Topology of polarization-ellipse strips in the light scattered by a dielectric nanosphere

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Numerical modeling of scattering of a plane elliptically polarized monochromatic wave on a silicon spherical nanoparticle is carried out. In the resulting light field near the particle, the topology of strips, formed by the axes of the polarization ellipses and the normal vectors to their planes, is studied. The strips may have one half-twist only if they enclose a circular polarization singularity line, while almost all other strips, even enclosing the linear polarization singularity lines, are trivial. The correlation between the twisting indices of different strips is found, and their relation to the topological features of points of the singular lines is analyzed.

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I. INTRODUCTION

Freund predicted [1] the possibility of manipulating the polarization of light, leading to the folding of its characteristic vectors into complex structures, e.g., cones, spirals, and Möbius strips. These Möbius strips were experimentally observed in tightly focused radiation, the transverse profile of which was specially modified by a liquid crystal device [2]. Later on, the method of generation of exotic Möbius strips of polarization characteristic vectors with varying twisting rate [3] via tight focusing was demonstrated and the possibility of generation of complex topological structures in the light field by manipulating its polarization singularities in nonparaxial light beams was shown [4]. Recent theoretical studies [5] show that changing eccentricity, azimuthal orientation, or centering of the trajectory used for the strip tracing can change their chirality and the number of twists.

Nonparaxial electromagnetic fields containing Möbius strips may be formed not only by tight focusing of the radiation but also by simple crossing of paraxial beams [6]. The rapidly developing branch of near-field nanophotonics is also of a significant interest for nonparaxial optics [7–9]. Singular lines in the proximity of nanoscale objects, irradiated by a plane monochromatic wave, initially containing no singularities, have been predicted both analytically [10] and numerically [11,12]. It was shown in [13] that Möbius strips formed by the axes of the polarization ellipse encircle the lines of polarization singularities emerging in the near field of nanoscale objects. Although the growth of interest in this topic is undoubted, the process of understanding the causes of the emergence and described behavior of the strips still remains in its earliest stages.

This paper describes the strips in a nonparaxial light field swept by the vectors characterizing its polarization ellipse. The example of this field is taken from previous work [12] in which the scattering of a plane elliptically polarized wave on a spherical silicon particle was modeled numerically. However, the goal in the present paper is to study the features of distribution of the total nonparaxial field, not just some of its projections on peculiar planes. The present paper also studies the interconnection between the parameters, characterizing the topology of polarization strips and the structure of polarization singularity lines of the electric field.

II. THEORETICAL BACKGROUND

In this paper we use classic methods to describe the polarization state of the nonparaxial light field [14]. In such fields, the polarization ellipses not only have arbitrary shapes and sizes, but also are arbitrarily oriented in space. The ellipse parameters are unambiguously defined by two scalar and two vector values: the intensity $|\mathbf{E}|^2$, the degree of ellipticity $M = |\mathbf{E}^* \times \mathbf{E}|/|\mathbf{E}|^2$, the normal vector to the plane of the ellipse $\mathbf{n} = \mathrm{Im}\{\mathbf{E}^* \times \mathbf{E}\}$, and the bidirectional vector (director) of its major axis $\overleftrightarrow{\mathbf{A}} = \pm\,\mathrm{Re}\{\mathbf{E}^*\sqrt{(\mathbf{E}\cdot\mathbf{E})/|\mathbf{E}\cdot\mathbf{E}|}\}$. In these formulas $\mathbf{E}$ is the vector complex amplitude of the oscillating monochromatic electric field $\tilde{\mathbf{E}}$, which is related to the amplitude as $\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})\exp(-i\omega t)\}$. The usage of the bidirectional vector $\overleftrightarrow{\mathbf{A}}$ appears more useful for the description of the ellipse axis orientation, because the axis of an ellipse is bidirectional by nature.

The vectors $\mathbf{n}$ and $\overleftrightarrow{\mathbf{A}}$ are not defined for all polarization states of the electromagnetic field. In the case when the polarization ellipse is a circle ($M = 1$), the director $\overleftrightarrow{\mathbf{A}}$ is not uniquely defined and in the other limit case of linear polarization ($M = 0$), the normal $\mathbf{n}$ to the ellipse plane loses its meaning. In the general case, the points in space where such behavior is taking place form isolated lines known as C^T and L^T lines or lines of circular and linear polarization singularity, respectively [14]. In their vicinity $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and the director of the minor axis of the ellipse $\overleftrightarrow{\mathbf{B}} = \pm\,\mathrm{Im}\{\mathbf{E}^*\sqrt{(\mathbf{E}\cdot\mathbf{E})/|\mathbf{E}\cdot\mathbf{E}|}\}$ have complex distributions of their spatial directions.

The distribution of the polarization ellipses near the singular points of circular polarization is comprehensively studied in the case of paraxial electromagnetic fields [15–18], when

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the normal vectors $\mathbf{n}$ of the polarization ellipse planes are considered parallel to the direction of the wave propagation. In the proximity of the singular points the directors $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$ thus change their orientation in a complex way, forming distinctive planar distributions (singularity patterns), known in the literature as lemons, stars, and monstars. The key characteristic of these distributions is the topological index, a number of full rotations of the major axis of the polarization ellipse in the positive direction when the ellipse is tracked along a small closed contour, encircling the singularity in the same direction [16].

The distribution of vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ near the singular lines in nonparaxial fields become significantly more complex and so its description requires another approach. Some characteristics of this kind of distribution may be obtained by studying the projections of the polarization ellipses of the field on certain selected planes. In this case, for the points of C^T and L^T lines, the isotropy parameters Υ_C and Υ_L, respectively, are introduced. These parameters generalize the idea of the topological index [19] and are related to the complex amplitude of the field and its spatial derivatives in the singular point as

$$\Upsilon_C = \frac{|\mathbf{E}^* \cdot \nabla(\mathbf{E}\cdot\mathbf{E})|^2 - |\mathbf{E}\cdot\nabla(\mathbf{E}\cdot\mathbf{E})|^2}{|\mathbf{E}^* \cdot \nabla(\mathbf{E}\cdot\mathbf{E})|^2 + |\mathbf{E}\cdot\nabla(\mathbf{E}\cdot\mathbf{E})|^2}, \tag{1}$$

$$\Upsilon_L = \frac{e_{ilm}e_{jpq}T_{pl}T_{qm}D_{ij}}{\sum\limits_{r,s=x,y,z} [(\delta_{rl} - D_{rl})(\delta_{sm} - D_{sm})T_{lm}]^2}. \tag{2}$$

In these expressions we use the vector differential operator $\nabla = \{\partial_x, \partial_y, \partial_z\}$, the tensor $D_{ij} = E_i E_j^*/|\mathbf{E}|^2$, the tensor $T_{ij} = \mathrm{Im}\{\sqrt{(\mathbf{E}\cdot\mathbf{E})}\partial_j E_i^*\}$ (the sign is chosen arbitrarily), and δ_{ij} and e_{ijk}, the Kronecker and Levi-Cività tensors, respectively. The summation is performed over repeated indices $i, j, l, m, p, q \in \{x, y, z\}$, where x, y, and z are the coordinates of an arbitrarily chosen Cartesian frame. The isotropy parameters have the same sign as the topological indices of the patterns, formed by the ellipse projections on a specially chosen plane. For the circular polarization singularities this plane coincides with the plane of $\tilde{\mathbf{E}}$ vector rotation and for the linear polarization singularities it is orthogonal to the direction of this vector oscillation.

The isotropy parameters introduced in [19] do not fully characterized the polarization singularities because only one (though in some sense preferred) plane for projection is taken into consideration. The distribution pictures of the polarization ellipse projections onto a plane significantly depend on the choice of its orientation to the extent that almost any point of the field may be viewed as some kind of "singularity" [20]. To obtain objective and unambiguous information of the three-dimensional distribution of the electromagnetic field polarization, one must analyze the change of the vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ along a small closed contour, encircling the singularity under consideration. During such motion, each of these vectors sweeps out a surface, known as a strip of the corresponding vector [21,22]. The complex behavior of the vector will define the topology of a strip, which may be orientable or nonorientable, twisted or nontwisted. The key parameters of the strip topology are the linking number $\mathcal{L}$ of its edges and its twist number $\mathcal{T}$. The linking number of two closed contours in space may be defined as the sum of the linking numbers of their projections onto an arbitrary plane, where the linking number of each intersection point is considered to be 1 if the first contour passes under the second one and crosses it from left to right, -1 if the first contour passes under the second one and crosses it from right to left, and 0 if the first contour passes over the second one. For the strips swept by a unit vector $\mathbf{v}$, tracked along a closed contour without self-intersections, and orthogonal to the tangent of that contour at each of its points, the twist number $\mathcal{T}$ is defined by

$$\mathcal{T} = \frac{1}{2\pi}\int_0^l \left(\left[\frac{d\mathbf{v}(\mathbf{g}(s))}{ds} \times \mathbf{v}(\mathbf{g}(s))\right]\cdot\frac{d\mathbf{g}}{ds}\right)ds. \tag{3}$$

Here $\mathbf{g}(s)$ is a function, parametrically defining the contour, where the arclength of the contour is generally taken as the parameter s. In this case $0 \leqslant s \leqslant l$ and $\mathbf{g}(0) = \mathbf{g}(l)$, where l is the full length of the contour. For a wide class of contours, including, in particular, all planar ones, the twist number $\mathcal{T}$ of any strip constructed on that contour is equal to the linking number of its edges $\mathcal{L}$ [23]. The main advantage of the integral (3) is that it can be evaluated in an arbitrarily chosen coordinate system, which is more versatile compared to the previously used methods of finding the $\mathcal{T}$ and $\mathcal{L}$ characteristics of the polarization ellipse strips [21,22].

The formula (3) is helpful in understanding the physical meaning of the twist number. The term in large parentheses represents a scalar product of the angular velocity of the vector $\mathbf{v}$ rotation while it is passing along the contour $\mathbf{g}(s)$ and a vector $d\mathbf{g}/ds$, tangent to that contour. Thus, the integral (3) is equal to the full number of revolutions of the vector $\mathbf{v}$ during the motion of its initial point along the contour $\mathbf{g}(s)$ in the positive direction in the plane, orthogonal to $d\mathbf{g}/ds$, that is, *around* the contour. In some sense, for the contours mentioned above, the parameter $\mathcal{T} = \mathcal{L}$ generalizes the definition of the topological index of singularity of the electric field. For the bidirectional vectors like $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$, the number of such revolutions may be a half-integer, without any discontinuities in the observable values. It is worth mentioning that the vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ considered in the present paper are not everywhere normalized nor are they generally orthogonal to the contours, chosen for construction of the corresponding strips. Nevertheless, for the analysis of their topology we can use the expression (3), in which the vector $\mathbf{v}$ is the projection of the considered vector onto the plane, orthogonal to the contour tangent, and additionally normalize this projection to unity. Except for some special cases, the field of the vector $\mathbf{v}$ may be uniquely constructed by these rules over all the selected contours and the linking number $\mathcal{L}$ of the strip edges for these projections coincides with the linking number of the strip edges of the original vectors. Five examples of the strips with different linking numbers $\mathcal{L}$ of their edges (or twisting numbers $\mathcal{T}$) are shown in Fig. 1. Three upper strips are orientable and they may be formed by all three vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$. Strips on the lower line are nonorientable (Möbius) strips and they may be formed only by the $\overleftrightarrow{\mathbf{A}}$ and

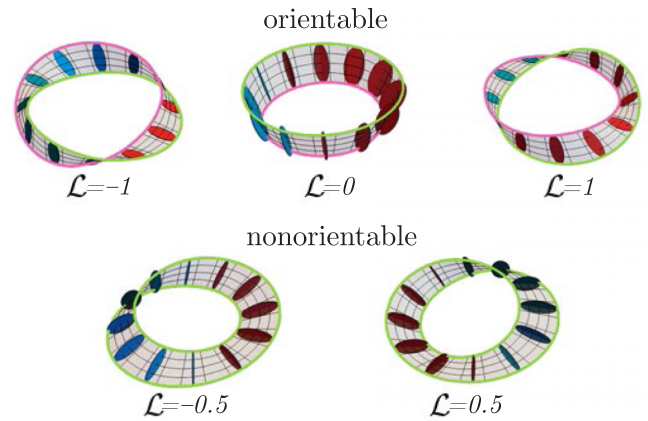

FIG. 1. Examples of the strips swept by a bidirectional vector $\overleftrightarrow{\mathbf{A}}$ of the major axis of the polarization ellipse with different linking numbers of their edges. Polarization ellipses are colored red if the electric field vector appears to the observer rotating clockwise and blue in the opposite case. The edges of the strip are shown in different colors if there are two of them (i.e., the strip is orientable).

$\overleftrightarrow{\mathbf{B}}$ axes directors, which, unlike usual vectors, are equivalent to themselves when rotated by 180°.

III. FORMULATION OF THE PROBLEM

The interaction between an elliptically polarized plane wave (degree of ellipticity $0 \leqslant M_0 \leqslant 1$ and wavelength $\lambda = 710$ nm) and a dielectric sphere with radius $R_0 = 90$ nm is studied, as in our earlier paper [12], with the help of the COMSOL MULTIPHYSICS package by the finite-element method. First, we set the values of the ellipticity degree M_0, the wavelength of the incident radiation, the radius of the sphere, and the properties of its material. Periodic boundary conditions are used at the boundaries of the computational space, which is split into 50 000 finite elements and is surrounded by a model medium able to absorb all the incident radiation (perfectly matched layers). This allows us to avoid possible "reflections" of the wave from the boundaries of the computational space. It was shown in [12] that for $0 < M_0 < 1$ there are two closed C^T lines in the near field of the light, scattered by the sphere, and the lines approach each other very closely for small values of M_0. For an almost linearly polarized incident wave ($M_0 \simeq 0$) there also are two L^T lines near the nanoparticle, but they gradually shrink until they eventually disappear with the increase of M_0.

When constructing the polarization ellipse strips, in this work we use only circular contours with a sufficiently small radius R such that the value R would not influence the topology of the strips. Each contour was given by the radius vector $\mathbf{r}_0$ of its center point, a normal vector $\mathbf{N}$ to the plane of the contour, and the radius R as follows:

$$\mathbf{g}(s) = \mathbf{r}_0 + R\left[\mathbf{e}\cos\left(\frac{s}{R}\right) + \mathbf{e}\times\mathbf{N}\sin\left(\frac{s}{R}\right)\right]. \qquad (4)$$

Here $\mathbf{e}$ is an arbitrarily chosen unit vector, orthogonal to $\mathbf{N}$, $s \in [0, 2\pi R)$. We also choose a small value of R in our computations to neglect second and higher spatial derivatives of the field. To make it possible R has to be less than $R_0 \ll \lambda$ by an order or more, but of course greater than the step of the computational grid. To construct the strips along the contours, commensurable with the step of the grid, we used biqubic interpolation of the field values. Since the integral (3) may take only integer or half-integer values, errors in its numeric computations of several percent order are not significant. That allows us to construct strips even for the very small values of R despite the growth of the numerical errors in this case.

IV. DISCUSSION OF THE RESULTS

Our study shows that nonparaxial light field, scattered by a dielectric sphere, contains strips with a variety of different topological characteristics. The strips of vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$, constructed along small planar contours, which do not encircle polarization singularity lines, are at the most part trivial (linking number $\mathcal{L} = 0$). There is an exception to this rule, when the vector, traced along the contour, lies in the plane of the contour in its vicinity. Figures 2(a) and 2(c) show the singular lines near the surface of a sphere, irradiated by a slightly elliptically polarized wave ($M_0 = 0.1$). The red thick stroke is used for the C^T lines and the blue thin stroke for the L^T lines.

A typical example of a strip swept by the vector $\mathbf{n}$ is shown in Fig. 2(a). Its edges are shown in pink and green. Figure 2(b) shows a close-up of the same strip. To make positioning of the strip clear with respect to the sphere, it is constructed on a contour with a radius of $R = 30$ nm. The derivative $d\mathbf{n}/ds$ in the integrand of Eq. (3) approaches zero with the decrease

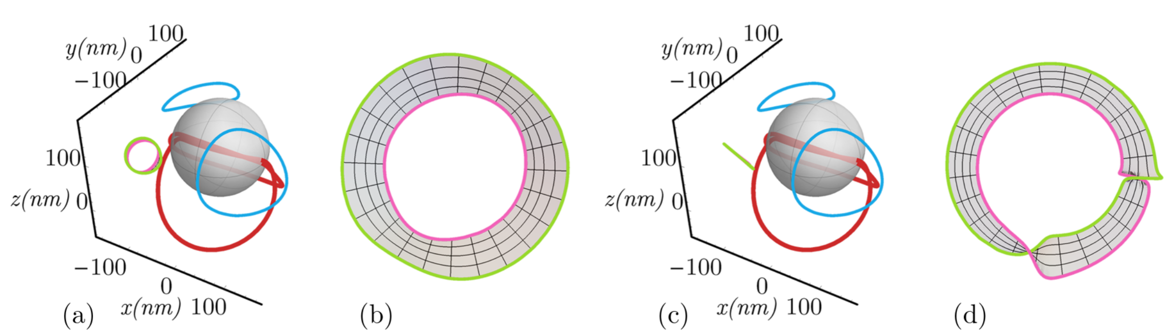

FIG. 2. Strip swept by the vector $\mathbf{n}$, entirely lying in the regular region of the field, near (a) a nanoparticle and (b) its close-up view in the case of the regular topology ($\mathcal{L} = 0$) and (c) and (d) with a single twisting ($\mathcal{L} = 1$). The degree of ellipticity of the incident wave $M_0 = 0.1$.

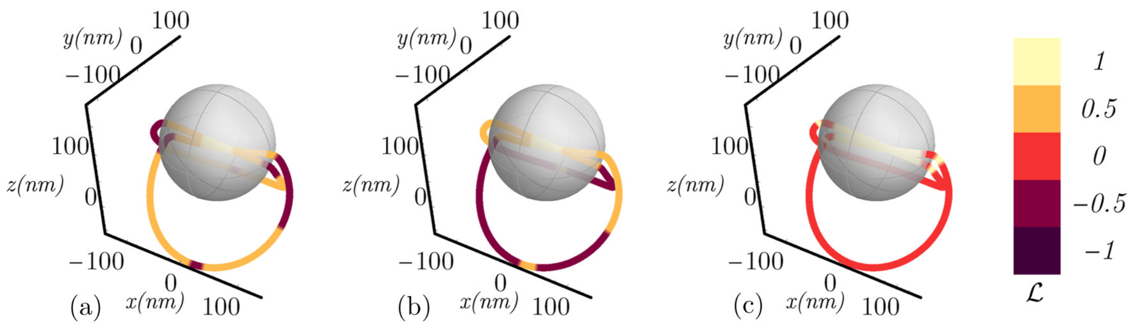

FIG. 3. Linking numbers of the strips of the vectors (a) $\overleftrightarrow{\mathbf{A}}$, (b) $\overleftrightarrow{\mathbf{B}}$, and (c) $\mathbf{n}$, enclosing a C^T line in the scattered field for the ellipticity degree $M_0 = 0.25$ of the incident wave.

of R; thus for a circle of a sufficiently small radius the integral inevitably turns to zero.

In some special cases, the strip, built on a small closed contour, which does not encircle polarization singularity lines, can be nontrivial [Figs. 2(c) and 2(d)]. Near such a point in space, the vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ are nearly constant, so if the normal to the bypass contour is orthogonal to the mean direction of these vectors, there exist at least two points on the contour where the mentioned vector is nearly parallel to the contour tangent. In this case, the topology of the corresponding strip is be determined by the behavior of the small components of the vector, which define its deviation from the mean direction. Even in the proximity of a regular point, the distribution of these small components is generally singular [20] and the linking number of the considered strip is nonzero. In the present research we discovered the strips with the linking number $\mathcal{L} = \pm 1$ in the near field of the nanoparticle, which did not encircle polarization singularity lines. As an example, Fig. 2(c) shows a nontrivial strip, entirely lying in the regular region of the field. It is swept by the vector $\mathbf{n}$, traced along a circle which lies in the plane that is parallel to the $\mathbf{n}$ vector, evaluated in the center of the circle. Figure 2(d) shows a close-up of this strip. If the vector being traced is strictly collinear to the contour tangent on some point of this contour, the construction of the strip faces principal difficulties, because the linking number may not be defined. However, that kind of situation is nongeneric and disappears with a small distortion of the contour or the fluctuation in the electric field.

To estimate the frequency of the occurrence of these nontrivial strips in the regular regions of the light field, we constructed and analyzed 2916 strips swept by the vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ during their motion along circles with a radius of $R = 10$ nm in a small region of 27 equally distributed regular points both inside and outside the nanoparticle, distanced no less than a couple of values of R from the nearest singular line of polarization. Three groups of strips swept by the vectors $\mathbf{n}$, $\overleftrightarrow{\mathbf{A}}$, and $\overleftrightarrow{\mathbf{B}}$ during their motion along 36 differently oriented circle contours were built. The direction of the normal to these contours planes was changed with a step of 5°. As a result, only 184 strips with the contours nearly coplanar to the considered vector had linking numbers distinct from zero. Positive and negative values were found in approximately equal proportion.

Analysis of the results of the numerical computation has confirmed that the strips of the directors $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$, constructed along the small contours, enclosing a C^T line, are nontrivial. The reason for this is the impossibility of an unambiguous choice of directions of $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$ at the points of circular polarization singularity. We considered the configurations of the field near the nanoparticle for a wide range of values of the incident wave ellipticity degree M_0. For each considered value of M_0 we selected 100 equidistant points on each of the two C^T lines near the dielectric sphere and constructed the strips on the contours, encircling the selected points and locally orthogonal to the lines. The topology of the strips swept by the vector $\mathbf{n}$ has also been analyzed; however, the majority of these strips were, as expected, trivial. The only exception made the contours lie nearly in the same plane as the $\mathbf{n}$ vector of the field in their center. As mentioned above, the same picture is observed near the points in space containing no polarization singularities of the electric field.

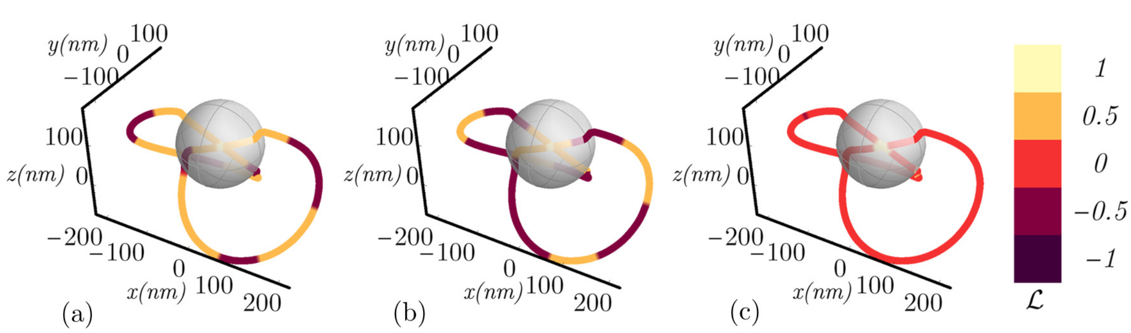

FIG. 4. Linking numbers of the strips of the vectors (a) $\overleftrightarrow{\mathbf{A}}$, (b) $\overleftrightarrow{\mathbf{B}}$, and (c) $\mathbf{n}$, enclosing a C^T line in the scattered field for the ellipticity degree $M_0 = 0.75$ of the incident wave.

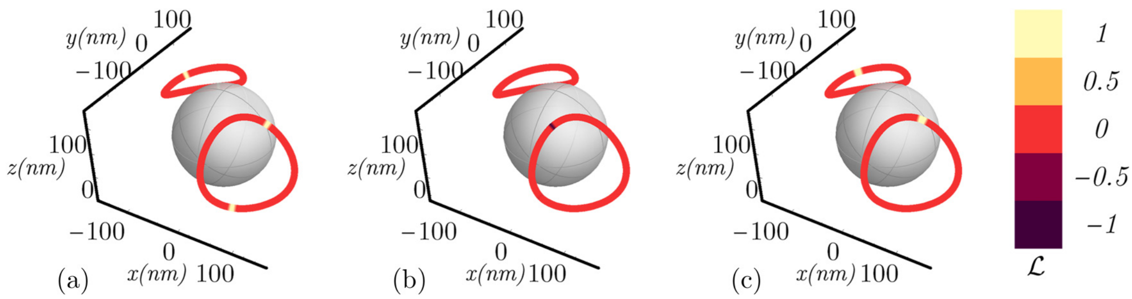

FIG. 5. Distribution of the linking numbers of strips of the vectors (a) $\overleftrightarrow{\mathbf{A}}$, (b) $\overleftrightarrow{\mathbf{B}}$, and (c) $\mathbf{n}$, encircling different parts of L^T lines, in the scattered field of the wave for the ellipticity degree $M_0 = 0.1$.

At the same time, strips swept by the directors $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$ are found to have nonzero linking numbers. Figures 3 and 4 show two examples of the C^T lines in the near field of the nanoparticle, the points of which are colored according to the linking number of the strips which are centered at these points. Figures 3(a), 4(a), 3(b), 4(b), 3(c), and 4(c) show the distribution of the number $\mathcal{L}$ for the strips of $\overleftrightarrow{\mathbf{A}}$, $\overleftrightarrow{\mathbf{B}}$, and the normal vector $\mathbf{n}$, respectively. In the majority of these lines the first two kinds of strips have half-integer linking number $\mathcal{L} = \pm\frac{1}{2}$. An exception made the strips, built around the points at which the two C^T lines are closest, and the contour, used for the strip construction, encircle both of the singular lines at the same time. In this case the linking number $\mathcal{L} = 1$. The strips with a half-integer linking number are nonorientable and are Möbius strips; we observed strips with both handedness $\mathcal{L} = \frac{1}{2}$ and $\mathcal{L} = -\frac{1}{2}$. The signs of the linking numbers of the $\overleftrightarrow{\mathbf{A}}$ and $\overleftrightarrow{\mathbf{B}}$ strips were found to be opposite at the most points of the C^T lines; however, the signs also coincided inside the sphere, where the lines approach each other.

It was shown in [12] that the isotropy parameter Υ_C of the points of an C^T line has different signs inside and outside the nanoparticle. The parameter Υ_C is calculated within the locally paraxial approximation, when all polarization ellipses around the point are assumed to lie in the same plane, coinciding with the plane of electric field rotation at the C^T point itself. This approximation allows us to understand, to some extent, the topological properties of the patterns of axes of the polarization ellipses in close proximity to the C^T line and in particular to distinguish its points with the topological types being used in paraxial optics. In the inner region of the sphere and in a small adjoining outer region the singularity has topological type lemon (positive isotropy parameter Υ_C), while in the outer region the topological type star prevails (negative isotropy parameter Υ_C). We have found no unambiguous interconnection between the sign of the Υ_C parameter and the topological type of the strip swept by the vectors $\overleftrightarrow{\mathbf{A}}$, $\overleftrightarrow{\mathbf{B}}$, and $\mathbf{n}$. This results agree with earlier work [24] and certainly shows that to define the topological type of a strip one must take into consideration even the small nonplanarity of the distribution of the axes of the polarization ellipses near the C^T point.

In the same manner, we have studied strips swept by the vectors $\overleftrightarrow{\mathbf{A}}$, $\overleftrightarrow{\mathbf{B}}$, and $\mathbf{n}$ which encircle the linear polarization singularity lines (L^T lines). The distribution of the linking numbers of these strips built around the L^T line at various points is shown in Fig. 5. To construct the strips we used small circular contours, locally orthogonal to the L^T line. We found that nearly all of the considered strips are trivial ($\mathcal{L} = 0$), with only exception being the ones constructed near the points where the isotropy parameter Υ_L changes its sign. These strips were found to have linking number $\mathcal{L} = \pm 1$, that is, they are twisted once clockwise or counterclockwise and, unlike the Möbius strips, are orientable. A more detailed study has shown that the special properties of these strips are caused not by the specificity of the distribution of polarization ellipses around the regarded points of the line, but by the orientation of the bypass contour. The reason is that the contour used is locally orthogonal to the L^T line and near the points, where Υ_L changes sign, the contour is nearly coplanar to the vector sweeping the strip. Such nontrivial strips are analogous to the ones found in the regular region of the field in the case of the specially chosen contours.

Despite the triviality, the strips of the vectors $\overleftrightarrow{\mathbf{A}}$, $\overleftrightarrow{\mathbf{B}}$, and $\mathbf{n}$ have specificity of their own. We demonstrate it with the example of the $\mathbf{n}$ vector strips, constructed near different points of the space and shown in Fig. 6. The black arrows indicate the distributions of the projections of the considered vector onto the contour plane. The strip in Fig. 6(a) is built in the region of the field containing no polarization singularities and the other two strips encircle L^T points with positive [Fig. 6(b)] and negative [Fig. 6(c)] values of the isotropy parameter. Despite all three strips being topologically trivial (which may be obtained by a continuous deformation of a simple ring) and their edges not being linked, the distributions of the components of the $\mathbf{n}$ vector in the contour plane

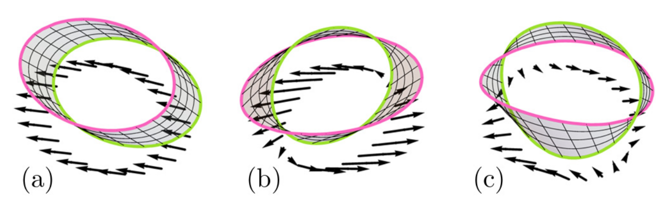

FIG. 6. Strips swept by the normal vector $\mathbf{n}$ of the polarization ellipse and components of this vector, in the plane of the point, (a) containing no polarization singularity, near the L^T points with (b) positive and (c) negative isotropy parameter.

are significantly different. Near the point in space where the electric field contains no polarization singularities, all vectors are nearly collinear [Fig. 6(a)]. Near the points of polarization singularities these distributions have a vortex [Fig. 6(b)] or saddle point [Fig. 6(c)] structure, depending on the sign of the isotropy parameter of the singular point encircled by the strip.

V. CONCLUSION

We have studied the strips swept by the vectors characterizing polarization ellipses along small closed planar contours in a nonparaxial electric field of a light wave scattered by a silicon sphere of subwavelength size. The twisting number of the strips was calculated by means of a contour integral, the evaluation of which did not require the usage of a specifically chosen coordinate system. Unlike Ref. [12] and analogous works [11,19], we considered the electric strength vector as is and not its projections onto some specifically chosen planes. We found the strips with linking numbers 0, $\pm\frac{1}{2}$, and ± 1, the topology of which depends not only on the presence or absence of singular points in the regarded region of space, but also on the orientation of the contours used for their construction with respect to the characteristic directions of the polarization ellipse of the scattered field. The topological difference of the strips which is irreparable by smooth deformation of them is found only for the strips of the axes of the polarization ellipse, constructed around the C^T lines. All other strips (built in the regular region of the field, around the L^T lines, and also the strips of the normal vector to the plane of the ellipse) were orientable and, in the vast majority of cases, trivial.

The strips of different vectors characterizing the polarization ellipse, traced along the same contour, can have different values of the linking number. The linking numbers of the strips enclosing a C^T line and swept by the major and minor axes of the polarization ellipses are in the most part opposite, which is an interesting case for separate theoretical investigation. However, we did not find an unambiguous relation between the linking numbers of the nontrivial strips and the isotropy parameters of the points of the C^T line, near which they were built. This all indicates that the linking numbers of the strips of polarization ellipses of the electric field appear to be distinct topological characteristics of the vector structure of the scattered electric field. The parameters characterizing the strip are not reducible to characteristics of the distributions of the projections of the polarization ellipse onto any particular plane.

Optical Möbius strips have been successfully measured experimentally in tightly focused nonparaxial field [2]. However, to observe analogous structures in the near field of nanoscale objects requires more complex experimental techniques. Experiments in which the polarization state of the near field of the nanoobjects is fully scanned are in the early stages of development. A few ideas on scanning the full polarization state near the nanoobjects were given in [25], all of which make use of quantum emitters of various types placed near the nanoobject. The far field of these emitters can be used to reconstruct the three-dimensional polarization state of the near field. Another potential method of full near-field measurements can be the extension of the ideas given in [8] by comparing the in-plane two-dimensional field distributions for several various orientations of the sample with respect to the phase-sensitive near-field microscope. Although almost any kind of near-field measurement somehow affects the theoretically predicted field distribution, the topological structures like polarization singularity lines and twisted polarization ellipse strips are known for their stability with respect to a certain degree of perturbation and are still likely to be observed in real-life experiments.

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