

Many-body dark solitons in one-dimensional hard-core Bose gases

Manuele Tettamanti

*Dipartimento di Fisica “Giuseppe Occhialini,” Università di Milano-Bicocca,
Piazza della Scienza 3, I-20126 Milano, Italy
and INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*Alberto Parola *Dipartimento di Scienza e Alta Tecnologia, Università dell’Insubria, Via Valleggio 11, 22100 Como, Italy*

(Received 14 May 2021; revised 7 October 2021; accepted 11 October 2021; published 27 October 2021)

The existence and stability of solitonic states in one-dimensional repulsive Bose-Einstein condensates is investigated within a fully many-body framework by considering the limit of infinite repulsion (Tonks-Girardeau gas). A class of stationary, shape-invariant states propagating at constant velocity are explicitly found and compared to the known solution of the Gross-Pitaevskii equation. The typical features attributed to nonlinearity are thus recovered in a purely linear theory, provided the full many-particle physics is correctly accounted for. However, the formation dynamics predicted by the Gross-Pitaevskii approximation considerably differs from the exact many-body evolution.

DOI: [10.1103/PhysRevA.104.043325](https://doi.org/10.1103/PhysRevA.104.043325)**I. INTRODUCTION**

Solitons are nonuniform, stationary solutions of a translationally invariant, nonlinear equation which can propagate with constant velocity (less than a critical value) without changing shape. Exact solitonic solutions are known in several frameworks, ranging from fluid dynamics to optics, including the one-dimensional Gross-Pitaevskii equation (GPE) with cubic nonlinearity, describing the dynamics of a Bose-Einstein condensate (BEC) [1]. Dark solitons are localized rarefaction in an asymptotically uniform fluid and they have been proven to exist and to be stable in the one-dimensional GPE with repulsive interactions [2]. By a Galileo transformation, a moving soliton in a fluid asymptotically at rest can be also seen as a soliton at rest in a flowing condensate. If the fluid is asymptotically at rest, the density profile of the dark soliton vanishes at the minimum, while if the fluid flows with a finite velocity the soliton becomes “gray;” i.e., its minimum density is nonzero. The maximum asymptotic fluid velocity supporting a solitonic solution coincides with the sound velocity in the asymptotic region.

In the laboratory, soliton-like structures have been created in BECs for more than 20 years [3,4] through techniques of phase imprinting, density engineering, and a combination of the two [5]. Experimental solitons are clearly related to the exact solution of the GPE and their physical origin traces back to the balance between dispersion and self-interaction, which stabilizes a nonuniform profile. However, although the qualitative features of a soliton are clearly reproduced in experiments, it is difficult to ascertain the precise solitonic nature of the observed density dip: Indeed, the underlying three dimensionality of the cigar-shaped traps, the finite lifetime of the condensate, the presence of longitudinal confine-

ment breaking translational invariance, and finite-temperature effects are only a few unavoidable perturbations which must be taken into account when comparing analytical and experimental results [5]. Solitonic states are often identified with a particular branch of excitation (dubbed “Lieb type II”) of the exact Lieb-Liniger solution [6,7] of a δ -interacting, one-dimensional Bose gas [8–12]. However, growing evidence [13–16] suggests that, starting from a superposition of these excited states, the exact dynamics drives particles into the density dip which eventually fades away. Solitons, thus, would emerge only as mean-field stationary solutions which do not carry over to the full many-body Schrödinger equation. On the other hand, it has been also argued that a single quantum measurement of a quasiuniform many-body state may still display a well-defined density dip. After an average over many different measures, however, the dip is bound to vanish due to fluctuations of the soliton position [17–19]. In any case, a well-defined, broken-symmetry, stationary solution of the full many-body problem has not been clearly identified yet. In this article, we ask whether it is possible to determine and dynamically generate “many-body solitons,” i.e., exact stationary solutions of an interacting BEC in one dimension.

Here we investigate the strong coupling limit of the Lieb-Liniger solution, the so-called Tonks-Girardeau (TG) gas, whose eigenstates and dynamics exactly maps onto those of a free Fermi gas [20–22]. Confining our attention to the TG model is not a serious limitation: It is known that the strong-coupling regime of a one-dimensional, δ -interacting Bose gas is not pathological, as opposed to the ideal gas limit. Its equilibrium properties, together with the spectrum of elementary excitations, are in fact smooth functions of the interaction strength all the way to the hard-core limit [6,7]. Most importantly, the long-time dynamics of a one-dimensional,

interacting Bose gas has been shown to be asymptotically described precisely by the TG model for any value of the (repulsive) coupling constant [23,24]. An obvious consequence of the strict one dimensionality of our model is the absence of true off-diagonal long-range order, i.e., the absence of a condensate and of macroscopic phase coherence [2]. Belonging to the class of Luttinger liquids, the TG gas still displays quasi-long-range order, characterized by a density matrix decaying as $r^{-1/2}$ and by a divergent momentum distribution [25]. Therefore, we believe that such a toy model may indeed serve as a useful guidance for the theoretical investigations of quasi-one-dimensional Bose-Einstein condensates, allowing, at the same time, for a full treatment of the many-body effects.

In this article, we show that a careful analysis of the solitonic GPE solution allows identification of the exact stationary states of the TG model representing a dark soliton propagating with constant velocity. A jump in the phase of the wave function goes along with the density modulation, in close analogy with the known GPE soliton. We also numerically investigate the possibility to dynamically generate TG solitons.

II. MANY-BODY SOLUTION

The GPE with cubic nonlinearity is known to describe a Bose gas in the weak coupling regime and thus is not expected to faithfully reproduce the exact many-body dynamics of the strongly interacting limit. However, by replacing the cubic term with a *quintic* nonlinearity and fixing the coupling g to the specific value $g = \frac{\hbar^2 \pi^2}{2m}$, the exact low-energy excitation spectrum of the TG gas is recovered. Hence, this modified version of the GPE is usually adopted to represent a strongly interacting Bose gas in one dimension [26,27] and its validity has thoroughly been tested [28]. In the following, we will make use of this equation, although the results obtained by means of the ordinary cubic GPE show no qualitative differences.

We start from the analytic, stationary solution of the quintic GPE representing a dark soliton in the comoving frame (i.e., the reference frame at rest with the soliton, while the fluid velocity is fixed at infinity), which reads [26,29]:

$$\Phi(x) = \sqrt{n(x)} e^{i\varphi(x)}, \quad (1)$$

$$\frac{n(x)}{n_\infty} = 1 - \frac{3\delta^2}{2 + \sqrt{4 - 3\delta^2} \cosh(2\pi \delta n_\infty x)}, \quad (2)$$

$$\varphi(x) = -\pi \sqrt{1 - \delta^2} n_\infty x - \frac{\operatorname{sgn} x}{2} \arccos \left[\frac{1 + \alpha \cosh(2\pi \delta n_\infty x)}{\alpha + \cosh(2\pi \delta n_\infty x)} \right], \quad (3)$$

where $\alpha = \frac{2-3\delta^2}{\sqrt{4-3\delta^2}}$ and the sign x function is required by the symmetry $\varphi(-x) = -\varphi(x)$. The parameter δ is related to the uniform mass flux j by:

$$\delta^2 = 1 - \frac{v_\infty^2}{c_\infty^2} = 1 - \left[\frac{j}{\pi \hbar n_\infty^2} \right]^2, \quad (4)$$

where v_∞ and c_∞ are the fluid and sound velocities at $|x| \rightarrow \infty$ respectively. The solution is then parametrized by the

asymptotic density n_∞ and the mass flux j . The asymptotic density n_∞ defines the length units for both the density profile $n(x)$ and the coordinate x , showing that, at fixed n_∞ , Eqs. (2) and (3) provide a family of solutions which only depends on the dimensionless parameter $0 \leq \delta \leq 1$ (4) or, equivalently, on the density drop

$$\bar{n}_0 = \frac{n(0)}{n_\infty} = \sqrt{4 - 3\delta^2} - 1. \quad (5)$$

With the adopted choice of signs the fluid is left moving (i.e., $j < 0$). The chemical potential is given by $\mu = \frac{\hbar^2 \pi^2 (2 - \delta^2) n_\infty^2}{2m}$, which equals the chemical potential of the uniform solution at the same asymptotic density n_∞ and conserved current j . Therefore, the soliton represents an excited state with finite excitation energy, as in the weakly interacting case [2].

Similarly, in the comoving frame, a many-body soliton should be an exact, stationary solution of the many-particle Schrödinger equation, breaking the translational symmetry of the Hamiltonian. In the case of a TG gas, it corresponds to a symmetry-breaking eigenstate of the free Fermi gas. A hint towards the identification of such a many-body state comes from the observation that a *half-soliton* [30] of the (cubic or quintic) GPE is precisely a steady-state solution of a Bose gas in an external step potential of height V_0 [29]. The particular *half-soliton* wave function has the unique feature of displaying a flat density profile beyond the step and requires a well-defined relation between the step height V_0 and the asymptotic density n_∞ :

$$\left(\frac{2\pi n_\infty}{Q} \right)^2 = \frac{18(1 - \delta^2)}{(4 - 3\delta^2)^2 - \sqrt{4 - 3\delta^2} (8 - 9\delta^2)} \quad (6)$$

with $\hbar Q = \sqrt{2mV_0}$. Analogously, the full soliton can be obtained by considering, instead of the step potential, a square well of depth $-V_0$ and width a and choosing the stationary solution with a flat density profile inside the well. This condition requires the same relation (6). Then, by letting the well width $a \rightarrow 0$ at fixed V_0 , the free solitonic solution (1) is recovered.

This procedure can be easily generalized to the free Fermi gas. Remarkably, it is possible to find a set of exact single-particle eigenstates of the square well potential characterized by a flat density profile inside the well, in close analogy to the relevant solution of the GPE equation previously identified. These states are suitable linear combinations of the scattering eigenfunctions of positive and negative momentum and, in the $a \rightarrow 0$ limit, they are simply given by

$$\psi_k(x) = \frac{1}{\sqrt{2\pi [1 + R_k^2]}} [e^{ikx} + R_k e^{-ikx}], \quad (7)$$

where $k \leq 0$, $p = \sqrt{k^2 + Q^2}$ and $R_k = \frac{k+p}{k-p}$. These combinations have a well-defined single-particle energy $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ and thus represent stationary states. In the comoving frame the many-body exact eigenstate can be written as a Slater determinant of these single-particle orbitals for momenta in the interval $-2k_F \leq k \leq 0$:

$$\Psi_B(x_1, \dots, x_N) = \left[\prod_{j>i} \operatorname{sgn}(x_j - x_i) \right] \det \psi_k(x_i), \quad (8)$$

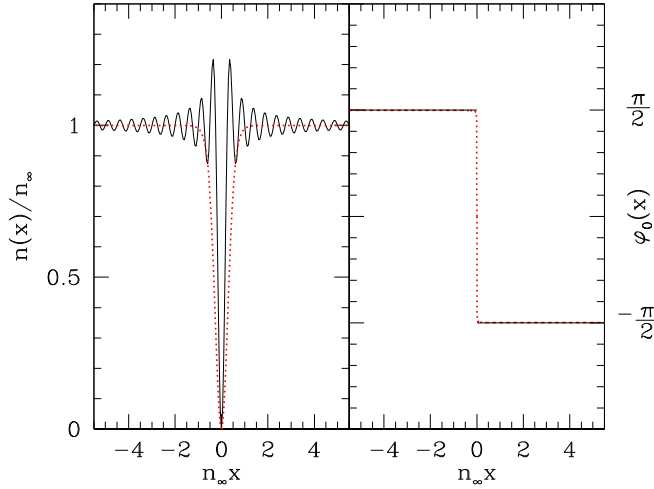


FIG. 1. Dark soliton solution for a strongly interacting Bose gas in the limit of vanishing current: $j \rightarrow 0$. The red dotted lines represent the quintic GPE solution (1) while the black solid ones the exact TG stationary eigenstate (8). Left panel, density profile; right panel, phase shift in the laboratory frame.

with $\psi_k(x_i)$ given by Eq. (7). In this case, the asymptotic density is just $n_\infty = \frac{k_F}{\pi}$ while the mass flux is

$$j = -\frac{\hbar Q^2}{8\pi} [\chi - \arctan \chi], \quad (9)$$

with $\chi = 2 \frac{2k_F}{Q} \sqrt{(\frac{2k_F}{Q})^2 + 1}$. The full density profile of this many-body eigenstate is not homogeneous and is analytically given by

$$n(x) = \int_{-2k_F}^0 \frac{dk}{2\pi} \left[1 + \frac{2R_k}{1+R_k^2} \cos(2kx) \right], \quad (10)$$

while the density drop is

$$\bar{n}_0 = \frac{n(0)}{n_\infty} = 1 - \frac{Q}{2\sqrt{2}k_F} \arctan \frac{2\sqrt{2}k_F}{Q}. \quad (11)$$

In summary, starting from a strongly interacting, flowing Bose gas in presence of an external potential well of given parameters (Q, a) we can find the stationary states with a constant density inside the well either by solving the quintic GPE or by use of the exact Bose-Fermi mapping. Each class of solutions depends on a single dimensionless quantity, which can be conveniently identified as the density at infinity $\frac{n_\infty}{Q} = \frac{k_F}{\pi Q}$. The mass current, the density drop and the full density profile are expressed in terms of such a parameter. Then, by taking the $a \rightarrow 0$ limit, the square well disappears and the dimensional parameter Q defining the well depth loses its physical meaning. Nevertheless, the density profile remains nonuniform in both cases and coincides with that of the the GPE soliton (2) and of the TG stationary state (10), respectively. Both states depend only on one dimensional and one dimensionless parameter: The asymptotic density n_∞ and the density drop \bar{n}_0 Eqs. (5) and (11). A comparison between the two profiles is shown in Figs. 1 and 2, where the (normalized) density and the phase of the solitonic solution is plotted for two representative choices of the density drop \bar{n}_0 [(5) and (11)]. The phase $\varphi(x)$ is defined as the integral of the local velocity, $\varphi(x) = \int_0^x dx' \frac{j}{\hbar n(x')}$, and to better appreciate

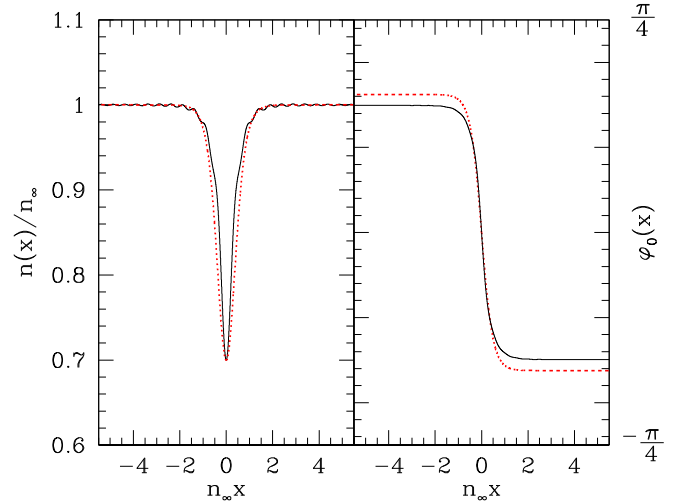


FIG. 2. Gray soliton solution for a strongly interacting Bose gas for $\bar{n}_0 = 0.7$. The red dotted lines represent the quintic GPE solution (1) while the black solid ones the exact TG stationary eigenstate (8). Left panel, density profile; right panel, phase shift in the laboratory frame.

the phase shift induced by the soliton, the figures display the phase of the wave function in the laboratory frame, where the fluid at infinity is at rest, $\varphi_0(x) = \varphi(x) - \frac{j}{\hbar n_\infty} x$. The agreement between the mean-field and the exact many-body solution is remarkable for both the density profile and the phase shift. Furthermore, the solutions display all the dark soliton properties: Indeed, even in the absence of any external potential, there is a notch in the density profile and a precise phase jump which do not change in time. Moreover, the exact stationary eigenstate and the GPE soliton are stationary solutions characterized by a well-defined constant velocity $v_{sol} = -\frac{j}{n_\infty}$, thus representing constantly moving rarefactions in the laboratory frame. The qualitative difference between the shape of the GPE soliton and the exact result is the presence of an oscillating power-law tail slowly approaching the asymptotic density in the TG solution, while the GPE soliton is exponentially localized [31]. Remarkably, in the limit $\frac{k_F}{Q} \rightarrow 0$ (Fig. 1), our expression reduces to the proposal put forward in Ref. [33].

III. EXPERIMENTAL REALIZATION

The next question we want to address is whether the previously identified solitonic solutions can be generated by means of some external perturbation applied on a homogeneous, flowing BEC, in order to set the system out of equilibrium, consequentially triggering the formation of stable structures. In the laboratory, two alternative methods are currently adopted to create solitonic structures: *phase imprinting* and *density engineering* [5]. In both cases, the starting configuration is a uniform (possibly flowing) BEC described by the trivial solution of the GPE equation $\Phi(x) = \sqrt{n_\infty} e^{-ik_0 x}$ or, in the Fermi representation, by the Slater determinant of plane waves $\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$ with $-k_F - k_0 \leq k \leq k_F - k_0$, where, in both cases, k_0 is related to the fluid velocity by $v = \frac{\hbar k_0}{m}$.

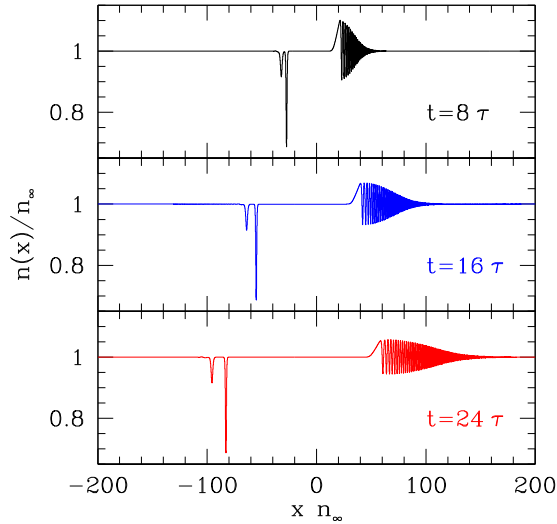


FIG. 3. Time evolution of the condensate density after phase imprinting. The three curves correspond to different dimensionless times with $\tau = \frac{m}{\hbar n_{\infty}^2}$.

A. Phase imprinting

For the phase imprinting procedure, the condensate wave function is initially perturbed by a nonuniform phase change of the form:

$$\Phi(x) = \sqrt{n_{\infty}} e^{-i(k_0 x - \frac{\pi}{2} \tanh \alpha x)} \quad (12)$$

(where α is a tuning parameter) and then the cloud is let evolve freely. The time evolution according to the quintic GPE is shown in Fig. 3 for $k_0 = n_{\infty}$ and $\alpha = n_{\infty}$.

The presence of a propagating dispersive shock wave (DSW) on the right is evident, while on the left two dark soliton-like structures appear, moving at different velocities. In Fig. 4, we enlarge the soliton region and we superimpose the analytical form previously obtained, for two different values of the density drop. The good agreement confirms the solitonic nature of the sharp features shown in Fig. 3.

The exact dynamics of a Tonks gas after phase imprinting can be evaluated by integrating the Schrödinger equation for a free Fermi gas. The initial condition at $t = 0$ is given by a Slater determinant of “phase imprinted” plane waves with $-k_F - k_0 \leq k \leq k_F - k_0$:

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{i(kx + \frac{\pi}{2} \tanh \alpha x)}. \quad (13)$$

A few snapshots of the TG dynamics are shown in Fig. 5 using the same set of parameters (k_0, α) of the GPE case. While similar structures are formed in the first stages of the evolution, the dark-soliton-like dip on the left is much smaller and further weakens in time. The exact dynamics does not appear to lead to the formation of stable, nonhomogeneous structures. Moreover, the dispersive shock waves clearly visible in the GPE dynamics (Fig. 3) are lacking in the exact TG evolution, being replaced by smooth density peaks which propagate at the same velocity of the DSW. Different sets of parameters (including smaller values of α) have been checked following a previous study [34] suggesting that the dynamics of a TG gas after phase imprinting strongly depends

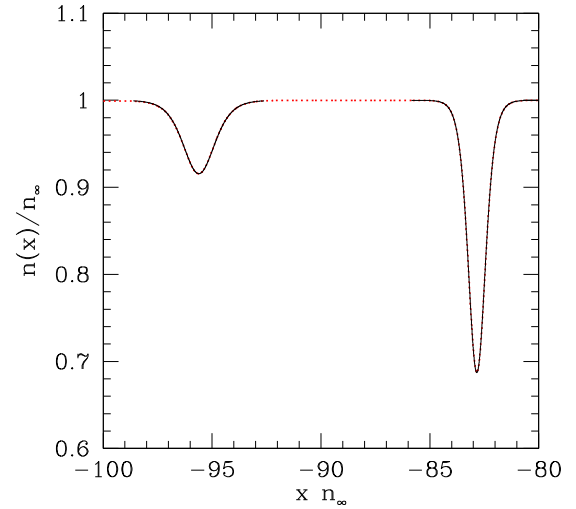


FIG. 4. Comparison between the shape of the two soliton-like structures at time $t = 24 \tau$ (red dotted lines) and the analytical solitonic solutions Eq. (2) (black solid lines) for $\bar{n}_0 = 0.9156$ (left curve) and $\bar{n}_0 = 0.6872$ (right curve). The analytical solitons have been shifted to match the position of the minima of the two numerical dips. The numerical and the analytical form are perfectly superimposed and the agreement is remarkable.

on the length scale α^{-1} governing the initial phase change. In these cases, although the ensuing GPE time evolution is smoother, the results are qualitatively similar to the cases previously illustrated: No solitonic structure forms according to the exact dynamics, while it is always present in mean-field approximation.

B. Density engineering

The density engineering method is another scheme commonly used in the laboratory to generate solitons. In this case,

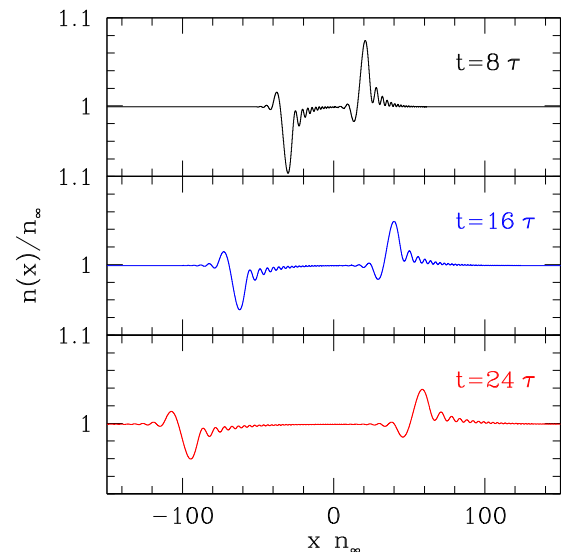


FIG. 5. Time evolution of the density of a TG gas after phase imprinting. The curves correspond to three different dimensionless times with $\tau = \frac{m}{\hbar n_{\infty}^2}$.

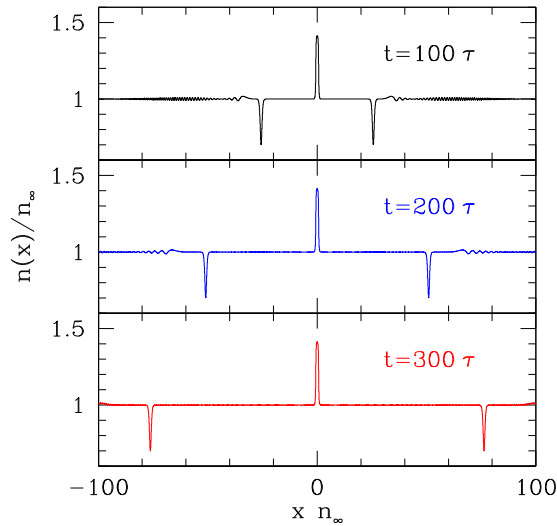


FIG. 6. Time evolution of the condensate density after the quench for the GPE case. The three curves correspond to different dimensionless times: With $\tau = \frac{m}{\hbar Q^2}$.

a quench in the density (rather than in the phase) is applied to a uniform condensate by means of a suitable external perturbation. The potential applied in order to obtain a solitary emission can be of different forms [5] and here we choose a square well of width a and depth $V_0 = \frac{\hbar^2 Q^2}{2m}$, which is suddenly generated at $t = 0$. In the following, we will take the atoms to be at rest before the quench (but the same technique can be applied to flowing gases), the initial density is set as $n = \frac{Q}{\pi}$, and the well width is given by $Qa = 2$.

Figure 6 displays a few snapshots of the (quintic) Gross-Pitaevskii equation dynamics at three different times with the aforementioned initial conditions. As the curves show, after the quench the gas is perturbed in the region of the potential well and solitons are emitted from both sides, propagating

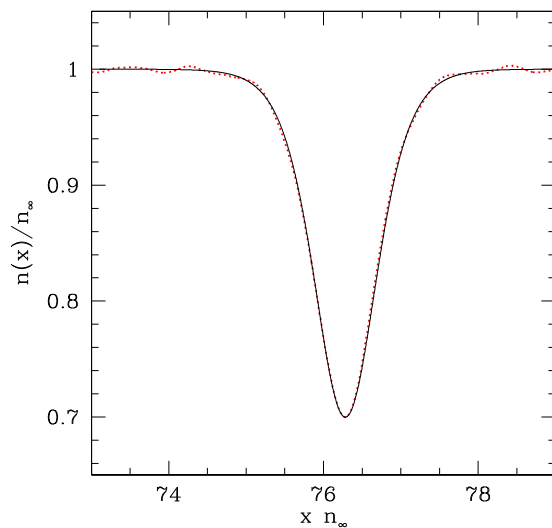


FIG. 7. Soliton structure of Fig. 6 at $t = 300 \tau$ (red dotted curve) superimposed with the analytic solution given in the article (black solid curve). The two curves are almost identical and thus difficult to distinguish on the canvas.

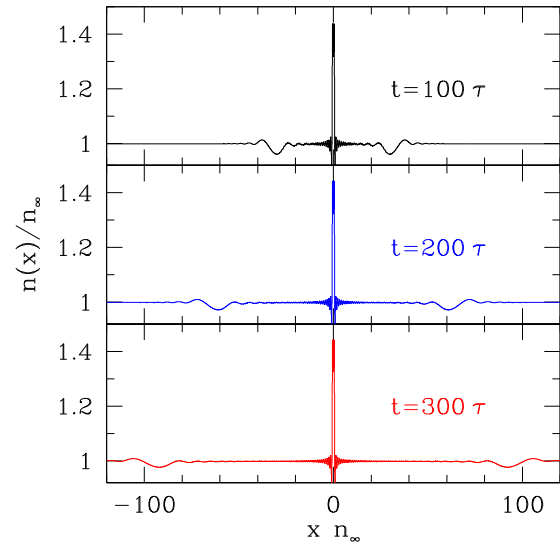


FIG. 8. Time evolution of the density of a Tonks-Girardeau gas after a quench. The three curves correspond to different dimensionless times with $\tau = \frac{m}{\hbar Q^2}$.

away from the potential. Dispersive shock waves are also evident near the soliton structures.

Figure 7 shows an enlargement of the soliton structure present in Fig. 6 at $t = 300 \tau$, superimposed with the analytic form previously detailed evaluated for the same density drop. The matching between the two curves is remarkable, demonstrating that the dip propagating away from the potential is indeed an actual soliton.

We now compare the results to the exact Tonks-Girardeau (TG) dynamics under the same conditions. Now the initial state at $t = 0$ is a Slater determinant of plane waves and the time evolution is governed by the Schrödinger equation with the same potential used in the GPE dynamics. A few snapshots of the TG exact evolution are given in Figs. 8 and 9 and we can see that the dynamics is qualitatively the same: A perturbation

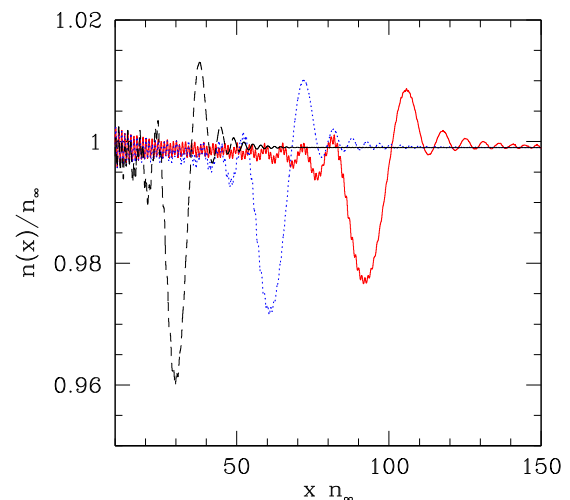


FIG. 9. An enlargement of Fig. 8 which shows the rarefaction created in the quench and its evolution in time. The three curves correspond to the three times plotted in Fig. 8: $t = 100 \tau$ (dashed black), $t = 200 \tau$ (dotted blue), and $t = 300 \tau$ (solid red) with $\tau = \frac{m}{\hbar Q^2}$.

appears in the region of the potential well while density dips propagate in both directions, analogously to the GPE case. Differently from the mean-field case, though, these structures do not behave as solitons as they suffer considerable dispersion upon evolution. Furthermore, DSW are absent. Thus we can conclude that, as for the phase imprinting case, density engineering is a valid technique to generate solitons at mean-field level but the same cannot be said for the many-body solution of the TG gas.

Finally, the proposal put forward in Ref. [35] has been studied and a time-dependent potential has been applied to the gas in order to excite the correct eigenstates; numerical simulations, however, show that not even in this case is a stable soliton solution for the TG gas reached.

IV. CONCLUSIONS

In summary, taking advantage of the known Bose-Fermi mapping in one dimension, an exact, stationary solution of the many-body problem for a Tonks-Girardeau gas in one dimension showing the same features attributed to a dark soliton has been found. This solution has a power-law oscillating decay and describes a density rarefaction which moves with constant speed (slower than the sound speed) in a uniform

TG gas. Both the density and the phase profile resemble the analytical solitonic solution of the (quintic) GPE. The GPE evolution after both phase imprinting and quench gives rise to soliton trains extremely well represented by the analytical solutions we found, at variance with the exact dynamics of a TG gas under the same initial conditions, which shows no evidence of solitonic structures. The absence of a dynamical mechanism to spontaneously generate solitons in a TG gas could be explained by recalling the analytical results of Ref. [36], where it was shown that the long-time dynamics of a Tonks-Girardeau gas always leads to the stationary state built with the exact single-particle scattering states of the external potential. Although the soliton-like wave function we have introduced in Eq. (8) is indeed a Slater determinant of suitable single-particle orbitals (7), they cannot be identified as pure scattering states.

This investigation has been specifically performed for the TG gas, whose properties are known to represent the generic behavior of one-dimensional, interacting Bose gas. However, the adopted procedure can be straightforwardly generalized to any one-dimensional fluid: The solitonic wave function is obtained starting from the exact symmetrical eigenstates of the interacting Bose gas in an external potential well and then letting the well width to zero.

-
- [1] M. Remoissenet, *Waves Called Solitons: Concepts and Experiments* (Springer, New York, 1999)
 - [2] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, UK, 2003).
 - [3] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, *Phys. Rev. Lett.* **83**, 5198 (1999).
 - [4] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, *Science* **287**, 97 (2000).
 - [5] D. J. Frantzeskakis, *J. Phys. A: Math. Theor.* **43**, 213001 (2010).
 - [6] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
 - [7] E. H. Lieb, *Phys. Rev.* **130**, 1616 (1963).
 - [8] M. Ishikawa and H. Takayama, *J. Phys. Soc. Jpn.* **49**, 1242 (1980).
 - [9] R. Kanamoto, L. D. Carr, and M. Ueda, *Phys. Rev. A* **81**, 023625 (2010).
 - [10] T. Karpiuk, T. Sowinski, M. Gajda, K. Rzazewski, and M. Brewczyk, *Phys. Rev. A* **91**, 013621 (2015).
 - [11] J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, *New J. Phys.* **18**, 075008 (2016).
 - [12] E. Kaminishi, T. Mori, and S. Miyashita, *J. Phys. B: At. Mol. Opt. Phys.* **53**, 095302 (2020).
 - [13] J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, *Phys. Rev. Lett.* **108**, 110401 (2012).
 - [14] S. S. Shamailov and J. Brand, *Phys. Rev. A* **99**, 043632 (2019).
 - [15] R. V. Mishmash and L. D. Carr, *Phys. Rev. Lett.* **103**, 140403 (2009).
 - [16] R. V. Mishmash, I. Danshita, Charles W. Clark, and L. D. Carr, *Phys. Rev. A* **80**, 053612 (2009).
 - [17] A. Syrwid and K. Sacha, *Phys. Rev. A* **92**, 032110 (2015).
 - [18] A. Syrwid, M. Brewczyk, M. Gajda, and K. Sacha, *Phys. Rev. A* **94**, 023623 (2016).
 - [19] D. Delande and K. Sacha, *Phys. Rev. Lett.* **112**, 040402 (2014).
 - [20] M. Girardeau, *J. Math. Phys.* **1**, 516 (1960).
 - [21] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, *Nature (London)* **429**, 277 (2004).
 - [22] T. Kinoshita, T. Wenger, and D. S. Weiss, *Science* **305**, 1125 (2004).
 - [23] D. Iyer and N. Andrei, *Phys. Rev. Lett.* **109**, 115304 (2012).
 - [24] D. Iyer, H. Guan, and N. Andrei, *Phys. Rev. A* **87**, 053628 (2013).
 - [25] H. G. Vaidya and C. A. Tracy, *Phys. Rev. Lett.* **42**, 3 (1979).
 - [26] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, *Phys. Rev. Lett.* **85**, 1146 (2000).
 - [27] C. Menotti and S. Stringari, *Phys. Rev. A* **66**, 043610 (2002).
 - [28] E. H. Lieb, R. Seiringer, and J. Yngvason, *Phys. Rev. Lett.* **91**, 150401 (2003).
 - [29] M. Tettamanti and A. Parola, *Universe* **6**, 105 (2020).
 - [30] P. Leboeuf, N. Pavloff, and S. Sinha, *Phys. Rev. A* **68**, 063608 (2003).
 - [31] These types of oscillations are expected in the case of fermions [32].
 - [32] M. Antezza, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. A* **76**, 043610 (2007).
 - [33] M. D. Girardeau and E. M. Wright, *Phys. Rev. Lett.* **84**, 5691 (2000).
 - [34] T. Karpiuk, M. Brewczyk, Ł. Dobrek, M. A. Baranov, M. Lewenstein, and K. Rzazewski, *Phys. Rev. A* **66**, 023612 (2002).
 - [35] H. Buljan, O. Manela, R. Pezer, A. Vardi, and M. Segev, *Phys. Rev. A* **74**, 043610 (2006).
 - [36] M. Tettamanti, S. L. Cacciatori, and A. Parola, *Phys. Rev. D* **99**, 045014 (2019).