Multiparty orthogonal product states with minimal genuine nonlocality

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Nonlocality without entanglement and its subsequent generalizations offer deep information-theoretic insights and subsequently find several useful applications. The concept of a genuinely nonlocal set of product states emerges as a natural multipartite generalization of this phenomenon. The existence of such sets eventually raises the problem concerning their entanglement-assisted discrimination. Here, we construct examples of genuinely nonlocal product states for an arbitrary number of parties. The strength of genuine nonlocality of these sets can be considered minimal as their perfect discrimination is possible with entangled resources residing in Hilbert spaces having the smallest possible dimensions. Our constructions lead to fully separable measurements that are impossible to implement even if all but one party come together. Furthermore, they also provide the opportunity to compare different multipartite states that otherwise are incomparable under single copy local manipulation.

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I. INTRODUCTION

Quantum entanglement has been established as a useful resource for numerous practical tasks, starting from an advanced means of communication [1–3], to improved metrology and estimation [4–7], to randomness processing [8–10]. For multipartite systems, entanglement appears in different inequivalent forms [11–13] and accordingly finds more exotic applications [14–17]. Characterization, quantification, and detection of quantum entanglement therefore have practical relevance and have vastly shaped the research direction in quantum information theory during the last three decades (see Refs. [18,19] and references therein). Entanglement also lies at the core of almost all foundational debates in quantum theory [20–27]. In particular, it is crucial to establish the puzzling nonlocal feature of quantum theory. Bell, in his seminal result [23,24], derived an experimentally testable criterion that any *local-realistic* theory must satisfy, whereas quantum statistics obtained from suitably chosen local measurements performed on a properly chosen entangled state can violate this inequality and hence establish a nonlocal feature of quantum theory. Several experiments with a variety of quantum systems have reported positive Bell tests and thus ensure the nonlocal nature of the quantum world [28–31].

Entanglement has also been proved to be advantageous in hypothesis testing and discrimination tasks [32–35]. A particular interest is the local state discrimination problem, where the aim is to identify a multipartite quantum state, drawn randomly from a known set of states, under the operational paradigm of local operation and classical communication (LOCC). In such a scenario, quantum theory exhibits a different kind of nonlocal behavior that involves no entanglement and is distinct from Bell nonlocality. In a seminal paper, Bennett *et al.* provide examples of orthogonal product bases for multipartite systems [36] that are locally indistinguishable. They coined the term "quantum nonlocality without entanglement" for this phenomenon as perfect discrimination of the states requires "nonlocal" (read as global or joint) measurement on the composite system. Subsequently this result motivates a plethora of research on the general local state discrimination problem [37–52] and in this paper our study will also deal with this particular kind of nonlocal behavior of quantum theory. A locally indistinguishable mutually orthogonal set of states can be distinguished perfectly if entangled states are provided as a resource along with LOCC. For instance, Bennett *et al.*'s [36] nonlocal product basis of the $(\mathbb{C}^3)^{\otimes 2}$ system can be perfectly distinguished if a maximally entangled state in this Hilbert space is provided as a resource. The seminal teleportation protocol [2] makes the discrimination task viable. Quite surprisingly, in subsequent work, Cohen showed that a two-qutrit maximally entangled state is not necessary for perfect discrimination of this nonlocal product basis; instead, a two-qubit maximally entangled state suffices for the purpose [53]. Cohen's protocol offers an efficient use of the costly entangled resource in the local state discrimination problem.

Recently, a stronger notion of nonlocality without entanglement phenomena is identified for multipartite quantum systems [54] which subsequently motivates renewed interest in constructing a nonlocal product set of states for multipartite systems as well as their entanglement assisted discrimination [55–59]. In this paper, we first present a set of tripartite product states which is locally indistinguishable given an arbitrary amount of entanglement shared between any two of the three parties. In other words, the set remains indistinguishable even if any two of the parties come together but do not share any entanglement with the third party. Therefore, the set requires a genuinely multipartite entangled resource for perfect discrimination when all the parties are spatially separated. Interestingly, we show that given a three-qubit Greenberger-Horne-Zeilinger (GHZ) state as resource the states can be perfectly distinguished although they live in a $\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ dimensional system. Note that the resource used here is much cheaper than a teleportation based resource (two copies of the two-qutrit maximally entangled state in this case). In fact our protocol uses the minimal dimensional genuinely entangled resource and hence the nonlocal strength of the constructed set of states can be considered minimal. We then generalize the construction for an arbitrary number of spatially separated parties and also discuss its entanglement assisted discrimination. For the n partite case, the construction lives in $\mathbb{C}^{n+1} \otimes (\mathbb{C}^3)^{\otimes n-1}$. Moreover, an n-qubit GHZ state suffices as resource for their perfect discrimination which again turns out to be the minimal dimensional resource. Our construction also provides an operational way to compare different classes of multipartite entanglement that otherwise are incomparable under LOCC.

II. PRELIMINARIES

Although the history of quantum state discrimination dates back to the early 1970's with an initial attempt to formulate information protocols using quantum optical devices [60–62], the local state discrimination problem gained research interest much later [36,63,64]. Given only one copy of the system, it asks one to identify the state chosen randomly from a known ensemble of states $\{p_i, |\psi_i\rangle\}_{i=1}^m$ under the restriction that the spatially separated parties can perform only LOCC, where $\forall\, i, |\psi_i\rangle \in \otimes_{j=1}^n \mathcal{H}_j$ with $\mathcal{H}_j$ being the Hilbert space of the jth subsystem. In a product local state discrimination problem, all $|\psi_i\rangle$'s are considered to be fully product states, i.e., $\forall\, i,\ |\psi_i\rangle = \otimes_{j=1}^n |\phi_i^j\rangle$ with $|\phi_i^j\rangle \in \mathcal{H}_j$.

Definition 1. Nonlocal product states (NPSs): A set of mutually orthogonal and fully product states $\mathbb{S} := \{|\psi_i\rangle\}_{i=1}^K \subset \otimes_{j=1}^n \mathcal{H}_j$ will be referred to as NPS if they cannot be perfectly distinguished under LOCC when all the parties are spatially separated.

Definition 2. Genuinely nonlocal product states (GNPSs): A set of mutually orthogonal and fully product states $\mathbb{S} := \{|\psi_i\rangle\}_{i=1}^K \subset \otimes_{j=1}^n \mathcal{H}_j$ will be referred to as GNPS if they cannot be locally distinguished in any possible bipartition.

Note that the above definition captures the strongest possible notion of nonlocality without entanglement phenomena for multipartite systems. The states of a GNPS can be neither locally distinguished in any "$n-1$ vs 1" bipartition nor locally distinguished in any "$n-k$ vs k" bipartition, with arbitrary k parties grouping together. Clearly every GNPS is a NPS, but the converse is not true in general. For instance, the Shifts UPB (unextendible product basis) of $(\mathbb{C}^2)^{\otimes 3}$ as constructed in Refs. [36,65] is a tripartite NPS but not a GNPS. In this paper, our primary aim is to construct GNPS and then study their entanglement assisted discrimination. Before discussing our construction, we first recall an example of bipartite NPS which is given by

$$\mathbb{S}_{\rm Ben} \equiv \{|0\rangle\,|\eta_\pm\rangle\,, |\eta_\pm\rangle\,|2\rangle\,, |2\rangle\,|\xi_\pm\rangle\,, |\xi_\pm\rangle\,|0\rangle\} \subset (\mathbb{C}^3)^{\otimes 2},$$

where $|\eta_\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\xi_\pm\rangle := (|1\rangle \pm |2\rangle)/\sqrt{2}$. As pointed out in Ref. [36], deletion of any state from $\mathbb{S}_{\rm Ben}$ makes the remaining set locally distinguishable, whereas if we add another orthogonal product state to it, for instance the state $|1\rangle\,|1\rangle$, the resulting set remains nonlocal. This fact can be further generalized. For this purpose, first note that two sets of states $\mathbb{S}$ and $\mathbb{S}'$ are called orthogonal if and only if $\langle\phi|\phi'\rangle = 0,\ \forall\ |\phi\rangle \in \mathbb{S},$ *and* $|\phi'\rangle \in \mathbb{S}'$; and they will be denoted as $\mathbb{S} \perp \mathbb{S}'$.

Observation 1. Let $\mathbb{S} \subset \otimes_{j=1}^n \mathcal{H}_j$ be a multipartite NPS or GNPS. The set of states $\mathbb{A} := \mathbb{S} \cup \mathbb{S}'$ is a NPS or GNPS for any set of mutually orthogonal states $\mathbb{S}'$ such that $\mathbb{S} \perp \mathbb{S}'$.

Proof of this observation trivially follows an argument of *reductio ad absurdum*. If $\mathbb{A}$ were a locally distinguishable set then for every $|\psi\rangle \in \mathbb{A}$ chosen at random it is possible to perfectly identify this state under LOCC. This should hold even when the state lies in the nonlocal set $\mathbb{S}$ which leads to a contradiction.

Given a set of states $\chi := \{|\beta\rangle_i \mid i = 1, \cdots K\}$ and another state $|\alpha\rangle$ let us define $\chi \otimes |\alpha\rangle := \{|\beta\rangle_i \otimes |\alpha\rangle \mid i = 1, \cdots K\}$. With this notation, we will now put our next observation which will be relevant in subsequent proofs.

Observation 2. Let $\mathbb{S} \subset \otimes_{j=1}^n \mathcal{H}_j$ be a multipartite NPS or GNPS. Consider the set $\mathbb{S}' := \mathbb{S} \otimes |\phi_0\rangle_{a_1\cdots a_m}$, where $|\phi_0\rangle_{a_1\cdots a_m}$ is some fully separable state with some of the subsystems $\{a_i\}$ in possession with the ith party. The resulting set $\mathbb{S}'$ is again an NPS or GNPS with respect to the same multipartite configuration.

Observation 2 follows from the fact that any fully separable state can always be prepared locally.

III. RESULTS

With the aforesaid observations in hand, in the following, we first construct a tripartite GNPS.

Proposition 1. The set of states $\mathbb{G}[4 \otimes 3^{\otimes 2}]$ defined below is a GNPS in $\mathbb{C}_A^4 \otimes \mathbb{C}_B^3 \otimes \mathbb{C}_C^3$:

$$\mathbb{G}[4 \otimes 3^{\otimes 2}] \equiv \left\{ \begin{array}{c} |\zeta_\pm^0\rangle := |\epsilon_\pm\rangle|1\rangle|2\rangle,\ \ |\zeta_\pm^{1,1}\rangle := |1\rangle|\gamma_\pm^1\rangle|2\rangle, \\ |\zeta_\pm^{1,2}\rangle := |\gamma_\pm^1\rangle|p\rangle|2\rangle,\ |\zeta_\pm^{1,3}\rangle := |p\rangle|\epsilon_\pm\rangle|2\rangle, \\ |\zeta_\pm^{2,1}\rangle := |2\rangle|1\rangle|\gamma_\pm^2\rangle,\ |\zeta_\pm^{2,2}\rangle := |\gamma_\pm^2\rangle|1\rangle|p\rangle, \\ |\zeta_\pm^{2,3}\rangle := |p\rangle|1\rangle|\epsilon_\pm\rangle \end{array} \right\}.$$

Here $\{|p\rangle, |q\rangle, |1\rangle, |2\rangle\}$ are mutually orthogonal states, $|\epsilon_\pm\rangle := \frac{1}{\sqrt{2}}(|p\rangle \pm |q\rangle)$ and $|\gamma_\pm^i\rangle := \frac{1}{\sqrt{2}}(|q\rangle \pm |i\rangle)$; $i = 1, 2$.

Proof. Consider the subset of states $\$^i \equiv \{|\zeta_\pm^0\rangle, |\zeta_\pm^{i,1}\rangle, |\zeta_\pm^{i,2}\rangle, |\zeta_\pm^{i,3}\rangle\} \subset \mathbb{G}[4 \otimes 3^{\otimes 2}]$, for $i \in \{1, 2\}$. The set $\$^1$ has an analogous structure as the set $\mathbb{S}_{\rm Ben}$ (with a notation change, $p \to 0, q \to 1, 1 \to 2$) between Alice and Bob while Charlie has the fixed state $|2\rangle$ (see Fig. 1). This, along with Observations 1 and 2, assure that the set $\mathbb{G}[4 \otimes 3^{\otimes 2}]$ cannot be locally discriminated even when Charlie groups with either Alice or Bob. Similarly the set $\$^2$ prohibits perfect local discrimination of $\mathbb{G}[4 \otimes 3^{\otimes 2}]$ even when Alice and Bob are grouped together. This completes the proof.

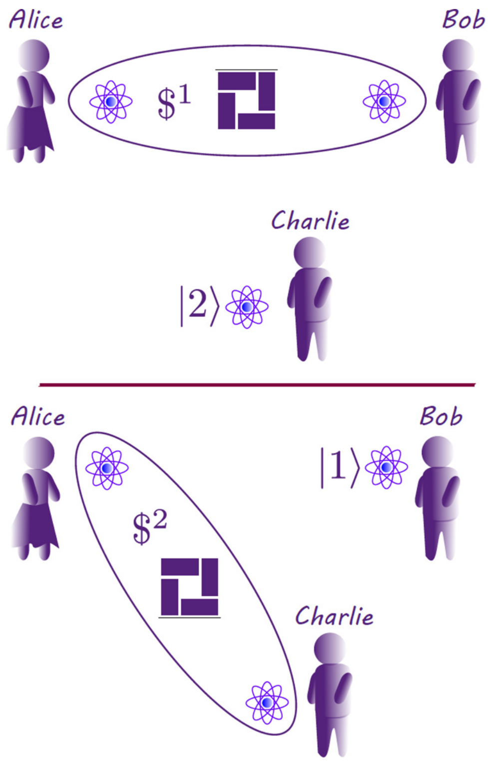

FIG. 1. Tripartite GNPS $\mathbb{G}[4\otimes 3^{\otimes 2}]$. Alice and Bob share the set of states $\$^1 \equiv \{|\zeta_\pm^0\rangle, |\zeta_\pm^{1,1}\rangle, |\zeta_\pm^{1,2}\rangle, |\zeta_\pm^{1,3}\rangle\}$ while Charlie's state is $|2\rangle$. Alice and Charlie share the set of states $\$^2 \equiv \{|\zeta_\pm^0\rangle, |\zeta_\pm^{2,1}\rangle, |\zeta_\pm^{2,2}\rangle, |\zeta_\pm^{2,3}\rangle\}$ while Bob's state is $|1\rangle$. Clearly, $\mathbb{G}[4\otimes 3^{\otimes 2}] \equiv \{\$^1_{AB}\otimes|2\rangle_C\}\cup\{\$^2_{AC}\otimes|1\rangle_B\}$.

At this point, a pertinent question is how to quantify the amount of genuine nonlocality without entanglement for a given GNPS? Note that, given a sufficient amount of entanglement among the spatially separated parties, any GNPS can be perfectly distinguished. For instance discrimination of $\mathbb{G}[4\otimes 3^{\otimes 2}]$ is possible given two copies of two-qutrit maximally entangled states—one shared between Alice-Bob and the other between Alice-Charlie. Since entanglement is a costly resource it is therefore relevant to go for a cost efficient discrimination protocol. Given two GNPSs a natural ordering of their strength of genuine nonlocality without entanglement can be made from the amount of entanglement required for their perfect discrimination. While the teleportation based discrimination of the $\mathbb{G}[4\otimes 3^{\otimes 2}]$ required two copies of two-qutrit maximally entangled states that live in the Hilbert space $\mathbb{C}^9_A\otimes\mathbb{C}^3_B\otimes\mathbb{C}^3_C$, we will now discuss a very cost efficient discrimination protocol. In particular we will show that an entanglement resource living in the Hilbert space $(\mathbb{C}^2)^{\otimes 3}$ will suffice for perfect discrimination of the set.

Theorem 1. The set of states $\mathbb{G}[4\otimes 3^{\otimes 2}]$ can be perfectly discriminated locally when the state $|g_3\rangle := (|000\rangle + |111\rangle)/\sqrt{2}$ is shared as a resource.

Proof. We will associate the block letter party index with the states that need to be distinguished and denote the resource state as $|g_3\rangle_{abc} = (|000\rangle_{abc} + |111\rangle_{abc})/\sqrt{2}$. Local distinguishability of the set $\mathbb{G}[4\otimes 3^{\otimes 2}]$ boils down to identifying the pairs $\{|\zeta_\pm\rangle\}$ preserving the postmeasurement orthogonality between $|\zeta_+\rangle$ and $|\zeta_-\rangle$, as the result in Ref. [37] assures local distinguishability between any two orthogonal states. The discrimination protocol proceeds as follows.

(1) Alice performs the measurement $\mathcal{M} \equiv \{M, \mathbb{I}-M\}$, where $M := \mathbb{P}[|p\rangle_A; |0\rangle_a] + \mathbb{P}[(|q\rangle, |1\rangle, |2\rangle)_A; |1\rangle_a]$. Here, we use the notation $\mathbb{P}[(|e\rangle, |f\rangle, \cdots)_K; (|x\rangle, |y\rangle, \cdots)_k] := (|e\rangle\langle e| + |f\rangle\langle f| + \cdots)_K \otimes (|x\rangle\langle x| + |y\rangle\langle y| + \cdots)_k$. Suppose the projector M clicks. The state $|\zeta\rangle_{ABC}\otimes|g\rangle_{abc}$ evolves to either $|\zeta\rangle_{ABC}\otimes|000\rangle_{abc}$ or $|\zeta\rangle_{ABC}\otimes|111\rangle_{abc}$, or it becomes entangled, where $|\zeta\rangle_{ABC}\in\mathbb{G}[4\otimes 3^{\otimes 2}]$. The complete list of the evolved states is given below:

$$\left\{\begin{array}{r} \left\{|\zeta_\pm^{1,3}\rangle, |\zeta_\pm^{2,3}\rangle\right\}_{ABC}\otimes|000\rangle_{abc}, \\ \left\{|\zeta_\pm^{1,1}\rangle, |\zeta_\pm^{1,2}\rangle, |\zeta_\pm^{2,1}\rangle, |\zeta_\pm^{2,2}\rangle\right\}_{ABC}\otimes|111\rangle_{abc}, \\ |\zeta_\pm^0\rangle_{ABC} \Rightarrow (|p\rangle_A|000\rangle_{abc} \pm |q\rangle_A|111\rangle_{abc})|1\rangle_B|2\rangle_C \end{array}\right\}.$$

(2) Bob and Charlie, respectively, perform the measurement

$$\begin{aligned} &\mathcal{K}\equiv\{K_1 := \mathbb{I}-K_2-K_3,\ K_2 := \mathbb{P}[|p\rangle_B; |1\rangle_b], \\ &K_3 := \mathbb{P}[(|p\rangle, |q\rangle)_B; |0\rangle_b],\ \}, \\ &\mathcal{N}\equiv\{N_1 := \mathbb{I}-N_2-N_3,\ N_2 := \mathbb{P}[|p\rangle_C; |1\rangle_c], \\ &N_3 := \mathbb{P}[(|p\rangle, |q\rangle)_C; |0\rangle_c].\}. \end{aligned}$$

If K_3 clicks the state is one of $\{|\zeta_\pm^{1,3}\rangle\}$, if K_2 clicks the state is one of $\{|\zeta_\pm^{1,2}\rangle\}$, if N_3 clicks the state is one of $\{|\zeta_\pm^{2,3}\rangle\}$, and if N_2 clicks the state is one of $\{|\zeta_\pm^{2,2}\rangle\}$. When both K_1 and N_1 click the state is one of $\{|\zeta_\pm^0\rangle, |\zeta_\pm^{1,1}\rangle, |\zeta_\pm^{2,1}\rangle\}$. Obtaining the outcome results from Bob and Charlie, Alice performs the following measurement:

$$\mathcal{M}'\equiv\left\{\begin{array}{c} M'_1 := \mathbb{P}[|1\rangle_A; \mathbb{I}_a],\quad M'_2 := \mathbb{P}[|2\rangle_A; \mathbb{I}_a], \\ M'_0 := \mathbb{I}-M'_1-M'_2. \end{array}\right\}.$$

If M'_1 clicks the state is one of $\{|\zeta_\pm^{1,1}\rangle\}$, if M'_2 clicks the state is one of $\{|\zeta_\pm^{2,1}\rangle\}$, otherwise it is one of $\{|\zeta_\pm^0\rangle\}$. If $\mathbb{I}-M$ clicks in step 1 then a similar protocol will follow. ■

Theorem 1 establishes nontrivial and efficient use of the three-qubit GHZ state in the product state discrimination problem under LOCC. The GNPS in Proposition 1 therefore possesses minimal genuine nonlocality without entanglement as the entangled resource required for its perfect discrimination lives in minimal dimensional Hilbert space. Although the discrimination resource is minimal in the sense of Hilbert-space dimension, here a question still remains open whether a state $\alpha|000\rangle+\beta|111\rangle\in(\mathbb{C}^2)^{\otimes 3}$ with $\alpha\neq\beta$ suffices for perfect discrimination of the set $\mathbb{G}[4\otimes 3^{\otimes 2}]$.[1] Such a resource is less costly as it has less three-tangle [66] than the state with $\alpha=\beta$. Our intuition is that possibly a state with $\alpha\neq\beta$ will not be sufficient for perfect discrimination of $\mathbb{G}[4\otimes 3^{\otimes 2}]$. It is extremely difficult to explore all the possible LOCC

[1]Our intuition, in fact, eventuates from the study of Cohen's work [53]. Following his technique, local distinguishability of the set $\mathcal{S}_{\rm Ben}$ can be analyzed with the resource state $\alpha|00\rangle+\beta|11\rangle$ $(\alpha\neq\beta)$, which seems not to provide a perfect success. However, a protocol independent proof of this conviction is not known yet.

TABLE I. Set of states $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$. Here $|\epsilon_{\pm}\rangle := \frac{1}{\sqrt{2}}(|p\rangle \pm |q\rangle)$ and $|\gamma^i_{\pm}\rangle := \frac{1}{\sqrt{2}}(|q\rangle \pm |i\rangle)$, with $i, j \in \{p, q, 1, \cdots, m\}$ and $\langle i|j\rangle = \delta_{ij}$.

Alice	Bob-1	Bob-2	$\cdots$	Bob-m	
$\|\epsilon_{\pm}\rangle$	$\|1\rangle$	$\|2\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^0_{\pm}\rangle$
$\|1\rangle$	$\|\gamma^1_{\pm}\rangle$	$\|2\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{1,1}_{\pm}\rangle$
$\|\gamma^1_{\pm}\rangle$	$\|p\rangle$	$\|2\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{1,2}_{\pm}\rangle$
$\|p\rangle$	$\|\epsilon_{\pm}\rangle$	$\|2\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{1,3}_{\pm}\rangle$
$\|2\rangle$	$\|1\rangle$	$\|\gamma^2_{\pm}\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{2,1}_{\pm}\rangle$
$\|\gamma^2_{\pm}\rangle$	$\|1\rangle$	$\|p\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{2,2}_{\pm}\rangle$
$\|p\rangle$	$\|1\rangle$	$\|\epsilon_{\pm}\rangle$	$\cdots$	$\|m\rangle$	$:= \|\zeta^{2,3}_{\pm}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\|m\rangle$	$\|1\rangle$	$\|2\rangle$	$\cdots$	$\|\gamma^m_{\pm}\rangle$	$:= \|\zeta^{m,1}_{\pm}\rangle$
$\|\gamma^m_{\pm}\rangle$	$\|1\rangle$	$\|2\rangle$	$\cdots$	$\|p\rangle$	$:= \|\zeta^{m,2}_{\pm}\rangle$
$\|p\rangle$	$\|1\rangle$	$\|2\rangle$	$\cdots$	$\|\epsilon_{\pm}\rangle$	$:= \|\zeta^{m,3}_{\pm}\rangle$

protocols assisted with such a resource. Therefore, answering this question requires a protocol independent argument which we leave here as an open question for future research.

We now move on to another consequence of the above construction. Note that the following set of 22 orthonormal product states

$$\mathbb{G}^C[4\otimes 3^{\otimes 2}] \equiv \left\{\begin{array}{c} |qq2\rangle,\ |qqq\rangle,\ |qqp\rangle,\ |q1q\rangle,\ |qpq\rangle,\ |ppp\rangle, \\ |1qq\rangle,\ |2qq\rangle,\ |pqq\rangle,\ |1qp\rangle,\ |2qp\rangle,\ |pqp\rangle, \\ |1pq\rangle,\ |2pq\rangle,\ |ppq\rangle,\ |1pp\rangle,\ |2pp\rangle,\ |qpp\rangle, \\ |11p\rangle,\ |11q\rangle,\ |2q2\rangle,\ |2p2\rangle \end{array}\right\}$$

spans the subspace orthogonal to the subspace spanned by $\mathbb{G}[4\otimes 3^{\otimes 2}]$, and hence the set of states $\mathbb{P}[4\otimes 3^{\otimes 2}] := \mathbb{G}[4\otimes 3^{\otimes 2}] \cup \mathbb{G}^C[4\otimes 3^{\otimes 2}]$ with adequate normalization constitutes an orthonormal product basis (ONPB) for the Hilbert space $\mathbb{C}^4\otimes(\mathbb{C}^3)^{\otimes 2}$; here $|xyz\rangle := |x\rangle_A\otimes|y\rangle_B\otimes|z\rangle_C$. Manifestly, this ONPB has the property of genuine nonlocality and accordingly it constitutes a fully separable measurement that cannot be implemented even when any two parties come together. It is not hard to argue that the discriminating resource of $\mathbb{G}[4\otimes 3^{\otimes 2}]$ suffices for discrimination of the set $\mathbb{P}[4\otimes 3^{\otimes 2}]$. However, at this point, a more difficult question is how much resource is necessary for implementation of the corresponding fully separable measurement. Presently we have no idea regarding the resource requirement and welcome further research in this direction. In the rest of the sections, we rather consider multipartite generalization of the above construction.

Proposition 2. Consider the set of states $\mathbb{G}[(m+2)\otimes 3^{\otimes m}] \equiv \{|\zeta^0_{\pm}\rangle, |\zeta^{i,1}_{\pm}\rangle, |\zeta^{i,2}_{\pm}\rangle, |\zeta^{i,3}_{\pm}\rangle\}_{i=1}^m$ given in Table I . This set is a GNPS in $\mathbb{C}^{m+2}\otimes(\mathbb{C}^3)^{\otimes m}$. Here, Alice possesses the subsystem in $\mathbb{C}^{m+2}$ and each Bob has a subsystem with qutrit Hilbert space.

Proof. Consider the subset of states $\$^i \equiv \{|\zeta^0_{\pm}\rangle, |\zeta^{i,1}_{\pm}\rangle, |\zeta^{i,2}_{\pm}\rangle, |\zeta^{i,3}_{\pm}\rangle\} \subset \mathbb{G}[(m+2)\otimes 3^{\otimes m}]$, for $i \in \{1, \ldots, m\}$. The set $\$^i$ has a similar structure as of the set $\mathbb{S}_{\mathrm{Ben}}$ between Alice and ith Bob while other Bobs have fixed states tagged with this set (see Fig. 2). This, along with Observation 1 and Observation 2, assure that the set $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$ cannot be locally distinguished in any bipartition. ∎

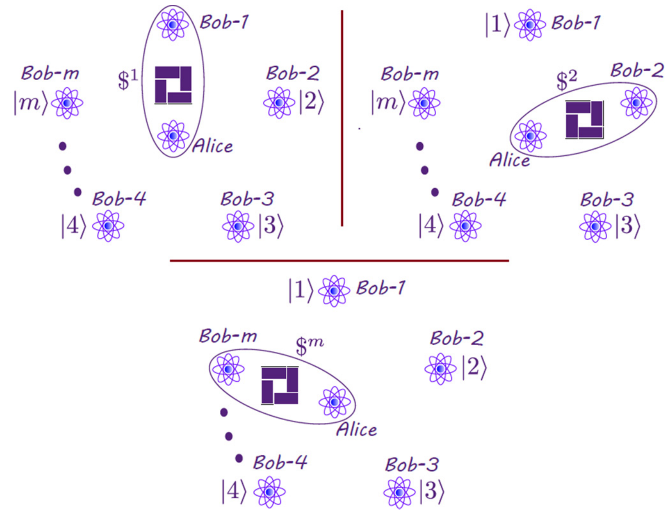

FIG. 2. Multiparty GNPS $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$. Alice and ith Bob share the set of states $\$^i \equiv \{|\zeta^0_{\pm}\rangle, |\zeta^{i,1}_{\pm}\rangle, |\zeta^{i,2}_{\pm}\rangle, |\zeta^{i,3}_{\pm}\rangle\}$ while jth Bob's state is $|j\rangle$, $j \neq i$. Clearly, $\mathbb{G}[(m+2)\otimes 3^{\otimes m}] \equiv \bigcup_i\{\bigotimes_{j\neq i}|j\rangle_{B_j}\otimes \$^i_{AB_i}\}$.

Our next result addresses entanglement assisted discrimination of the set $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$.

Theorem 2. The set of states $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$ can be perfectly discriminated locally given the genuine resource state $|g_{m+1}\rangle_{ab_1\cdots b_m} := \frac{1}{\sqrt{2}}(|0^{\otimes m+1}\rangle + |1^{\otimes m+1}\rangle)_{ab_1\cdots b_m}$.

This proof follows straightforwardly by generalizing the discrimination strategy discussed in the proof of Theorem 1. For completeness we provide the proof in Appendix A. It is important to note that here also the discriminating resource lives in the minimal Hilbert-space dimension, i.e., in $(\mathbb{C}^2)^{\otimes m+1}$, and hence the genuine nonlocality of the GNPS $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$ can be considered minimal.

We will now discuss another important implication of our construction. In the multipartite scenario, one of the most pertinent problems is the resource comparison among different entangled states. One possible way is to check the possible interconversion between two states under LOCC. However, there exist states that are not comparable in this sense. For instance, consider the states $|g_3\rangle \in (\mathbb{C}^2)^{\otimes 3}$ and $|\psi\rangle \equiv |\chi\rangle\otimes|\eta\rangle$, with $|\chi\rangle \in \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$ having Schmidt rank greater than 2 and $|\eta\rangle \in \mathbb{C}^{d_3}$. Clearly, $|g_3\rangle$ being a genuine entangled state cannot be obtained from the biseparable state $|\psi\rangle$ under LOCC. On the other hand, neither a deterministic [67] nor a probabilistic [68] transformation from the state $|g_3\rangle$ to the state $|\psi\rangle$ is possible even if entanglement of $|\psi\rangle$ is strictly less than unity.[2] At this point a task (τ) based ordering relation $\succ_\tau$ might be of interest. We will say the state ρ is

[2]Although $|g_3\rangle$ and $|\psi\rangle$ are LOCC incomparable under one-copy manipulation (which is the topic we are focusing on here), given many copies of $|g_3\rangle$ one can, however, obtain $|\psi\rangle$ under LOCC, but the converse is a strict impossibility.

better than the state σ in performing the task τ, if the task can be perfectly done with the state ρ as a resource but not with σ and hence it induces an operational ordering between the states represented as $\rho \succ_\tau \sigma$. In that sense, our construction suggests the following ordering relation.

Corollary 1. The task (τ_m) of entanglement assisted discrimination of the set $\mathbb{G}[(m+2)\otimes 3^{\otimes m}]$ induces the ordering relation $|g_{m+1}\rangle \succ_{\tau_m} \rho := \sum p_i\chi^i \otimes \eta^i$, where $\forall\, i$, $\chi^i \in \mathcal{D}(\otimes_{j=1}^m \mathcal{H}_j)$ and $\eta^i \in \mathcal{D}(\mathcal{H})$ with $\mathcal{H}_j$'s and $\mathcal{H}$ having arbitrary dimension, and $p_i \geqslant 0$, $\sum p_i = 1$.

Here, $\mathcal{D}(X)$ denotes the set of density operators acting on X. Note that the state ρ can have at most m partite entanglement which makes the proof of Corollary 1 immediate. Consider now the tripartite resource state $|\psi_3\rangle := |g_3\rangle_{abc}^{\otimes 2}$, i.e., two copies of the three-qubit GHZ state shared among three parties, and the state $|\phi_3\rangle := |\phi^+\rangle_{ab}\otimes|\phi^+\rangle_{bc}\otimes|\phi^+\rangle_{ca}$, i.e., three copies of the two-qubit maximally entangled state $|\phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ symmetrically shared among three parties. Both $|\psi_3\rangle$ and $|\phi_3\rangle$ contain tripartite genuine entanglement and both the states have the same single party marginal. Moreover, these two resources are incomparable under LOCC [69]; in fact it is not possible to convert $2N$ three-party GHZ states into $3N$ singlets even in an asymptotic sense [70]. At this point, consider the task $\tau^\star$ of distinguishing the ordered pair of states $(|\zeta_i\rangle, |\zeta_j\rangle)$ chosen randomly from the Cartesian product set $\mathbb{G}[4\otimes 3^{\otimes 2}] \times \mathbb{G}[4\otimes 3^{\otimes 2}]$. Our next result brings a bona fide ordering between the locally incomparable genuine resource states $|\psi_3\rangle$ and $|\phi_3\rangle$.

Corollary 2. The tripartite product state discrimination problem $\tau^\star$ induces the ordering relation $|\psi_3\rangle \succ_{\tau^\star} |\phi_3\rangle$.

Proof. The task $\tau^\star$ considers discrimination of the ordered tuple $(|\zeta_i\rangle, |\zeta_j\rangle)$ chosen randomly from $\mathbb{G}[4\otimes 3^{\otimes 2}] \times \mathbb{G}[4\otimes 3^{\otimes 2}]$. Clearly, the task cannot be done under LOCC. An additional resource $|\phi_3\rangle$ also fails to achieve the desired objective perfectly. The set $\mathbb{G}[4\otimes 3^{\otimes 2}]$ being a GNPS necessitates consumption of at least two of the three symmetrically distributed Einstein-Podolsky-Rosen (EPR) states for perfect discrimination of the first element of the ordered pair $(|\zeta_i\rangle, |\zeta_j\rangle)$. Since identification of the first element does not provide any information regarding the second, therefore it cannot be perfectly discriminated using the remaining one EPR state. However, given the resource $|\psi_3\rangle$, two copies of three-qubit GHZ, the players can use the first and second copy, respectively, to perfectly identify $|\zeta_i\rangle$ and $|\zeta_j\rangle$. This can be done by following the protocol discussed in Theorem 1. This completes the proof. ■

Furthermore, following the construction of bipartite unextendible product bases of Ref. [48], the present construction can be further generalized for higher dimensional Hilbert spaces. For the explicit construction we refer to the Appendices. There we construct a GNPS in $\mathbb{C}^6\otimes(\mathbb{C}^5)^{\otimes 2}$. It might be interesting to see whether a resource efficient discrimination protocol is possible for this set.

IV. DISCUSSIONS

We have constructed genuinely nonlocal product bases for an arbitrary many number of parties. We then argued that the strength of nonlocality of those sets can be considered minimal as they require an entangled resource of minimal dimension for their perfect discrimination. The constructions also lead to fully separable measurements the implementation of which requires all the parties to either come together or share some multipartite resource that contains entanglement across all possible bipartite cuts.

Our paper also motivates some interesting questions for further research. While we have considered the entangled resource of minimal dimension, the question remains open which particular entangled state in this minimum dimensional Hilbert space turns out to be the optimal resource. In this respect, constructing a tripartite GNPS that can be perfectly distinguished with the resource of the three-qubit W state might be of particular interest.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF THEOREM 2

Proof. (1) Alice performs the measurement $\mathcal{M} \equiv \{M, \mathbb{I} - M\}$, where

$$M := \mathbb{P}[|p\rangle_A; |0\rangle_a] + \mathbb{P}[(|q\rangle, |1\rangle, \dots, |m\rangle)_A; |1\rangle_a].$$

The evolved states are given by

$$\left\{\begin{array}{r} \{|\zeta_\pm^{1,3}\rangle, \dots, |\zeta_\pm^{m,3}\rangle\} \otimes |0^{\otimes m+1}\rangle_{ab_1\cdots b_m}, \\ \{|\zeta_\pm^{1,1}\rangle, |\zeta_\pm^{1,2}\rangle, \dots, |\zeta_\pm^{m,1}\rangle, |\zeta_\pm^{m,2}\rangle\} \otimes |1^{\otimes m+1}\rangle_{ab_1\cdots b_m}, \\ |\zeta_\pm^0\rangle \Rightarrow |\tilde{\zeta}_\pm^0\rangle \end{array}\right\},$$

where

$$|\tilde{\zeta}_\pm^0\rangle = \big(|p\rangle_A|0^{\otimes m+1}\rangle_{ab_1\cdots b_m} \pm |q\rangle_A|1^{\otimes m+1}\rangle_{ab_1\cdots b_m}\big) \otimes|1\rangle_{B_1}\cdots|m\rangle_{B_m}. \tag{A1}$$

(2) ith Bob performs a similar measurement as in Theorem 1. If K_3^i clicks the state is one of $\{|\zeta_\pm^{i,3}\rangle\}$, if K_2^i clicks the state is one of $\{|\zeta_\pm^{i,2}\rangle\}$, and if all K_1^i's click the state is one of $\{|\zeta_\pm^0\rangle, |\zeta_\pm^{i,1}\rangle\}_{i=1}^m$. Alice then performs the measurement

$$\mathcal{M}' \equiv \left\{\begin{array}{r} M_1' := \mathbb{P}[|1\rangle_A; \mathbb{I}_a], \cdots M_m' := \mathbb{P}[|m\rangle_A; \mathbb{I}_a], \\ M_0' := \mathbb{I} - (M_1' + \cdots + M_2'). \end{array}\right\}.$$

If M_i' clicks the state is one of $\{|\zeta_\pm^{i,1}\rangle\}$, otherwise it is one of $\{|\zeta_\pm^0\rangle\}$. Now the result in Ref. [37] assures local distinguishability between any two orthogonal states. ■

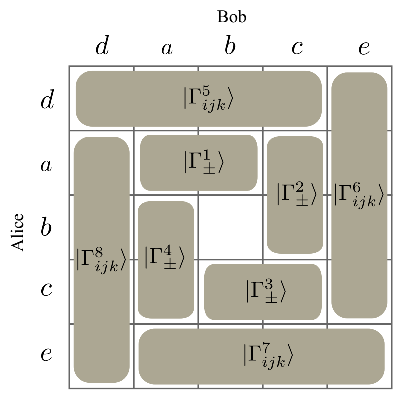

FIG. 3. Tile structure of the set $\mathbb{S}_{\text{Ben}}[5\otimes 5]$. Cardinality of the set is 24. Each inner layered tile contains two mutually orthonormal states, while each outer layered tile contains four mutually orthonormal states. Orthogonality among the states from different tiles is evident from the structure.

APPENDIX B: CONSTRUCTION OF GNPS IN HIGHER-DIMENSIONAL HILBERT SPACES

The NPS $\mathbb{S}_{\text{Ben}} \subset \mathbb{C}^3\otimes\mathbb{C}^3$ can be expressed in the following generic form:

$$\mathbb{S}_{\text{Ben}} \equiv \mathbb{S}_{\text{Ben}}[3\otimes 3] \equiv \{|a\rangle|s_\pm\rangle, |s_\pm\rangle|c\rangle, |c\rangle|t_\pm\rangle, |t_\pm\rangle|a\rangle\}, \tag{B1}$$

where $\{|a\rangle, |b\rangle, |c\rangle\}$ are pairwise orthonormal states and $|s_\pm\rangle := \frac{1}{\sqrt{2}}(|a\rangle \pm |b\rangle)$ and $|t_\pm\rangle := \frac{1}{\sqrt{2}}(|b\rangle \pm |c\rangle)$. A generalization of $\mathbb{S}_{\text{Ben}}[3\otimes 3]$ in $\mathbb{C}^5\otimes\mathbb{C}^5$ is given by

$$\mathbb{S}_{\text{Ben}}[5\otimes 5] \equiv \left\{\begin{array}{c} |\Gamma^1_\pm\rangle := |a\rangle|s_\pm\rangle, \quad |\Gamma^2_\pm\rangle := |s_\pm\rangle|c\rangle, \\ |\Gamma^3_\pm\rangle := |c\rangle|t_\pm\rangle, \quad |\Gamma^4_\pm\rangle := |t_\pm\rangle|a\rangle, \\ |\Gamma^5_{ijk}\rangle := |d\rangle|u_{ijk}\rangle, |\Gamma^6_{ijk}\rangle := |u_{ijk}\rangle|e\rangle, \\ |\Gamma^7_{ijk}\rangle := |e\rangle|v_{ijk}\rangle, |\Gamma^8_{ijk}\rangle := |v_{ijk}\rangle|d\rangle \end{array}\right\}, \tag{B2}$$

where $\{|a\rangle, |b\rangle, |c\rangle, |d\rangle, |e\rangle\}$ is an orthonormal basis of $\mathbb{C}^5$ and $|u_{ijk}\rangle \in \mathcal{S}_{abcd}$ and $|v_{ijk}\rangle \in \mathcal{S}_{abce}$, with

$$\bar{\mathcal{S}}_{\alpha\beta\delta\gamma} \equiv \left\{\begin{array}{l} |\alpha\rangle + (-1)^i|\beta\rangle + (-1)^j|\delta\rangle + (-1)^k|\gamma\rangle, \\ \text{with } i, j, k \in \{0, 1\} \textit{ and } i \oplus_2 j \oplus_2 k = 0 \end{array}\right\}.$$

$\bar{\mathcal{S}}_{\alpha\beta\delta\gamma}$ contains the un-normalized states of $\mathcal{S}_{\alpha\beta\delta\gamma}$. The NPS $\mathbb{S}_{\text{Ben}}[5\otimes 5]$ has a *layered* tile structure (see Fig. 3). This has been recently studied to understand the intricate geometrical structure of the set of bipartite states having positive partial transpose, i.e., the *Peres set* [48]. Furthermore, from Ref. [71] it is evident that the set (B2) can be locally distinguished if a two-qutrit maximally entangled state is shared as resource. Note that the protocol in Ref. [71] is resource efficient compared to the teleportation based protocol as the later requires a maximally entangled state of $\mathbb{C}^5\otimes\mathbb{C}^5$.

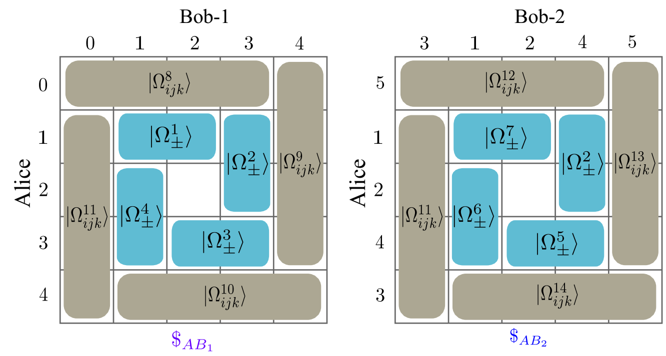

FIG. 4. Tile structure of the set $\$_{AB_1}$ (left) and $\$_{AB_2}$ (right). With all the states in outer layered (gray) tiles of $\$_{AB_1}$ Bob-2's state is $|4\rangle_{B_2}$, while for the inner layer (blue) his state is $|3\rangle_{B_2}$. In $\$_{AB_2}$, Bob-1's state tagged with the outer layer is $|0\rangle_{B_1}$ and for the inner layer it is $|3\rangle_{B_1}$.

Consider now the following set of states in $\mathbb{C}^6_A\otimes\mathbb{C}^5_{B_1}\otimes\mathbb{C}^5_{B_2}$:

$$\mathbb{G}[6\otimes 5^{\otimes 2}]$$

$$\equiv \left\{\begin{array}{l} |\Omega^1_\pm\rangle := |1\rangle|\alpha_\pm\rangle|4\rangle, \ |\Omega^2_\pm\rangle := |\alpha_\pm\rangle|3\rangle|4\rangle, \\ |\Omega^3_\pm\rangle := |3\rangle|\beta_\pm\rangle|4\rangle, \ |\Omega^4_\pm\rangle := |\beta_\pm\rangle|1\rangle|4\rangle, \\ |\Omega^5_\pm\rangle := |4\rangle|3\rangle|\gamma_\pm\rangle, \ |\Omega^6_\pm\rangle := |\gamma_\pm\rangle|3\rangle|1\rangle, \\ |\Omega^7_\pm\rangle := |1\rangle|3\rangle|\alpha_\pm\rangle, \ |\Omega^8_{ijk}\rangle := |0\rangle|\Psi_{ijk}\rangle|3\rangle, \\ |\Omega^9_{ijk}\rangle := |\Psi_{ijk}\rangle|4\rangle|3\rangle, |\Omega^{10}_{ijk}\rangle := |4\rangle|\Phi_{ijk}\rangle|3\rangle, \\ |\Omega^{11}_{ijk}\rangle := |\Phi_{ijk}\rangle|0\rangle|3\rangle, |\Omega^{12}_{ijk}\rangle := |5\rangle|0\rangle|\Phi_{ijk}\rangle, \\ |\Omega^{13}_{ijk}\rangle := |\Upsilon_{ijk}\rangle|0\rangle|5\rangle, |\Omega^{14}_{ijk}\rangle := |3\rangle|0\rangle|\Upsilon_{ijk}\rangle \end{array}\right\}, \tag{B3}$$

where

$$|\alpha_\pm\rangle := \frac{1}{\sqrt{2}}|1\pm 2\rangle, \quad |\beta_\pm\rangle := \frac{1}{\sqrt{2}}|2\pm 3\rangle,$$

$$|\gamma_\pm\rangle := \frac{1}{\sqrt{2}}|2\pm 4\rangle,$$

$$|\Psi_{ijk}\rangle \in \mathcal{S}_{0123}, |\Phi_{ijk}\rangle \in \mathcal{S}_{1234}, |\Upsilon_{ijk}\rangle \in \mathcal{S}_{1245}.$$

Before proceeding further, let us first analyze the structure of the set $\mathbb{G}[6\otimes 5^{\otimes 2}]$. The subset $\$_{AB_1} \equiv \{|\Omega^1_\pm\rangle, |\Omega^2_\pm\rangle, |\Omega^3_\pm\rangle, |\Omega^4_\pm\rangle, |\Omega^8_{ijk}\rangle, |\Omega^9_{ijk}\rangle, |\Omega^{10}_{ijk}\rangle, |\Omega^{11}_{ijk}\rangle\}$ has a kind of analogous structure as of (B2) between Alice and Bob-1 (see Fig. 4). Please note here an important point: Bob-2 has the state $|4\rangle_{B_2}$ tagged with $\{|\Omega^1_\pm\rangle, |\Omega^2_\pm\rangle, |\Omega^3_\pm\rangle, |\Omega^4_\pm\rangle\}$, while with $\{|\Omega^8_{ijk}\rangle, |\Omega^9_{ijk}\rangle, |\Omega^{10}_{ijk}\rangle, |\Omega^{11}_{ijk}\rangle\}$ Bob-2's state $|3\rangle_{B_2}$ is tagged. Similarly, $\$_{AB_2} \equiv \{|\Omega^2_\pm\rangle, |\Omega^5_\pm\rangle, |\Omega^6_\pm\rangle, |\Omega^7_\pm\rangle, |\Omega^{11}_{ijk}\rangle, |\Omega^{12}_{ijk}\rangle, |\Omega^{13}_{ijk}\rangle, |\Omega^{14}_{ijk}\rangle\}$ has a kind of similar structure as of (B2) between Alice and Bob-2 with Bob-1 having the tagged state $|3\rangle_{B_1}$ with $\{|\Omega^2_\pm\rangle, |\Omega^5_\pm\rangle, |\Omega^6_\pm\rangle, |\Omega^7_\pm\rangle\}$ and having the tagged state $|0\rangle_{B_1}$ with $\{|\Omega^{11}_{ijk}\rangle, |\Omega^{12}_{ijk}\rangle, |\Omega^{13}_{ijk}\rangle, |\Omega^{14}_{ijk}\rangle\}$. This structure, along with Observations 1 and 2 discussed in this paper, leads us to the following proposition.

Proposition 3. The set of states $\mathbb{G}[6\otimes 5^{\otimes 2}]$ is a GNPS in $\mathbb{C}^6\otimes\mathbb{C}^5\otimes\mathbb{C}^5$.

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