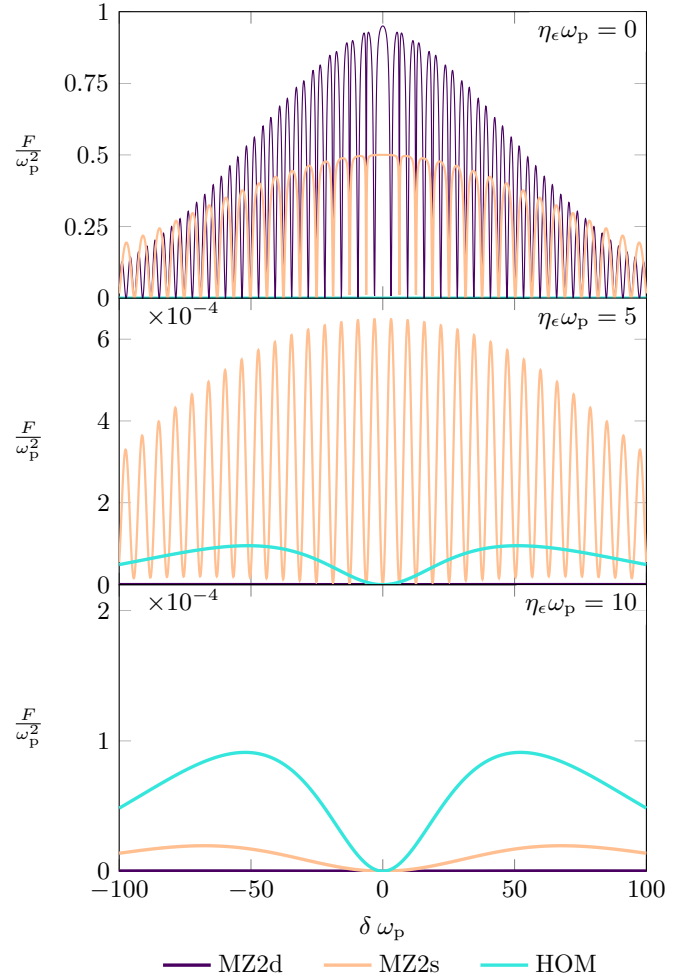


FIG. 8. Comparison of the two-photon protocols, in the frequency-independent mode-correlated noise configuration with $\sigma = \frac{\omega_p}{100}$ and $\alpha = 0.9$. We see behavior qualitatively similar to the frequency-entangled case depicted in Fig. 3. As also seen in Fig. 6(b), information decays at large δ even in the absence of noise, and we now note that once noise is introduced the Fisher information is generally slightly lower than the frequency-entangled equivalent. We also see the wider HOM dip of an independent photon input; specifically, the Fisher information is equivalent to that of frequency-entangled HOM case with the reduced spectral width $\sigma/\sqrt{2}$.

Combining these two effects, we will now always see zero information.

APPENDIX D: NOISE RESILIENCE OF INDEPENDENT PHOTONS

For comparison to Fig. 3, we plot in Fig. 8 the Fisher information at different frequency-dependent noise values when our input photons are now independent. The behavior is qualitatively similar but the value of the resulting Fisher information is slightly reduced.

APPENDIX E: NOISE DISTRIBUTIONS

In Eq. (25) we derive our noisy probabilities by integrating our fixed shift probabilities with Gaussian weightings. While

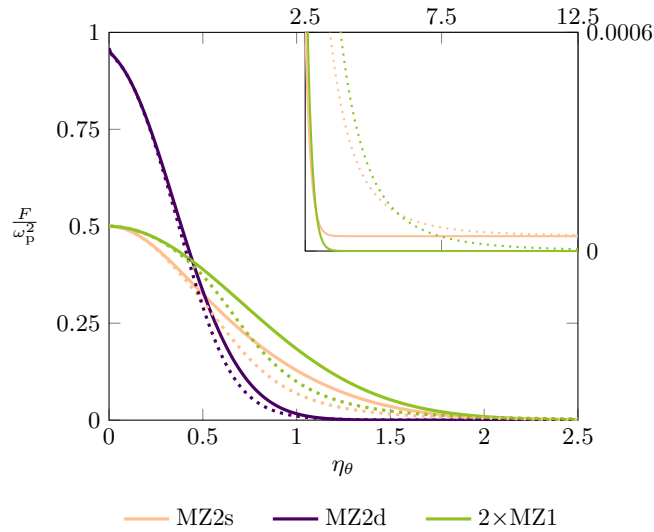


FIG. 9. Comparison of the maximal information against frequency-independent noise modeled with a (wrapped) Gaussian noise distribution (solid) and a von Mises noise distribution (dotted). We choose frequency-entangled photons and mode-correlated noise with $\sigma = \frac{\omega_p}{100}$ and $\alpha = 0.9$. At low noise, values are similar. As noise increases the curves diverge, the von Mises curves dropping faster. At higher noise the curves converge again. In the inset, we see that for large noise values ($\eta_\theta > 2.5$) the Gaussian curves, which initially sat above, have now dropped below the von Mises curves. Information plateaus to the same values for both distributions [the residual HOM-like information from Fig. 4(a) can be seen for the MZ2s protocol], but with the von Mises distribution this happens over a notably larger range of noise values.

the choice of a Gaussian noise distribution is the most natural for frequency-dependent (ϵ) noise, being equivalent to some jitter in the delay, for frequency-independent noise (where the θ shifts will always lie within an interval of width 2π) a more conventional choice would be some circular distribution for θ shifts.

One such common circular distribution can be obtained by wrapping the standard Gaussian distribution around the circle. If we define an arbitrary Gaussian distribution

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad (\text{E1})$$

then the wrapped Gaussian distribution is given by [42]

$$W(x) = \sum_{k=-\infty}^{\infty} G(x + 2\pi k). \quad (\text{E2})$$

Then, suppose some periodic function $f(x)$ such that $f(x + 2\pi k) = f(x)$ for all integers k . We can then demonstrate that integrating this function with a Gaussian distribution over $(-\infty, \infty)$ is equivalent to integrating with the wrapped Gaussian over a 2π window:

$$\int_{-\infty}^{\infty} dx f(x)G(x) = \sum_{k=-\infty}^{\infty} \int_{2\pi k-\pi}^{2\pi k+\pi} dx f(x)G(x)$$

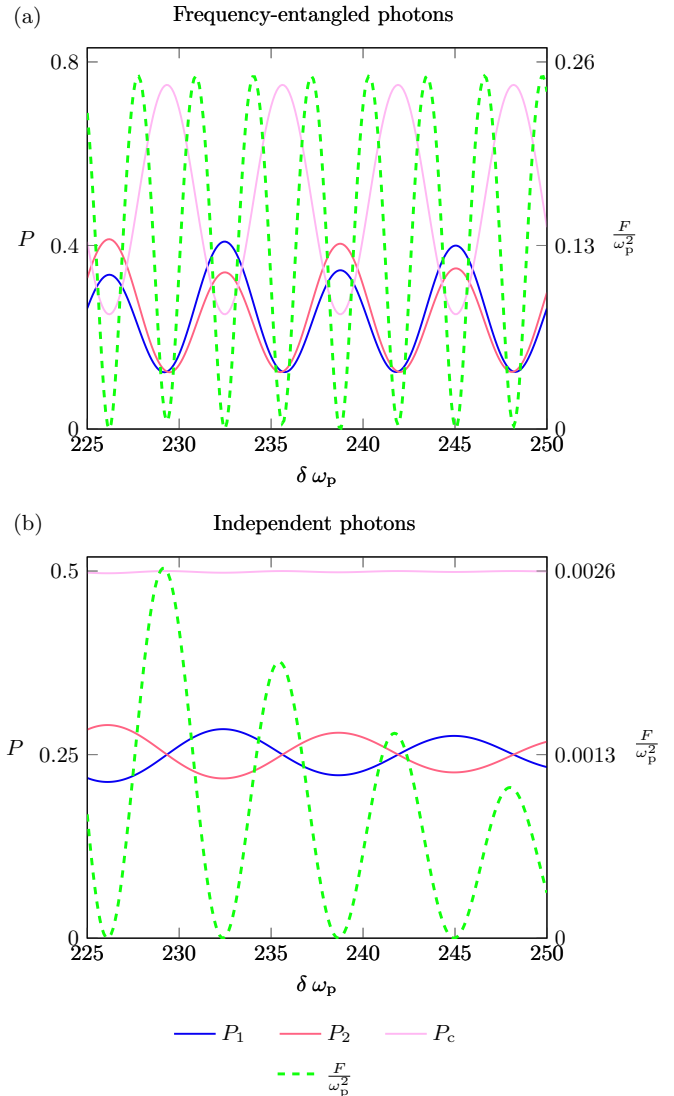


FIG. 10. Comparison of the detected signal to the resulting Fisher information for the MZ2s protocol in the absence of noise. Solid lines are the three probabilities (detection at detector 1, at detector 2, or a coincidence at both detectors) and the dashed line is the Fisher information. We choose $\sigma = \frac{\omega_p}{100}$ and examine behavior at high $|\delta|$. In (a) (frequency-entangled input photons) we see regular oscillations in the detected signal produce Fisher information that likewise oscillates but does not decay. In (b) (frequency-independent input photons) we see the detected probabilities, still slightly oscillating, are tending towards constants. Thus the Fisher information is decaying to zero.

$$\begin{aligned} &= \int_{-\pi}^{\pi} dx \sum_{k=-\infty}^{\infty} f(x + 2\pi k)G(x + 2\pi k) \\ &= \int_{-\pi}^{\pi} dx f(x) \sum_{k=-\infty}^{\infty} G(x + 2\pi k) \\ &= \int_{-\pi}^{\pi} dx f(x)W(x). \end{aligned} \quad (\text{E3})$$

We now note that all of our fixed shift probabilities given in Sec. II have an appropriate periodic dependence on θ , i.e.,

$P(\theta + 2\pi k) = P(\theta)$ for all integers k . Therefore, the integral in Eq. (25) will produce the same results as if we had used a wrapped Gaussian distribution.

An alternative circular distribution, the von Mises distribution, is given by [42]

$$\frac{e^{\kappa \cos(x-\mu)}}{2\pi I_0(\kappa)}, \quad (\text{E4})$$

with $I_0(\kappa)$ the modified Bessel function of the first kind. Here κ is analogous to the Gaussian distribution's $1/\sigma^2$, so for our purposes we write the von Mises weighting in the form

$$J_\theta(\vartheta) = \frac{e^{\cos(\vartheta)/\eta_\theta^2}}{2\pi I_0\left(\frac{1}{\eta_\theta^2}\right)}. \quad (\text{E5})$$

This can then replace the second expression in Eq. (24), and the ϑ integral from Eq. (25) is now performed over the region $[-\pi, \pi]$.

Employing the von Mises distribution produces the same qualitative results but with a slightly different dependence on the noise parameter η_θ . Considering only frequency-independent noise, we plot in Fig. 9 the maximal information for our protocols with both noise distributions. Values are most similar for high and low η_θ , attaining the same limits at $\eta_\theta = 0$ and $\eta_\theta \rightarrow \infty$, while the curves are furthest separated at moderate noise values. Also notable, from the inset, is that the von Mises model takes notably longer to decay to its high- η_θ limits.

For the protocols in Fig. 9, the Supplemental Material [40] contains the full probability and Fisher information expressions where a von Mises noise distribution was chosen for frequency-independent noise. Also included are all initial fixed shift probabilities from Sec. II, plus those for the Sec. IV model variations, so that all results can be easily reproduced for an alternative choice of noise distributions.

APPENDIX F: OSCILLATIONS IN THE DETECTED SIGNAL

In Sec. V we discussed how, at high $|\delta|$, the Fisher information oscillates without decay when the input photons are frequency entangled. If the photons are independent, the Fisher information instead decays to zero. To further demonstrate this, Fig. 10 plots the detection probabilities together with the Fisher information in a high- $|\delta|$ region, for the MZ2s protocol.

Generally, we expect a high Fisher information when probabilities change most quickly with respect to the parameter of interest, while it vanishes at the extrema of the probabilities as there is no local information at these points; Eq. (26) shows that the Fisher information must vanish whenever the derivative of all probabilities is zero. In practice, a degree of prior information (such as that obtained through some initial coarse calibration) enables one to tune the setup and operate in a region in which the Fisher information is high [11].

The relation between detected signal and resulting Fisher information is similar for the MZ2d protocol.

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