Enhanced quantumness via non-Markovianity

Haiping Li, Jian Zou,* and Bin Shao
Key Laboratory of Advanced Optoelectronic Quantum Architecture and Measurement, Ministry of Education, School of Physics, Beijing Institute of Technology, Beijing 100081, China

(Received 25 March 2021; accepted 15 October 2021; published 1 November 2021)

In this paper we address the issue of the interplay between non-Markovianity and nonclassicality of a system. We consider the action of non-Markovian environments on a qubit in terms of two models, a classical noise model and a quantum microscopic collision model. For both models we use the violation of the Leggett-Garg inequality (LGI) as a criterion of quantumness, and use the conditional past-future (CPF) correlation to measure non-Markovianity. Besides for the collision model, we also use violation of the nondiscord generating and detecting (NDGD) dynamics to indicate the nonclassicality. And LGI, NDGD, and CPF are probed sequentially by three projective measurements of some observable. In this paper we do not use the assumption of the quantum regression theorem and especially for the collision model the measurement backaction on the environment is explicitly considered. We find that compared with the Markovian dynamics the LGI can be violated for a longer time interval, i.e., non-Markovianity can preserve the quantumness of the system. For the collision model we find that for a specific measurement process, although the non-Markovianity cannot be detected, the effect of non-Markovianity of the system dynamics on nonclassicality still exists, and for some specific measurement operator the violation of both LGI and NDGD will never occur for the Markovian dynamics while with the enhancement of non-Markovianity the quantumness can appear. We also find that the NDGD can be violated for a much wider parameter regime than the LGI.

DOI: 10.1103/PhysRevA.104.052201

I. INTRODUCTION

No system is isolated because of unavoidable coupling to its environment. In the study of open quantum systems, Markovian approximation [1–3] is usually adopted, and the evolution of the system density matrix can be given by Lindblad equations [4,5], such that the system dynamics is easier to handle and describe, while in some cases the Markovian approximation breaks down, such as strong coupling between system and environment and nonvanishing initial system-environment correlation [3]. In addition, it has been found that non-Markovianity can lead to a significant variety of physical effects in the dynamics of open quantum systems [6–12] and can serve as a resource in information theory [13–17]. Recently, the quantification of non-Markovianity has been widely studied, such as measures based on the monotonicity of trace-distance distinguishability [18–21], positivity of quantum maps [22–26], the changes of quantum Fisher information [27], the detection of initial correlations [28], channel capacities, information flow [29–31], and so forth.

In contrast to previous definitions of non-Markovianity, an operational criterion of a quantum process introduced in Ref. [32] is based on the process tensor formalism, which coincides with the definition of condition probability distributions in classical Markovian dynamics. The conditional past-future (CPF) correlation proposed in Ref. [33] relies on a similar formulation of classical Markovianity, i.e., the statistical independence of past and future events when conditioned on a given state at the present time. Also, an ensemble of three time-ordered (random) system events provides a minimal basis for detecting classical Markovianity. In the quantum regime, the three events correspond to the outcomes of three (system) measurement processes and this approach depends on postselection techniques [34] and retrodicted quantum measurements [35,36]. The related CPF correlation becomes a univocal indicator of departure from a memoryless regime. And its experimental implementation has been recently realized [37,38].

Since the birth of quantum mechanics the discussion about the boundary between classical and quantum realms has never stopped. Central to quantum mechanics are concepts such as coherence and entanglement caused by the superposition principle [39,40]. Bell's inequalities [41] explore the nonlocal nature of entanglement between spatially separated quantum systems and have laid the necessary and sufficient conditions for local realism [42–45]. Different from Bell's inequality, the Leggett-Garg inequality (LGI) is about the time correlation of a system under continuous measurement and has been developed to test quantum coherence at the macroscopic level [46,47]. The LGI is based on two assumptions, i.e., macrorealism per se (MRps), in which the system remains in one of its macroscopically distinguishable states, and noninvasive measurability (NIM), in which it is possible to determine the state of the macrosystem without affecting its subsequent dynamics. However, the two assumptions are incompatible with quantum statistics. Thus, the dynamics of a quantum system might violate the LGI [48]. A series of theoretical studies has

*zoujian@bit.edu.cn

been put forward [49–59], and since then many experiments have supported this conclusion [60–65].

As we know, whether in the study of non-Markovianity or in observing quantumness for a system mentioned above, the physics of consecutive measurements on the same quantum system promises to hold fascinating insights into the nature of quantum mechanics. Generally it is easy to obtain the multitime statistics by using the master equation for Markovian dynamics [3,66,67]. But when it comes to the non-Markovian cases, a multitime statistics has to be obtained by referring to the full system-environment dynamics. Only in this way can we trace the correlations between the open quantum system and its environment which affect the subsequent dynamics of the open system, but it is very difficult to handle mathematically. Some authors [68,69] tried to deal with this problem with the assumption of the quantum regression theorem (QRT) [67,70–72], while for non-Markovian dynamics the QRT is not always satisfied. A stochastic Hamiltonian model and a central spin model have been studied without the assumption of QRT [33,73], but a simple situation, where all spins of the environment start in the same state and all the couplings between each spin of the environment and the qubit are equal, was considered. By using the collision model [74–77], where the environment is modelled by an ensemble of individual ancillas with which the system sequentially interacts, we can solve this problem. Moreover, for a collision model, various ways of the system-environment and environment-environment interactions can result in both Markovian and non-Markovian dynamics. A non-Markovian collision model consists of local interactions between system and subenvironment interspersed with subenvironment-subenvironment coupling [75]. But a memoryless collision model assumes that the reservoir consists of a large number of noninteracting subunits [74–76]. It is not difficult for a collision model to obtain multitime statistics due to its ability to tackle its environment, and the reduced system state can be obtained easily. And also collision models provide a physically transparent way to introduce measurement for the open system.

Recently, much attention has been paid to the quantum-to-classical transition in open quantum systems [55,68,69,78]. In the non-Markovian dynamics, Ref. [68] provided a direct connection between nonclassicality and a nondiscord generating and detecting (NDGD) of the system-environment dynamics. Besides there were a few studies on the quantum-to-classical transition for some specific open quantum systems in the non-Markovian environment. Reference [79] considered a qubit embedded in a leaky cavity and controlled by a classical field and drew a conclusion that the enhancement of quantumness is usually accompanied by a disappearance of non-Markovianity, while for a system in a non-Markovian dephasing environment [80] a completely different conclusion from above was obtained.

In the previous studies of quantumness and non-Markovianity, the master equation was generally adopted, which means that the measurement backaction on the environment has been ignored. In this paper, we consider a qubit interacting with two kinds of environments, a classical noise model and an environment consisting of a collection of identical ancillas (collision model). We choose CPF correlation to describe the quantum memory effects and the violation of LGI to indicate the nonclassicality of the system. In addition, we also use the NDGD to discuss the quantumness for the collision model, and make a comparison between the NDGD and the LGI. It is noted that CPF correlation, LGI, and NDGD are all based on the system dynamics interrupted by three consecutive measurements. It is found that in the Markovian dynamics the LGI can be violated only within a short time interval between two measurements. But for the non-Markovian dynamics, the LGI can be violated even for longer time intervals owing to the information backflow from the system to the environment, i.e., non-Markovianity can preserve the quantumness of the system. In addition, for the microscopic collision model, we find that the violation of NDGD can survive longer than LGI. Above we study the relationship between nonclassicality and non-Markovianity for the whole dynamical process; specifically, we take the maximum value of CPF and maximum violations of LGI and NDGD for three measurement operators σ_x, σ_y, and σ_z and different initial system states. Then for the microscopic collision model we investigate CPF correlation, LGI, NDGD, and the relationship among them for a specific measurement process, i.e., for the same measurement operator and the same initial state. Generally the above conclusions for the whole dynamics are still valid for most of the specific measurement process. But in some special cases, the conclusion is different. For a given measurement process, when the initial state is the eigenstate of the measurement operator, $C_{pf} = 0$ while quantumness is not affected. In other words, C_{pf} depends on the initial state, while $\delta_{\rm LG}$ does not. Thus the non-Markovianity cannot be detected by C_{pf} in this case, and the effect of non-Markovianity on quantumness will not change with this undetectability. But when the measurement operator is σ_z, neither the violation of LGI nor the violation of NDGD will occur for the Markovian dynamics, while with the enhancement of non-Markovianity the quantumness can appear.

II. CPF CORRELATION AND LGI

Now we introduce CPF correlation and LGI. A measurement of observable M that acts on the system only at three successive times $t_a < t_b < t_c$ obtains outcomes a, b, and c. The corresponding measurement operators are defined as M_a, M_b, and M_c, respectively, satisfying $\sum_a M_a^\dagger M_a = \sum_b M_b^\dagger M_b = \sum_c M_c^\dagger M_c = I$, where I is the identity matrix in the system Hilbert space, and the sum indices run over all possible outcomes at each stage. The system and environment evolve between consecutive measurements, and the corresponding evolving operator between measurement times t_i and t_j is expressed as $U_{i,j}$. For a Markovian dynamics, the probability of the future outcome only depends on the latest measurement, being independent of the earlier operations. The transition between Markovian and non-Markovian regimes can be visualized by CPF correlation, which is defined as [33]

$$C_{pf} = \sum_{ca} [P(c, a|b) - P(c|b)P(a|b)]ac. \tag{1}$$

The indices a and c run over all the possible outcomes occurring at times t_a and t_c correspondingly, and b is a definite particular outcome of the second measurement. And the CPF correlation can be considered as an indicator of non-Markovianity. For a Markovian system $C_{pf} = 0$, and non-Markovian effects break CPF correlation independence and are present whenever $C_{pf} \neq 0$. In general, the probability of previous outcome a and future outcome c with the given present outcome b can be expressed as

$$\begin{aligned} P(c, a|b) &= P(c|b, a)P(a|b) \\ &= \mathrm{Tr}\big[M_c U_{b,c} \rho_b^{SE} U_{b,c}^\dagger M_c^\dagger\big] \\ &\quad \times \frac{\mathrm{Tr}\big[M_b U_{a,b} M_a \rho_0^{SE} M_a^\dagger U_{a,b}^\dagger M_b^\dagger\big]}{\sum_{a'} \mathrm{Tr}\big[M_b U_{a,b} M_{a'} \rho_0^{SE} M_{a'}^\dagger U_{a,b}^\dagger M_b^\dagger\big]}, \end{aligned} \quad (2)$$

where $P(a|b)$ is a retrodicted quantum probability of outcome a given b, ρ_0^{SE} is the initial system-environment state, and ρ_b^{SE} is the system-environment state after a measurement M_b on the system only at t_b. And $P(c|b) = \sum_a P(c, a|b)$, and $P(a|b) = \sum_c P(c, a|b)$.

For a macroscopic system with two ontic states undergoing an arbitrary dynamic process, the system will evolve from one state to the other and at any particular moment the system is found to be in a definite macroscopic state. Based on MRps and NIM, i.e., a dichotomic observable M can produce definite outcomes $+1$ or -1 and the confirmation of the macrosystem state will not influence the state itself and its subsequent dynamics, the standard LGI is obtained to test the microrealism in the view of quantum mechanics:

$$\delta_{\mathrm{LG}} = \langle M_a M_b\rangle + \langle M_b M_c\rangle - \langle M_a M_c\rangle \leqslant 1, \quad (3)$$

where

$$\begin{aligned} \langle M_i M_j\rangle &= \sum_{ij} P(i, j) M_i M_j \\ &= \mathrm{Tr}\big[M_j U_{i,j} M_i \rho_i^{SE} M_i^\dagger U_{i,j}^\dagger M_j^\dagger\big] M_i M_j, \end{aligned} \quad (4)$$

and ρ_i^{SE} is the joint state of system and environment at time t_i.

III. CLASSICAL NOISE MODEL

Now we consider a classical noise model. It is known that the interaction between a qubit and its environmental extra degrees of freedom can be mimicked by a classical colored noise [81]. In Ref. [73], a classical noise model ignoring the free Hamiltonian of the qubit has been considered. In this paper, we consider the free Hamiltonian of the qubit itself, therefore the Hamiltonian of the system can be described by

$$H = [\xi(t) + \omega]\sigma_z, \quad (5)$$

where $\xi(t)$ is a classical noise, ω is the qubit frequency, and σ_z is the system Pauli matrix in the z direction. Thus, the system evolves under this Hamiltonian following the dynamics,

$$\frac{d\rho_t^{st}}{dt} = -i[\xi(t) + \omega]\big[\sigma_z, \rho_t^{st}\big], \quad (6)$$

and the system state ρ_t can be obtained from averaging ρ_t^{st} over all the noise realizations $\xi(t)$, i.e., $\rho_t = \overline{\rho_t^{st}}$. For a pure initial system state, it can be described by a wave function $|\psi_t\rangle$, and $\rho_t^{st} = |\psi_t\rangle\langle\psi_t|$, thus the evolution of the system can be rewritten as

$$\frac{d}{dt}|\psi_t\rangle = -i[\xi(t) + \omega]\sigma_z|\psi_t\rangle. \quad (7)$$

In this section, we suppose that the initial system state is in its ground state, i.e., $|\psi(0)\rangle = |0\rangle$, and all the three measurements M_a, M_b, and M_c project the system state onto $|\pm\rangle = \frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$, i.e., $M_a = M_b = M_c = \Pi_{\pm 1}$ and $\Pi_{\pm 1} = |\pm\rangle\langle\pm|$. For our initial state after the first measurement, the system state turns into $|\psi^a(0)\rangle = \frac{|0\rangle + a|1\rangle}{\sqrt{2}}$, and the probability with the measurement outcome a is $P_{st}(a) = \langle\psi(0)|\Pi_a|\psi(0)\rangle = \frac{1}{2}$, where $a = \pm 1$ is the outcome of the first measurement. After evolving for a time interval t, the state of the system becomes

$$|\psi^x(t)\rangle = \frac{e^{-i(\int_0^t dt'\xi(t') + \omega t)}|0\rangle + e^{i(\int_0^t dt'\xi(t') + \omega t)}a|1\rangle}{\sqrt{2}}. \quad (8)$$

After the second measurement, the probability of outcomes $b = \pm 1$, given the previous outcomes a, is

$$\begin{aligned} P_{st}(b|a) &= \langle\psi^a(t)|\Pi_{\hat{a}=b}|\psi^a(t)\rangle \\ &= \tfrac{1}{2}\{1 + ba\mathrm{Re}\big[e^{-2i(\int_0^t dt'\xi(t') + \omega t)}\big]\}. \end{aligned} \quad (9)$$

From Eq. (9), it is easy to find that the conditional probability $P_{st}(b|a)$ depends on each particular noise realization. For the initial state $|0\rangle$, the joint probability of outcomes a and b is

$$\begin{aligned} P_{st}(a, b) &= P_{st}(b|a)P_{st}(a) \\ &= \tfrac{1}{4}\big\{1 + ba\mathrm{Re}\big[e^{(-2i(\int_0^t dt'\xi(t') + \omega t)}\big]\big\}. \end{aligned} \quad (10)$$

After the second measurement, the wave function changes into $|\psi^{ab}(t)\rangle = \frac{|0\rangle + b|1\rangle}{\sqrt{2}}$. It is noted that $|\psi^{ab}(t)\rangle$ is only dependent on the outcome b, but is irrelevant to t and the particular noise realization. In the next step, after a time interval τ, the system state evolves into

$$|\psi^{ab}(t + \tau)\rangle = \frac{e^{-i(\int_t^{t+\tau} dt'\xi(t') + \omega\tau)}|0\rangle + e^{i(\int_t^{t+\tau} dt'\xi(t') + \omega\tau)}b|1\rangle}{\sqrt{2}}. \quad (11)$$

The conditional probability of the third measurement outcomes $c = \pm 1$, given the previous outcomes a and b, is

$$\begin{aligned} P_{st}(c|b, a) &= \langle\psi^{ab}(t + \tau)|\Pi_z|\psi^{ab}(t + \tau)\rangle \\ &= \tfrac{1}{2}\big\{1 + cb\mathrm{Re}\big[e^{-2i(\int_t^{t+\tau} dt'\xi(t') + \omega\tau)}\big]\big\}. \end{aligned} \quad (12)$$

From Eq. (10), we get $P_{st}(b) = \sum_a P_{st}(b, a) = \frac{1}{2}$, and from Bayes's rule the retrodicted probability can be written as

$$\begin{aligned} P_{st}(a|b) &= \frac{P_{st}(b|a)P_{st}(a)}{P_{st}(b)} \\ &= \tfrac{1}{2}\big\{1 + ba\mathrm{Re}\big[e^{-2i(\int_0^t dt'\xi(t') + \omega t)}\big]\big\}, \end{aligned} \quad (13)$$

and

$$\begin{aligned} P_{st}(c, a|b) &= P_{st}(c|b, a)P_{st}(a|b) \\ &= \tfrac{1}{4}\big\{1 + ba\mathrm{Re}\big[e^{-2i(\int_0^t dt'\xi(t') + \omega t)}\big]\big\} \\ &\quad \big\{1 + cb\mathrm{Re}\big[e^{-2i(\int_t^{t+\tau} dt'\xi(t') + \omega\tau)}\big]\big\}. \end{aligned} \quad (14)$$

From Eq. (1), the CPF correlation can be obtained:

$$C_{pf} = \sum_{ac}[P(c,a|b) - P(c|b)P(a|b)]M_a M_c$$
$$= f(t,\tau) - f(t)f'(\tau), \tag{15}$$

where

$$f(t) = \overline{\mathrm{Re}\big[e^{-2i(\int_0^t dt'\xi(t')+\omega t)}\big]}, \tag{16}$$

$$f'(\tau) = \overline{\mathrm{Re}\big[e^{-2i(\int_t^{t+\tau} dt'\xi(t')+\omega\tau)}\big]}. \tag{17}$$

In this paper we consider the stationary noise, which implies $f'(\tau) = f(\tau)$. Finally,

$$f(t,\tau) = \overline{\mathrm{Re}\big[e^{-2i(\int_0^t dt'\xi(t')+\omega t)}\big]\mathrm{Re}\big[e^{-2i(\int_t^{t+\tau} dt'\xi(t')+\omega\tau)}\big]}. \tag{18}$$

Thus Eq. (15) can be written as

$$C_{pf} = f(t,\tau) - f(t)f(\tau). \tag{19}$$

Notably, the CPF correlation is conditional on the definite outcome b. Analogous to C_{pf}, the left-hand side of Eq. (3) can be written as

$$\delta_{\mathrm{LG}} = -f(t+\tau) + f(t) + f(\tau). \tag{20}$$

Now, we consider the Gaussian noise as a specific example to study the connection between the nonclassicality and the non-Markovianity. In order to calculate $f(t)$ and $f(t,\tau)$ we first introduce the characteristic noise function [71]:

$$G[k] = \overline{\exp\bigg[i\int_0^\infty k(t')\xi(t')dt'\bigg]}, \tag{21}$$

where $k(t)$ is an arbitrary test function. For a Gaussian noise with $\overline{\xi(t)} = 0$, Eq. (21) can be written as

$$G[k] = \exp\bigg[-\frac{1}{2}\int_0^\infty dt_2\int_0^\infty dt_1 k(t_2)k(t_1)\chi(t_2,t_1)\bigg], \tag{22}$$

where $\chi(t_2,t_1) \equiv \overline{\xi(t_2)\xi(t_1)} = \chi(|t_2 - t_1|)$ is the noise correlation function. For the Gaussian noise $f(t)$ and $f(t,\tau)$ can be determined through the characteristic noise function $G[k]$. Using $\mathrm{Re}[a] = (a + a^*)/2$ from Eqs. (21) and (22) $f(t)$ in Eq. (16) can be obtained with $k(t') = \theta(t - t')$, and $f(t,\tau)$ in Eq. (18) can be obtained by taking $k(t') = \theta(t + \tau - t')\theta(t' - t) \pm \theta(t - t')$ [73]. For an exponential correlation noise, $\chi(t_2,t_1) = g^2\exp(-|t_2 - t_1|/\tau_c)$, where τ_c is the characteristic correlation time of the noise, and g^2 measures its initial width, we obtain

$$f(t) = \cos(2\omega t)e^{-4(g\tau_c)^2[\frac{t}{\tau_c}-(1-e^{-\frac{t}{\tau_c}})]}, \tag{23}$$

$$f(t,\tau) = \tfrac{1}{2}\cos[2\omega(t+\tau)]e^{4(g\tau_c)^2(-\frac{t+\tau}{\tau_c}+1-e^{-\frac{t+\tau}{\tau_c}})} + \tfrac{1}{2}\cos[2\omega(t-\tau)]e^{4e^{-\frac{t+\tau}{\tau_c}}(g\tau_c)^2[-e^{\frac{t+\tau}{\tau_c}}(\frac{t+\tau}{\tau_c}-3)+1-2e^{\frac{t}{\tau_c}}-2e^{\frac{\tau}{\tau_c}}]}.$$

And based on Eqs. (19), (20), and (23) the expressions of C_{pf} and δ_{LG} can be obtained. And from Eq. (23) it can be seen that the CPF and LGI functions are symmetric in the t-τ plane. For the white noise, $\chi(t_2,t_1) = \gamma\delta(t_2 - t_1)$, we can obtain

$$f(t) = \cos(2\omega t)e^{-2\gamma t},\ f(t,\tau) = f(t)f(\tau). \tag{24}$$

Then we can obtain $C_{pf} = 0$, and

$$\delta_{\mathrm{LG}} = -\cos[2\omega(t+\tau)]e^{2\gamma(t+\tau)} + \cos(2\omega\tau)e^{-2\gamma\tau}$$
$$+ \cos(2\omega t)e^{-2\gamma t}. \tag{25}$$

In this case, the Markovian limit is obtained. Then we consider another case of infinite correlation-time noise, $\chi(t_2,t_1) = g^2$, and obtain

$$f(t) = \cos(2\omega t)e^{-2(gt)^2},\quad f(t,\tau) = \tfrac{1}{2}[f(t+\tau) + f(t-\tau)]. \tag{26}$$

Therefore, we can obtain

$$C_{pf} = \tfrac{1}{2}e^{-2g^2(t+\tau)^2}\{\cos[2\omega(t+\tau)]$$
$$-2\,cos(2\omega t)\cos(2\omega\tau)\}$$
$$+\tfrac{1}{2}e^{-2g^2(t-\tau)^2}\cos[2\omega(t-\tau)], \tag{27}$$

$$\delta_{\mathrm{LG}} = -\cos([2\omega(t+\tau)]e^{-2g^2(t+\tau)^2} + \cos(2\omega\tau)e^{-2g^2\tau^2}$$
$$+ \cos(2\omega t)e^{-2g^2t^2}. \tag{28}$$

Note that in the limit $\tau_c \to 0$, Eq. (23) gives the white-noise limit. And in the limit $\tau_c \to \infty$, Eq. (23) reduces to Eq. (26).

This means that, in the limit $\tau_c \to 0$, C_{pf} and δ_{LG} reduce to those in the case of white noise, and in the limit $\tau_c \to \infty$ the results of the infinite correlation-time limit are obtained.

We plot C_{pf} as a function of time t for equal time $\tau = t$ in Fig. 1. Different values of correlation time τ_c are chosen for fixed $\gamma_\omega = 2g^2\tau_c = 0.1$. From Fig. 1, we can find that with the increase of τ_c the amplitude of CPF correlation increases and its decay will slow down as expected; i.e., the bigger τ_c is, the stronger the non-Markovianity is. Specifically, for small correlation times ($\omega\tau_c = 0.001$), CPF correlation rapidly reaches its maximum, then vanishes quickly, while for larger correlation time ($\omega\tau_c = 2$) the CPF correlation is not null even for longer time intervals between two measurements. From Fig. 1(c), the violation of CPF will last a very long time interval for large correlation times $\omega\tau_c = 1000$. In fact, in the limit $\omega\tau_c \to \infty$, the corresponding CPF correlation reduces to

$$C_{pf} = \tfrac{1}{2}[\cos(4\omega t) + 1] - [\cos(2\omega t)]^2,$$

and will not decay. In Fig. 2, δ_{LG} is plotted with the same parameters as in Fig. 1. From Fig. 2, it can be seen that LGI can always be violated for a short time interval ωt, and the longer the correlation time τ_c the longer the duration of the violation of LGI. Moreover, when the correlation time $\tau_c \to \infty$, δ_{LG} becomes

$$\delta_{\mathrm{LG}} = -\cos(4\omega t) + 2\cos(2\omega t),$$

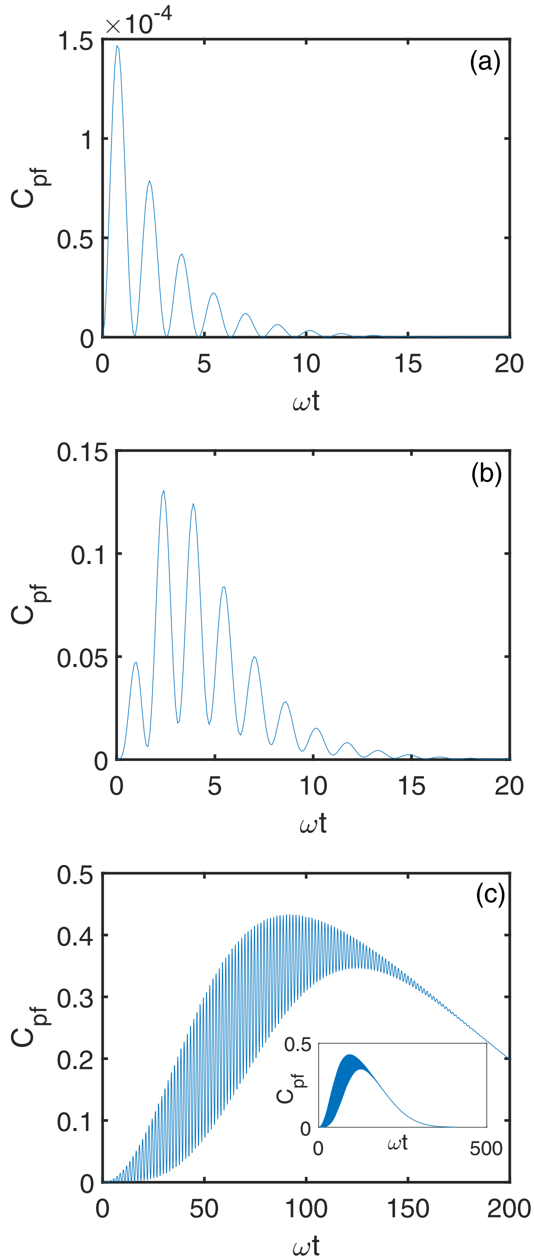

FIG. 1. C_{pf} for the classical noise $2(\frac{g}{\omega})^2(\tau_c\omega) = 0.1$ with different characteristic correlation times (a) $\omega\tau_c = 0.001$, (b) $\omega\tau_c = 2$, and (c) $\omega\tau_c = 1000$.

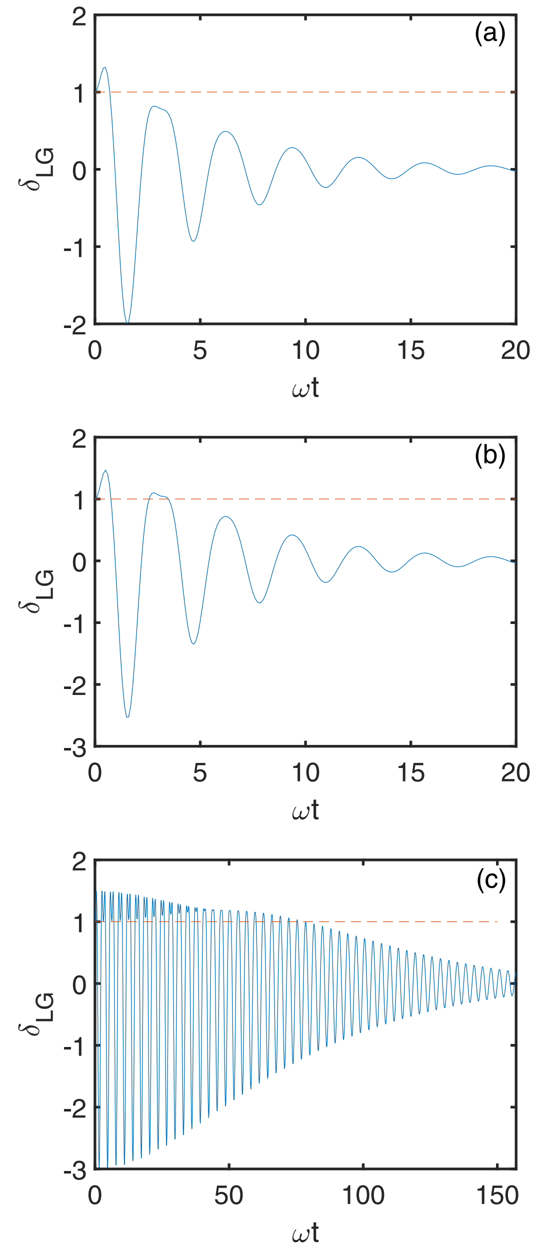

FIG. 2. $\delta_{\rm LG}$ for the classical noise with different characteristic correlation times for $2(\frac{g}{\omega})^2(\tau_c\omega) = 0.1$ and (a) $\omega\tau_c = 0.001$, (b) $\omega\tau_c = 2$, and (c) $\omega\tau_c = 1000$.

thus the violation of LGI ($\delta_{\rm LG} > 1$) will always exist. From Fig. 2, we notice that LGI can be violated both in Markovian and in non-Markovian regimes for a short time interval because the coherence of the system has not flowed to the environment completely, but it does not mean that the nonclassicality is irrelevant to the non-Markovianity. Conversely, in the non-Markovian case, the range of violation of LGI will increase due to the backflow of coherence from the environment to the system, so the connection between nonclassicality and non-Markovianity is built. In addition, it should be noted that due to the oscillatory nature of $\delta_{\rm LG}$ the dynamics of the system can also be quantum even though LGI is not violated in some time intervals. In order to clearly show this, we introduce the Leggett-Garg-type inequalities [78,82,83]:

$$L_+ = -\langle M_aM_b\rangle - \langle M_bM_c\rangle - \langle M_aM_c\rangle \leqslant 1, \tag{29}$$

$$L_- = \delta_{\rm LG} = \langle M_aM_b\rangle + \langle M_bM_c\rangle - \langle M_aM_c\rangle \leqslant 1. \tag{30}$$

It should be noted that $L_- = \delta_{\rm LG}$. Figure 3 displays the Leggett-Garg-type inequalities L_+ and L_- as functions of ωt for the two-level system introduced in Eq. (5) but ignoring the effect of classical noise. It can be seen from Fig. 3 that L_+ and L_- are complementary; i.e., if one of them is not violated, the other is violated and vice versa. However, the effect of classical noise leads to a damping of the oscillations of $\delta_{\rm LG}$,

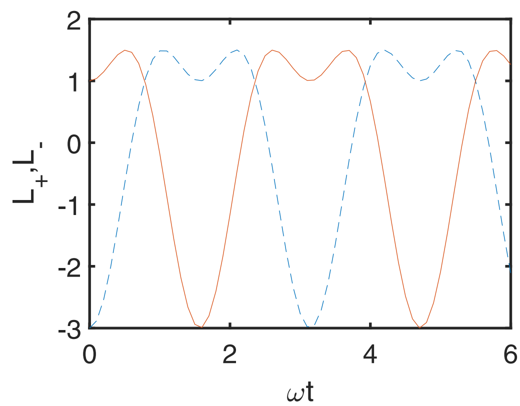

FIG. 3. Leggett-Garg-type inequalities L_+ (blue dashed line) and L_- (orange solid line) as a function of time ωt for a two-level system in the absence of noise.

and after a certain maximal measurement interval no further violations of LGI can be observed, and the dynamics of the system will be classical. In addition, it is noted that for the coherent evolution of the isolated system δ_{LG} is periodic and the amplitude of its oscillations will not decay (see Fig. 3).

IV. COLLISION MODEL

Now we explore the relationship between nonclassicality and non-Markovianity by focusing on a qubit interacting with a large number of subunits, i.e., the collision model. The free Hamiltonian of the qubit system is the same as that of the classical noise model, i.e., $H_s = \omega\sigma_z$. The corresponding free evolution operator is $U_0(\theta) = e^{-i\theta\sigma_z}$, where $\theta = \omega\delta t$ and δt is the free evolution time of the system. In Fig. 4, we give the operational definition of the collision model [20] used in this paper: The system undergoes a unitary evolution $U_0(\theta)$, it interacts with E_1, and E_1 interacts with E_2. In the next step, the system moves forward, it undergoes a unitary evolution $U_0(\theta)$, it interacts with E_2, E_2 interacts with E_3, and so on.

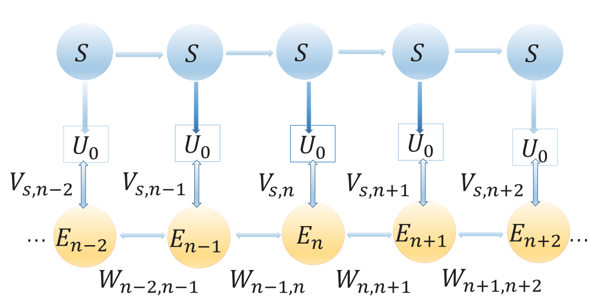

FIG. 4. Schematic of the collision model with S-E and E-E interaction. In the nth step of the dynamics, after local dynamics given by unitary operator U_0, S collides with E_n, which can be described by $V_{s,n}$, the next E_n collides with E_{n+1} with the nearest-neighbor E-E collision denoted by $W_{n,n+1}$, then S shifts by one site. In the next step, S has local unitary dynamics, then collides with E_{n+1}, and E_{n+1} collides with E_{n+2}, and at the end of this step S moves forward and so on.

Next, we concentrate on the condition that all the elements of the environment are dichotomous systems with logical states $\{|0\rangle, |1\rangle\}$ and assume that they are initialized at the ground state $|0\rangle$.

The collision between the system and the nth element of the environment is assumed to be described by a unitary operator

$$V_{S,n}(\gamma) = \cos\gamma I_{S,n} + i\sin\gamma S_{S,n}, \tag{31}$$

where $I_{S,n}$ is the identity operator, $\gamma \in R$ is the dimensionless interaction strength, and a swap gate $S_{S,n}$ in the eigenbasis of the system and the nth subenvironment is introduced:

$$S_{S,n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

After the interaction of the system and nth subenvironment, the nearest-neighbor interaction between the nth and $(n+1)$th subenvironment is considered and the unitary evolution governing this interaction dynamics is chosen:

$$W_{n,n+1}(\delta) = \cos\delta I_{n,n+1} + i\sin\delta S_{n,n+1}. \tag{33}$$

It is noted that the collisions between subenvironments introduce a memory mechanism. As for $\delta = 0$, the interaction between subenvironments (E-E interaction) is switched off so that the quantum process has no information backflow from the environment and the dynamics is fully Markovian. For $\delta = \frac{\pi}{2}$, the fresh subenvironment that the system will interact with at the nth step carries full information obtained at $n-1$ collisions. Thus the joint system of S and E appears like an iterated two-qubit system and a strong non-Markovian dynamics is obtained. The interactions introduced above can also be expressed as dynamic maps:

$$\Phi_{S,n}[\rho] = V_{S,n}(\gamma)\rho V_{S,n}^{\dagger}(\gamma), \tag{34}$$

$$\Psi_{n,n+1}[\rho] = W_{n,n+1}(\delta)\rho W_{n,n+1}^{\dagger}(\delta), \tag{35}$$

where ρ is an arbitrary state. Furthermore, in this paper we retain the nth subenvironment freedom until the system interacts with the $(n+1)$th subenvironment, i.e., the correlation established after S has collided with E_{n-1} is erased only after E_{n-1} has collided with E_n. The S-E initial state ρ_0^{SE} is supposed be a factorized product state. Here, in order to simplify later discussion, let us first introduce a concatenation of dynamics maps $\{\Gamma_{n-1,n}\}$ to characterize the evolution of a composite system composed of a system and the environment between the time t_{n-1} and t_n:

$$\rho^{SE}_{n,n+1} = \Gamma_{n-1,n}\big(\rho^{SE}_{n-1,n}\big) = \Psi_{n,n+1}\big(\Phi_{S,n}\big\{U_0(\theta)\big[\mathrm{Tr}_{n-1}\big(\rho^{SE}_{n-1,n}\big)\otimes(|0\rangle\langle 0|)_{n+1}\big]U_0(\theta)^\dagger\big\}\big), \tag{36}$$

where $\rho^{SE}_{n-1,n}$ is the joint state of the system E_{n-1} and E_n after the interaction between E_{n-1} and E_n, $\mathrm{Tr}_{n-1}(\cdot)$ means that E_{n-1} is traced out prior to the collision between S and E_n, and $U_0(\theta)$ is defined at the beginning of this section. In this way, the reduced state of the system after the collision with the nth subenvironment can be obtained:

$$\rho^S_n = \mathrm{Tr}_{n+1,n}\big[\Gamma_{n-1,n}\big(\rho^{SE}_{n-1,n}\big)\big] = \mathrm{Tr}_{n+1,n}\big[\Psi_{n,n+1}\big(\Phi_{S,n}\big\{U_0(\theta)\big[\mathrm{Tr}_{n-1}\big(\rho^{SE}_{n-1,n}\big)\otimes(|0\rangle\langle 0|)_{n+1}\big]U_0(\theta)^\dagger\big\}\big)\big]. \tag{37}$$

Through simple calculations, we find that unlike classical noise, in the collision model, neither C_{pf} nor δ_{LG} is symmetrical in the plane of the time interval t between the first two measurements and the time interval τ between the last two measurements. This is because in the case of classical noise we suppose it to be stationary, while in the case of the microscopic collision model the process is transient. Moreover the backaction of measurement is explicitly considered. In Fig. 5, we plot C_{pf} against n for $\theta = \pi/2$, $\gamma = 0.1\pi/2$, and different E-E interaction strengths ($\delta = 0.1\pi/2, 0.5\pi/2$). We consider the system state is initialized at $|+\rangle$, and the three successive measurements are chosen as projective ones, being performed in the y direction. For convenience we assume that the time interval between the first two measurements and that between the last two measurements are equal. It is noted that in this model E-E interaction strength δ quantifies the information backflow from the environment to the system. From Fig. 5, we can find that both the time interval for $C_{pf} \neq 0$ and its amplitude increase with the E-E interaction strength δ. For $\delta = 0.1\pi/2$, the time interval for $C_{pf} \neq 0$ is small. For larger $\delta = 0.5\pi/2$, the time interval for $C_{pf} \neq 0$ is larger than the former. With the increasing of δ, the time interval for $C_{pf} \neq 0$ increases. And in the limit of an E-E complete swap $\delta = \pi/2$, C_{pf} will not vanish no matter how large the time interval is. In Fig. 6, for the same parameters as in Fig. 5, we plot δ_{LG} as a function of n. From Fig. 6 we can see that with the increase of E-E interaction strength δ the violation range of LGI ($\delta_{\mathrm{LG}} > 1$) will increase. For small E-E interaction strength ($\delta = 0.1\pi/2$), LGI is violated only for short time interval while it can be violated for a longer time interval for a larger E-E interaction strength δ. For example, for $\delta = 0.5\pi/2$, the violation of LGI will exist for a longer time by comparing with

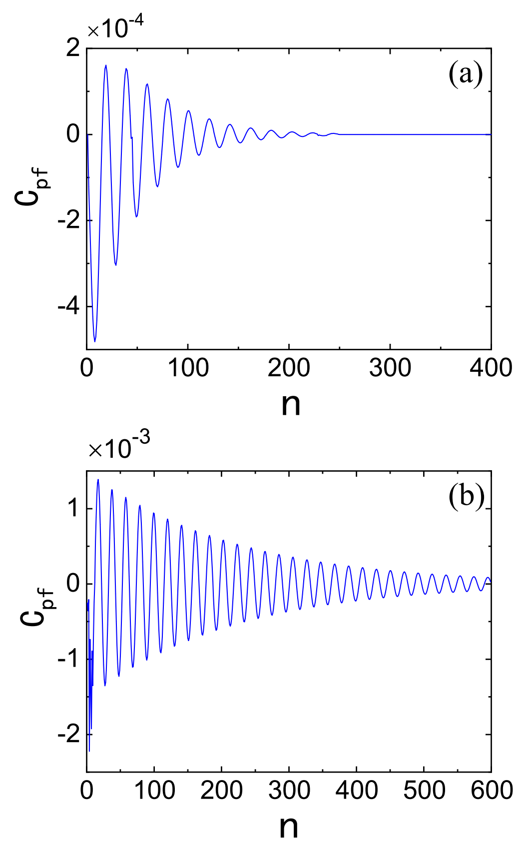

FIG. 5. The n dependence of CPF correlation for the collision model with $\theta = \pi/2$, $\gamma = 0.1\pi/2$, and different E-E interaction strengths (a) $\delta = 0.1\pi/2$ and (b) $\delta = 0.5\pi/2$.

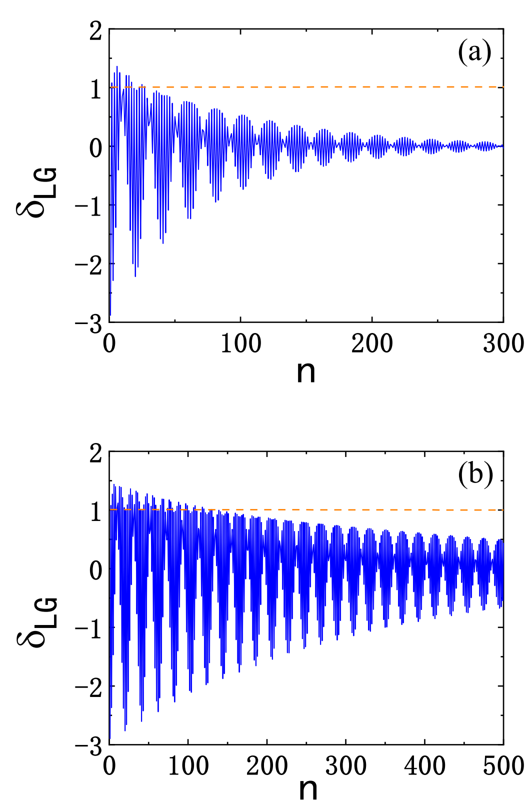

FIG. 6. The n dependence of δ_{LG} for the collision model with $\theta = \pi/2$, $\gamma = 0.1\pi/2$, and different E-E interaction strengths (a) $\delta = 0.1\pi/2$ and (b) $\delta = 0.5\pi/2$.

$\delta = 0.1\pi/2$. In addition, in the limit of a complete swap, no matter how large the time interval is, the LGI will always be violated in this circumstance. In addition, it should be noticed that we choose $\theta = \pi/2$ as an example to demonstrate the properties of CPF and LGI in Figs. 5 and 6, respectively. In fact, for any definite value of θ, the conclusion that both the violations of C_{pf} and $\delta_{\rm LG}$ will be stronger with the increasing of E-E interaction strength δ still holds.

By comparing Figs. 5 and 6, similar to the classical noise model, we can find that LGI can be violated in a short period of time from the beginning due to the initial coherence of the system whether the system dynamics is Markovian or non-Markovian. Moreover, with the increase of n, the information of the open system will flow into its environment and LGI will oscillate with a damped amplitude. However, for the non-Markovian process, the decrease of $\delta_{\rm LG}$ will slow down owing to the information backflow from the environment to the system. Especially on the condition of an E-E full state swap ($\delta = \pi/2$), $\delta_{\rm LG}$ oscillates periodically and will not decay with n.

Recently, a link was built between NDGD and the nonclassicality of the multitime statistics in the non-Markovian dynamics [68]. If the process is NDGD, for a classical observer, doing nothing cannot be distinguished from a measurement in the classical basis averaging over the outcome at any point in time. The quantum-to-classical transition can also fully be quantified by a new physical quantity generating and detecting (DGD). The calculation of NDGD needs to consider the overall evolution of system and environment, and in general the master equation is hard to deal with. In this case the discrete nature of the collision model together with their tractability make it standout. A vanishing DGD dynamics cannot create discord that can be detected by the observer at the next time [68], concretely; for the collision model used in this paper a NDGD dynamics is defined as

$$\sum_n(\mathcal{M}_n\bigotimes\mathcal{I}^e)\circ\Gamma_{m,n}\circ\sum_m(\mathcal{M}_m\bigotimes\mathcal{I}^e)\circ\Gamma_{l,m}\circ\sum_l(\mathcal{M}_l\bigotimes\mathcal{I}^e)$$
$$=\sum_n(\mathcal{M}_n\bigotimes\mathcal{I}^e)\circ\Gamma_{m,n}\circ\mathcal{I}^{se}_m\circ\Gamma_{l,m}\circ\sum_l(\mathcal{M}_l\bigotimes\mathcal{I}^e)\ \forall t_n\geqslant t_m\geqslant t_l, \tag{38}$$

with

$$\mathcal{M}_i[\rho] = M_i\rho M_i^\dagger, \tag{39}$$

where $\Gamma_{m,n} = \Gamma_{n-1,n}\circ\Gamma_{n-2,n-1}\circ\ldots\circ\Gamma_{m+1,m+2}\circ\Gamma_{m,m+1}$ is defined as the dynamical map of the total system evolution between any two times t_m and t_n, $\Gamma_{n-1,n}$ has been defined in Eq. (36), "$\circ$" represents the composition of maps, $\{M_i\}$ is a positive operator-valued measure on the system alone satisfying $\sum_i M_i^\dagger M_i = I$ where the sum indices run over all possible measurement outcomes at each stage, $\mathcal{I}^e$ represents the identity channel of the environment space only, and $\mathcal{I}^{se}_m$ represents the identity channel on the joint state of the system and environment. Then the DGD can be quantified by $\Delta_{\rm DGD}$, which is defined as the trace distance between the left-hand side and right-hand side of Eq. (38) acting on an arbitrary discord-zero S-E state ρ^{SE}:

$$\Delta_{\rm DGD} = \Big\|\sum_n(\mathcal{M}_n\bigotimes\mathcal{I}^e)\circ\Gamma_{m,n}\circ\sum_m(\mathcal{M}_m\bigotimes\mathcal{I}^e)\circ\Gamma_{l,m}\circ\sum_l(\mathcal{M}_l\bigotimes\mathcal{I}^e)[\rho^{SE}]$$
$$-\sum_n(\mathcal{M}_n\bigotimes\mathcal{I}^e)\circ\Gamma_{m,n}\circ\mathcal{I}^{se}_m\circ\Gamma_{l,m}\circ\sum_l(\mathcal{M}_l\bigotimes\mathcal{I}^e)[\rho^{SE}]\Big\|_1. \tag{40}$$

For a classical dynamic process, $\Delta_{\rm DGD} = 0$ can be obtained. As long as $\Delta_{\rm DGD} \neq 0$, the process is nonclassical. We plot $\Delta_{\rm DGD}$ as a function of n in Fig. 7, where the initial system state, environment state, and measurement operator are the same as those introduced above, and the remaining parameters are the same as those in Fig. 6. Comparing Figs. 7(a) and 7(b), we find that when $\delta = 0.5\pi/2$ the violation of NDGD will survive for a longer time interval than $\delta = 0.1\pi/2$, i.e., a large E-E interaction strength δ corresponds to a larger violation range of NDGD. Thus a conclusion that nonclassicality can be maintained by non-Markovianity is obtained. Also, the violation of NDGD exists for a longer time interval than the violation of LGI as shown in Figs. 6(b) and 7(b). For example, for $\delta = 0.5\pi/2$, NDGD can be violated for $n = 450$–600 where LGI is not violated, i.e., NDGD can be violated for a wider parameter regime than the LGI.

According to the definition of CPF correlation for a dynamics process, non-Markovianity is defined as the existence of at least one set of measurement processes such that the CPF correlation does not vanish. In the above discussion, we focus on the non-Markovianity, the quantumness, and the interplay between them for the dynamics of the system. This means that in the calculation of C_{pf}, $\delta_{\rm LG}$, and $\delta_{\rm DGD}$ we take the maximum values of them over three measurement operators σ_x, σ_y, and σ_z, respectively. And it is found that, for this collision model, the non-Markovianity and quantumness are maximized when the measurement operator is the eigenstate of $\sigma_y(\sigma_x)$ and the corresponding initial system state is the eigenstate of $\sigma_x(\sigma_y)$, which we have used in the above discussion. In this sense we conclude that non-Markovianity can enhance the quantumness of the system, while it should be noted that the CPF correlation, LGI, and NDGD all depend

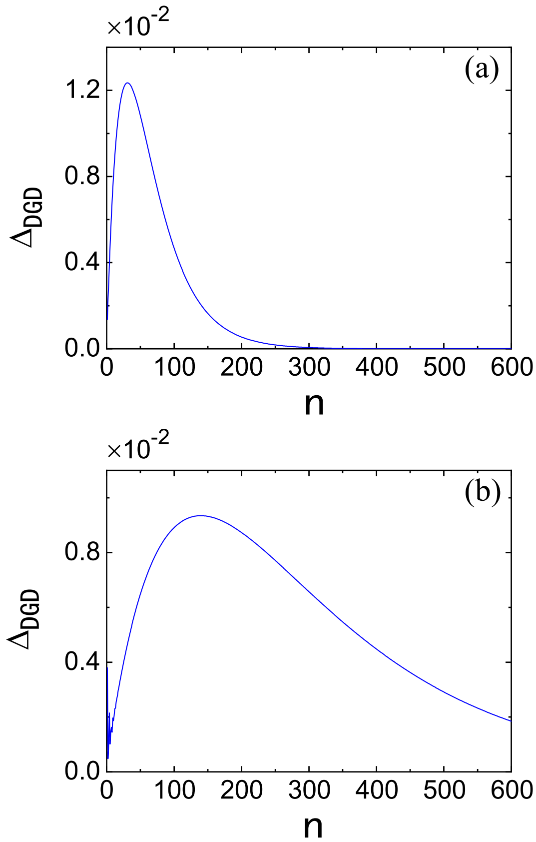

FIG. 7. The n dependence of $\Delta_{\rm DGD}$ for the collision model with $\theta = \pi/2$, $\gamma = 0.1\pi/2$, and different E-E interaction strengths (a) $\delta = 0.1\pi/2$ and (b) $\delta = 0.5\pi/2$.

on a quantum process along a definite evolution interrupted by three time-ordered successive measurements. And now we are interested in the relationship among the CPF correlation, LGI, and NDGD for a specific quantum process, i.e., for the same initial state and the same specific measurement operator. As we will see in the following, for a particular measurement operator and initial state in some measurement time interval, the non-Markovianity of the dynamics might not be detected by CPF correlation although the dynamics is definitely a non-Markovian one. For a given measurement operator, C_{pf} depends on the initial state and from its definition it can be found that when the initial system state is the same as the eigenvector of the corresponding measurement operator C_{pf} will equal to zero, which means that the non-Markovianity cannot be detected in this case. If the initial state is orthogonal to the eigenvector of the corresponding measurement operator, C_{pf} arrives at its maximum value. Therefore, the non-Markovianity of a dynamics process should not be determined by the C_{pf} of a particular measurement process but should take all possible measurement operators and initial system states. However, from our numerical calculation, we find that when measuring the system in x, y, and z directions the initial state of the system does not affect the violation of LGI while NDGD remains qualitatively unchanged despite the quantitative differences. In an experiment about a definitely non-Markovian dynamics for a fixed measurement setting, although the non-Markovianity cannot be detected, the impact of non-Markovianity of the system dynamics on quantumness of the system will not change. In addition, when measuring the system in the z direction, compared with measuring in the x and y directions, the values of C_{pf}, $\delta_{\rm LG}$, and $\delta_{\rm DGD}$ are smaller. Especially for E-E interaction strength $\delta = 0$, neither the violation of LGI nor the violation of NDGD will occur. This can be explained as follows: The measurement operator σ_z commutes with the system Hamiltonian, and the interaction between the system and the element of the environment does not create any coherence of the system. Thus, for this measurement process there is no coherence to be produced. And in the case of $\delta = 0$, which means a Markovian dynamics, the non-coherence-generating-and-detecting dynamics is satisfied and the process is classical [69]. However, when $\delta \neq 0$, the violation of NDGD can be observed. Different from $\delta = 0$ where the system collides with a new ancilla of the environment at a vacuum state, when $\delta \neq 0$, the system collides with a new ancilla with some coherence due to the interaction between the subunits of the environment, so nonclassicality appears. This means that for the specific measurement process the non-Markovianity can make the nonclassicality appear. And when δ is small the violation of LGI cannot occur and only δ is large enough, and the violation of LGI begins to appear. A similar conclusion to that for the whole dynamics as discussed above can be obtained: If NDGD is violated, the violation of LGI is obtained, i.e., in some cases, quantumness cannot be detected by $\delta_{\rm LG}$ but can be detected by $\delta_{\rm DGD}$.

V. CONCLUSION

In this paper, we have considered a qubit coupled to two different environments, i.e., a classical noise model and a collision model, and investigated the relationship between nonclassicality and non-Markovianity. We have characterized the non-Markovianity of dynamics by the CPF correlation and the nonclassicality by the LGI. In addition, another measurement of nonclassicality, NDGD, has been introduced for the collision model. It is noted that an ensemble of three time-ordered (random) system events provides a minimal basis for all CPF correlation, LGI, and NDGD.

For both models we have found that in the Markovian regime the LGI can be violated when the time interval between the two sequence measurements is short, while when the degree of non-Markovianity increases the LGI can be violated for a longer time interval because of the information backflow from the environment to the system. So we have obtained a conclusion that the non-Markovianity can enhance the nonclassicality of the system. In previous studies the master equation was usually used, and sometimes the QRT was adopted, so the measurement backaction on the environment cannot be explicitly considered. All the CPF correlations, LGI, and NDGD are related to a multitime statistics, and for a non-Markovian dynamics in order to obtain such statistics the whole system-environment dynamics should be considered. In general, a measurement on a system inevitably changes the system dynamics between consecutive measurements, which brings difficulties to the evaluation of the CPF correlations, LGI, and NDGD. By using the microscopic collision model we can trace the correlations between the open quantum system and its environment, which affect the subsequent dynamics of the open system, and the measurement backaction on the environment can be explicitly considered. The above conclusion that the non-Markovianity can enhance the nonclassicality of the system is for the dynamics of the system, which means that when we evaluate the CPF correlation, LGI, and NDGD we take the maximum values of them over three measurement operators σ_x, σ_y, and σ_z, respectively. For the collision model, we have also considered the CPF correlation, LGI, and NDGD for a specific measurement process, i.e., for the same measurement operator and initial state. The CPF correlation is related to the initial state of the system, and when the initial state of the system is the eigenstate of the measurement operator the non-Markovianity of the system cannot be detected by the CPF correlation, while the effect of non-Markovianity of dynamics on quantumness will not be affected by this initial state. And when the measurement operator is in the eigenstate of σ_z neither LGI nor NDGD will be violated in the Markovian dynamics, and as the degree of the non-Markovianity increases the violation of NDGD and LGI can be detected. This means that for this special case the non-Markovianity can make the nonclassicality of the system appear. We have also found that the violation of NDGD is more likely to occur than that of LGI: For the measurement operator σ_z, for very weak non-Markovianity, the violation of NDGD can occur, while only when the non-Markovianity becomes strong enough the violation of LGI begins to occur. Generally the NDGD can be violated for a much wider parameter regime than the LGI, which means that the violation of NDGD is tighter for detecting the nonclassicality than the LGI.

ACKNOWLEDGMENT

This work is financially supported by the National Natural Science Foundation of China (Grants No. 11775019 and No. 11875086).

[1] C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).

[2] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (John Wiley & Sons, New York, 1993).

[3] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, London, 2002).

[4] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. **17**, 821 (1976).

[5] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys **48**, 119 (1976).

[6] B. Bylicka, M. Tukiainen, D. Chruściński, J. Piilo, and S. Maniscalco, Thermodynamic power of non-Markovianity, Sci. Rep. **6**, 1 (2016).

[7] Z. X. Man, Y. J. Xia, and R. Lo Franco, Validity of the Landauer principle and quantum memory effects via collisional models, Phys. Rev. A **99**, 042106 (2019).

[8] S. Lorenzo, R. McCloskey, F. Ciccarello, M. Paternostro, and G. M. Palma, Landauer's Principle in Multipartite Open Quantum System Dynamics, Phys. Rev. Lett. **115**, 120403 (2015).

[9] S. Lorenzo, A. Farace, F. Ciccarello, G. M. Palma, and V. Giovannetti, Heat flux and quantum correlations in dissipative cascaded systems, Phys. Rev. A **91**, 022121 (2015).

[10] A. Kutvonen, T. Ala-Nissila, and J. Pekola, Entropy production in a non-Markovian environment, Phys. Rev. E **92**, 012107 (2015).

[11] M. Pezzutto, M. Paternostro, and Y. Omar, An out-of-equilibrium non-Markovian quantum heat engine, Quantum Sci. Technol. **4**, 025002 (2019).

[12] P. Marco, P. Mauro, and O. Yasser, Implications of non-Markovian quantum dynamics for the Landauer bound, New J. Phys. **18**, 123018 (2016).

[13] B. Bylicka, D. Chruściński, and S. Maniscalco, Non-Markovianity and reservoir memory of quantum channels: A quantum information theory perspective, Sci. Rep. **4**, 5720 (2014).

[14] Z. X. Man, Y. J. Xia, and R. Lo Franco, Cavity-based architecture to preserve quantum coherence and entanglement, Sci. Rep. **5**, 13843 (2015).

[15] R. Lo Franco, Nonlocality threshold for entanglement under general dephasing evolutions: A case study, Quantum Inform. Process. **15**, 2393 (2016).

[16] L. Aolita, F. de Melo, and L. Davidovich, Open-system dynamics of entanglement: A key issues review, Rep. Prog. Phys. **78**, 042001 (2015).

[17] A. Mortezapour and R. L. Franco, Protecting quantum resources via frequency modulation of qubits in leaky cavities, Sci. Rep. **8**, 14304 (2018).

[18] H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian behavior of Quantum Processes in Open Systems, Phys. Rev. Lett. **103**, 210401 (2009).

[19] E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, Phys. Rev. A **81**, 062115 (2010).

[20] R. McCloskey and M. Paternostro, Non-Markovianity and system-environment correlations in a microscopic collision model, Phys. Rev. A **89**, 052120 (2014).

[21] S. Campbell, F. Ciccarello, G. M. Palma, and B. Vacchini, System-environment correlations and Markovian embedding of quantum non-Markovian dynamics, Phys. Rev. A **98**, 012142 (2018).

[22] A. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and Non-Markovianity of Quantum Evolutions, Phys. Rev. Lett. **105**, 050403 (2010).

[23] S. C. Hou, X. X. Yi, S. X. Yu, and C. H. Oh, Alternative non-Markovianity measure by divisibility of dynamical maps, Phys. Rev. A **83**, 062115 (2011).

[24] Z. He, H.-S. Zeng, Y. Li, Q. Wang, and C. Yao, Non-Markovianity measure based on the relative entropy of coherence in an extended space, Phys. Rev. A **96**, 022106 (2017).

[25] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Assessing Non-Markovian Quantum Dynamics, Phys. Rev. Lett. **101**, 150402 (2008).

[26] D. Chruściński, A. Rivas, and E. Størmer, Divisibility and Information Flow Notions of Quantum Markovianity for Noninvertible Dynamical Maps, Phys. Rev. Lett. **121**, 080407 (2018).

[27] X.-M. Lu, X. Wang, and C. P. Sun, Quantum fisher information flow and non-Markovian processes of open systems, Phys. Rev. A **82**, 042103 (2010).

[28] S. Luo, S. Fu, and H. Song, Quantifying non-Markovianity via correlations, Phys. Rev. A **86**, 044101 (2012).

[29] B. Bylicka, M. Johansson, and A. Acín, Constructive Method for Detecting the Information Backflow of Non-Markovian Dynamics, Phys. Rev. Lett. **118**, 120501 (2017).

[30] D. Chruściński and S. Maniscalco, Degree of Non-Markovianity of Quantum Evolution, Phys. Rev. Lett. **112**, 120404 (2014).

[31] L. Mazzola, C. A. Rodriguez-Rosario, K. Modi, and M. Paternostro, Dynamical role of system-environment correlations in non-Markovian dynamics, Phys. Rev. A **86**, 010102(R) (2012).

[32] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Operational Markov Condition for Quantum Processes, Phys. Rev. Lett. **120**, 040405 (2018).

[33] A. A. Budini, Quantum Non-Markovian Processes Break Conditional Past-Future Independence, Phys. Rev. Lett. **121**, 240401 (2018).

[34] Y. Aharonov and L. Vaidman, Properties of a quantum system during the time interval between two measurements, Phys. Rev. A **41**, 11 (1990).

[35] S. Gammelmark, B. Julsgaard, and K. Mølmer, Past Quantum States of a Monitored System, Phys. Rev. Lett. **111**, 160401 (2013).

[36] D. Tan, S. J. Weber, I. Siddiqi, K. Mølmer, and K. W. Murch, Prediction and Retrodiction for a Continuously Monitored Superconducting Qubit, Phys. Rev. Lett. **114**, 090403 (2015).

[37] S. Yu, A. A. Budini, Y. T. Wang, Z. J. Ke, Y. Meng, W. Liu, Z. P. Li, Q. Li, Z. H. Liu, J. S. Xu, J. S. Tang, C. F. Li, and G. C. Guo, Experimental observation of conditional past-future correlations, Phys. Rev. A **100**, 050301(R) (2019).

[38] T. d. L. Silva, S. P. Walborn, M. F. Santos, G. H. Aguilar, and A. A. Budini, Detection of quantum non-Markovianity close to the Born-Markov approximation, Phys. Rev. A **101**, 042120 (2020).

[39] E. Schrodinger, Die gegenw*ä*rtige Situation in der Quantenmechanik, Naturwissenschaften **23**, 823 (1935).

[40] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. **47**, 777 (1935).

[41] J. S. Bell, On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. **38**, 447 (1966).

[42] A. Fine, Hidden Variables, Joint Probability, and the Bell Inequalities, Phys. Rev. Lett. **48**, 291 (1982).

[43] S. J. Freedman and J. F. Clauser, Experimental Test of Local Hidden-Variable Theories, Phys. Rev. Lett. **28**, 938 (1972).

[44] A. Aspect, J. Dalibard, and G. Roger, Experimental Test of Bell's Inequalities using Time-Varying Analyzers, Phys. Rev. Lett. **49**, 1804 (1982).

[45] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Violation of Bell's Inequality under Strict Einstein Locality Conditions, Phys. Rev. Lett. **81**, 5039 (1998).

[46] A. J. Leggett and A. Garg, Quantum Mechanics versus Macroscopic Realism: Is the Flux There When Nobody Looks? Phys. Rev. Lett. **54**, 857 (1985).

[47] C. Emary, N. Lambert, and F. Nori, Leggett-Garg inequalities, Rep. Prog. Phys. **77**, 016001 (2014).

[48] A. J. Leggett, Realism and the physical world, Rep. Prog. Phys. **71**, 022001 (2008).

[49] D. Avis, P. Hayden, and M. M. Wilde, Leggett-Garg inequalities and the geometry of the cut polytope, Phys. Rev. A **82**, 030102(R) (2010).

[50] A. Montina, Dynamics of a Qubit as a Classical Stochastic Process with Time-Correlated Noise: Minimal Measurement Invasiveness, Phys. Rev. Lett. **108**, 160501 (2012).

[51] J. Kofler and C. A. Brukner, Condition for macroscopic realism beyond the Leggett-Garg inequalities, Phys. Rev. A **87**, 052115 (2013).

[52] C. Emary, Decoherence and maximal violations of the Leggett-Garg inequality, Phys. Rev. A **87**, 032106 (2013).

[53] G. Y. Chen, S. L. Chen, C. M. Li, and Y. N. Chen, Examining non-locality and quantum coherent dynamics induced by a common reservoir, Sci. Rep. **3**, 2514 (2013).

[54] S. Kumari and A. K. Pan, Inequivalent Leggett-Garg inequalities, EPL **118**, 50002 (2017).

[55] T. Chanda, T. Das, S. Sen, and U. Sen, Canonical Leggett-Garg inequality: Nonclassicality of temporal quantum correlations under energy constraint, Phys. Rev. A **98**, 022138 (2018).

[56] J. Naikoo, A. K. Alok, and S. Banerjee, Study of temporal quantum correlations in decohering b and k meson systems, Phys. Rev. D **97**, 053008 (2018).

[57] J. Naikoo, A. K. Alok, S. Banerjee, and S. U. Sankar, Leggett-Garg inequality in the context of three flavor neutrino oscillation, Phys. Rev. D **99**, 095001 (2019).

[58] J. Naikoo, S. Banerjee, and R. Srikanth, Effect of memory on the violation of Leggett-Garg inequality, Quantum Inform. Process **19**, 408 (2020).

[59] J. Naikoo and S. Banerjee, Entropic Leggett-Garg inequality in neutrinos and $b(k)$ meson systems, Eur. Phys. J. C **78**, 602 (2018).

[60] J. P. Groen, D. Ristè, L. Tornberg, J. Cramer, P. C. de Groot, T. Picot, G. Johansson, and L. DiCarlo, Partial-Measurement Backaction and Nonclassical Weak Values in a Superconducting Circuit, Phys. Rev. Lett. **111**, 090506 (2013).

[61] M. E. Goggin, M. P. Almeida, M. Barbieri, B. P. Lanyon, J. L. O'Brien, A. G. White, and G. J. Pryde, Violation of the Leggett-Garg inequality with weak measurements of photons, Proc. Natl. Acad. Sci. USA **108**, 1256 (2011).

[62] J. Dressel, C. J. Broadbent, J. C. Howell, and A. N. Jordan, Experimental Violation of Two-Party Leggett-Garg Inequalities with Semi-Weak Measurements, Phys. Rev. Lett. **106**, 040402 (2011).

[63] V. Athalye, S. S. Roy, and T. S. Mahesh, Investigation of the Leggett-Garg Inequality for Precessing Nuclear Spins, Phys. Rev. Lett. **107**, 130402 (2011).

[64] A. M. Souza, I. S. Oliveira, and R. S. Sarthour, scattering quantum circuit for measuring Bell's time inequality: A nuclear magnetic resonance demonstration using maximally mixed states, New J. Phys. **13**, 053023 (2011).

[65] H. Katiyar, A. Shukla, K. R. K. Rao, and T. S. Mahesh, Violation of entropic Leggett-Garg inequality in nuclear spins, Phys. Rev. A **87**, 052102 (2013).

[66] E. B. Davies, *Quantum Theory of Open Systems* (Academic, New York, 1976).

[67] H. Carmichael, *An Open Systems Approach to Quantum Optics: Lectures Presented at the Université Libre de Bruxelles* (Springer-Verlag, Berlin, 1993).

[68] S. Milz, D. Egloff, P. Taranto, T. Theurer, M. B. Plenio, A. Smirne, and S. F. Huelga, When is a noN-Markovian Quantum Process Classical?, Phys. Rev. X **10**, 041049 (2020).

[69] A. Smirne, D. Egloff, M. G. Díaz, M. B. Plenio, and S. F. Huelga, Coherence and non-classicality of quantum Markov processes, Quantum Sci. Technol. **4**, 01LT01 (2018).

[70] M. Lax, Quantum noise. XI. Multitime correspondence between quantum and classical stochastic processes, Phys. Rev. **172**, 350 (1968).

[71] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, New York, 1992).

[72] C. Gardiner, P. Zoller, and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer-Verlag, Berlin, 2000).

[73] A. A. Budini, Conditional past-future correlation induced by non-Markovian dephasing reservoirs, Phys. Rev. A **99**, 052125 (2019).

[74] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, Thermalizing Quantum Machines: Dissipation and Entanglement, Phys. Rev. Lett. **88**, 097905 (2002).

[75] M. Ziman, P. Štelmachovič, and V. Bužek, Description of quantum dynamics of open systems based on collision-like models, Open Syst. Inf Dyn. **12**, 81 (2005).

[76] M. Ziman and V. Bužek, All (qubit) decoherences: Complete characterization and physical implementation, Phys. Rev. A **72**, 022110 (2005).

[77] F. Ciccarello, S. Lorenzo, V. Giovannetti, and G. M. Palma, Quantum collision models: Open system dynamics from repeated interactions, arXiv:2106.11974.

[78] A. Friedenberger and E. Lutz, Assessing the quantumness of a damped two-level system, Phys. Rev. A **95**, 022101 (2017).

[79] H. Gholipour, A. Mortezapour, F. Nosrati, and R. L. Franco, Quantumness and memory of one qubit in a dissipative cavity under classical control, Ann. Phys. (NY) **414**, 168073 (2020).

[80] P.-W. Chen and M. M. Ali, Investigating Leggett-Garg inequality for a two level system under decoherence in a non-Markovian dephasing environment, Sci. Rep. **4**, 6165 (2014).

[81] P. W. Anderson and P. R. Weiss, Exchange narrowing in paramagnetic resonance, Rev. Mod. Phys. **25**, 269 (1953).

[82] S. F. Huelga, T. W. Marshall, and E. Santos, Proposed test for realist theories using Rydberg atoms coupled to a high-q resonator, Phys. Rev. A **52**, R2497(R) (1995).

[83] J. Naikoo, S. Banerjee, and A. M. Jayannavar, Violation of Leggett-Garg-type inequalities in a driven two-level atom interacting with a squeezed thermal reservoir, Phys. Rev. A **100**, 062132 (2019).