Reverse quantum speed limit: How slowly a quantum battery can discharge

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We introduce the notion of a reverse quantum speed limit for arbitrary quantum evolution which answers a fundamental question: How slowly can a quantum system evolve in time? Using the geometrical approach to quantum mechanics, the reverse speed limit follows from the fact that the gauge-invariant length of the reference section is always greater than the Fubini-Study distance on the projective Hilbert space of the quantum system. We illustrate the reverse speed limit for two-level quantum systems with an external driving Hamiltonian and show that our results hold well. We find several examples where our bound is tight. We also find one practical application of the reverse speed limit in the discharging process of quantum batteries, which answers the following question: How slowly can quantum batteries discharge? Our result provides a lower bound on the average discharging power of quantum batteries.

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I. INTRODUCTION

Quantum physics has several inherent limitations. These limitations could be either operational or dynamical limitations. From a dynamical point of view, the quantum speed limit (QSL) has played a pivotal role in quantum computation, quantum control, and even in quantum thermodynamics. These limitations are not only crucial for theoretical questions but also have practical relevance, as recent years have witnessed rapidly developing quantum technologies.

The question of how fast a quantum system can evolve in time was first addressed in Ref. [1]. The notion of speed of transportation of a quantum system, what is now known as the "quantum speed limit," was first introduced by Aharonov and Anandan [2] using the Fubini-Study metric. The speed of transportation of a quantum system for unitary and nonunitary evolutions was introduced in Refs. [3–5] using the Riemannian metric. The QSL is defined as the maximal dynamical evolution speed of the quantum system. It determines the minimal dynamical evolution time required for a quantum state of a quantum system to evolve from an initial state to target state [1,2,6–45]. Since QSL determines how fast quantum systems evolve it is natural to ask if there is a reverse speed limit for quantum evolution. To answer this question, we formally introduce the notation of reverse quantum speed limit (RQSL) for arbitrary quantum evolution. This is defined as the minimal evolution speed of closed quantum systems and it sets a bound on the maximal evolution time required for a quantum state of a closed quantum system to evolve from an initial state to a target state. This is an upper bound on the quantum evolution time. The quantum evolution will be slower if the reverse quantum speed limit time increases. It may have several meaningful applications in the field of quantum physics, ranging from quantum information, quantum computation, to quantum optimal control and the quantum battery.

The quantum battery (QB) was originally introduced by Alicki and Fannes [46]. It consists of small quantum systems with many degrees of freedoms in which we can store energy or extract energy from them. Quantum batteries are much better than classical batteries in many ways because they offer several quantum mechanical advantages [47,48]. In recent years, lots of theoretical models of quantum batteries have been studied by several groups to enhance the feature of quantum batteries by exploiting the nonclassical resources of quantum mechanics. Also, several charging and discharging protocols have been proposed to enhance the power [49–54], work storage [46,55], stability [56–58], and so on. With the help of QSL, how fast we can charge the quantum batteries has been found [50,59–61]. Here with the help of RQSL, we will answer the question "how slowly can quantum batteries discharge?" QSL and RQSL also set the bound on the average charging or discharging power of quantum batteries. We believe that it will help us in the practical realization of quantum batteries, which can maintain its power for longer duration. Ideally, one should design quantum batteries which discharge slowly while operating and hence our RQSL play a pivotal role in deciding the figure of merit of such quantum batteries.

Our paper is organized as follows. In Sec. II, we provide a basic framework to appreciate various geometric structures that will be used for proving the reverse quantum speed limit. In Sec. III, we apply RQSL for two-level systems. In Sec. IV, we study the slow discharging of quantum batteries by employing RQSL. Finally, we summarize our results in Sec. V.

II. GEOMETRICAL REVERSE QUANTUM SPEED LIMIT FOR UNITARY EVOLUTIONS

Before we prove the reverse quantum speed limit, we need some background on the geometry of quantum evolution.

A. For pure initial state

Let us consider a set of vectors $\{\psi\}$ of $(D+1)$-dimensional quantum system that belongs to a Hilbert space $\mathcal{H}^{D+1}$. If these vectors are not normalized, we can consider $\{\psi/||\psi||\}$ to be a

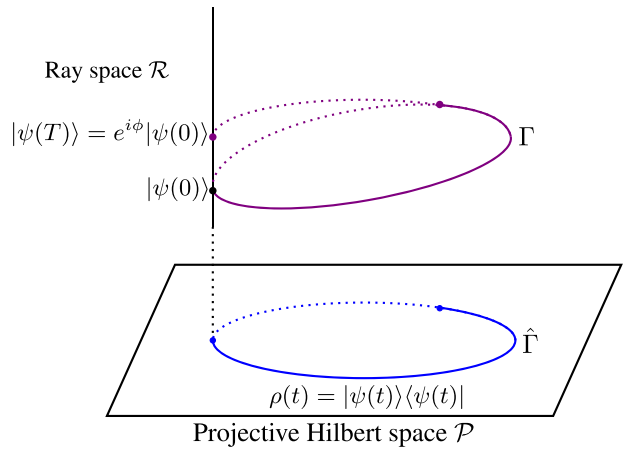

FIG. 1. In this schematic diagram the violet line represents the curve Γ in $\mathcal{H}$. The curve Γ is the actual trajectory of the noncyclic evolution of quantum system whose state vector $|\psi(t)\rangle \in \mathcal{H}$ evolves according to the Schrödinger equation. For cyclic evolution the curve Γ begins and ends on the same ray but at different points. The blue line represents the curve $\hat{\Gamma}$ in $\mathcal{P}$. The curve $\hat{\Gamma}$ is basically a projection of curves Γ in $\mathcal{P}$. Note here that there may be infinite number of such curves Γ in $\mathcal{H}$, which can give rise to the same curve $\hat{\Gamma}$ in $\mathcal{P}$.

set of vectors of norm 1 that belongs to a unit-normed Hilbert space $\mathcal{L}$. The state of a quantum system is represented by a ray in the ray space $\mathcal{R} = \mathcal{L}/U(1)$. Two normalized state vectors $|\psi\rangle$ and $|\psi'\rangle$ are equivalent if they belong to the same ray, i.e., they merely differ by a phase factor $(|\psi'\rangle \equiv e^{i\phi}|\psi\rangle$, where $e^{i\phi} \in U(1))$. The set of rays of $\mathcal{H}$ via a projection map is known as the projective Hilbert space $\mathcal{P}$. The projection map $\Pi : \mathcal{L} \to \mathcal{P}$ is a principal fiber bundle $\mathcal{L}[\mathcal{P}, U(1), \Pi]$, with structure group $U(1)$. This can be observed by considering the action of the multiplicative group C^* of nonzero complex numbers on the space $C^{D+1} - \{0\}$ given by the equivalence relation $(z_1, z_2, \ldots, z_{D+1})\lambda := (z_1\lambda, z_2\lambda, \ldots, z_{D+1}\lambda)\ \forall\ \lambda \in C^*$. This is a free action and the orbit space is the space $\mathcal{CP}^D$ of the complex lines in the Hilbert space $\mathcal{H} = \mathcal{C}^{D+1}$. Thus, we get the principal bundle $C^* \to C^{D+1} - \{0\} \to \mathcal{CP}^D = \mathcal{P}$ in which the projection map associates with each $(D+1)$ tuple $(z_1, z_2, \ldots, z_{D+1})$ the point in $\mathcal{CP}^D$ with the homogeneous coordinates. Thus, any pure quantum state at given instant of time is represented by a point in $\mathcal{P}$ and the evolution of the quantum system is represented by a curve Γ in $\mathcal{H}$, which projects to a curve $\hat{\Gamma} = \Pi(\Gamma)$ in $\mathcal{P}$ [3–5]. See Fig. 1 for a geometric depiction.

During a noncyclic evolution of a quantum system, the initial state and the final state belong to two different rays of the Hilbert space. Thus, the evolution curve $\hat{\Gamma}$ is an open path in $\mathcal{P}$ where the initial and final points lie on two different rays. Using the Pancharatnam connection [62], we can compare the relative phases of state vectors belonging to two different rays. If a quantum system evolves from an initial state to the final state, then the relative phase difference between these states is given by

$$e^{i\Phi} = \frac{\langle\psi(0)|\psi(t)\rangle}{|\langle\psi(0)|\psi(t)\rangle|}. \tag{1}$$

Here, the initial and final states should not be orthogonal. If $\langle\psi(t)|\psi(0)\rangle$ is complex, then the quantum system does acquire a relative phase during the evolution of the system. In this case if we map the open path $\hat{\Gamma}$ in $\mathcal{P}$ to $\mathcal{L}$, there are many open curves in $\mathcal{L}$ corresponding to this open curve in $\mathcal{P}$. Among all of them, there exists one special open curve, which is traced out by the reference state. This reference state is a vector that depends on the initial state vector of the system. To define this special open curve Γ_0, let us construct the "reference section" $|\chi(t)\rangle$ of the bundle covering $\rho(t) = \Pi[\psi(t)]$. It is a map $s : \mathcal{P} \to \mathcal{L}$ such that the image of each point $\rho(t) \in \mathcal{P}$ lies in the fiber $\Pi(\rho)$ over ρ, i.e., $\Pi os = id_p$, for details see Ref. [63]. The "reference section" defined with respect to the initial point is a mapping of the state curve Γ_0 through the section s and is given by [4,5]

$$|\chi(t)\rangle = \frac{\langle\psi(t)|\psi(0)\rangle}{|\langle\psi(t)|\psi(0)\rangle|}|\psi(t)\rangle. \tag{2}$$

It has the following properties: (i) $s\Pi(|\psi(0)\rangle) = |\chi(0)\rangle = |\psi(0)\rangle$; (ii) $\Pi(|\psi(0)\rangle) = \Pi(|\chi(0)\rangle)$; and (iii) $\langle\chi(0)|\chi(t)\rangle$ is always real and positive, i.e., $|\chi(0)\rangle$ and $|\chi(t)\rangle$ remain in phase throughout the quantum evolution. Also, it insures that the length of the curve traced by $|\chi(t)\rangle$ is invariant under $U(1)$ gauge transformation. The "reference section" defined above plays an important role in the theory of geometric phases. Using this we can prove that the geometric phase acquired by a quantum system for an arbitrary noncyclic evolution is given by the integral over the connection-form, i.e., $\Phi_G = i\int\langle\chi|d\chi\rangle$ [4,5].

Now, we will need two geometric structures to prove the reverse quantum speed limit. Consider two curves $\Gamma_0 : [0, t] \to \mathcal{L}$ and $\bar{\Gamma} : [0, t] \to \mathcal{L}$ as traced out by the "reference section" $|\chi(t)\rangle$ and the horizontal curve $|\bar{\psi}(t)\rangle$, respectively. To define the later, let us consider the evolution of a quantum system as described by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

where $H(t)$ is the driving Hamiltonian of the system. In general, when the system evolves in time from $t = 0$ to $t = T$, it will acquire a dynamical phase $\Phi_D = -\frac{1}{\hbar}\int_0^T i\langle\psi(t)|H(t)|\psi(t)\rangle\,dt$ and a geometric phase $\Phi_G = \Phi - \Phi_D = i\int\langle\chi|d\chi\rangle$, where $\Phi = \mathrm{Arg}\langle\psi(0)|\psi(T)\rangle$ is the total phase. However, if the system undergoes a parallel transport then it will acquire only the geometric phase. The significance of the parallel-transported vector is that locally it does not undergo any rotation, but globally it picks up a phase which is geometric in nature. The parallel-transported vector is given by $|\bar{\psi}(t)\rangle = \exp[i/\hbar\int_0^t\langle\psi(t')|H(t')|\psi(t')\rangle dt']|\psi(t)\rangle$ and this is also called the horizontal vector. We can check that it satisfies the parallel transport condition $\langle\bar{\psi}(t)|\dot{\bar{\psi}}(t)\rangle = 0$, i.e., the tangent vector is orthogonal to the vector itself at any point in time.

Now, we need to define length of the curves for these two vectors. The inner product in $\mathcal{H}$ induces a metric in $\mathcal{P}$ and the presence of metric allows the definition of the length of a differentiable curve in $\mathcal{L}$.

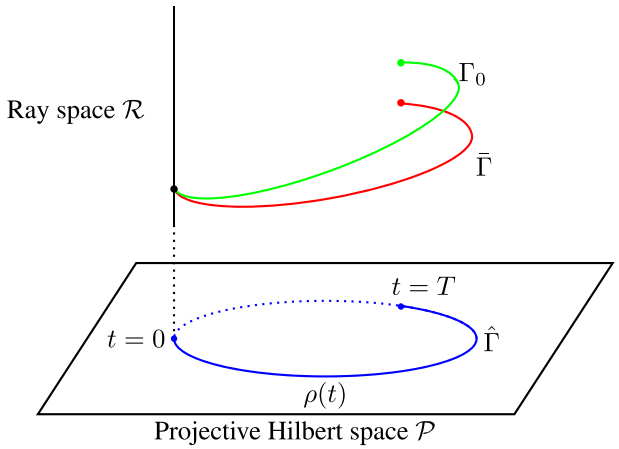

FIG. 2. In this schematic diagram the blue line represents the curve $\hat{\Gamma}$ in $\mathcal{P}$ which is a projection of $|\psi(t)\rangle$ (as described in Fig. 1). The reference section curve Γ_0 and horizontal curve $\bar{\Gamma}$ are represented by green and red lines, respectively. Γ_0 and $\bar{\Gamma}$ traced by unit vectors $|\chi(t)\rangle$ and $|\bar{\psi}(t)\rangle$. These two curves do not depend on the actual curve Γ traced by $|\psi(t)\rangle$ and they are gauge-invariant.

1. Definition (Length of reference section)

Let $t \to |\chi(t)\rangle$ be a curve $\Gamma_0(t)$ during an arbitrary evolution of a quantum system. The total length of the differentiable curve Γ_0 from a point $|\chi(0)\rangle$ to a point $|\chi(t)\rangle$ is a real number defined as

$$l(\chi(t))|_0^T = \int_0^T \langle\dot{\chi}(t)|\dot{\chi}(t)\rangle^{\frac{1}{2}} dt, = \int_{\Gamma_0} ||d\chi||, \tag{3}$$

where $|\dot{\chi}(t)\rangle$ is the velocity vector in $\mathcal{L}$ of the curve Γ_0 at time t along the path of evolution (relative to the initial point $|\chi(0)\rangle$). See Fig. 2 for various length of the curves discussed in the paper.

2. Definition (Length of horizontal curve)

Let $t \to |\bar{\psi}(t)\rangle$ be a curve $\bar{\Gamma}(t)$ during an arbitrary evolution of a quantum system. The total length of the differentiable curve $\bar{\Gamma}(t)$ from a point $|\bar{\psi}(0)\rangle$ to a point $|\bar{\psi}(t)\rangle$ is a real number defined as

$$l(\bar{\psi}(t))|_0^T = \int_0^T \langle\dot{\bar{\psi}}(t)|\dot{\bar{\psi}}(t)\rangle^{\frac{1}{2}} dt, = \int_{\bar{\Gamma}} ||d\bar{\psi}||, \tag{4}$$

where $|\dot{\bar{\psi}}(t)\rangle$ is the velocity vector in $\mathcal{L}$ of the curve Γ at time t along the path of the evolution of the horizontal curve.

The length of the reference curve and the length of the horizontal curves are two fundamental geometric structures associated with any quantum evolution. Some properties of these lengths of the curves are in the follwoing order. First, we note that the integrals in Eqs. (3) and (4) exist in the interval $[0, T]$ since the integrand is continuous and the resulting integrals yield real numbers. These two lengths respect an important property of reparametrization invariance, i.e., all the curves deduced from Γ_0 and $\bar{\Gamma}$ by a change of parameter t to t' with $\frac{dt}{dt'} > 0$, the length of these curves remain unaltered. Furthermore, they are also gauge-invariant, i.e., when $|\psi(t)\rangle \to e^{i\alpha(t)}|\psi(t)\rangle$, then $l(\chi(t))$ and $l(\bar{\psi}(t))$ remain the same. This was proved in Ref. [5]. Thus, they qualify to be called as geometric structures as these lengths are also independent of the particular Hamiltonian used to evolve the quantum system. There may be an infinite number of Hamiltonians which can give rise to the same $l(\chi(t))$ and $l(\bar{\psi}(t))$. The length of the horizontal curve is actually (up to a factor of 2) the total distance traveled by the quantum state as measured by the Fubini-Study metric.

To see this, consider the Bargmann angle, which measures the distance between two arbitrary pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by

$$\tfrac{1}{2}S_o(|\psi_1\rangle, |\psi_2\rangle) = \cos^{-1}(|\langle\psi_1|\psi_2\rangle|). \tag{5}$$

If two pure states $|\psi(t)\rangle$ and $|\psi(t+dt)\rangle$ are separated by an infinitesimal distance, then we have the infinitesimal Fubini-Study metric on $\mathcal{P}$, which is defined as [2]

$$dS^2 = 4(1 - |\langle\psi(t)|\psi(t+dt)\rangle|^2). \tag{6}$$

Let $|\psi(t)\rangle$ be a state of the system that evolves according to the Schrödinger equation. The distance between $|\psi(0)\rangle$ and $|\psi(T)\rangle$ along the evolution curve is determined by integrating the Fubini-Study metric [2], which is given by

$$S = \frac{2}{\hbar}\int_0^T \Delta H(t)\, dt, \tag{7}$$

where $\Delta H(t)^2 = \langle\psi(t)|H(t)^2|\psi(t)\rangle - \langle\psi(t)|H(t)|\psi(t)\rangle^2$ is the energy fluctuation during the quantum evolution. Note that the length of the horizontal curve for the Schrödinger evolution is given by $l(\bar{\psi}(t))|_0^T = \int_0^T \frac{\Delta H(t)}{\hbar} dt$, i.e., $S = 2l(\bar{\psi}(t))|_0^T$. The standard quantum speed limit (QSL) follows from the fact that the total distance traveled by the quantum system as measured by the Fubini-Study metric is always greater than or equal to the shortest distance connecting the initial and the final points, i.e., $S \geqslant S_0$. Similarly, here we will show how the geometry of the quantum state space dictates that there is indeed a reverse quantum speed limit.

One fundamental result in the geometry of the quantum evolution is that the length of the reference section is greater than the length of the horizontal curve. In fact, we can prove that $dl(\chi(t))^2 \geqslant dl(\bar{\psi}(t))^2$ and hence the gauge-invariant length $l(\chi(t))|_0^T$ is always greater than the length of the horizontal curve $l(\bar{\psi}(t))|_0^T$.

We can write Eq. (2) as $|\chi(t)\rangle = \xi(t)|\psi(t)\rangle$, where $\xi(t) = \frac{\langle\psi(0)|\psi(t)\rangle}{|\langle\psi(0)|\psi(t)\rangle|}$. Then Eq. (3) can be expressed as

$$||d\chi||^2 = ||d\xi||^2 + 2d\xi^*\xi\langle\psi|d\psi\rangle + ||d\psi||^2.$$

On using Eq. (4) and the following expressions:

$$||d\xi||^2 = [i\langle\chi|d\chi\rangle - i\langle\psi|d\psi\rangle]^2,$$
$$d\xi^*\xi = [\langle\psi|d\psi\rangle - \langle\chi|d\chi\rangle],$$

we obtain

$$||d\chi||^2 - ||d\bar{\psi}||^2 = [i\langle\chi|d\chi\rangle]^2.$$

Since $i\langle\chi|d\chi\rangle$ is real, we have

$$||d\chi||^2 \geqslant ||d\bar{\psi}||^2,$$
$$\text{i.e.,}\quad l(\chi(t))|_0^T \geqslant l(\bar{\psi}(t))|_0^T.$$

The difference between the length and the distance plays a significant role, which is essentially the connection form that gives rise to the geometric phase for the arbitrary quantum evolution [4,5]. Viewed differently, the existence of the intrinsic curvature in the quantum state space gives rise to the inequality, i.e., $l(\chi(t))|_0^T \geqslant l(\bar{\psi}(t))|_0^T$.

If the Hamiltonian is time-independent, then the above conditions provides a nontrivial bound for the reverse speed limit, which can be expressed as an inequality

$$T \leqslant \frac{\hbar l(\chi(t))|_0^T}{\Delta H}. \tag{8}$$

This is the fundamental reverse quantum speed limit (RQSL). If the speed of transportation of the state vector is slow and if the total length of the "reference-section" curve is more, then the system will evolve more slowly.

For the time-dependent Hamiltonian, we can obtain the reverse speed limit for the quantum system as given by

$$T \leqslant \frac{\hbar l(\chi(t))|_0^T}{\overline{\Delta H}}, \tag{9}$$

where $\overline{\Delta H}$ is the time average of the fluctuation over the time for which evolution occurs, i.e., $\overline{\Delta H} = \frac{1}{T}\int_0^T \Delta H(t)dt$. Thus, Eq. (9) provides the reverse speed limit bound, i.e., the upper bound of the speed limit for time as given by

$$T_{\text{RQSL}} = \frac{\hbar l(\chi(t))|_0^T}{\Delta H}. \tag{10}$$

Since $l(\chi(t))|_0^T \geqslant l(\bar{\psi}(t))|_0^T \geqslant \frac{1}{2}S_o[|\psi(0)\rangle, |\psi(T)\rangle]$, we can write the following inequality:

$$T_{\text{RQSL}} \geqslant T \geqslant T_{\text{QSL}}. \tag{11}$$

Thus, the geometric structures of the quantum evolution imposes fundamental bound on the evolution time as it is upper bounded by reverse speed limit time. Equation (12) suggests that, in quantum mechanics, evolution time is both upper and lower bounded. It is worth stressing that the standard quantum speed limit as well as the reverse quantum speed limit obtained here, both owe their existence to the geometry of quantum state space. Equations (9) and (10) constitute the central result of our paper.

B. For mixed initial state

The reverse speed limit can be generalized for mixed initial states undergoing unitary time evolution. Any mixed state of a quantum system can be viewed as a reduced state of an enlarged pure entangled state. We can purify the mixed state ρ_S in $\mathcal{H}_S$ of a system (S) by attaching an ancillary system (A). Hence the state of the enlarged system $(S+A)$ described by the pure state $|\Psi\rangle_{SA} \in \mathcal{H}_S \otimes \mathcal{H}_A$ and by tracing out A, we retrieve the mixed state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ of S. The purified state is given by

$$|\Psi\rangle_{SA} = \sum_k \sqrt{p_k}|k\rangle_S|k\rangle_A, \tag{12}$$

where $\{|k\rangle_S\}$ and $\{|k\rangle_A\}$ are the basis of a system Hilbert space $\mathcal{H}_S$ and ancillary system Hilbert space $\mathcal{H}_A$, respectively.

If $|\Psi(0)\rangle_{SA}$ is the initial state of joint system at time $t=0$ and it is transformed to $|\Psi(t)\rangle_{SA}$ by a local unitary operator $U_{SA}(t) = U_S(t)\otimes I_A$ [45,45,64] then we have

$$|\Psi(t)\rangle_{SA} = \sum_k \sqrt{p_k}U_S(t)|k\rangle_S|k\rangle_A, \tag{13}$$

where $U_S(t) = e^{\frac{-it}{\hbar}H_S}$ and H_S is the Hamiltonian of the system. This unitary evolution is equivalent to $\rho_S(0) \to \rho_S(t) = U_S(t)\rho_S(0)U_S(t)^\dagger$ as during the evolution at each time $t \in [0,T]$, $|\Psi(t)\rangle_{SA}$ satisfies the condition

$$\text{Tr}_A[|\Psi(t)\rangle_{SA}\langle\Psi(t)|] = \rho_S(t) = U_S(t)\rho(0)U_S(t)^\dagger, \tag{14}$$

where $\rho_S(0)$ is the initial state of the system and $\text{Tr}_A[|\Psi(0)\rangle_{SA}\langle\Psi(0)|] = \rho_S(0)$.

Since, the purification is not unique, any state $|\Psi(t)\rangle_{SA} = \sum_k \sqrt{p_k}U_S(t)|k\rangle_S V|k\rangle_A$, where $V \in \mathcal{H}_A$ denotes a unitary operator in $\mathcal{H}_A$, is also a valid purification. Geometrically, this can be thought of as a right action of $V(\mathcal{H}_A)$ on $\mathcal{H}_S\otimes\mathcal{H}_A$ along the fibers of the projection from $\mathcal{H}_S\otimes\mathcal{H}_A$ to the space of the density operators. This projection is uniquely characterized by the equality $\text{Tr}(\rho_S O_S) = \langle\Psi(t)|(O_S\otimes I_A)|\Psi(t)\rangle_{SA}$ satisfied by every operator O. This can be regarded as a principal fiber bundle $\mathcal{H}_S\otimes\mathcal{H}_A$ over mixed states with a structure group $V(\mathcal{H}_A)$ and a well-defined principal connection for mixed states [65,66].

In the purified Hilbert space $\mathcal{H}_S\otimes\mathcal{H}_A$, the entangled "reference section" with respect to the initial state is defined as [64,67]

$$|\chi(t)\rangle_{SA} = \frac{\text{Tr}[U_{SA}(t)\rho_{SA}(0)]}{|\text{Tr}[U_{SA}(t)\rho_{SA}(0)]|}|\Psi(t)\rangle_{SA}, \tag{15}$$

where $\rho_{SA}(0) = |\Psi(0)\rangle_{SA}\langle\Psi(0)|$ and $U_{SA}(t) = e^{\frac{-it}{\hbar}H_S}\otimes I_A$.

Now, if we carry over the notion of the reference section and horizontal curve to the joint system. The length of the reference section can be defined as

$$l(\chi(t)_{SA})|_0^T = \int_0^T \langle\dot{\chi}(t)|\dot{\chi}(t)\rangle_{SA}^{\frac{1}{2}}dt, \tag{16}$$

where $|\dot{\chi}(t)\rangle_{SA}$ is the velocity vector of the curve Γ_0 at time t along the path of evolution the path of evolution (relative to the initial point $|\chi(0)\rangle_{SA}$) in $\mathcal{H}_S\otimes\mathcal{H}_A$.

The length of the horizontal curve can be defined as

$$l(\bar{\Psi}(t)_{SA})|_0^T = \int_0^T \langle\dot{\bar{\Psi}}(t)|\dot{\bar{\Psi}}(t)\rangle_{SA}^{\frac{1}{2}}dt, \tag{17}$$

where $|\dot{\bar{\Psi}}(t)\rangle_{SA}$ is the velocity vector of the curve $\bar{\Gamma}$ at time t along the path of evolution of the horizontal curve in $\mathcal{H}_S\otimes\mathcal{H}_A$.

During the evolution of the mixed state, we will have $l(\chi(t)_{SA})|_0^T \geqslant l(\bar{\Psi}(t)_{SA})|_0^T$. Now, realizing the fact that $l(\bar{\Psi}(t)_{SA})|_0^T = \text{Tr}[\rho_S(t)H_S^2] - \text{Tr}(\rho_S(t)H_S)^2 = \Delta H_S^2$, we have the fundamental upper bound of quantum speed limit time for mixed states which can be expressed as

$$T \leqslant \frac{\hbar l(\chi(t)_{SA})|_0^T}{\Delta H_S}. \tag{18}$$

Indeed, we can check that the reverse quantum speed limit for mixed states given in Eq. (19) reduces to the pure-state case given in Eq. (9), if the system is initially prepared in a pure state and undergoes a unitary time evolution.

III. RQSL FOR THE TWO-LEVEL QUANTUM SYSTEMS

In the dynamics of a two-level quantum system, we can easily examine the reverse speed limit bounds. The simplest models of two-level systems are spin-half particles in a time-dependent external field and the Jaynes-Cummings model. In the sequel, we will illustrate the reverse speed limit bounds for these two quantum systems.

A. Spin in time-dependent external field

Consider the atom (which has two energy levels) in the time-dependent external field whose Hamiltonian is given as

$$H = H_{\text{atom}} + H_{\text{field}}(t) = J_1\sigma_z + J_2(t)\sigma_x,$$

where $J_2(t)$ is defined as $J_2(t) = 0$ for $t = 0$ and J_2 for $t > 0$. Here, $|0\rangle$ and $|1\rangle$ are the excited and ground states of the atom with eigenvalues $\pm J_1$. If the initial state of a system is $|0\rangle$, then the time evolution of the state of the system at an arbitrary time is given by

$$|\psi(t)\rangle = \left[\cos\left(\frac{\Omega}{\hbar}t\right) - \frac{iJ_1}{\Omega}\sin\left(\frac{\Omega}{\hbar}t\right)\right]|0\rangle - \frac{iJ_2}{\Omega}\sin\left(\frac{\Omega}{\hbar}t\right)|1\rangle,$$

where $\Omega = \sqrt{{J_1}^2 + {J_2}^2}$. The fluctuation in the energy of the system during quantum evolution is given by $\Delta H = J_2$. In the time interval $[0, T = \frac{\pi\hbar}{2\Omega}]$, the system reaches to the target state $\frac{J_1|0\rangle + J_2|1\rangle}{\sqrt{J_1^2+J_2^2}}$ (for simplicity, lets assume $J_1 = J_2 = \hbar$, it implies $T = \frac{\pi}{2\sqrt{2}}$). To obtain the speed limit bounds, we need to calculate $\frac{S_0}{2}$, $l(\bar{\psi}(t))|_0^T$ and $l(\chi(t))|_0^T$. We find that $S_o[|\psi(0)\rangle, |\psi(T)\rangle] = \pi$, and $l(\bar{\psi}(t))|_0^T = \frac{J_2}{\hbar}\frac{\pi}{2\sqrt{2}} = 1.1107$. For the calculation of the length of the curve "$l(\chi(t))|_0^T$," first we need $|\chi(t)\rangle$. This is given by

$$|\chi(t)\rangle = \frac{\cos(\frac{\Omega}{\hbar}t) + \frac{iJ_1}{\Omega}\sin(\frac{\Omega}{\hbar}t)}{\sqrt{\cos^2(\frac{\Omega}{\hbar}t) + (\frac{J_1}{\Omega})^2\sin^2(\frac{\Omega}{\hbar}t)}}\left\{\left[\cos\left(\frac{\Omega}{\hbar}t\right) - \frac{iJ_1}{\Omega}\sin\left(\frac{\Omega}{\hbar}t\right)\right]|0\rangle - \frac{iJ_2}{\Omega}\sin\left(\frac{\Omega}{\hbar}t\right)|1\rangle\right\}.$$

Using Eq. (3), we find the length of the reference curve as given by

$$l(\chi(t))|_0^T = \frac{1}{\hbar}\int_0^T \sqrt{\Omega^2 + J_1^2\frac{1 - 2b^2 - 2(1-b^2)\cos^2(at)}{[\cos^2(at) + b^2\sin^2(at)]^2}}\,dt,$$

where $a = \frac{\Omega}{\hbar}$ and $b = \frac{J_1}{\Omega}$. We can simplify the above expression further by substituting the values of J_1 and J_2. This leads to

$$l(\chi(t))|_0^T = \int_0^{\frac{\pi}{2\sqrt{2}}} \sqrt{2 - 4\frac{\cos^2[\sqrt{2}t]}{[\cos^2(\sqrt{2}t) + 1]^2}}\,dt.$$

The value of above integral is $l(\chi(t))|_0^T = 1.2526$ and $l(\chi(t))|_0^T > l(\bar{\psi}(t))|_0^T > \frac{S_0}{2}$ indeed holds. Using these values of $l(\chi(t))|_0^T$ and $\frac{S_0}{2}$, the reverse speed limit and the standard speed limit are given by

$$T_{\text{RQSL}} = \frac{\hbar l(\chi(t))|_0^T}{\Delta H} = 1.2526,$$
$$T_{\text{QSL}} = \frac{\hbar S_0(T)}{2\Delta H} = 0.7853.$$

Thus, we obtained the desired upper and lower speed limit bounds on the evolution time of a given system, which completely agree with Eq. (12).

As of now we obtained the speed limit bounds for pure initial state. Let us consider the case, when the initial state of the system is a mixed state, i.e., $\rho_S = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ with $0 \leqslant p \leqslant 1$. After the purification, the state of the system and the ancillary is given by

$$|\psi(0)\rangle_{SA} = \sqrt{p}|0\rangle_S|a_0\rangle_A + \sqrt{1-p}|1\rangle_S|a_1\rangle_A,$$

where $|a_0\rangle$ and $|a_1\rangle$ are the orthonormal bases of the ancillary system. The time evolution of the joint system at a later time t is given by

$$|\Psi(t)\rangle_{SA} = U_{SA}(t)|\Psi(0)\rangle_{SA},$$

where $U_{SA} = e^{\frac{-it}{\hbar}[J_1\sigma_z + J_2(t)\sigma_x]} \otimes I_A$. The joint state at time t is given by

$$\begin{aligned}|\Psi(t)\rangle_{SA} = \sqrt{p}\Bigg\{&\left[\cos\left(\frac{\Omega t}{\hbar}\right) - \frac{iJ_1}{\Omega}\sin\left(\frac{\Omega t}{\hbar}\right)\right]|0\rangle \\ &- \frac{iJ_2}{\Omega}\sin\left(\frac{\Omega t}{\hbar}\right)|1\rangle\Bigg\}|a_0\rangle \\ &+ \sqrt{1-p}\Bigg[-\frac{iJ_2}{\Omega}\sin\left(\frac{\Omega t}{\hbar}\right)|0\rangle \\ &+ \left[\cos\left(\frac{\Omega t}{\hbar}\right) + \frac{iJ_1}{\Omega}\sin\left(\frac{\Omega t}{\hbar}\right)\right]|1\rangle\Bigg\}|a_1\rangle,\end{aligned}$$

where $\Omega = \sqrt{{J_1}^2 + {J_2}^2}$. The fluctuation of the Hamiltonian for the system is given by $\Delta H_S = \sqrt{J_1^2 + 4p(1-p)J_2^2}$.

Now, the reference section $|\chi(t)\rangle_{SA}$ can be expressed as

$$|\chi(t)\rangle_{SA} = \frac{\cos(\frac{\Omega}{\hbar}t) + \frac{iJ_1(2p-1)}{\Omega}\sin(\frac{\Omega}{\hbar}t)}{\sqrt{\cos^2(\frac{\Omega}{\hbar}t) + (\frac{J_1(2p-1)}{\Omega})^2\sin^2(\frac{\Omega}{\hbar}t)}}|\Psi(t)\rangle_{SA}.$$

In time interval $[0, T = \frac{\pi\hbar}{2\Omega}]$, the system reaches to the target state $|\Psi(T)\rangle_{SA}$ (for simplicity, let us assume that $J_1 = J_2 = \hbar$ and this implies $T = \frac{\pi}{2\sqrt{2}}$) with

$$|\Psi(T)\rangle_{SA} = \sqrt{p}|+\rangle|a_0\rangle + \sqrt{1-p}|-\rangle|a_1\rangle,$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The state of the system at time T is given by

$$\rho_S(T) = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \left(p - \tfrac{1}{2}\right)(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

To obtain the reverse speed limit bound, first we need to calculate the length of the curve. For the purpose of illustration, we assume $p = \frac{1}{3}$ and the length is given by

$$l(\chi(t)_{SA})|_0^T = \int_0^{\frac{\pi}{2\sqrt{2}}} \sqrt{2 - \frac{8[1 + 17\cos(2\sqrt{2}t)]}{[19 + 17\cos(2\sqrt{2}t)]^2}}\,dt.$$

The value of the above integral is $l(\chi(t)_{SA})|_0^T = 2.2458$. Therefore, the reverse quantum speed limit bound is given by $T_{\mathrm{RQSL}} = \frac{\hbar l(\chi(t)_{SA})|_0^T}{\Delta H_S} = 1.6341$. We also find that the quantum speed limit and reverse speed limit respect the bound. Thus, the desired upper and lower speed limit bounds on the evolution time for the mixed initial state completely agree with our new bound.

B. Jaynes-Cummings model

One may ask how tight is the reverse quantum speed limit? Is there any physical system for which RQSL saturates the bound? We will show that the Jaynes-Cummings (JC) model [68] which describes the interaction of a two-level atom with a single quantized mode of an optical cavity's electromagnetic field indeed saturates the reverse quantum speed limit. The Hamiltonian of the JC model with rotating wave approximation can be expressed as

$$\begin{aligned} H &= H_{\mathrm{atom}} + H_{\mathrm{field}} + H_{\mathrm{int}}(t) \\ &= \frac{\hbar\omega}{2}\sigma_z + \hbar\omega a^\dagger a + \lambda(t)(\sigma_+ a + \sigma_- a^\dagger), \end{aligned}$$

where $\lambda(t)$ is defined as $\lambda(t) = 0$ at $t = 0$ and λ for $t > 0$. Here, $|e\rangle$ and $|g\rangle$ are the excited and ground states of the atom with eigenvalues $\frac{\hbar\omega}{2}$ and $-\frac{\hbar\omega}{2}$, respectively. The cavity has "$n+1$" number of photons. If the initial state of the total system $|g\rangle|n+1\rangle$, then the time evolution of the total system at arbitrary time t is given as

$$|\psi(t)\rangle = \cos(\lambda t\sqrt{n+1})|g\rangle|n+1\rangle - i\sin(\lambda t\sqrt{n+1})|e\rangle|n\rangle,$$

where $|n\rangle$ and $|n+1\rangle$ are states of the field. The energy fluctuation of the system during evolution is given by $\Delta H = \lambda\hbar\sqrt{n+1}$. Since initially the atom in the ground state $|g\rangle$, then in time interval $[0, T = \frac{\pi}{2\lambda\sqrt{n+1}}]$, it evolves to target state $|e\rangle$. To obtain speed limit bounds for evolution, let us evaluate $\frac{S_0}{2}$, $l(\bar{\psi}(t))|_0^T$ and $l(\chi(t))|_0^T$. One can check that the geodesic distance $S_0 = \pi$. The total distance as measured by the horizontal curve during the time evolution is given by $l(\bar{\psi}(t))|_0^T = \frac{\pi}{2}$.

In fact, the system undergoes parallel transport during the quantum evolution, i.e., it satisfies the condition $\langle\psi(t)|\dot{\psi}(t)\rangle = 0$. In this case, we find that $|\chi(t)\rangle = |\psi(t)\rangle = |\bar{\psi}\rangle$ and hence the length, distance, and the geodesic distance all are equal during the evolution of the quantum system. In this scenario, the quantum speed limit bounds on evolution time saturates, i.e., it completely satisfies Eq. (12) as well, i.e., $T_{\mathrm{RQSL}} = T = T_{\mathrm{QSL}}$.

IV. DISCHARGING OF QUANTUM BATTERIES

Future technology aims to develop strategies to store energy which can be later consumed by quantum devices. This motivates to design quantum batteries by using quantum mechanical features. The simplest model of quantum batteries consist of an array of N two-level quantum systems [49,50,59–61,69]. However, their charging and discharging procedures differ because they use a different external field to charge and discharge quantum batteries. Once a quantum battery is successfully charged we need to decouple it from the external field. In this connection, it has been shown that how fast a quantum battery can be charged is governed by the quantum speed limit for the evolution of the system. In this section, we will show that the reverse quantum speed limit provides a fundamental lower bound on the average discharging power like the quantum speed limit provides the upper bound on the average charging or discharging power.

A. Charging and discharging protocol

To deposit work (charging) into an array of N atoms or extract work (discharging) from the array of the N atom, we apply a time-dependent external field $H(t)$ for period $[0, T]$ such that it must be a reversible cyclic operation [50]. The Hamiltonian during this process is given by

$$H_0 \to H_0 + H(t) \to H_0,$$

where H_0 is the Hamiltonian of the quantum battery. The norm of the total Hamiltonian must be less than $E_{\max}$ [61], i.e.,

$$||H_0 + H(t)|| \leqslant E_{\max},$$

where $E_{\max}$ is the maximum energy eigenvalue difference or gap of H_0. There are two ways to charge and discharge, the first one is charging and discharging each atom individually known as the parallel protocol and the second one is to charge and discharge all the atoms together known as the collective protocol [50,60,61]. Although, the charging and discharging process are similar, we are only interested in the discharging process. Because, while charging, we always want to charge it fast but that is not the case during the discharging process. We always want to extract energy according to our necessity. Then, the natural question is how slowly can a quantum battery discharge? Of course we have to remember that we cannot extract more than the stored energy. Here, we will show that the fundamental reverse speed limit can answer the question of how long it takes to discharge a quantum battery.

B. Average discharging power

The average discharging power is defined as

$$\begin{aligned} \bar{P} &= \frac{1}{T}\int_0^T \frac{dW(t)}{dt}dt, \\ &= \frac{W(T)}{T}, \end{aligned} \tag{19}$$

where $W(T)$ is the ergotropy and T is the time required to discharge the QB, which is lower bounded by the QSL time and upper bounded by the RQSL time. The ergotropy is the amount of energy deposited in quantum systems or extractable from the quantum system [70] and defined as

$$W(T) = \langle\psi(T)|H_0|\psi(T)\rangle - \langle\psi(0)|H_0|\psi(0)\rangle, \tag{20}$$

where H_0 is the Hamiltonian of QB, $|\psi(0)\rangle$ and $|\psi(T)\rangle$ are the initial and final states of the QB, respectively. This is the maximum work that can be extracted unitarily from a given quantum state with respect to the Hamiltonian H_0.

The reverse speed limit provides a nontrivial lower bound to the average discharging power, i.e., we have

$$\bar{P} \geqslant \frac{W(T)\Delta H}{\hbar l(\chi(t))|_0^T}, \tag{21}$$

where $l(\chi(t))|_0^T$ is the length of the reference section and ΔH is the energy fluctuation during the quantum evolution.

We will illustrate the average discharging power of the quantum battery for two cases.

C. Harmonic and square-wave discharging

Consider N-independent batteries consisting of N two-level atoms which we can discharge though the classical harmonic field [69,71]. The total Hamiltonian of the discharging process is described as

$$H = \frac{\varepsilon}{2}\sum_{i=1}^{N}\sigma_i^z + \frac{A}{2}\cos(\omega t)\sum_{i=1}^{N}\sigma_i^x,$$

where the first term in the Hamiltonian denotes an array of N two-level atoms and the second term denotes the classical harmonic field. Here, $|e\rangle$ and $|g\rangle$ are the excited and ground states of a single-atom battery with eigenvalues $\frac{\varepsilon}{2}$ and $-\frac{\varepsilon}{2}$, respectively.

The effective Hamiltonian in the rotating wave approximation can be written as [69,71]

$$\bar{H} = \frac{\bar{\varepsilon}}{2}\sum_{i=1}^{N}\sigma_i^z + \bar{A}\sum_{i=1}^{N}\sigma_i^x,$$

where $\bar{A} = \frac{A}{2}(1-\frac{\zeta}{\sqrt{N}})$ and $\bar{\varepsilon} = \varepsilon J_0(\frac{A}{\omega\sqrt{N}}\zeta) - \omega$ (effective detuning). Here $\zeta \in [0,1]$ is an undetermined parameter and $J_0(\frac{A}{\omega\sqrt{N}}\zeta)$ denotes the Bessel function of order zero [69,71].

The above Hamiltonian is similar to N batteries coupled with square wave charger or discharger. In the discharging process of N batteries, individual atoms discharge independently (parallel discharging). If the initial state of a single quantum battery is $|e\rangle$, then during the discharging process the time evolution of the wave function of the single-atom battery system at arbitrary time t is given by

$$|\psi(t)\rangle = [\cos(\frac{\Omega_R}{2\hbar}t) - \frac{i\bar{\varepsilon}}{\Omega_R}\sin(\frac{\Omega_R}{2\hbar}t)]|e\rangle - \frac{i2\bar{A}}{\Omega_R}\sin(\frac{\Omega_R}{2\hbar}t)|g\rangle,$$

where $\Omega_R = \sqrt{\bar{\varepsilon}^2 + 4\bar{A}^2}$. The energy fluctuation of the system during the evolution is given by $\Delta H = \bar{A}$ and hence it evolves with a speed $\frac{2\bar{A}}{\hbar}$.

During the time interval $[0, \frac{\pi\hbar}{\Omega_R}]$ the initial state $|e\rangle$ evolves to target state $|\psi(T)\rangle$. To calculate maximum and minimum discharging time of the quantum battery, first we need to calculate $\frac{S_0}{2}$, $l(\bar{\psi}(t))|_0^T$ and $l(\chi(t))|_0^T$. The geodesic distance $\frac{S_0}{2}$ is given by

$$\frac{1}{2}S_0[|\psi(0)\rangle, |\psi(T)\rangle] = \cos^{-1}\left[|\cos\left(\frac{\Omega_R}{2\hbar}T\right) - \frac{i\bar{\varepsilon}}{\Omega_R}\sin\left(\frac{\Omega_R}{2\hbar}T\right)|\right].$$

The total length of the horizontal curve is given by $l(\bar{\psi}(t))|_0^T = \frac{\bar{A}}{\hbar}T$.

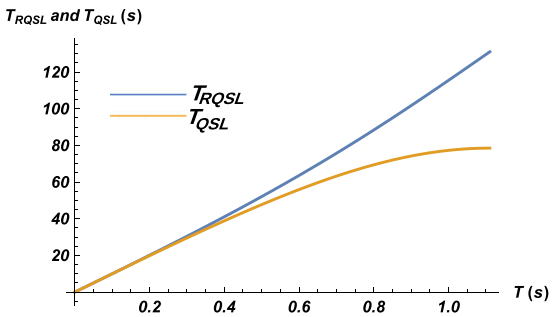

FIG. 3. Here, we depict T_{RQSL} and T_{QSL} versus T for $N = 100$ atom battery for harmonic and square wave discharging. Here we take ΔH in the unit of Joule-second; hence T_{RQSL}, T_{QSL}, and T are in seconds.

For the calculation of length $l(\chi(t))|_0^T$, first we need the reference section $|\chi(t)\rangle$ which can be expressed as

$$|\chi(t)\rangle = \frac{\cos(\frac{\Omega_R}{2\hbar}t) + \frac{i\bar{\varepsilon}}{\Omega_R}\sin(\frac{\Omega_R}{2\hbar}t)}{\sqrt{\cos^2(\frac{\Omega_R}{2\hbar}t) + (\frac{\bar{\varepsilon}}{\Omega_R})^2\sin^2(\frac{\Omega_R}{2\hbar}t)}}\left\{\left[\cos\left(\frac{\Omega_R}{2\hbar}t\right) - \frac{i\bar{\varepsilon}}{\Omega_R}\sin\left(\frac{\Omega_R}{2\hbar}t\right)\right]|e\rangle - \frac{i2\bar{A}}{\Omega_R}\sin\left(\frac{\Omega_R}{2\hbar}t\right)|g\rangle\right\}.$$

Now, the length of the reference section curve is given by

$$l(\chi(t))|_0^T = \frac{1}{\hbar}\int_0^T \sqrt{\frac{{\Omega_R}^2}{4} + \frac{\bar{\varepsilon}^2[1-2b^2-2(1-b^2)\cos^2(at)]}{4[\cos^2(at)+b^2\sin^2(at)]^2}}\,dt,$$

where $a = \frac{\Omega_R}{2\hbar}$ and $b = \frac{\bar{\varepsilon}}{\Omega_R}$.

In the parallel discharging protocol, QSL and RQSL of the discharging N atoms battery is N times of QSL and RQSL of single-atom battery, respectively. Thus, the reverse speed limit bound on the discharging time of N atoms is defined as

$$T_{\mathrm{RQSL}} = N\frac{\hbar l(\chi(t))|_0^T}{\Delta H}.$$

The above Fig. 3 shows that the upper and the lower bounds of the discharging time of 100 atoms ($N = 100$) quantum battery. Here, we assumed $\bar{A} = \hbar$ and $\bar{\varepsilon} = 2\hbar$. In the plot, range of T is 0 to $\frac{\pi}{2\sqrt{2}}$, each value T represents a different final state $|\psi(T)\rangle$. The total amount of the work extracted from the single-atom battery during interval $[0, \frac{\pi\hbar}{\Omega_R}]$ is [69]

$$W(T) = \frac{4\varepsilon\bar{A}}{\bar{\varepsilon}^2 + 4\bar{A}^2}.$$

Therefore, the upper and lower bounds on the average power of the quantum battery are given by

$$\frac{4\varepsilon\bar{A}^2}{(\bar{\varepsilon}^2+4\bar{A}^2)l(\chi(t))|_0^T} \leqslant \bar{P} \leqslant \frac{8\varepsilon\bar{A}^2}{(\bar{\varepsilon}^2+4\bar{A}^2)S_0}.$$

In time interval $[0, \frac{\pi\hbar}{2\bar{A}}]$, initial state $|e\rangle$ evolves to target state $|\psi(T)\rangle = |g\rangle$ (up to a phase), when we modulate ω such that $\bar{\varepsilon} = 0$ (tuned case). In this case $|\chi(t)\rangle = |\bar{\psi}(t)\rangle$, which implies that the length, distance, and geodesic distance all are equal. This means that the speed limit bounds saturates when $\bar{\varepsilon} = 0$. Thus, the average power bound also saturates.

Geometrically, for the tuned case, the system evolves along a shortest geodesic and obeys the parallel transport condition. However, for the detuned case, i.e., when the $\bar{\varepsilon} \neq 0$ system may not evolve along the geodesic and then quantum battery may take a longer time to discharge. This can be harnessed in the future to design efficient quantum batteries which will take more time to discharge. This shows that the geometry of the quantum state space also dictates the discharging power of the quantum battery.

D. Cavity-assisted discharging

Consider the model of the quantum battery [49,59,61,72], as an array of N two-level atoms inside the optical cavity. These atoms do not interact with each other. The total Hamiltonian with rotating wave approximation that describes the discharging process of the quantum battery is given by

$$H = \frac{\hbar\omega}{2}\sum_{j=1}^{N}\sigma_z^j + \hbar\omega a^\dagger a + \lambda(t)\sum_{j=1}^{N}(\sigma_+^j a + \sigma_-^j a^\dagger),$$

where the first term denotes the Hamiltonian of N atoms, the second term in the Hamiltonian denotes the single quantized mode of an optical cavity's electromagnetic field, and the third term denotes the interaction between the atoms and cavity [68]. The cavity has n number of photons, $\lambda(t)$ is a time-dependent coupling constant set to be λ during the charging or discharging period $[0, T]$ and 0 otherwise.

In the parallel discharging, we extract work form individual atoms independently using the external field. The time evolution of the single-atom battery system is given by

$$|\psi(t)\rangle = \cos(\lambda t\sqrt{n+1})|e\rangle|n\rangle - i\sin(\lambda t\sqrt{n+1})|g\rangle|n+1\rangle,$$

where $|e\rangle$ and $|g\rangle$ denotes the excited and ground states of the atom, respectively. $|n\rangle$ and $|n+1\rangle$ are states of the field. In the time interval $[0, \frac{\pi}{2\lambda\sqrt{n+1}}]$ the initial state of single atom $|e\rangle$ evolves to final state $|g\rangle$. In this model of the quantum battery, we find that the length of the reference curve, the length of the horizontal curve, and the geodesic distance all are equal for discharging or charging of this quantum battery, i.e., $l(\chi(t))|_0^T = l(\tilde{\psi}(t))|_0^T = \frac{S_0}{2}$. This suggests that the reverse quantum speed limit bound saturates for this quantum battery model, thereby suggesting that such a quantum battery takes a fixed amount of time to discharge. It also suggests that the lower and upper average power bounds also saturate during the charging and discharging processes.

V. CONCLUSION

In summary, we proved a reverse quantum speed limit for arbitrary unitary evolutions of pure as well as mixed states. The reverse speed limit arises due to the geometry of the quantum state space, i.e., the total length of the reference curve is always greater than the length of the horizontal curve. The difference between these two lengths gives rise to the notion of the curvature in the state space. Therefore, the reverse quantum speed limit owes its existence due to the intrinsic curvature on the projective Hilbert space of the quantum system. This is similar in spirit to the fact that the standard quantum speed limit also follows from the geometric consideration, i.e., the total distance traveled by the quantum state as measured by the Fubini-Study metric is always greater or equal to the shortest distance joining the initial and the final points on the projective Hilbert space of the quantum system.

Even though the QSL and the RQSL bounds are fundamentally of geometrical nature, they differ in some important ways. First, to compute QSL one needs to know much less about the path followed by the system than to compute RQSL. Second, evaluating QSL may be much easier than evaluating the actual time that the system takes to evolve from the initial to the final state, while evaluating the RQSL bound may be difficult in physical situations. The QSL depends only on the energy uncertainty and on the initial and final states, while the RQSL depends on the energy fluctuation and details of the path length followed by the reference state of the quantum system. Nevertheless, the RQSL will play an important role similar to the QSL.

We also find physical systems for which the upper bound for the reverse quantum speed limit actually saturates. We successfully presented examples in support of our results. As an important application, we showed how our result for the reverse speed limit answers a pertinent question: how long does it take for a quantum battery to discharge? Our results show that the geometry of the quantum state space will play a key role in the future design of the quantum battery. We also showed that the reverse speed limit is tight by revealing the cases when the discharging and charging times saturate in two different proposed models of quantum batteries. In future, we can generalize the reverse speed limit for open quantum systems and apply it to the aging problem in the quantum battery, i.e., what is the upper bound for the lifetime of an open quantum battery? This will provide a route towards future extension and the usability of our results in the context of battery stabilization. We believe that the fundamental reverse speed limit will have a host of applications in quantum computing, quantum measurement, quantum control, quantum metrology, and a variety of other areas.

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