Entropic uncertainty relations for mutually unbiased periodic coarse-grained observables resembling their discrete counterparts

Łukasz Rudnicki[1,2,*] and Stephen P. Walborn[3,4]

[1] *International Centre for Theory of Quantum Technologies (ICTQT), University of Gdańsk, 80-308 Gdańsk, Poland*

[2] *Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, PL-02-668 Warsaw, Poland*

[3] *Departamento de Física, Universidad de Concepción, 160-C Concepción, Chile*

[4] *ANID, Millennium Science Initiative Program, Millenium Institute for Research in Optics, Universidad de Concepción, 160-C Concepción, Chile*

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One of the most important and useful entropic uncertainty relations concerns a d-dimensional system and two mutually unbiased measurements. In such a setting, the sum of two information entropies is lower bounded by $\ln d$. It has recently been shown that projective measurements subject to operational mutual unbiasedness can also be constructed in a continuous domain, with the help of periodic coarse graining. Here we consider the whole family of Rényi entropies applied to these discretized observables and prove that such a scheme does also admit the entropic uncertainty relations mentioned above.

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I. INTRODUCTION

Uncertainty relations are often cited as a key deviation between classical and quantum physics, describing the simultaneous unpredictability of two or more properties of a quantum system. Since the development of the concept of entropy to characterize information or the lack thereof, entropic uncertainty relations (EURs) have taken on a fundamental and useful role in quantum physics and quantum information [1–5]. They can be associated with secret quantum key rates [6–8] and used as identifiers of quantum correlations [9–16], for example. Additional applications can be found in recent review articles [3,5].

EURs exist for systems described by either discrete variables [17–20], continuous variables [21,22], or some combination of the two [23–25]. A key feature of discrete systems is that EURs for two *mutually unbiased* observables give lower bounds that are a function of the dimension d alone. To be more precise, two d-dimensional operators X and Y with all eigenstates $(i, j = 0, \dots, d-1)$ satisfying $|\langle X_i|Y_j\rangle| = 1/\sqrt{d}$ render two mutually unbiased measurements. For this case, an EUR involving Rényi entropies (with natural logarithm) of orders α and β, such that $1/\alpha + 1/\beta = 2$, is given by [18]

$$H_\alpha[X] + H_\beta[Y] \geqslant \ln d, \tag{1}$$

where

$$H_\alpha[X] = \frac{1}{1-\alpha} \ln \sum_{i=0}^{d-1} p_i^\alpha[X], \tag{2}$$

and $p_i[X] = \langle X_i|\rho|X_i\rangle$. As usual, ρ represents the density matrix describing the system.

In the continuous-variable scenario, similar types of EURs have been developed. However, the crucial difference between the discrete and the continuous case is that the finite dimension d is "lost" within a standard treatment, being replaced by a scaling parameter, related to the observables in question. To understand that effect we shall first observe that a system of mutually unbiased measurements can be characterized by two *a priori* independent parameters: the number of possible measurement outcomes and the uniform "overlap" between different measurements. The first parameter is formally the same as the number of projectors forming the resolution of the identity and is assumed here to be the same for both measurements. The latter one has a clear operational meaning [26,27] for all projective measurements (for general positive operator-valued measures it is more complicated [28]), being equal to the true overlap between the eigenstates, in a special case of rank 1 projectors. Clearly, if the first parameter is finite (therefore discrete), conservation of probability fixes the value of the latter one, as explained above in Eq. (1). However, for continuous variables, the number of outcomes is usually considered to be infinite, either countably or uncountably. As a consequence, the overlap becomes a free, setup-dependent, scaling parameter. This point is discussed further in Sec. II below.

The partitioning of a continuous variable into discretized bins, due to the finite precision of a measurement device or otherwise, introduces the bin width as an additional scaling parameter (see also Sec. II for more details). In principle, this discretization leads to a countable—but infinite—number of measurement outcomes, which can be further hashed into a finite set. One approach is to arrange the infinite number of bins periodically, giving a finite set of projective measurement operators with rank >1. Periodic coarse graining of this type is interesting in that it allows one to define truly mutually unbiased measurements for an appropriate choice

*lukasz.rudnicki@ug.edu.pl

of bin widths [26,29]. Though they have a finite number of outcomes, these observables are not discrete in any rigorous sense. For example, they do not reproduce the well-known results for a maximum number of mutually unbiased bases in finite dimensions [30]. Thus, it is an open question as to whether PCG observables obey discretelike or continuouslike EURs. That is, can the sum of entropies be lower bounded by only the logarithm of the number of measurement outcomes, or do measurement-dependent scaling parameters appear?

Here, we show that PCG observables do follow a discrete EUR analogous to relation (1). The paper is organized as follows. In Sec. II we discuss in more detail the role of scaling parameters in continuous variables. Discretization of continous variables and PCG observables are introduced in Secs. III and IV. In Sec. V we prove our main EUR for the special case of the position and momentum PCG pair. We also state the same result for arbitrary phase-space variables. In addition, in Sec. VI we study the continuous limit of the EURs considered.

II. MUTUAL UNBIASEDNESS AND CONTINUOUS VARIABLES

To better illustrate the issue of scaling parameter in the continuous regime, let us consider phase-space quadrature variables, given by $\mathsf{q}_\theta = \cos\theta \mathsf{x} + \sin\theta \mathsf{p}$, where x and p, recovered for $\theta = 0$ and $\theta = \pi/2$, respectively, are the usual position and momentum operators obeying $[\mathsf{x}, \mathsf{p}] = i\hbar$. The commutator $[\mathsf{q}_\theta, \mathsf{q}_{\theta'}] = i\hbar \sin\Delta\theta$ clearly depends upon the relative angle between the operators $\Delta\theta = \theta - \theta'$. Therefore, uncountably many eigenstates of these operators are mutually unbiased, with overlaps given by [31]

$$|\langle q_\theta | q_{\theta'}\rangle| = (2\pi\hbar|\sin\Delta\theta|)^{-1/2}. \tag{3}$$

Mutual unbiasedness of both measurements is encoded in the fact that the above overlap depends neither on q_θ nor on q'_θ.

In other words, the indicator of systems' dimension d is replaced by $2\pi\hbar|\sin\Delta\theta|$—the continuous parameter which depends on both the underlying structure of the phase space (presence of $\hbar$) and the interrelation between the involved operators, quantified by $\sin\Delta\theta$. As a natural consequence, the EURs expressed in terms of continuous Rényi [22,33] and Shannon [21,32] entropies do depend on both parameters. We go back to these types of EURs in Sec. VI. From now on we also set $\hbar \equiv 1$.

An additional scaling factor arises when one takes into account that the above eigenstates describe a nonphysical scenario of infinite energy, and consequently, physical scenarios involve some sort of coarse graining. That is, the eigenstates $|q_\theta\rangle$ are approximated by "smeared" quantum states $\int dq'_\theta Q(q_\theta - q'_\theta)|q'_\theta\rangle$, where $Q(q_\theta - q'_\theta)$ is a square integrable function that is localized around q_θ with some finite width parameter δ_θ. Likewise, though this is just an analogy rather than a formal continuation of the previous argument, physical measurement devices (detectors) cannot be described by uncountably many rank-one projectors $|q_\theta\rangle\langle q_\theta|$, but rather by countably many (though, still infinite number of) integrated projective measurements of the form

$$\int_{q_\theta-\delta_\theta/2}^{q_\theta+\delta_\theta/2} dq'_\theta |q'_\theta\rangle\langle q'_\theta|. \tag{4}$$

Adequate consideration of coarse graining in this context leads to uncertainty relations (URs) with lower bounds that, in addition, depend explicitly on the width parameters δ_θ [22,23,34,36,37]. Improper attention to this inherent coarse graining can have detrimental consequences [38–40]. An overview of URs for coarse-grained continuous variables (CVs) can be found in Ref. [4].

III. DISCRETE SETTINGS IN CONTINUOUS-VARIABLE SYSTEMS

Table I summarizes the cases discussed in the previous section. As can be seen, only settings with an infinite number of outcomes have so far been successfully considered in the continuous scenario, even though only a discrete one can lead to a counterpart of the EUR in Eq. (1). Therefore, as emphasized in the bottom right cell of Table I, the aim of this paper is to provide a setting which obeys Eq. (1) for continuous variables.

To this end we need an alternative approach to the standard coarse graining described by Eq. (4), i.e., other methods of binning together the rank-1 projectors. A number of strategies have been adopted in this direction [41–46]. With the goal of defining truly mutual unbiased measurements in CV systems, periodic coarse graining (PCG) has been a successful approach. That is, two sets of CV phase-space projectors $\Pi_k[\theta]$ and $\Pi_l[\theta']$ (like before $k, l = 0, \dots, d-1$) can be defined such that their eigenstates give equal probability outcomes when the other measurement operator is applied [26,29]. This may seem to suggest that one can define a discrete variable system within a CV one, which may be loosely true, but not in any rigorous sense. For example, it was shown that these PCG observables, though mutually unbiased, do not follow the known conditions concerning the number of allowed mutually unbiased bases for discrete systems. Rather, depending on the number of outcomes d, they can mimic either the discrete or continuous cases, or neither [30].

Here we explore another way to benchmark PCG observables, that is, through the corresponding EURs, and show that

TABLE I. Different types of settings relevant for continuous-variable systems and associated, known EURs. Here we fill the gap of a discrete setting.

Number of measurements' outcomes	Overlap between the measurements	Entropic URs
Uncountably infinite	$(2\pi\hbar|\sin\Delta\theta|)^{-1/2}$	[21,22,32,33]
Countably infinite	Additionally depends on coarse-graining widths	[22,23,34,35]
Discrete, equal to d	Always $1/\sqrt{d}$	Present paper

they indeed mimic the discrete case in that they obey the entropic URs from Eq. (1). This applies to PCG of usual position and momentum operators, as well as arbitrary phase-space operators. In this way, we realize our main goal and derive a family of entropic uncertainty relations for PCG observables. Moreover, the state-independent lower bound that depends only on the number of measurement outcomes d implies that PCG observables could be an interesting route for quantum information protocols, since system-dependent parameters, such as relative phase-space direction or measurement bin width, are absent.

IV. PERIODIC COARSE-GRAINED OBSERVABLES

In order to construct coarse-grained mutually unbiased projective measurements, we group rank-1 projectors according to periodic bin functions ($k = 0, \dots, d-1$) [47]:

$$M_k(z;T) = \begin{cases} 1, & k\,s \leqslant z (\text{mod } T) < (k+1)s, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

The bin functions can be thought of as continuous square waves with spatial period T and bin width $s = T/d$.

While for simplicity, in Sec. V, we first consider the special case of position and momentum, we now introduce notation which covers a general pair of phase-space directions. Let

$$\Pi_k[\theta] = \int_{\mathbb{R}} dq_\theta\, M_k(q_\theta; T_\theta)|q_\theta\rangle\langle q_\theta|, \tag{6}$$

for $k = 0, \dots, d-1$, be a set of d projectors rendering PCG in the θ direction of the phase space. In Refs. [26,29], additional displacement parameters setting the origin of the phase space have been introduced. However, as these degrees of freedom do not at all influence the present discussion, they are omitted here. One just needs to remember that all arguments remain valid independent of the choice of the origin of the phase space.

Given a mixed state ρ, we further define the probabilities

$$p_k[\theta] = \text{Tr}(\rho\Pi_k[\theta]). \tag{7}$$

Operational mutual unbiasedness of two measurements has been defined for pure states in Ref. [26] (see also Ref. [27]); however, one can easily realize that this definition extends to the case of mixed states by convexity. To be more precise, we call both θ and θ' measurements as mutually unbiased if, for all states ρ such that $p_k[\theta]$ is a permutation of $(1, 0, \dots, 0)$ with $d-1$ zeros, we find that $p_l[\theta'] = 1/d$ for all l, and *vice versa*.

It is quite straightforward to realize that for $\rho = \sum_n \lambda_n |\Psi_n\rangle\langle\Psi_n|$, with all $\lambda_n \geqslant 0$ and $\sum_n \lambda_n = 1$, the requirement $p_k[\theta] = 1$ for some k enforces $\langle\Psi_n|\Pi_k[\theta]|\Psi_n\rangle = 1$ for all n. Consequently $\langle\Psi_n|\Pi_l[\theta']|\Psi_n\rangle = 1/d$.

Note that the above operational definition of mutual unbiasedness, as well as its natural extension to the case of mixed states, applies to any pair of projective measurements, not necessarily being the PCG, which we use here for the sake of illustration and further discussion.

In Ref. [29], it has been proven that, if

$$\frac{T_\theta T_{\theta'}}{2\pi} = \frac{d|\sin\Delta\theta|}{M}, \quad M \in \mathbb{N}, \quad \forall_{n=1,\dots,d-1}\ \frac{M\,n}{d} \notin \mathbb{N}, \tag{8}$$

with M being a natural number ($M \neq 0$) such that $M\,n/d \notin \mathbb{N}$ for all $n = 1, \dots, d-1$ (i.e., M is not co-prime with d), then both sets of the PCG projectors are mutually unbiased.

V. ENTROPIC URS FOR PCG

We are interested in an entropic UR of the general form

$$H_\alpha[\theta] + H_\beta[\theta'] \geqslant -2\ln\mathcal{C}, \tag{9}$$

where as usual $1/\alpha + 1/\beta = 2$ and the Rényi entropy is defined in Eq. (2). Our aim is to show that $\mathcal{C} \leqslant 1/\sqrt{d}$. To this end we partially follow Refs. [34,48]. We adapt, to the case of PCG observables, the methodology presented therein, which was tailored to "standard" coarse graining as in Eq. (4). Initially, this requires a replacement of finite intervals by periodic sets. The replacement propagates to complete sets of functions and probability distributions. Later on, Eq. (8), which encodes the property of mutual unbisedness of the PCG, is utilized in order to further work out the bound relevant for the current scenario. In comparison, in Refs. [34,48], eigenvalues of an integral equation known from signal processing theory have instead been discussed.

We first introduce a few pieces of notation. Let $O_k[\theta]$ be sets defined as

$$O_k[\theta] = \{z \in \mathbb{R} : \quad M_k(z; T_\theta) = 1\}, \tag{10}$$

and note that

$$\Pi_k[\theta] = \int_{O_k[\theta]} dq_\theta |q_\theta\rangle\langle q_\theta|. \tag{11}$$

From now on we focus our attention on the position-momentum couple, further denoting $O_k[x] \equiv O_k[0]$, $O_k[p] \equiv O_k[\pi/2]$, $T_x \equiv T_0$, and $T_p \equiv T_{\pi/2}$. We define $\varphi_{km}(x)$ and $\xi_{ln}(p)$ to be orthonormal and complete sets of functions on $O_k[x]$ and $O_l[p]$, respectively, i.e.,

$$\int_{O_k[x]} dx\, \varphi_{k_1 m}(x)\varphi^*_{k_2 m'}(x) = \delta_{k_1 k}\delta_{k_2 k}\delta_{m' m}, \tag{12a}$$

$$\int_{O_l[p]} dp\, \xi_{l_1 n}(p)\xi^*_{l_2 n'}(p) = \delta_{l_1 l}\delta_{l_2 l}\delta_{n' n}. \tag{12b}$$

Such complete sets are guaranteed to exist, since functions supported on, e.g., $O_k[x]$ form a subspace of the Hilbert space of square integrable functions, which is separable (so is every subspace).

Moreover, without loss of generality we restrict our attention to pure states $\rho = |\Psi\rangle\langle\Psi|$, since they are known to cover extreme points of information entropies. We therefore define amplitudes as follows:

$$a_{km} = \int_{O_k[x]} dx\, \psi(x)\varphi^*_{km}(x), \tag{13a}$$

$$b_{ln} = \int_{O_l[p]} dp\, \tilde{\psi}(p)\xi^*_{ln}(p), \tag{13b}$$

where as usual $\psi(x) = \langle x|\Psi\rangle$ and $\tilde{\psi}(p) = \langle p|\Psi\rangle$.

Generalizing Eqs. (A7)– (A9) from Ref. [34], by replacing the intervals appearing there by the sets $O_k[x]$ and $O_l[p]$, and with a slight adjustment of the notation concerning arguments of the Rényi entropies, we immediately get the result

$$H_\alpha[|a|^2] + H_\beta[|b|^2] \geqslant -2\ln\mathcal{C}, \tag{14}$$

where

$$\mathcal{C} = \sup_{(k,l,m,n)} \left| \int_{O_k[x]} dx \int_{O_l[p]} dp \frac{e^{ipx}}{\sqrt{2\pi}} \varphi_{km}^*(x)\xi_{ln}(p) \right|. \tag{15}$$

Arguments of the Rényi entropies in Eq. (14) are not denoted as the directions on the phase space, but as the probability distributions entering Eq. (2). These distributions are more fine-grained than Eq. (7), since

$$p_k[0] = \sum_m |a_{km}|^2, \quad p_l[\pi/2] = \sum_n |b_{ln}|^2. \tag{16}$$

If we further apply the Cauchy-Schwarz inequality to the $\int_{O_k[x]} dx$ integral and use normalization of $\varphi_{km}(x)$, we arrive at the bound $\mathcal{C} \leqslant \sup_{(k,l,n)} W_{kl}^n$, where

$$W_{kl}^n = \sqrt{\int_{O_k[x]} dx \int_{O_l[p]} dp \int_{O_l[p]} dp' \frac{e^{i(p-p')x}}{2\pi} \xi_{ln}(p)\xi_{ln}^*(p')}. \tag{17}$$

Our task is therefore to compute the kernel

$$\int_{O_k[x]} dx \frac{e^{i(p-p')x}}{2\pi}. \tag{18}$$

The periodic bin function can be decomposed in the Fourier series

$$M_k(z;T) = \frac{1}{d} + \sum_{N\in\mathbb{Z}/\{0\}} f_{k,N} e^{\frac{2\pi iN}{T}z}, \tag{19}$$

where

$$f_{k,N} = \frac{1 - e^{-\frac{2\pi iN}{d}}}{2\pi iN} e^{-\frac{2\pi iN}{d}k}. \tag{20}$$

Using Eq. (19) to calculate the kernel, we find

$$\int_{O_k[x]} dx \frac{e^{i(p-p')x}}{2\pi} = \frac{1}{d}\delta(p-p') + \sum_{N\in\mathbb{Z}/\{0\}} f_{k,N}\delta\left(p - p' + \frac{2\pi}{T_x}N\right). \tag{21}$$

Consequently, we obtain the result

$$\begin{aligned} &\int_{O_k[x]} dx \int_{O_l[p]} dp \int_{O_l[p]} dp' \frac{e^{i(p-p')x}}{2\pi} \xi_{ln}(p)\xi_{ln}^*(p') \\ &= \frac{1}{d} + \sum_{N\in\mathbb{Z}/\{0\}} f_{k,N} \int_{O_l[p]} dp \int_{O_l[p]} dp' \delta \\ &\times \left(p - p' + \frac{T_p MN}{d}\right) \xi_{ln}(p)\xi_{ln}^*(p'), \end{aligned} \tag{22}$$

where we have utilized normalization of $\xi_{ln}(p)$ to integrate the first Dirac δ contribution, and we have applied condition (8) while changing arguments of the remaining Dirac δ's. Due to the last step, every Dirac δ in the second expression leads to an autocorrelation term, which is nonvanishing only when MN/d is an integer. However, due to the further requirement established in Eq. (8), we find $MN/d \in \mathbb{Z}$ if and only if $N/d \in \mathbb{Z}$. But in this special case the factor $1 - e^{-\frac{2\pi iN}{d}}$ present in $f_{k,N}$ becomes equal to 0, so that all terms in the sum over $N \in \mathbb{Z}/\{0\}$ disappear, leaving the bare contribution $1/d$. As a result, $\mathcal{C} \leqslant 1/\sqrt{d}$, as expected.

Finally, we observe [34] that a particular choice

$$\varphi_{k0}(x) = \langle x|\Pi_k[0]|\Psi\rangle/\sqrt{p_k[0]}, \tag{23a}$$

$$\xi_{l0}(p) = \langle p|\Pi_l[\pi/2]|\Psi\rangle/\sqrt{p_l[\pi/2]}, \tag{23b}$$

with other functions in both complete sets being orthogonal to Eqs. (23) leads to the probabilities $|a_{km}|^2 = p_k[0]\delta_{m0}$ and $|b_{ln}|^2 = p_l[\pi/2]\delta_{n0}$. Therefore, the bound $\mathcal{C} \leqslant 1/\sqrt{d}$, which consequently gives $-2\ln\mathcal{C} \geqslant \ln d$, is also valid for our main UR under consideration; namely, Eq. (1) for the position and momentum pair of PCG observables, denoted by angles $\theta = 0$ and $\theta = \pi/2$, respectively, is proven. It is easy to recognize that this bound, due to the property of mutual unbiasedness, is saturated if a state is localized in either of the sets $O_k[x]$ or $O_l[p]$, for a fixed value of the index k or l.

A. Extension to any two directions in phase space

In order to extend the above result to two arbitrary phase-space observables, q_θ and $q_{\theta'}$, i.e., to show that the general EUR

$$H_\alpha[\theta] + H_\beta[\theta'] \geqslant \ln d, \tag{24}$$

holds (as always with $1/\alpha + 1/\beta = 2$), we first observe that several steps of the previous derivation can immediately be repeated with minor modifications. To be more precise, Eqs. (12)–(20) from Sec. V just require a slight adjustment of the notation, which boils down to a replacement of labels "0" or "x" by "θ" and "$\pi/2$" or "p" by "θ'," as well as function arguments by q_θ and q'_θ, respectively. Moreover, the Fourier transform in Eq. (15) must be replaced by the fractional Fourier transform [49] which gives the generalized overlap $\langle q_\theta|q_{\theta'}\rangle = \mathcal{F}(q_\theta, q_{\theta'})$ and reads (as before $\Delta\theta = \theta - \theta'$)

$$\mathcal{F}(q_\theta, q_{\theta'}) = \sqrt{\frac{-ie^{i\Delta\theta}}{2\pi\sin\Delta\theta}} e^{i\frac{\cot\Delta\theta}{2}(q_\theta^2 + q_{\theta'}^2) - i\frac{q_\theta q_{\theta'}}{\sin\Delta\theta}}. \tag{25}$$

Note that $|\mathcal{F}(q_\theta, q_{\theta'})|$ reduces to the overlap in Eq. (3). Consequently, the Fourier kernel (18) is replaced by

$$\int_{O_k[\theta]} dq_\theta \mathcal{F}(q_\theta, q_{\theta'})\mathcal{F}(\tilde{q}_{\theta'}, q_\theta). \tag{26}$$

Since

$$\mathcal{F}(q_\theta, q_{\theta'})\mathcal{F}(\tilde{q}_{\theta'}, q_\theta) = \frac{e^{i\frac{\cot\Delta\theta}{2}(q_{\theta'}^2 - \tilde{q}_{\theta'}^2)}}{2\pi|\sin\Delta\theta|} e^{i\frac{q_\theta}{\sin\Delta\theta}(\tilde{q}_{\theta'} - q_{\theta'})}, \tag{27}$$

we easily generalize Eq. (21) as

$$\begin{aligned} &\int_{O_k[\theta]} dq_\theta \mathcal{F}(q_\theta, q_{\theta'})\mathcal{F}(\tilde{q}_{\theta'}, q_\theta) \\ &= \frac{1}{d}\delta(\tilde{q}_{\theta'} - q_{\theta'}) + \sum_{N\in\mathbb{Z}/\{0\}} f_{k,N} e^{i\frac{\cot\Delta\theta}{2}(q_{\theta'}^2 - \tilde{q}_{\theta'}^2)} \delta \\ &\times \left(\tilde{q}_{\theta'} - q_{\theta'} + \frac{2\pi\sin\Delta\theta}{T_\theta}N\right). \end{aligned} \tag{28}$$

The remaining part of the derivation follows exactly the same way as for the particular case of position and momentum. The only difference is that due to the MUB condition Eq. (8), the term $2\pi \sin\Delta\theta/T_\theta$ inside the Dirac δ is replaced by $\pm T_{\theta'}M/d$, where the sign $\pm$ depends on the order of θ and θ' on the phase space. In Eq. (22) we find the plus sign as the angle difference for position and momentum is in $[0,\pi]$. Thus, we have an uncertainty relation of the form (1) for PCG observables corresponding to any two nonparallel phase-space quadratures.

VI. CONTINUOUS LIMIT

At the end we would briefly like to elaborate on the continuous limit for PCG observables. To this end we recall that $d = T_\theta/s_\theta$, for all variables q_θ, where s_θ is the bin width. Then, using Eq. (8), we can write $d = 2\pi|\sin\Delta\theta|/s_\theta s_{\theta'}$. Plugging this into Eq. (24), we have

$$H_\alpha[\theta] + H_\beta[\theta'] + \ln(s_\theta s_{\theta'}) \geqslant \ln 2\pi|\sin\Delta\theta|. \qquad (29)$$

Each Rényi entropy can be rewritten as follows:

$$H_\alpha[\theta] = -\ln s_\theta + \frac{1}{1-\alpha}\ln\left[\sum_{i=0}^{d-1} s_\theta\left(\frac{p_i[\theta]}{s_\theta}\right)^\alpha\right]. \qquad (30)$$

In the continuous limit $d\to\infty$, we set $T_\theta \sim \sqrt{d}$, so that $T_\theta\to\infty$ while at the same time $s_\theta\to 0$. In this limit, the sum multiplied by s_θ tends to the integral $\int_0^\infty dq_\theta$, while the term in parentheses in Eq. (30) becomes a continuous probability distribution supported on $[0,\infty)$. This specific probability distribution takes into account two points on the real line, one on the positive side and one the negative side (though not symmetrically). To explain it a bit better we can for the moment restrict ourselves to a box $[-L, L]$ and, given a function $f(x)$ supported on that box, consider the function $g(x) = f(x) + f(x-L)$, which is supported on $[0, L]$. In our limiting procedure, the continuous probability distributions on the real line, which normally are the arguments of the Rényi entropies, will be of the f type, while $p_i[\theta]/s_\theta$ tends to the distribution of the g type. Obviously, the g-type probability distributions will always have entropy smaller than that of f-type distributions. Therefore, the continuous Rényi entropy $h_\alpha[\theta]$ will also be bigger than the continuous limit of the entropy in Eq. (30):

$$h_\alpha[\theta] \geqslant \lim_{d\to\infty}(H_\alpha[\theta] + \ln s_\theta). \qquad (31)$$

As a consequence, using Eq. (29), we obtain the continuous UR

$$h_\alpha[\theta] + h_\beta[\theta'] \geqslant \ln 2\pi|\sin\Delta\theta|. \qquad (32)$$

The uncertainty relation obtained is clearly weaker than the best-known URs for continuous variables [22,33]. This is because the latter follow from a completely different mathematical machinery, namely, "p-q norm" inequalities for the Fourier transform. Our result is, on the contrary, closest in spirit with standard finite-dimensional treatment of mutually unbiased bases which, while powerful, does not know much about sophisticated properties of the Fourier transform.

VII. DISCUSSION

We have provided an entropic uncertainty relation for a discrete set of mutually unbiased, periodic coarse-grained observables. Different from the underlying continuous observables, or other discretization schemes, here the uncertainty limit is bounded only by the number of measurement outcomes, which plays the role of dimension. We extend our results to apply to observables constructed from eigenstates of any two nonparallel quadrature operators and show that a meaningful (though not optimal) continuous limit can be obtained. The bound in the continuous limit is meaningful because it reproduces the result by Hirschman [50], which is a natural relative of the Maassen-Uffink bound (see Eq. (21) in Ref. [18]). Since the optimal bound relevant for the continuous variables [21] is associated with a different mathematical toolbox (p-q norm inequalities for the Fourier transform), it is clear why we do not reach it in the continuous limit.

The main motivation for our work is the overall question concerning the behavior of discretized observables constructed within a continuous Hilbert space, and whether these observables are "more continuous" or "more discrete" in their characteristics. Our results show in the case of periodic coarse graining that the discretization is indeed manifest in the desired way, whereas entropic uncertainty relations are applied.

A number of possible applications and open questions exist. First, it is tempting to ask whether these results can be extended to include more than two observables, as in Refs. [29,30]. This remains an open question, since to date an entropic uncertainty relation for more than two continuous operators has been conjectured but not proven [5,51]. What has been proven for the continuous case concerns Gaussian states and loses relevance in the PCG scenario. Moreover, the established results for discrete systems [52–54] do not seem to be directly applicable. As an application of our results, the EUR derived here for PCG observables can be adapted to identify entanglement or, more specifically, as a criteria for EPR-steering correlations [55] between two parties. For example, it is straightforward to follow the recipe in Refs. [14,56], which, for $\alpha = \beta = 1$ (Shannon entropies), leads to

$$H_1[q_\theta|q_\phi] + H_1[q_{\theta'}|q_{\phi'}] \geqslant \ln d, \qquad (33)$$

where $H_1[r|s]$ is the conditional Shannon entropy and r and s refer to measurement directions of Alice and Bob (the two parties). Violation of the above inequality indicates EPR steering correlations in Alice and Bob's bipartite system. We expect that interesting applications of EURs for PCG observables will be found, in particular, for quantum states with a natural periodicity.

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