Detection of entanglement during pure dephasing evolutions for systems and environments of any size

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(Received 27 May 2021; revised 12 August 2021; accepted 15 September 2021; published 11 October 2021)

We generalize the scheme for detection of qubit-environment entanglement to qudit-environment systems. This is of relevance for many-qubit systems and the quantification of the operation of quantum algorithms under the influence of external noise, since only decoherence that is not entangling in its nature can be effectively described by quantum channels and similar methods in more complicated scenarios. The generalization involves an increase in the class of entangled states which are not detected by the scheme, but the type of entanglement which cannot be detected is also least likely to qualitatively influence decoherence. We exemplify the operation of the scheme on a realistically modeled nitrogen-vacancy-center spin qutrit interacting with an environment of nuclear spins.

DOI: 10.1103/PhysRevA.104.042411

I. INTRODUCTION

Qubits are the simplest quantum systems, since their Hilbert space contains only two states. The result of this is that some complex measures of quantumness or quantum correlations are much easier to study for qubits than for larger quantum systems. One example of this is mixed-state entanglement [1,2], which can be found directly from the density matrix for a system of two qubits [3] but otherwise requires minimization over all possible preparations of a state [2,4] or the use of measures which do not quantify all types of entanglement [5–8]. Similarly, bound entanglement [9,10], a type of entanglement which is not detected by the Peres-Horodecki criterion [11,12], does not exist for systems of two qubits.

The number of coherences (off-diagonal elements of the density matrix) grows quadratically with the size of the system [as $N(N-1)/2$ to be precise; obviously a density matrix is Hermitian, hence only half of the coherences are independent variables], so the single-qubit coherence is replaced by three coherences for a qutrit, six coherences for a system of size $N = 4$, and so on. Furthermore, the dependencies between the different coherences are relevant. An example here is the simple task of checking whether a matrix is a density matrix and can therefore describe a physical state. This requires checking three conditions: Hermitianity, unit trace, and positivity. Only the third condition is problematic, as it requires diagonalization of the matrix, which can only be done numerically for larger matrices. For a 2×2 Hermitian matrix, only the absolute value of the coherence is relevant for positivity, which is not the case already for a qutrit.

The consequence is that there is a qualitative difference when studying larger systems as opposed to studies restricted to qubits and conclusions drawn for qubits rarely translate seamlessly to larger systems. This is also the case for asymmetric bipartite systems composed of a qudit (N-dimensional system of interest) and its environment. In particular, entanglement formed between a qudit and its environment is much harder to study than in the case of a qubit. This is evident for pure-dephasing interactions, the only type of system-environment couplings for which simple, general formulas for qualification of a state obtained during the evolution as entangled or not exist [13,14]. Qualifying qubit-environment entanglement (QEE) requires checking a single condition [13] and this allowed an entanglement measure tailored specifically to quantify this type of entanglement to be proposed [15], which yields substantial computational advantage with respect to standard entanglement measures [2,4,16]. Qualifying system-environment entanglement (SEE), on the other hand, requires checking $N-1$ conditions which are analogous to the QEE conditions and, additionally, $(N-1)(N-2)/2$ conditions which are qualitatively different [14]. This rapid growth in complexity with system size N precludes the possibility of an analogous SEE measure to be proposed.

The creation of entanglement with the environment throughout the evolution is relevant because the behavior of the environment is qualitatively different when entanglement is formed and when the evolution is separable [13,14,17]. In many situations QEE can lead to effects which cannot be explained by decoherence modeled classically [18]. This backaction, the situation where entanglement manifests itself in the state of the environment, which in turn influences the evolution of the qubit, is the reason why QEE can be measured with little effort [19–22]. In fact the first experimental demonstration of the operation of the scheme in Ref. [19] has just been reported [23].

In the following we study a qudit for which the interaction with the environment leads to pure dephasing, in order to generalize a scheme for the detection of QEE by operations

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and measurements only on the qubit [19]. This type of interaction is the dominating decoherence mechanism for many solid state qubits [24–31]. One motivation for the importance of SEE is that most solid state qubits are in fact only approximations of qubits (e.g., where two states are energetically distinct and can therefore be addressed separately). The more relevant one is that ensembles of qubits are of vital importance for any type of quantum data processing and ensembles of qubits interacting with an environment can no longer be treated with the methods for studying QEE. From the point of view of entanglement with an environment they are in fact qudits and display the whole range of complexity relating to many coherences and phase relations between them.

We show that one method for the detection of QEE [19] can in fact be generalized to detect SEE. The complexity of the procedure only grows linearly with the size of the qudit, so it does not reflect the quadratic growth of the number of SEE criteria [14]. The price paid is the growing number of entangled states that cannot be detected by the procedure. In addition to the type of entanglement which cannot be detected by the qubit procedure, there is now a second class of entanglement which cannot be witnessed for larger systems. Optimistically, the type of entanglement which is detected by the procedure is the type which is most likely to influence the operation of quantum algorithms [20].

The paper is organized as follows. In Sec. II we introduce the type of system-environment density matrix which can be classified in terms of SEE by the proposed scheme and the conditions on the Hamiltonian and initial state of the system and the environment to guarantee this form throughout the evolution. In Sec. III we restate the criteria for separability of such density matrices. We introduce the proposed scheme for the detection of entanglement in Sec. IV and study the limitations of applicability of the scheme in Sec. V. In Sec. VI we study the working of the scheme on a nitrogen-vacancy (NV)-center spin qutrit interacting with a nuclear environment. Section VII concludes the paper.

II. CLASS OF PROBLEMS STUDIED

In the following we present a scheme which allows to detect entanglement between a quantum system of interest (with no limitation on the dimension of its Hilbert space) and its environment. The method can only be used for system-environment density matrices of the form (N is the dimension of the system, and the dimension of the environment is unspecified and arbitrary)

$$\hat{\sigma} = \sum_{k,l=0}^{N-1} c_k c_l^* |k\rangle\langle l| \otimes \hat{R}_{kl}. \tag{1}$$

Here the states on the left side of the tensor product correspond to some basis $\{|k\rangle\}$ in the system subspace, while the matrices $\hat{R}_{kl}$ describe the environment. For the full matrix, (1), to be a density matrix, the diagonal environmental matrices $\hat{R}_{kk}$ have to be density matrices, but there is no such limitation for off-diagonal matrices, with $k \neq l$.

Although density matrices of the form of (1) are, at certain instants in time, encountered in evolutions governed by different Hamiltonians [32], the prevailing situation in which they are encountered is when the system-environment Hamiltonian can only lead to pure-dephasing decoherence of the system. A system-environment Hamiltonian of this class can always be written in the form [14]

$$\hat{H} = \sum_{k=0}^{N-1} \varepsilon_k |k\rangle\langle k| + \hat{H}_{\rm E} + \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes \hat{V}_k, \tag{2}$$

where $\{|k\rangle\}$ is the same system basis as used in Eq. (1) and is now specified as the pointer basis of the system [33,34]. Obviously the first term in the Hamiltonian, (2), is the free Hamiltonian of the system, the second term is the (arbitrary) free Hamiltonian of the environment, and the third term describes the evolution. The first and last terms commute, which is the necessary and sufficient condition for the Hamiltonian to lead to pure dephasing for all initial states (such Hamiltonians cannot describe processes which involve energy exchange between the system and the environment).

A Hamiltonian of this type is diagonal in the subspace of the system, and the corresponding evolution operator retains this property:

$$\hat{U}(t) = \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes \hat{w}_k(t). \tag{3}$$

The environmental operators $\hat{w}_k(t)$ can be understood as evolution operators of the environment conditional on the pointer state of the system and are given by

$$\hat{w}_k(t) = e^{-\frac{i}{\hbar}\varepsilon_k t} e^{-\frac{i}{\hbar}(\hat{H}_{\rm E}+\hat{V}_k)t}. \tag{4}$$

The free evolution of each pointer state is included in the operators, (4), but it has no bearing on entanglement and as such is irrelevant for the results presented here.

Using Eq. (3) on any initial system-environment state will yield its joint density matrix at time t, but to obtain a density matrix of the form of (1), restriction on the initial state is needed. First, the state must be of product form with respect to the environment, and second, the initial system state must be pure,

$$\hat{\sigma}(0) = |\psi\rangle\langle\psi| \otimes \hat{R}(0), \tag{5}$$

with $|\psi\rangle = \sum_{k=0}^{N-1} c_k|k\rangle$. The initial state of the environment $\hat{R}(0)$ is arbitrary. Acting with the evolution operator, (3), on the initial state, (5), we obtain a system-environment density matrix of the form of (1) for all times t, and the environmental matrices, which are the only time-dependent element, are given by

$$\hat{R}_{kl}(t) = \hat{w}_k(t)\hat{R}(0)\hat{w}_l^\dagger(t). \tag{6}$$

III. CRITERIA FOR SYSTEM-ENVIRONMENT SEPARABILITY

In general, quantifying SEE or even QEE is a hard problem when mixed states of the environment are involved and requires strictly numerical analysis for Hilbert spaces of dimension larger than 6 [2,4,35,36]. Such studies have been undertaken for specific systems and interactions [37–42], but although many interesting behaviors of entanglement have been observed, no general conclusions could be drawn. In this case, contrary to quantum discord [17,43,44], it is just as strenuous to qualify a system-environment state only as entangled or separable.

In Ref. [14] it has been shown that to qualify a system-environment state of the form of (1) as entangled or separable it is enough to check two classes of criteria, which have been derived from the Peres-Horodecki criterion [11,12] and the definition of mixed-state separability.

The following are criteria of separability, and if any of them is violated, then there is entanglement between the system and its environment. The full set of independent criteria of the first type and of the second type constitutes an if and only if condition of separability [14]. The first class of criteria is a generalization of the (single) separability criterion of QEE [13] and states that separability requires that for all $k \neq l$ we have

$$\hat{R}_{kk}(t) = \hat{R}_{ll}(t). \tag{7}$$

There are $N-1$ independent criteria of this type [14], where N is the dimension of the system and it is enough to check (7) with l set constantly to a given value, e.g., $l = 0$. Physically, if criterion (7) is fulfilled for a given k and l, it means that the evolution of the environment is indistinguishable regardless of whether the system is in pointer state $|k\rangle$ or $|l\rangle$. If all such criteria are met, then the environment evolves in exactly the same way for the system in any of the pointer states. Contrary to pure initial states of the environment, this does not preclude decoherence of the system which is not initially in a pointer state (or mixture thereof) [13,14,37].

The second class of criteria requires commutation between products of different conditional evolution operators of the environment, (4); namely, for separability we must have

$$[\hat{w}_i(t)\hat{w}_j^\dagger(t), \hat{w}_k(t)\hat{w}_l^\dagger(t)] = 0 \tag{8}$$

for all i, j, k, and l. Only $(N-1)(N-2)/2$ of these conditions are independent [14].

The second class of separability criteria lacks the straightforward physical interpretation characteristic for the first class, which correlates SEE with information about the system state that has been transferred into the environment. This correlation allows for the detection of entanglement, at least in principle, by measurements performed on the environment. There exist states of the form of (1) for which all of the separability criteria of the first type are fulfilled, but not all of the criteria of the second type; such states are entangled [14]. Hence, determining that a state is separable requires checking in total $N(N-1)/2$ criteria for a system of size N regardless of the size of the environment, but if any individual criterion of either type is violated, then there must be entanglement present.

IV. SCHEME FOR DETECTION OF SEE

For a qubit system, there exists only one separability criterion and it is of the first type, (7). In this case the distinguishability of entangled and separable states by measurements on the environment alone can be used to design schemes for entanglement detection which are operated solely on the qubit [19,20]. This is a result of the backaction of the environment on the evolution of the qubit and the possibility of preparing a state of the environment by allowing it to evolve in the presence of the system in one of its pointer states. If the qudit environment state, (1), is entangled in such a way that it violates any of the separability criteria, (7), then this type of entanglement can also by detected by operations and measurements restricted to the system.

The procedure for the detection of QEE described in Ref. [19] is particularly straightforward to generalize. To detect whether there is entanglement in the qudit-environment state given by Eq. (1) at time τ which is obtained using the evolution operator, (3), on the initial state, (5), one must prepare and measure modified qudit-environment states, which involve preparation of the environment prior to exciting a superposition system state. The idea is as follows. At time $t = 0$ the system is prepared in one of its pointer states $|k\rangle$ and allowed to evolve for time τ. This does not change the state of the system but the environment does evolve, so the system-environment state is given by

$$\hat{\sigma}(\tau) = |k\rangle\langle k| \otimes \hat{R}_{kk}(\tau). \tag{9}$$

If the system is now (at time τ) prepared in a superposition state $|\psi\rangle = \sum_{k=0}^{N-1} c_k|k\rangle$, then it will evolve according to Eq. (1), but with a new initial state, $\hat{R}_{kk}(\tau)$ instead of $\hat{R}(0)$. Further evolution will lead to pure dephasing of the qudit and each of its coherences will evolve according to

$$\rho_{ij}^{(k)}(\tau, t) = c_i c_j^* \mathrm{Tr}(\hat{w}_i(t)\hat{w}_k(\tau)\hat{R}(0)\hat{w}_k^\dagger(\tau)\hat{w}_j^\dagger(t)), \tag{10}$$

where t is the time elapsed from time τ. An ideal test state $|\psi\rangle$ is an equal superposition of all pointer states, as it maximizes the chances of determining entanglement.

If the procedure is repeated for a different initial system pointer state $|l\rangle$ and any of the coherences, (10), show a different evolution at any point after time τ, $\rho_{ij}^{(k)}(\tau, t) \neq \rho_{ij}^{(l)}(\tau, t)$, this signifies that at time τ the criterion, (7), is not fulfilled for states $|k\rangle$ and $|l\rangle$. This further means that if the system was initialized in any superposition which contains pointer states $|k\rangle$ and $|l\rangle$ and the environment was initialized in the state $\hat{R}(0)$, then at time τ the joint system-environment state would be entangled.

Otherwise the procedure has to be repeated for a different choice of system pointer state $|k\rangle$ and again compared with the evolution for $|l\rangle$. Only when all possible values of $k \neq l$ are exhausted can one be sure that no entanglement can be witnessed by the procedure. The procedure is schematically represented in Fig. 1, where we illustrate only the operations performed on the qudit. The corresponding behavior of the environment is described by Eq. (9) in the first step, while the buildup of correlations (either quantum or classical) in the second step is witnessed by Eq. (10).

V. LIMITATIONS OF APPLICABILITY

The method described above is an entanglement witness [45–48], so a negative result does not signify separability. There are two situations in which entanglement is present but cannot be witnessed here. The first is the same as in the case where the system is a qubit [14]; namely, the witness will not detect entanglement if all of the conditional evolution operators of the environment, $\hat{w}_k(t)$, commute. In this case the

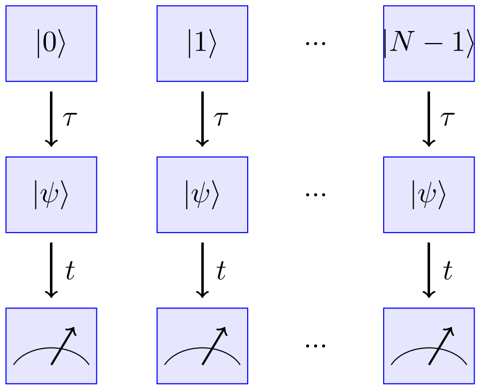

FIG. 1. Schematic of the scheme for the detection of SEE showing only the operations and measurements performed on the qubit. The system is prepared in each of its pointer states consecutively and allowed to evolve for time τ. Afterwards the same superposition state is excited and the time dependence of system coherences is measured. Any difference in the evolution of coherence for preparation in different pointer states signifies that a superposition state would be entangled with its environment at time τ of undisturbed evolution.

preparation of the environment for time τ does not change the resulting evolution of the system coherences, which are now always given by

$$\rho_{ij}^{(k)}(\tau,t) = c_i c_j^* \mathrm{Tr}(\hat{w}_i(t)\hat{R}(0)\hat{w}_j^\dagger(t)). \tag{11}$$

This type of entanglement could still be detected by measurements on the environment since we still have

$$\hat{R}_{kk}(t) \neq \hat{R}_{ll}(t)$$

if and only if the state, (1), is entangled, but there is no effect on the evolution of the qudit.

Note that if only some of the conditional evolution operators of the environment mutually commute, then entanglement for some initial states of the system can still be detected, in many cases even for all system states. This is because the number of independent criteria, (7), is $N-1$ [14], compared to the $N(N-1)/2$ nontrivial combinations of indices k and l. If criterion (7) is broken for a given k and l, this means that a superposition with $c_k \neq 0$ and $c_l \neq 0$ will be entangled with its environment at time τ. This works analogously with indices k and l', but if the criterion is shown to be broken (using the scheme described in the previous section) for both sets of indices, the consequence is that a superposition with $c_l \neq 0$ and $c_{l'} \neq 0$ will also be entangled with its environment at time τ. Hence even if $\hat{w}_l(t)$ and $\hat{w}_{l'}(t)$ commute, it is possible to check entanglement for an initial state with $c_l \neq 0$ and $c_{l'} \neq 0$ using the proposed scheme.

The other situation is when no entanglement of the type witnessed by criterion (7) is generated during the evolution. If only separability criteria of the second type, (8), are violated, this type of entanglement does not manifest itself in the conditional evolution of the environment and cannot be detected using this simple scheme. In fact, detecting such entanglement would most likely require tomography of the system-environment state.

VI. EXAMPLE: NV-CENTER SPIN QUTRIT

To exemplify the operation of the scheme described above, we use it to detect entanglement between an NV-center spin interacting with an environment of partially polarized nuclear spins of the spinful carbon isotope ^{13}C in the diamond lattice [49–52]. The dominant carbon isotope ^{12}C is spinless and does not contribute to the NV-center spin decoherence, so that the environment is sparse. The lowest energy level of the NV center is effectively a spin qutrit, with $S = 1$, so the dimension of the system is $N = 3$ and only two entanglement criteria of the first type, (7), need to be checked to show that entanglement would be present for any initial superposition state of the system.

For the majority of values of the applied magnetic field, the pure dephasing approximation can be used to describe this system and environment [26,27], so the Hamiltonian is of the form given by Eq. (2). For convenience we change the summation over system states to $k = -1, 0, 1$, so the index k corresponds to the three lowest level spin states of the qutrit, which are also its pointer states [33,34]. The energies which describe the free evolution of the qutrit in the Hamiltonian, (2), are equal to $\varepsilon_0 = 0$ and $\varepsilon_{\pm1} = \Delta \pm \gamma_e B_z$. Here B_z is the magnetic field applied in the z direction, so $\gamma_e B_z$ is the magnetic-field-induced splitting of the qutrit levels ($\gamma_e = 28.08$ MHz/T is the electron gyromagnetic ratio). The zero-field splitting, $\Delta\hat{S}_z^2$ with $\Delta = 2.87$ GHz, determines the z direction, which is dependent on the geometry of the NV center. The term is responsible for the uneven energy splitting of the qutrit states. The free evolution of the environment is given by $\hat{H}_\mathrm{E} = \sum_j \gamma_n B_z \hat{I}_j^z$, where j labels the ^{13}C spins, $\gamma_n = 10.71$ MHz/T is the gyromagnetic ratio for ^{13}C nuclei, and $\hat{I}_j^z$ are operators of the z component of the nuclear spins.

The hyperfine interaction between the qutrit and the nuclear environment yields operators which describe the response of the environment to a given pointer state of the qutrit. They are given by $\hat{V}_0 = 0$ and $\hat{V}_{\pm1} = \pm\hat{V}$, with

$$\hat{V} = \sum_j \left(\mathbb{A}_j^{z,x}\hat{I}_j^x + \mathbb{A}_j^{z,y}\hat{I}_j^y + \mathbb{A}_j^{z,z}\hat{I}_j^z\right). \tag{12}$$

Here the coupling constants for each direction are of the form

$$\mathbb{A}_j^{z,i} = \frac{\mu_0}{4\pi}\frac{\gamma_e\gamma_n}{r_j^3}\left(1 - \frac{3(\mathbf{r}_j\cdot\hat{\mathbf{i}})(\mathbf{r}_j\cdot\hat{\mathbf{z}})}{r_j^2}\right), \tag{13}$$

where $\mathbf{r}_j$ is a displacement vector between the jth nucleus and the qutrit, while $\hat{\mathbf{i}} = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors in three distinct directions. μ_0 is the magnetic permeability of the vacuum.

The conditional evolution operators of the environment, (4), which enter the full system-environment evolution operator, (3), are now straightforward to compute (see Ref. [16] for details) and are given by

$$\hat{w}_{\pm1}(t) = \bigotimes_j \Bigg[\cos\left(M_\pm^j t\right)\mathbb{I}_j \frac{i\sin\left(M_\pm^j t\right)}{M_\pm^j} \times\left(\pm\mathbb{A}_j^{z,\mathrm{pl}}\hat{I}_j^x + \left(\gamma_e B_z \pm \mathbb{A}_j^{z,z}\right)\hat{I}_j^z\right)\Bigg], \tag{14}$$

$$\hat{w}_0(t) = \bigotimes_j \left[\cos(\gamma_e B_z t)\mathbb{I}_j - i\sin(\gamma_e B_z t)\hat{I}_j^z\right], \tag{15}$$

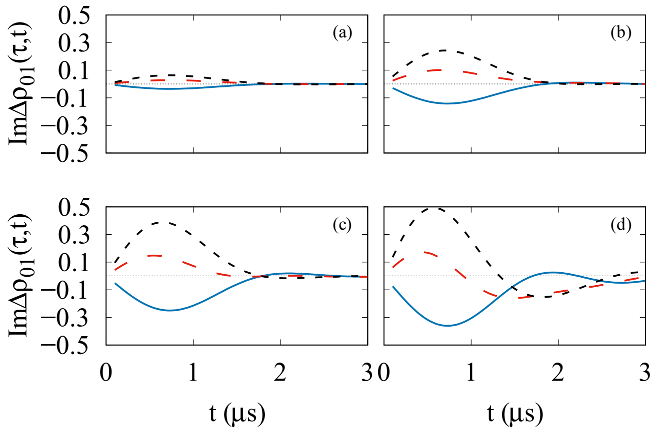

FIG. 2. Imaginary part of the evolution postpreparation stage of the difference between a single NV-center qutrit coherence for different pointer states in the preparation stage, $\rho_{01}^{(k)}(\tau,t)-\rho_{01}^{(q)}(\tau,t)$. Dashed red lines: $k=0$, $q=1$. Solid blue lines: $k=0$, $q=-1$. Dashed green lines: $k=-1$, $q=1$. The preparation stage lasted for $\tau=3\,\mu$s. Applied magnetic field $B_z=0.02\,$T. Details of the coupling are listed in Table I. Different panels correspond to different initial polarizations of the environment: (a) $p=0.1$; (b) $p=0.4$; (c) $p=0.7$; (d) $p=1$.

with $M_\pm=\sqrt{A_x^2+(\gamma_n B_z\pm A_z)^2}$ and $\mathbb{A}_j^{z,\mathrm{pl}}=\sqrt{(\mathbb{A}_j^{z,y})^2+(\mathbb{A}_j^{z,y})^2}$.

Given the initial state of the environment, we can find the evolution of the coherences which are needed to detect QEE, namely, Eq. (10). The thermal-equilibrium state of this environment (with respect to its free Hamiltonian) is effectively proportional to unity due to the small value of the gyromagnetic ratio for ^{13}C nuclei. As such a state will not lead to entanglement, we consider a dynamically polarized nuclear environment [53–55], so that

$$\hat{R}(0)=\bigotimes_j \tfrac{1}{2}\big(\mathbb{I}_j+p_j\hat{I}_j^z\big), \tag{16}$$

where $p_j\in[-1,1]$ is the polarization of the jth nucleus.

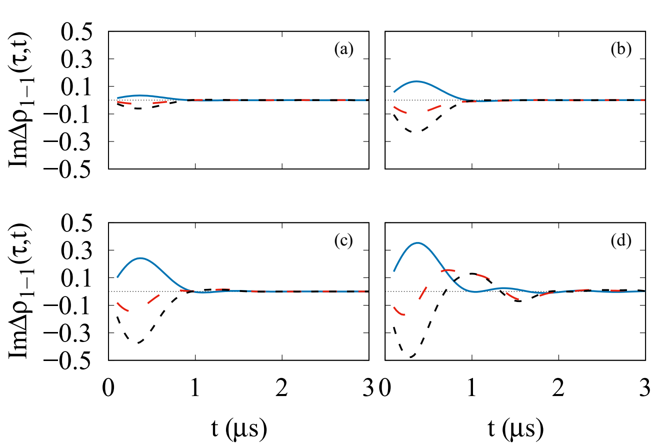

FIG. 3. As Fig. 2, but for a different qutrit coherence, $\rho_{1-1}^{(k)}(\tau,t)-\rho_{1-1}^{(q)}(\tau,t)$.

TABLE I. Table of calculated coupling constants for 14 environmental spins at randomly generated locations around the NV center.

k	r_k (nm)	$\mathbb{A}_k^{z,x}$ (1/μs)	$\mathbb{A}_k^{z,y}$ (1/μs)	$\mathbb{A}_k^{z,z}$ (1/μs)
1	0.504422	1.37617	0	0.973096
2	0.563961	0.196941	0.682223	−0.417774
3	0.563961	−0.689293	0.170556	−0.417774
4	0.563961	0.492352	−0.511667	−0.417774
5	0.617788	0.499393	0	−0.353124
6	0.636801	0.469395	−0.487809	−0.0189664
7	0.636801	−0.0134113	−0.116145	−0.47416
8	0.667287	−0.297224	0.220631	−0.300241
9	0.667287	−0.169842	0.58835	0.0600483
10	0.667287	−0.382145	0.220631	0.660531
11	0.667287	0	0	−0.420338
12	0.684928	0.326087	0.242057	−0.22399
13	0.684928	0.251553	−0.0484114	−0.329398
14	0.684928	−0.372671	0.161371	−0.22399

In Figs. 2 and 3 we plot the difference of the evolution of a single chosen coherence of the qutrit (as a function of the time t) for different pointer states in the preparation part of the procedure (up to time τ), $\rho_{ij}^{(k)}(\tau,t)-\rho_{ij}^{(q)}(\tau,t)$. Each figure contains curves corresponding to all three combinations of pointer states in the preparation stage, $k,q=-1,0,1$ (any two would suffice to determine whether entanglement would be present for any initial system superposition state after time τ). Figure 2 shows the difference in evolution for the coherence between the $|0\rangle$ and the $|1\rangle$ qutrit states; Fig. 3, for the coherence between the $|-1\rangle$ and the $|1\rangle$ states. The evolution of the third coherence is not shown, as it would be superfluous. Furthermore, we only show the imaginary part of the difference of the evolution, because the results are more striking in this case, and the real part would not contribute anything relevant to the discussion (since the scheme has already witnessed entanglement).

The preparation time $\tau=3\,\mu$s was chosen long so that the presence of the qutrit in a pointer state has the strongest possible effect on the new (postpreparation) state of the environment and consequently the states differ most notably for different pointer states. This in turn enhances the differences observed in qutrit evolution. The magnetic field is $B_z=0.02\,$T. Each plot contains four panels corresponding to four different initial states of the environment, characterized by different polarizations. As in the NV-center spin qubit case [19], an unpolarized environment does not entangle with the qutrit and the magnitude of the observed effect increases with higher initial polarization. The results are given for an environment consisting of 14 nuclear spins placed at randomly generated locations. Table I lists the distances between each nuclear spin and the NV center, as well as of the coupling constants used, which were calculated using Eq. (13).

VII. CONCLUSION

We have proposed a scheme for the indirect detection of entanglement between a system of any dimensionality and an environment interacting via a Hamiltonian which leads to pure dephasing of the qudit. This is a generalization of a scheme proposed for a system composed of a single qubit [19], but even though the number of separability criteria increases quadratically with the size of the system [14], the complexity of the scheme only increases linearly. The price paid is that the set of states for which entanglement cannot be detected using the scheme is also larger, and entanglement connected with breaking separability criteria based on commutation between products of conditional evolution operators of the environment (which does not exist for a qubit system) cannot be detected.

On the other hand, the scheme only requires straightforward operations and measurements on the system of interest and allows for detection of entanglement in systems too large for any type of state tomography to be feasible. It detects entanglement which manifests itself in the evolution of the environment and, as such, is most likely to have an effect on the evolution of the system. The mechanism of the scheme directly relies on the influence of SEE on the evolution of the system, so it will detect the type of entanglement which is bound to be most detrimental to the system (or a description of system evolution which assumes separability, such as using quantum channels [1]).

ACKNOWLEDGMENT

The authors would like to thank Łukasz Cywiński and Damian Kwiatkowski for sharing the data listed in Table I.

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