Leggett-Garg inequalities and decays of unstable systems

Francesco Giacosa[1,2] and Giuseppe Pagliara[3,4]
[1]*Institute of Physics, Jan Kochanowski University, ulica Uniwersytecka 7, 25-406 Kielce, Poland*
[2]*Institute for Theoretical Physics, Goethe University, Max-von-Laue-Strasse 1, 60438 Frankfurt am Main, Germany*
[3]*Dipartimento di Fisica e Scienze della Terra, Università di Ferrara, Via Saragat 1, 44122 Ferrara, Italy*
[4]*INFN Sezione di Ferrara, Via Saragat 1, 44122 Ferrara, Italy*

(Received 16 April 2021; accepted 10 November 2021; published 29 November 2021)

We apply the Leggett-Garg inequalities (LGIs) to the cases of classical and quantum unstable systems. For classical systems the two assumptions of macroscopic realism and noninvasive measurements imply that the three-measurement string K_3 is identically equal to 1. Also, for quantum-mechanical systems, for which the two assumptions are, in general, not valid, we find that $K_3 = 1$ for purely exponential decays ($K_3 \leqslant 1$ is the general LGI). On the other hand, the necessary deviations from the exponential decay law at short and long times predicted by quantum mechanics lead to values of $K_3 \neq 1$. Moreover, a strict violation $K_3 > 1$ typically occurs at short times. Thus, we conclude that experiments in which such deviations from the exponential decay law have been observed should also have in their data violations of the LGIs.

DOI: 10.1103/PhysRevA.104.052225

I. INTRODUCTION

Correlations between spatially separated entangled states are at the core of quantum mechanics (QM) and are a necessary consequence of the linear superposition principle. Such quantum-mechanical correlations have no analogs in classical physics and lead to violations of Bell's inequalities [1,2]. In 1985, Leggett and Garg (LG) [3] derived similar inequalities (LGIs) for the correlations of the outcomes of measurements of the same observable of a system at different times. Interestingly, those violations have been seen in different experimental setups; see the review in [4], which also includes the case of neutrino oscillations [5].

The LGIs are based on two assumptions which definitely hold true in classical systems: (i) macroscopic realism (MR), according to which macroscopic properties are uniquely defined, (see also [6–9]), and (ii) noninvasive measurement (NIM), implying that a measurement does not affect in any way the system under investigation.

A natural question that we shall address in this work concerns the violation of the LGIs for unstable quantum systems. For such systems, the so-called survival probability $p(t)$ is defined as the probability that the state has not yet decayed at time $t > 0$, assuming that it was prepared at $t = 0$ [thus, $p(0) = 1$]. We recall that an actual decay implies that $p(\infty) = 0$; that is, the Poincaré time is genuinely infinite.

Quite interestingly, the survival probability can also be defined for strictly classical systems, such as the probability that a mouse trap is undecayed; see Fig. 1 for a schematic presentation. In this case, the function $p(t)$ depends on the particular system under study and can have [besides the constraints $p'(t) < 0$ and $p(\infty) = 0$] any form. For a classic decay both MR and NIM are clearly fulfilled, and as expected, no violation of the LGI takes place, regardless of the particular classic decay function $p(t)$. In particular, we shall concentrate on the LG correlator K_3, which in general fulfills the LGI $-3 \leqslant K_3 \leqslant 1$. In the case of classic decays, as we will show, it turns out that $K_3 = 1$; thus, the LGI reduces to a LG *equality* in this special case.

Concerning quantum decays, the survival probability $p(t)$ is usually very well approximated by an exponential function [10,11], but it is well established that the exponential behavior is never exact [12]. In particular, the deviations are enhanced at short and long times (see also the experimental confirmations in Refs. [13–16]).

At short times, the decay law can usually be described by a quadratic function, $p(t) \simeq 1 - t^2/\tau_Z^2$, where τ_Z is the Zeno time. As a consequence, the so-called quantum Zeno effect (QZE), which is the freezing of the decay by subsequent repeated measurements at sufficiently small time intervals, is possible [17,18]. Note that the QZE was originally verified as a slowing down of certain transitions in systems involving Rabi oscillations between energy levels [19–21], but it was later verified for an actual quantum decay in Ref. [14]. A related phenomenon is the inverse Zeno effect (IZE): this is an increase in the decay rate that may take place in certain systems when an appropriate time interval between subsequent measurements is chosen. As argued in Refs. [22,23] the IZE might be as relevant as the QZE (see also the theoretical works in Refs. [24–27], as well as the experimental verification in [14]).

In this paper we show, via the correlator K_3 mentioned above, that the LGIs are violated for quantum decays. In particular, such violations are enhanced when short times are involved, thus when the QZE and/or the IZE are also possible. Only in the (unphysical) limit in which the decay is exactly exponential at all times are the LGIs not violated, and they reduce to the LG equality $K_3 = 1$, which holds for classical decays.

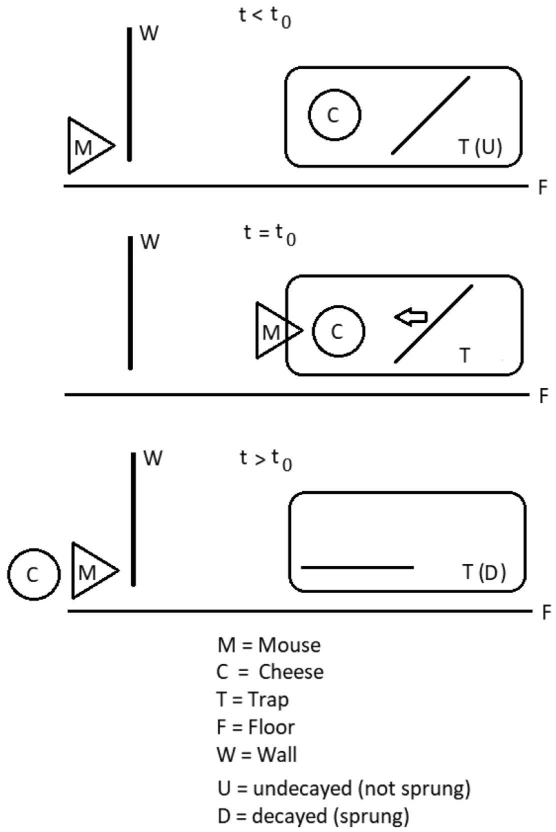

FIG. 1. A classical decay via a mouse trap. Top: a trap is placed at the initial time $t = 0$, and the mouse notices it; the trap is undecayed (U). Middle: at a certain time $0 < t_0 < \infty$ the mouse very rapidly snaps the cheese; the traps decays at this time. Bottom: the trap is decayed (D) for any time $t > t_0$.

Once it has been verified that the LGIs are violated for quantum systems, the natural question is the origin of such deviations. Namely, in QM the collapse of the wave function necessarily implies a strong violation of the NIM. Moreover, the MR is also violated since any quantum decay implies a superposition of decayed and undecayed components, the "quantum version" of the mouse trap mentioned above, due to the linearity of QM, just as Schrödinger's cat enters in a corresponding superposition of sprung and not sprung. Yet as we shall discuss later, the breaking of the NIM (and not of MR) is at the core of the violation of the LGI inequalities for quantum decays.

This paper is organized as follows: in Sec. II we present the derivation of the LGIs for classical and quantum unstable systems; in Sec. III we present some numerical examples that make use of a toy model as well modeling of quantum tunneling as realized in experiments. Finally, in Sec. IV, we present our conclusions.

II. LEGGETT-GARG INEQUALITIES FOR UNSTABLE SYSTEMS

The starting point of the LGIs is the n-measurement LG string K_n, which is built from the two-time correlation functions C_{ij}, which read

$$C_{ij} = \sum_{Q_i,Q_j} Q_iQ_jP_{ij}(Q_i, Q_j), \tag{1}$$

where Q_i (Q_j) represents the outcome of a measurement at time t_i (t_j) with $t_j > t_i$, which we set to 1 if the system is found to be still undecayed (or "alive") and -1 if it is found to be decayed (or "dead"). The quantity $P(Q_i, Q_j)$ is the joint probability associated with the four possible events $Q_i = \pm 1$ and $Q_j = \pm 1$. For instance, $P_{ij}(1, 1)$ is the joint probability that after the first and second measurements the system has been found to be undecayed (with similar notation for the other three joint probabilities). Thus, in the study of (both classic and quantum) decays the correlation quantity C_{ij} takes the explicit form

$$C_{ij} = P_{ij}(1, 1) + P_{ij}(-1, -1) - P_{ij}(1, -1) - P_{ij}(-1, 1).$$

In the following, we shall concentrate our discussion on the LG string K_3, given by [4]

$$K_3 = C_{12} + C_{23} - C_{13}, \tag{2}$$

which is constrained to fulfill the LGIs:

$$-3 \leqslant K_3 \leqslant 1. \tag{3}$$

A. Classical case

Let us discuss now how to compute the correlation function for a "classical unstable system," namely, a system and a measurement of it that obey the MR and the NIM assumptions. Within this context, it is easy to realize that $P_{ij}(-1, 1) = 0$ since, if the system is decayed at t_i, it cannot be alive at t_j. This holds true in all cases (classical or quantum) in which the measurements are sequential since the second choice of the measurement does not influence the first outcome.

Then, we can set $P_{ij}(-1, -1)$ equal to $P_i(-1)$, which is the probability that the system has decayed at time t_i. Namely, if the system decayed at t_i, then it is surely still decayed at $t_j > t_i$, and thus, the joint probability corresponds to the single-measurement probability.

Alternatively, one can write

$$P_{ij}(-1, 1) + P_{ij}(-1, -1) = P_i(-1), \tag{4}$$

which corresponds to summing over the two possible states at $t = t_j$, and since $P_{ij}(-1, 1) = 0$, one obtains

$$P_{ij}(-1, -1) = P_i(-1). \tag{5}$$

Notice that this is strictly true under the MR hypothesis since the state of the particle is decayed or undecayed regardless of the measurement. Similarly,

$$P_{i,j}(1, 1) = P_j(1), \tag{6}$$

which is a direct consequence of the NIM hypothesis. In fact, it corresponds to the assumption that the mouse is *not* affected by the measurement that has occurred at $t_i < t_j$. We will see that in the quantum case this joint probability is different since the NIM is not fulfilled.

The joint probability $P_{ij}(1, -1)$ can be obtained with the normalization $\sum_{Q_i,Q_j} P_{ij}(Q_i, Q_j) = 1$; thus, $P_{ij}(1, -1) =$

TABLE I. Probabilities of two measurements at t_i and t_j (U = undecayed, D = decayed).

Sequence	Classic (MR+NIM)	QM (collapse)
$P_{ij}(1,1) \equiv$ UU	$p(t_j)$	$p(t_i)p(t_j - t_i)$
$P_{ij}(1,-1) \equiv$ UD	$p(t_i) - p(t_j)$	$p(t_i)[1 - p(t_j - t_i)]$
$P_{ij}(-1,1) \equiv$ DU	0	0
$P_{ij}(-1,-1) \equiv$ DD	$1 - p(t_i)$	$1 - p(t_i)$
Sum	1	1

$1 - P_j(1) - P_i(-1)$. Finally, from $P_i(1) + P_i(-1) = 1$ it follows that

$$C_{ij} = 1 + 2P_j(1) - 2P_i(1). \tag{7}$$

It is also useful to reexpress Eq. (7) by introducing the classic survival probability $p(t)$ that the system has not decayed at time t. One has

$$P_j(1) = p(t_j),\ P_j(-1) = 1 - p(t_j), \tag{8}$$

out of which the quantity C_{ij} takes the form

$$C_{ij} = 1 + 2p(t_j) - 2p(t_i). \tag{9}$$

The summary of all classic probabilities is displayed in Table I, where a comparison with the QM case (which will be discussed later) can be found.

Out of Eq. (2) we obtain, via a straightforward calculation,

$$K_3 = 1, \tag{10}$$

which holds for *each* classical unstable system. This is a quite remarkable result since it does not even depend on the specific functional form of the classical decay law $p(t)$, which could very well be different from an exponential.

The classical decay can be explained with a simple example. Following a certain established tradition for QM-related topics, we pick an animal, a mouse. In a given room adjacent to the mouse's lair, an old-fashioned mouse trap with cheese is placed at $t = 0$. The mouse is associated with a certain probability $p(t)$ that it has not yet come in contact with the trap. Of course, $p(t)$ is a given function related to the complicated and stochastic algorithm of the mouse's brain and is not known *a priori*. We simply assume that $p(t)$ tends to zero for large times; hence, at a certain (unknown) time the mouse will steal the cheese; see Fig. 1 for a pictorial representation of the mouse-trap sequence. Note that in the spirit of the time and according to the animal-friendly attitude of the authors, we assume that our mouse—even if it is only imaginary—is not injured in the process: it takes the cheese and runs away, content. Yet through the mouse's actions the trap is sprung. The observer (within this saga an unpleasant old-fashioned farmer dealing with old mouse traps) opens the room at times $t_i > 0$ and $t_j > t_i$ to see whether the mouse was there: the farmer checks whether the trap is still undecayed (U) or decayed (D) at both times and studies the correlation C_{ij}. Clearly, if the trap has sprung at t_i, it is also sprung at t_j: this is sequence DD. In contrast, if it is not sprung at t_j, it was not sprung also at t_i (sequence UU). Since DU is zero, the last sequence is UD: the trap is intact at t_i but sprung at t_j. The probability of UD is calculated as the probability that the system decays between t_i and t_j. By denoting with $h(t) = -p'(t)$ the probability density of decaying at time t, UD is given by

$$\int_{t_i}^{t_j} h(t)dt = p(t_i) - p(t_j). \tag{11}$$

Summarizing (see also Table I),

$$\begin{aligned} C_{ij} &= \text{UU} + \text{DD} - \text{UD} \\ &= p(t_j) + [1 - p(t_i)] - [p(t_i) - p(t_j)], \end{aligned} \tag{12}$$

in agreement with Eq. (7), out of which K_3 is easily evaluated to be 1, independent of the mouse function $p(t)$.

B. Quantum systems

Let us now discuss the case of an unstable quantum system. Both MR and NIM are violated; thus, the system is, if not observed, in a superposition of undecayed and decayed configurations (no MR); moreover, the act of observing or measuring the system perturbs its decay law by resetting the clock (no NIM).

However, even if MR cannot be assumed to hold, each measurement generates a collapse of the system into either decayed or not decayed. In this respect, the collapse is equivalent to MR since it is not possible, within the present setup, to distinguish MR from the collapse. Note that the collapse is intended here as an effective phenomenon whose deep understanding has still not been achieved (it is not clear whether it is a physical collapse or not, e.g., Ref. [28]). However, as matter of fact, for each observer the outcome of the measurement is univocal, either decayed or not, and this is enough for the following discussion. In other words, in the study of decays we need to work with the decayed-undecayed basis, and we cannot *rotate* to another basis to test the QM superposition.

Next, we turn to the evaluation of the three joint probabilities [$P_{ij}(-1,1) = 0$ as before, since if the system decayed at $t = t_i$, it cannot be alive at $t = t_j > t_i$]. It is useful to introduce the conditional probability $P(jQ_j|iQ_i)$, which is the probability of obtaining Q_j provided that at t_i the system had a value Q_i. Through the conditional probability one can write $P_{ij}(1,1) = P(j1|i1)P_i(1)$. Now, if the system was alive at t_i, the probability that it is still alive at t_j is $p(t_j - t_i)$ since the system collapses onto the undecayed state after the first measurement. This is the crucial difference between the classical and quantum cases: the measurement "resets" the clock to the initial time; this process is a clear violation of NIM [29]. The same features of QM are at the origin of the QZE and IZE. We can thus write

$$P_{ij}(1,1) = p(t_i)p(t_j - t_i). \tag{13}$$

Let us compare this expression with the same joint probability in the classical case, which was previously derived as $P_{ij}(1,1) = P_j(1) = p(t_j)$. That result can be reobtained by considering that in the classical case (MR and NIM hold true) the conditional probability $P(j1|i1) = P_j(1)/P_i(1)$ since the condition that the system is undecayed at t_i is necessary for it to be undecayed at t_j (namely, the set of cases in which the system is undecayed at t_j is a subset of the set of cases in which the system is undecayed at t_i). Note that the quantum-mechanical $P(j1|i1) = p(t_j - t_i)$ reduces to the classical one in the case of a purely exponential decay law.

Next, similar to the classical case, $P_{ij}(-1,-1) = P_i(-1) = 1 - p(t_i)$. As before $P_{ij}(1,-1)$ can be determined by the normalization condition, and finally, the quantum two-time correlation function reads

$$C_{ij}^q = 1 + 2p(t_i)p(t_j - t_i) - 2p(t_i), \tag{14}$$

which should be compared with Eqs. (7) and (9). If the decay law of a quantum-mechanical unstable system were purely exponential, $p(t) = e^{-\gamma t}$, then one would obtain a result in agreement with Eq. (9). We summarize the quantum results and compare them to the classical outcomes in Table I.

In QM, the quantity K_3 takes the form

$$K_3 \equiv K_3(t_1,t_2,t_3) = 1 + 2p(t_1)[p(t_2 - t_1) - p(t_3 - t_1)] + 2p(t_2)p(t_3 - t_2) - 2p(t_2). \tag{15}$$

A specific choice, which will be useful later, is obtained by setting $t_1 = 0$:

$$K_3(0,t_2,t_3) = 1 + 2[p(t_2) - p(t_3)] + 2p(t_2)p(t_3 - t_2) - 2p(t_2). \tag{16}$$

For $p(t) = e^{-\gamma t}$ one gets $K_3(t_1,t_2,t_3) = 1$. This is consistent with the fact that the exponential decay carries no memory. Yet, as discussed in the Introduction, the actual QM decay law is never exactly exponential, even if the exponential law can be a very good approximation. For a general discussion, let us consider the following simplified schematic form for $p(t)$ [16,30]:

$$p(t) \simeq \begin{cases} 1 - \frac{t^2}{\tau_Z^2} \text{ for small } t, \\ Ze^{-\gamma t} \text{ for intermediate } t, \\ kt^{-\alpha} \text{ for large } t, \end{cases} \tag{17}$$

where τ_Z (Zeno time), Z, and k are appropriate factors whose numerical values depend on the specific system under study.

The short-time deviations, which were already discussed in the Introduction, allow for QZE. At intermediate times, the behavior is exponential, but a constant Z different from 1 enters into the expression. It can be either larger or smaller than 1, depending on the particular quantum decay. Indeed, $Z > 1$ implies QZE, and $Z < 1$ is a manifestation of IZE. (In fact, within the exponential regime, a single measurement at T gives the survival probability $Ze^{-\gamma T}$, while two measurements at $T/2$ and T correspond to $Z^2 e^{-\gamma T}$; thus, one has QZE for $Z > 1$ and IZE for $Z < 1$.) At long times, as shown already in the seminal paper in Ref. [31], the function $p(t)$ shows a power-law behavior, the reason for which is the necessary existence of a ground state: the decay law at large t is determined by the behavior of the spectral function at energies close to the ground-state energy. Typically, the power-law behavior occurs after many lifetimes, on the order of 10 (see Ref. [15], where such a challenging measurement was performed).

If we choose all three times, t_1, t_2, t_3, within the intermediate "exponential" region, we find

$$K_3 \equiv K_3(t_1,t_2,t_3) \simeq 1 + 2Z(Z-1)e^{-\gamma t_2} \neq 1, \tag{18}$$

which depends only on the intermediate times t_2. It implies that $K_3 > 1$ when $Z > 1$ and vice versa. Anyway, in both cases one obtains a result different from the classic LG result $K_3 = 1$. The classic result is obtained only for large enough t_2 (but still within the exponential interval).

If, instead, we choose $t_1 = 0$ and t_2 and t_3 within the exponential domain, we get a quite analogous result:

$$K_3 \equiv K_3(0,t_2,t_3) \simeq 1 + 2Z(Z-1)e^{-\gamma t_3}, \tag{19}$$

where, in this case, the final time t_3 enters into the expression.

Before showing in the next section some numerical results for some specific models, we briefly discuss the origin of the LGI violations: while a quantum decay violates both the MR and the NIM, only the latter is relevant for the LGI violation. This feature is already clear from our discussion above about the for all practical purposes equivalence between the classic MR and the collapse in QM. In addition, it can also be understood through the following arguments:

(i) In the purely exponential limit, the quantum decay does not break the LGIs. In this particular case, there is no difference between an invasive measurement and a noninvasive measurement (see Table I): the NIM *de facto* applies (even if a collapse takes place when a measurement is performed but the reset of the clock is invisible in the exponential limit). However, the quantum state (also known as the quantum version of the mouse trap) is in a superposition of decayed and undecayed and thus breaks MR. The nonviolation of the LGIs in this case implies that the violation of MR alone is not sufficient.

(ii) Conversely, let us consider the classical example of the mouse trap in which, however, the mouse is affected by the observer checking the status of the trap at a given time. For instance, the mouse may reset its own internal clock when someone opens the room by looking at the trap. Then, it is clear that the NIM is broken in this classical example, but MR is not since the trap is always in one unique state, either decayed or not. The resulting equations are the same as in the quantum case described above, and the LGIs are violated. Thus, this example shows that the violation of MR is also not necessary for violating LGIs.

Both arguments (i) and (ii) show that the NIM alone is responsible for the breaking of the LGIs in the case under study. The important aspect is that in QM the NIM is never fulfilled (apart from the limiting case of an exponential), while in a classical system this can be, in principle, always realized (in a classical world, we can always find a way to check whether the trap is sprung or not *without* the mouse noticing it).

III. NUMERICAL EXAMPLES ON NONEXPONENTIAL DECAY LAWS

In this section we describe some specific numerical examples. Let us first introduce a toy decay law which features both the short- and long-time deviations from the exponential:

$$p(t) = \frac{1}{2}\left(e^{-\gamma \frac{t^2}{t+1}} + \frac{1}{1+t^\alpha}\right). \tag{20}$$

The corresponding temporal behavior is shown in Fig. 2 (red solid line) for $\gamma = 1$ and $\alpha = 2$. To compare it with the exponential decay law, we have fitted $p(t)$ with an exponential and found an effective lifetime τ (that we use as the unit of time).

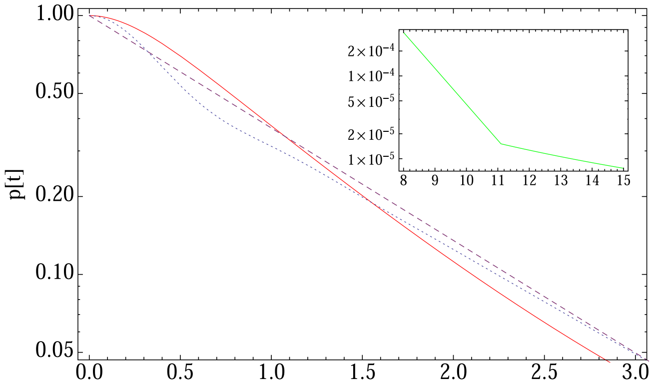

FIG. 2. The three different nonexponential functions used for $p(t)$. The red solid line corresponds to Eq. (20), the blue dotted line corresponds to Eq. (21) (the one reported in Ref. [13]), and the green line in the inset corresponds to Eq. (22) (see Ref. [15]). The dashed line is the exponential function.

Next, let us consider a more realistic, and hence interesting, decay law which has been found in the experimental setup of cold atoms devised in Refs. [13,14]. In particular, we make use of the analytical approximation of the tunneling process used in those works:

$$\ln[p(t)] = -\int_0^t d\tau(t-\tau)W(\tau),$$
$$W(\tau) = \frac{a^2}{2V_0}\int_{-\infty}^{\infty} ds \frac{1}{1+(s-a\tau/V_0)^2}\frac{1}{1+s^2} \times \cos\left(\frac{V_0^2}{a}\int_{s-a\tau/V_0}^{s}\sqrt{1+z^2}dz\right). \tag{21}$$

This $p(t)$ depends on two parameters, the acceleration of the trap a and the potential well depth V_0. To show a numerical example we fix $V_0 = 100$ kHz/h and $a = 4200$ m s^{-2}, and the time dependence of the survival probability is shown in Fig. 2 (blue dotted line). The clearly visible deviations from the exponential at short times are in agreement with the experimental findings of Refs. [13,14].

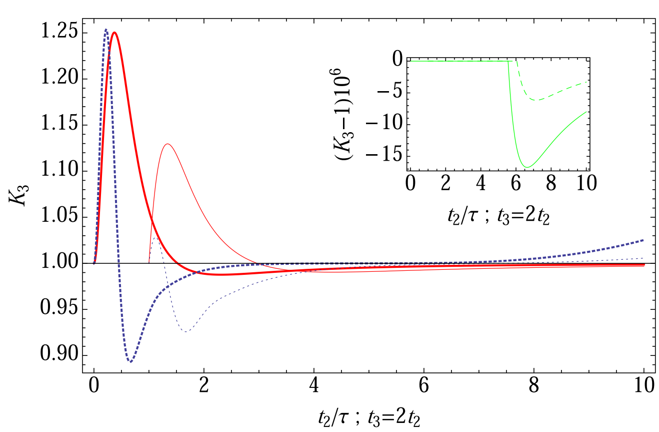

FIG. 3. K_3 as a function of t_2 with $t_3 = 2t_2$ for the three different $p(t)$. The solid red lines refer to $t_1 = 0$ (thick) and $t_1 = \tau$ (thin) for $p(t)$ from Eq. (20). The dotted blue lines refer to $t_1 = 0$ (thick) and $t_1 = \tau$ (thin) for $p(t)$ From Eq. (21). The green lines in the inset show the difference $K_3 - 1$ (magnified by a factor of 10^6) for the long-time-deviation case of Eq. (22) for $t_1 = 0$ (solid line) and $t_1 = \tau$ (dashed line).

For completeness, we also consider example long-time deviations from the exponential. We use the results of Ref. [15] for the decays of molecules of polyfluorene ($\tau = 0.35$ ns, power-law index $\alpha = -2.26$, and turnover time $\tau^{\text{turnover}} = 11.1\tau$), whose survival probability can be modeled as

$$p(t) = \begin{cases} e^{-t/\tau} \text{ for } t \leqslant \tau^{\text{turnover,}} \\ kt^{\alpha} \text{ for } t > \tau^{\text{turnover,}} \end{cases} \tag{22}$$

where k is a normalization constant. (Note that in comparison with Eq. (17) we neglect the initial quadratic time and set

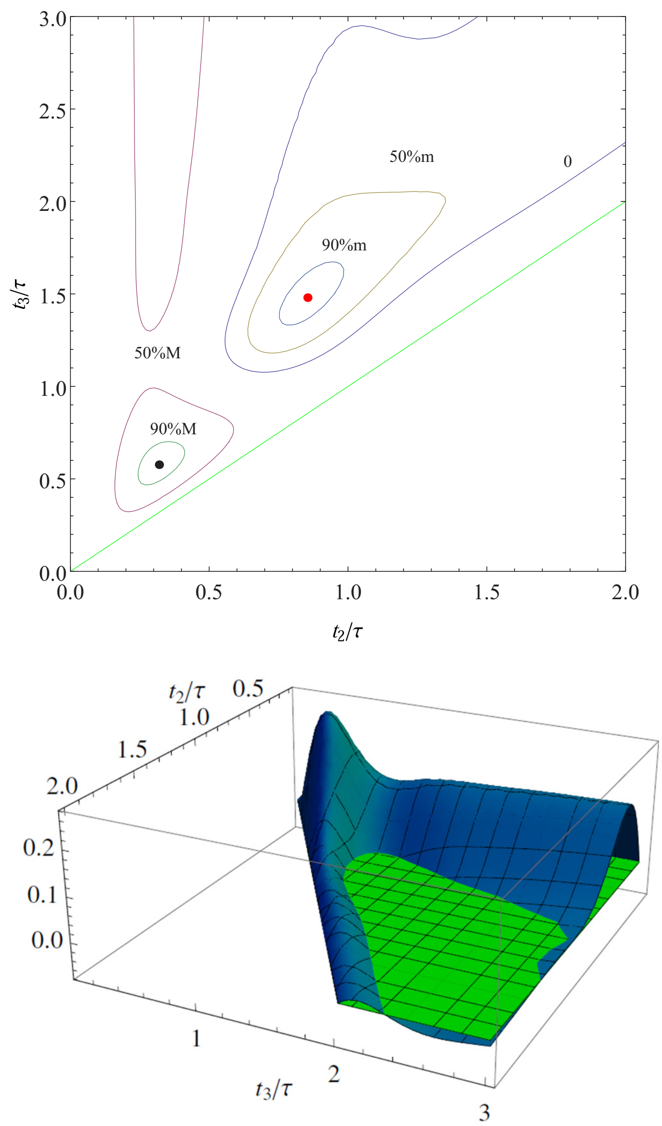

FIG. 4. Top: Contours of $K_3 - 1$ for $t_1 = 0.1\tau$ for $p(t)$ from Eq. (9). The dots correspond to the maximum ($M = 0.28$ for this example, black) and the minimum [$m = -0.08$ for this example, red (gray)]. The green line delimits the region of the plane for which $t_3 > t_2$. The LGI is violated, and oscillations around zero are visible. Bottom: 3D plot of $K_3 - 1$ [blue (dark gray) surface] corresponding to the top panel. In green (light gray), the plane $K_3 - 1 = 0$.

$Z = 1$.) The deviations from the classical result are displayed in the inset of Fig. 2 (green line).

It is interesting to observe that, in principle, the QZE or IZE is also possible for measurements performed at very long time intervals that range into the power-law behavior. However, one would need to detect the same unstable system at least twice and find it undecayed in both cases. This is a quite improbable event that would require a very large statistic that is not reachable at present.

In Fig. 3 we show K_3 as a function of t_2 for $t_1 = 0, \tau$ and for $t_3 = 2t_2$. In all three cases deviations from the classical limit $K_3 = 1$ are found: the deviations are only from above for the first example of $p(t)$, from above and below in the case of the $p(t)$ of Ref. [13], and only from below for the case of long-time deviations. Strictly speaking, the general LGI $K_3 \leqslant 1$ is violated by the first and second $p(t)$, but all of them violate the LG equality $K_3 = 1$ that holds for classic decays. The magnitudes of the departures from $K_3 = 1$ are quite different: they amount to 10% or more for [13], while they are very small, on the order of 10^{-5}, for the case of the long-time deviations of Ref. [15]. Moreover, we observe that the violations are of the order of a few percent even in the case in which $t_1 \sim \tau$ (see Fig. 3). This property may be interesting for investigating nonexponential decays and QZE and IZE and also when studying other unstable systems.

For instance, as computed in Ref. [32] for the electromagnetic transition of the hydrogen atoms, $\tau_Z \sim 10^{-15}$ s, while $\tau \sim 10^{-9}$ s; thus, a direct experimental detection of QZE and IZE would be very challenging. On the other hand, our results suggest that the correlation functions built for testing the LGIs could show sizable and potentially detectable deviations from the classical or exponential case. A viable possibility would also be to investigate the functions $K_3(t_1, t_2, t_3)$ and $K_3(0, t_2, t_3)$, which are different from unity even when the times belong to the exponential domain [see Eqs. (18) and (19), respectively]. The study of the correlator K_3 allows us to investigate at the same time the deviations from the exponential decay as well as QZE and IZE. Moreover, it does so by using only two or three intermediate measurements, which can be a simplification in actual future realizations.

As a final example, we also display the contour plots of $K_3(0.1\tau, t_2, t_3) - 1$ for the survival probability function in Eq. (21) with $t_1 = 0.1\tau$ as well as the corresponding three-dimensional (3D) plot (see Fig. 4). In this way the landscape of departures from $K_3 = 1$ is visible.

IV. CONCLUSIONS

In this work, we have studied the LGIs for classic and quantum decays by focusing on the three-time correlator K_3. The latter equals unity for any classic decay. In the quantum case, it is unity *only* in the (unphysical) limit in which the decay law is purely exponential but is different as soon as deviations are taken into account. Since the quantum decay law is never purely exponential and displays deviations at short and long times, $K_3 \neq 1$. Interestingly, such violations are enhanced at short times and are connected to the QZE and IZE but are also present at long times.

We have provided numerical examples of such violations of the LGIs also by using data from decays already measured in experiments aiming at testing the short- and long-time behaviors of the quantum decay law. The study presented in this work offers an additional tool to test the nonexponential behavior of the quantum decay law by measuring correlation functions. In particular, detecting such violations could be easier than detecting the QZE and IZE within the initial quadratic regime since the departures from $K_3 = 1$ may last longer.

In the future, the study of LGIs can be applied to various systems, such as the one described in Ref. [16]. Moreover, due to the ability to model potentials, it can be applied to novel tunneling experiments. Another interesting extension concerns the case of multiple decays [33–35], in which more than a single decay channel is considered.

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