Geometrical interpretation of the photon position operator with commuting components

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It is shown that the photon position operator $\hat{\vec{X}}$ with commuting components can be written in the momentum representation as $\hat{\vec{X}} = i\hat{\vec{D}}$, where $\hat{\vec{D}}$ is a flat connection in the tangent bundle $T(\mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \geqslant 0\})$ over $\mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \geqslant 0\}$ equipped with the Cartesian structure. Moreover, $\hat{\vec{D}}$ is such that the tangent 2-planes orthogonal to the momentum are propagated parallel with respect to $\hat{\vec{D}}$ and also $\hat{\vec{D}}$ is an anti-Hermitian (i.e., $\hat{\vec{X}}$ is Hermitian) operator with respect to the scalar product $\langle \mathbf{\Psi} | \hat{H}^{-2s} | \mathbf{\Phi} \rangle$. The eigenfunctions $\mathbf{\Psi}_{\vec{X}}(\vec{x})$ of the position operator $\hat{\vec{X}}$ are found.

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I. INTRODUCTION

Newton and Wigner, in their distinguished paper on localized states of relativistic quantum particles [1], arrived at the result (among others) that for massless particles with the spin greater than or equal to 1 no localized states in the sense explained in the paper exist. The authors concluded the following: "This is an unsatisfactory, if not unexpected, feature of our work." Wightman [2] also found that the photon is not localizable. In contrast, using some slightly weaker requirements for localizability, one can show [3,4] that the photon is localizable. Pryce, in his pioneering work devoted to the mass center in relativistic field theory [5], found an operator which he considered as the photon position operator. It is the Hermitian operator, but unfortunately its components do not commute. Hawton introduced the photon position operator with commuting components [6]. She demonstrated that this operator differs from the Pryce operator only by one term, which turned out to be closely related to the Berry potential leading to the Berry phase whose appearance has been proved experimentally [7,8] and discussed in detail from group-theoretic and geometrical points of view by Białynicki-Birula and Białynicka-Birula in [9]. Hawton's position operator for the photon was then widely investigated in subsequent works [10–15]. Our work follows this path. In Sec. II we present a simple derivation of the Hawton position operator for the photon. We show that this operator in the momentum representation, in terms of differential geometry, can be interpreted as some covariant derivative (connection) multiplied by an imaginary unit i. This covariant derivative is defined in the Cartesian tangent bundle $T(M)$, where M is a dense open submanifold of the Cartesian momentum space $\mathbb{R}^3$; the respective curvature tensor is zero (the connection is flat), 2-planes orthogonal to the momentum are propagated parallel, and the covariant derivative is an anti-Hermitian operator with respect to a given scalar product. We are able to find the general forms of operators arising in this way and show that they are all unitarily equivalent. Then we find a phase-space image of the photon position operator in the sense of the Weyl-Wigner-Moyal formalism. In Sec. III we find the eigenfunctions $\widetilde{\mathbf{\Psi}}_{\vec{X}}(\vec{k})$ of the photon position operators in the momentum representation and we calculate the inverse Fourier transforms of those eigenfunctions, which give us the wave eigenfunctions $\mathbf{\Psi}_{\vec{X}}(\vec{x})$. It is demonstrated that $\mathbf{\Psi}_{\vec{X}}(\vec{x})$ is localized in a small neighborhood of $\vec{x} = \vec{X}$. This result is compatible with the interpretation of the photon wave function $\mathbf{\Psi}(\vec{x})$ given in [16–18]. Section IV is devoted to analysis of the Berry potential and the Berry phase related to Hawton's position operator. Although several results of our paper have been found previously by other authors, the geometrical interpretation given here provides a different perspective on the photon position operator.

II. RELATION BETWEEN HAWTON'S POSITION OPERATOR OF THE PHOTON AND A COVARIANT DERIVATIVE (CONNECTION)

In this section we show that the photon position operator introduced by Hawton [6] is closely related to some covariant derivative (connection) in the tangent bundle over the differential manifold $\mathbb{R}^3 \setminus \{(0,0,c) \in \mathbb{R}^3 : c \geqslant 0\}$ with the Cartesian structure. The construction presented here refers to a particular selection of 3-axis [half-line starting at (0,0,0)] excluded from $\mathbb{R}^3$, which allows us to obtain explicit expressions for the photon position operator and its eigenstates. However, as will be seen later, the different options for selecting the 3-axis are equivalent and therefore physically irrelevant. In our work we adopt quantum mechanics of the photon as developed

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by Białynicki-Birula [16,17] and Sipe [18] and reconstructed in [19]. Using the notation of [19], we conclude that the photon wave function in the Cartesian momentum coordinates is represented by the complex vector function (an arrow is put over three-component quantities, while boldface denotes matrix objects)

$$\widetilde{\mathbf{\Psi}}(\vec{k}) = \begin{pmatrix} \widetilde{\Psi}_1(\vec{k}) \\ \widetilde{\Psi}_2(\vec{k}) \\ \widetilde{\Psi}_3(\vec{k}) \end{pmatrix}, \quad \vec{k} = (k_1, k_2, k_3) \in \mathbb{R}^3 \tag{2.1}$$

perpendicular to the vector $\vec{k}$,

$$k_j \widetilde{\Psi}_j(\vec{k}) = 0 \tag{2.2}$$

(summation over j from 1 to 3).

Since the components of the metric tensor in momentum space are given by the Kronecker delta δ_{jl}, there is no difference between covariant and contravariant components in the Cartesian coordinates (k_1, k_2, k_3). We are looking for the photon position operator in the momentum representation. In standard quantum mechanics we have

$$\hat{\vec{x}} = i\vec{\nabla}, \quad \vec{\nabla} \equiv \left(\frac{\partial}{\partial k_1}, \frac{\partial}{\partial k_2}, \frac{\partial}{\partial k_3}\right) = (\partial_1, \partial_2, \partial_3). \tag{2.3}$$

However, we quickly note that with (2.2) satisfied,

$$k_j(\hat{x}_l \widetilde{\mathbf{\Psi}}(\vec{k}))_j = -i\widetilde{\Psi}_j(\vec{k})\partial_l k_j = -\widetilde{\Psi}_l(\vec{k}), \quad l = 1, 2, 3. \tag{2.4}$$

Therefore, $\hat{\vec{x}}\widetilde{\mathbf{\Psi}}(\vec{k})$ does not fulfill the condition (2.2) for $\widetilde{\mathbf{\Psi}} \neq 0$ although $\widetilde{\mathbf{\Psi}}(\vec{k})$ does. This means that the standard position operator (2.3) is definitely not the photon position operator and consequently a generalization of (2.3) is needed. We propose to assume that the position operator of the photon $\hat{\vec{X}}$ has the form

$$\hat{\vec{X}} = i\hat{\vec{D}} \Longrightarrow \hat{X}_l = i\hat{D}_l, \quad l = 1, 2, 3, \tag{2.5}$$

where $\hat{\vec{D}} = (\hat{D}_1, \hat{D}_2, \hat{D}_3)$ is an operator in the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$ determined by a suitable covariant derivative (the connection) in the tangent bundle $T(M)$ over $M \subset \mathbb{R}^3$, with M some dense open submanifold of $\mathbb{R}^3$ endowed with the Cartesian structure. This connection we also denote by $\hat{\vec{D}}$. Since we require the components $\hat{X}_1, \hat{X}_2, \hat{X}_3$ of $\hat{\vec{X}}$ to commute, we must assume that the operators $(\hat{D}_1, \hat{D}_2, \hat{D}_3)$ mutually commute

$$[\hat{D}_l, \hat{D}_m] = 0, \quad l, m = 1, 2, 3. \tag{2.6}$$

This last condition implies the vanishing of the curvature of the connection $\hat{\vec{D}}$,

$$R^j_{lmn} = 0, \quad j, l, m, n = 1, 2, 3. \tag{2.7}$$

It means that the connection $\hat{\vec{D}}$ is flat [20,21]. Then we expect that the operators $\hat{X}_l$, $l = 1, 2, 3$, acting on a photon wave function $\widetilde{\mathbf{\Psi}}(\vec{k})$ give also a photon wave function. This assumption implies that the connection $\hat{\vec{D}}$ has to fulfill the conditions

$$k_j(\hat{D}_l \widetilde{\mathbf{\Psi}}(\vec{k}))_j = 0, \quad l = 1, 2, 3, \tag{2.8}$$

for any section $\widetilde{\mathbf{\Psi}}(\vec{k})$ of the complexified tangent bundle $T_{\mathbb{C}}(M)$ satisfying the relation (2.2).

Finally, restriction imposed on the connection $\hat{\vec{D}}$ follows from the fact that the photon position operator $\hat{\vec{X}}$ should be a Hermitian operator with respect to the Białynicki-Birula scalar product [6,10,17,19]

$$\langle \widetilde{\mathbf{\Phi}} | \widetilde{\mathbf{\Psi}} \rangle_{\text{BB}} := \langle \widetilde{\mathbf{\Phi}} | \hat{H}^{-1} | \widetilde{\mathbf{\Psi}} \rangle = \int \frac{d^3k}{(2\pi)^3 |\vec{k}|} \widetilde{\mathbf{\Phi}}^\dagger(\vec{k}) \widetilde{\mathbf{\Psi}}(\vec{k}), \tag{2.9}$$

where $\hat{H}$ is the photon Hamiltonian operator, which in momentum representation reads

$$\hat{H} = c\hbar |\vec{k}|. \tag{2.10}$$

Consequently, $\hat{X}_l$ for $l = 1, 2, 3$ has to satisfy the relations

$$\left(\int \frac{d^3k}{(2\pi)^3 |\vec{k}|} \widetilde{\mathbf{\Phi}}^\dagger(\vec{k}) \hat{X}_l \widetilde{\mathbf{\Psi}}(\vec{k})\right)^* = \int \frac{d^3k}{(2\pi)^3 |\vec{k}|} \widetilde{\mathbf{\Psi}}^\dagger(\vec{k}) \hat{X}_l \widetilde{\mathbf{\Phi}}(\vec{k}),$$
$$l = 1, 2, 3, \tag{2.11}$$

for any photon wave functions $\widetilde{\mathbf{\Phi}}(\vec{k})$ and $\widetilde{\mathbf{\Psi}}(\vec{k})$. Hence, by (2.5), we assume that the covariant derivative $\hat{\vec{D}}$ is an anti-Hermitian operator with respect to the scalar product (2.9). So

$$\left(\int \frac{d^3k}{(2\pi)^3 |\vec{k}|} \widetilde{\mathbf{\Phi}}^\dagger(\vec{k}) \hat{D}_l \widetilde{\mathbf{\Psi}}(\vec{k})\right)^*$$
$$= -\int \frac{d^3k}{(2\pi)^3 |\vec{k}|} \widetilde{\mathbf{\Psi}}^\dagger(\vec{k}) \hat{D}_l \widetilde{\mathbf{\Phi}}(\vec{k}), \quad l = 1, 2, 3, \tag{2.12}$$

for sections $\widetilde{\mathbf{\Phi}}(\vec{k})$ and $\widetilde{\mathbf{\Psi}}(\vec{k})$ of the complexified tangent bundle $T_{\mathbb{C}}(M)$.

Now we are going to construct the covariant derivative (the connection) which satisfies the above conditions (2.6) [or equivalently (2.7)], (2.8), and (2.12). First we need to determine what the submanifold $M \subset \mathbb{R}^3$ is. We show that $M \neq \mathbb{R}^3$. To this end, we assume the opposite, that $M = \mathbb{R}^3$. The set of real tangent vectors at the point $(k_1, k_2, k_3) \neq (0, 0, 0)$ fulfilling the orthogonality condition (2.2) constitutes the 2-plane $\Pi^\perp(k_1, k_2, k_3)$ perpendicular to the vector $\vec{k} = (k_1, k_2, k_3)$. Thus we obtain a two-dimensional differential distribution

$$\mathcal{D}_2 : \mathbb{R}^3 \ni (k_1, k_2, k_3) \mapsto \Pi^\perp(k_1, k_2, k_3). \tag{2.13}$$

Then the condition (2.8) means that the planes $\Pi^\perp(k_1, k_2, k_3)$ are propagated parallel with respect to the connection $\hat{\vec{D}}$ and the condition (2.6) [or equivalently (2.7)] says, as has been pointed out above, that $\hat{\vec{D}}$ is flat. The integral manifolds of the distribution $\mathcal{D}_2$ are the 2-spheres $|\vec{k}| = \text{const}$. We can easily conclude that our assumptions about the connection $\hat{\vec{D}}$ imply, among other things, that for any 2-sphere $|\vec{k}| = \text{const} > 0$ and for any point (k_1, k_2, k_3) of this sphere and any nonzero vector tangent at (k_1, k_2, k_3) to the sphere we can propagate this vector parallel with respect to $\hat{\vec{D}}$ on the whole sphere, thus obtaining a nowhere vanishing tangent vector field on the 2-sphere. As is well known, this is impossible for a topological reason (the Euler characteristic of the 2-sphere is nonzero [21,22]). Consequently, M cannot be considered as equal to $\mathbb{R}^3$. To get M we must remove at least one point from each 2-sphere $|\vec{k}| = \text{const} > 0$. We decide to follow this minimal restriction and we remove the north pole for each

2-sphere $|\vec{k}| = \text{const} > 0$ and also point (0,0,0) of $\mathbb{R}^3$. Thus the submanifold $M \subset \mathbb{R}^3$ is assumed as

$$M = \widetilde{\mathbb{R}}^3 := \mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \geqslant 0\} \tag{2.14}$$

and we also define the Cartesian structure on M. Therefore, in geometrical language, our task is to find a general flat connection $\hat{\hat{D}}$ in the tangent bundle $T(\widetilde{\mathbb{R}}^3)$, anti-Hermitian with respect to the scalar product (2.9) and such that 2-planes of the two-dimensional differential distribution $\mathcal{D}_2$ defined by (2.13) are propagated parallel with respect to $\hat{\hat{D}}$. To solve this problem, we first choose a basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)_{\vec{k}^{(0)}}$ of the tangent space $T_{\vec{k}^{(0)}}(\widetilde{\mathbb{R}}^3)$ at some point $\vec{k}^{(0)} = (k_1^{(0)}, k_2^{(0)}, k_3^{(0)}) \in \widetilde{\mathbb{R}}^3$ so that $\vec{e}_1, \vec{e}_2 \in \Pi^\perp$. Since $\widetilde{\mathbb{R}}^3$ is simply connected and we assume that the connection $\hat{\hat{D}}$ is flat, the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)_{\vec{k}^{(0)}}$ can be propagated parallel with respect to $\hat{\hat{D}}$ on the entire $\widetilde{\mathbb{R}}^3$ giving a triad of the pointwise independent vector fields $(\vec{e}_1(\vec{k}), \vec{e}_2(\vec{k}), \vec{e}_3(\vec{k}))$ on $\widetilde{\mathbb{R}}^3$. Moreover, if $\hat{\hat{D}}$ satisfies the condition (2.8), i.e., the planes $\Pi^\perp(k_1,k_2,k_3)$ of the distribution $\mathcal{D}_2$ are propagated parallel with respect to $\hat{\hat{D}}$, then

$$\vec{e}_1(\vec{k}), \vec{e}_2(\vec{k}) \in \Pi^\perp(\vec{k})\, \forall\, \vec{k} \in \widetilde{\mathbb{R}}^3. \tag{2.15}$$

Define

$$\vec{e}_\mu(\vec{k}) \equiv \vec{e}_\mu = e_{\mu j}\frac{\partial}{\partial k_j}, \quad \mu = 1,2,3. \tag{2.16}$$

Let $(\vec{e}^{\,1}(\vec{k}), \vec{e}^{\,2}(\vec{k}), \vec{e}^{\,3}(\vec{k}))$ be the triad of 1-forms dual to $(\vec{e}_1(\vec{k}), \vec{e}_2(\vec{k}), \vec{e}_3(\vec{k}))$,

$$\vec{e}^{\,\mu}(\vec{k}) \equiv \vec{e}^{\,\mu} = e^\mu_j dk_j, \quad \mu = 1,2,3. \tag{2.17}$$

Thus we have

$$\vec{e}^{\,\mu}(\vec{e}_\nu) = \delta^\mu_\nu \Longleftrightarrow e^\mu_j e_{\nu j} = \delta^\mu_\nu, \quad \mu,\nu = 1,2,3. \tag{2.18}$$

From (2.18) we quickly infer that

$$e^\mu_j e_{\mu l} = \delta_{jl}. \tag{2.19}$$

The metric tensor $g_{\mu\nu}$ in the space tangent to the momentum space reads, in the basis $\vec{e}_\mu$,

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu = e_{\mu j} e_{\nu j}. \tag{2.20}$$

From (2.18) and (2.20) we easily find the relations

$$e_{\mu j} = g_{\mu\nu} e^\nu_j, \quad e^\mu_j = g^{\mu\nu} e_{\nu j}, \tag{2.21}$$

where, as usual, $g^{\mu\nu}$ is the inverse tensor to $g_{\mu\nu}$,

$$g_{\mu\rho} g^{\rho\nu} = \delta^\mu_\nu. \tag{2.22}$$

From the assumption that the vector fields $\vec{e}_\mu$, $\mu = 1,2,3$, are propagated parallel with respect to the connection $\hat{\hat{D}}$ we have

$$\hat{D}_l \vec{e}_\mu = 0 \Longrightarrow \partial_l e_{\mu j} + \Gamma_{jml} e_{\mu m} = 0, \tag{2.23}$$

where Γ_{jml} are the connection coefficients. Multiplying both sides of Eq. (2.23) by $e_{\mu n}$, summing over μ from 1 to 3, and using (2.19), we get the connection coefficients as

$$\Gamma_{jnl} = -e^\mu_n \partial_l e_{\mu j} = e_{\mu j} \partial_l e^\mu_n. \tag{2.24}$$

Altogether, Eq. (2.24) gives the general form of the connection coefficients of the flat connection $\hat{\hat{D}}$ in $T(\widetilde{\mathbb{R}}^3)$ for which the 2-planes $\Pi^\perp(\vec{k})$ are propagated parallel. Therefore, it remains only to study the condition (2.12). The left-hand side of Eq. (2.12) with the use of the well-known formula for the covariant derivative

$$(\hat{D}_l \widetilde{\boldsymbol{\Psi}})_j = \partial_l \widetilde{\Psi}_j + \Gamma_{jml} \widetilde{\Psi}_m \tag{2.25}$$

[and analogous formula for $(\hat{D}_l \widetilde{\boldsymbol{\Phi}})_j$], after integrating by parts and after employing (2.24), (2.21), and (2.19), gives

$$\begin{aligned}\left(\int \frac{d^3k}{(2\pi)^3|\vec{k}|} \widetilde{\boldsymbol{\Phi}}^\dagger(\vec{k}) \hat{D}_l \widetilde{\boldsymbol{\Psi}}(\vec{k})\right)^* &= -\int \frac{d^3k}{(2\pi)^3|\vec{k}|} \widetilde{\boldsymbol{\Psi}}^\dagger(\vec{k}) \hat{D}_l \widetilde{\boldsymbol{\Phi}}(\vec{k}) \\ &\quad - \int \frac{d^3k}{(2\pi)^3|\vec{k}|} \widetilde{\Psi}^*_n e^\mu_n e^\nu_j (\partial_l g_{\mu\nu} + g_{\mu\nu}|\vec{k}|\partial_l|\vec{k}|^{-1}) \widetilde{\Phi}_j.\end{aligned} \tag{2.26}$$

Comparing this with the right-hand side of (2.12), we conclude that $\hat{\hat{D}}$ is an anti-Hermitian operator with respect to the scalar product (2.9) if and only if

$$\begin{aligned}\partial_l g_{\mu\nu} + (|\vec{k}|\partial_l|\vec{k}|^{-1}) g_{\mu\nu} = 0 &\Longleftrightarrow \partial_l(|\vec{k}|^{-1} g_{\mu\nu}) \\ &= 0 \Longleftrightarrow |\vec{k}|^{-1} g_{\mu\nu} = \text{const}_{\mu\nu}.\end{aligned} \tag{2.27}$$

Without any loss of generality we can choose the triad $\vec{e}_\mu(\vec{k})$ and its dual $\vec{e}^{\,\mu}(\vec{k})$ as [see (2.20), (2.24), and (2.27)]

$$\vec{e}_\mu(\vec{k}) = |\vec{k}|^{1/2} \vec{E}_\mu(\vec{k}), \quad \vec{e}^{\,\mu}(\vec{k}) = |\vec{k}|^{-1/2} \vec{E}^\mu(\vec{k}), \quad \mu = 1,2,3, \tag{2.28}$$

where $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ is the orthonormal right-oriented triad of vector fields on $\widetilde{\mathbb{R}}^3$,

$$\begin{aligned}&\vec{E}_\mu(\vec{k}) \cdot \vec{E}_\nu(\vec{k}) = \delta_{\mu\nu}, \quad \vec{E}_1(\vec{k}), \vec{E}_2(\vec{k}) \in \Pi^\perp(\vec{k}), \\ &\vec{E}_3(\vec{k}) = \frac{\vec{k}}{|\vec{k}|}, \quad \vec{E}_1(\vec{k}) \times \vec{E}_2(\vec{k}) = \vec{E}_3(\vec{k}),\end{aligned} \tag{2.29}$$

and $\vec{E}^\mu(\vec{k})$ is the triad of 1-forms dual to $\vec{E}_\mu(\vec{k})$. Inserting (2.28) into (2.24) we find the connection coefficients

$$\Gamma_{jnl} = -\frac{k_l}{2|\vec{k}|^2}\delta_{jn} + E_{\mu j}\partial_l E_{\mu n} = -\frac{k_l}{2|\vec{k}|^2}\delta_{jn} - E_{\mu n}\partial_l E_{\mu j}. \tag{2.30}$$

(Note that $E_{\mu n} = E^\mu_n$.) Therefore,

$$\begin{aligned}(\hat{D}_l \widetilde{\boldsymbol{\Psi}})_j &= \partial_l \widetilde{\Psi}_j - \frac{k_l}{2|\vec{k}|^2}\widetilde{\Psi}_j + E_{\mu j}(\partial_l E_{\mu n})\widetilde{\Psi}_n \\ &= \left(\delta_{jn}\partial_l - \delta_{jn}\frac{k_l}{2|\vec{k}|^2} + E_{\mu j}(\partial_l E_{\mu n})\right)\widetilde{\Psi}_n.\end{aligned} \tag{2.31}$$

Finally, using (2.5), we find the photon position operator as

$$\begin{aligned}&\hat{\hat{X}} = (\hat{X}_1, \hat{X}_2, \hat{X}_3), \\ &(\hat{X}_l \widetilde{\boldsymbol{\Psi}})_j = i\left(\delta_{jn}\partial_l - \delta_{jn}\frac{k_l}{2|\vec{k}|^2} + E_{\mu j}(\partial_l E_{\mu n})\right)\widetilde{\Psi}_n,\end{aligned} \tag{2.32}$$

which can be rewritten in the matrix form

$$\hat{X}_l \widetilde{\mathbf{\Psi}} = i(\mathbf{1}\partial_l + \mathbf{\Gamma}_l)\widetilde{\mathbf{\Psi}}, \quad \mathbf{\Gamma}_l := -\frac{k_l}{2|\vec{k}|^2}\mathbf{1} + \mathbf{A}_l,$$
$$\mathbf{A}_l := \mathbf{E}\partial_l \mathbf{E}^{-1}, \quad (\mathbf{E})_{j\mu} := E_{\mu j}. \tag{2.33}$$

The formulas (2.32) or (2.33) define the Hawton position operator for the photon when the Białynicki-Birula scalar product (2.9) is assumed to apply [6,10,11]. Performing analogous calculations, we can easily show that, under the assumption that the scalar product has the form

$$\langle \widetilde{\mathbf{\Phi}} | \widetilde{\mathbf{\Psi}} \rangle' \sim \int \frac{d^3k}{(2\pi)^3 |\vec{k}|^{2s}} \widetilde{\mathbf{\Phi}}^\dagger(\vec{k}) \widetilde{\mathbf{\Psi}}(\vec{k}), \tag{2.34}$$

the formula (2.27) now has to read

$$|\vec{k}|^{-s} g_{\mu\nu} = \mathrm{const}_{\mu\nu} \tag{2.35}$$

and (2.28) become

$$\vec{e}_\mu(\vec{k}) = |\vec{k}|^s \vec{E}_\mu(\vec{k}), \quad \vec{e}^{\,\mu}(\vec{k}) = |\vec{k}|^{-s} \vec{E}^\mu(\vec{k}). \tag{2.36}$$

Consequently, the photon position operator in this case reads

$$\hat{X}_l \widetilde{\mathbf{\Psi}} = i\left(\mathbf{1}\partial_l - s\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \mathbf{A}_l\right)\widetilde{\mathbf{\Psi}}. \tag{2.37}$$

Thus we recover the general form of the photon position operator given by Hawton [6]. We can quickly show that the action of $\hat{X}_l$ on $\widetilde{\mathbf{\Psi}}$ can be written in a compact form

$$(\hat{X}_l \widetilde{\mathbf{\Psi}})_j = i e_{\mu j} \partial_l \left(e^\mu_n \widetilde{\Psi}_n\right). \tag{2.38}$$

This corresponds to the last formula of Sec. V in Hawton's work [6].

We are going now to study the form of connection $\hat{\tilde{D}}$ in more detail and find an explicit expression for $\hat{\tilde{D}}$. First, we introduce a coordinate system on $\widetilde{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \geqslant 0\}$. Let $(k_1,k_2,k_3) \in \widetilde{\mathbb{R}}^3$. We project this point on the unit 2-sphere S^2 with center (0,0,0) along the ray defined by $\vec{k} = (k_1,k_2,k_3)$. This projection determines the point $\vec{k}'$ on S^2 of the Cartesian coordinates $(\frac{k_1}{|\vec{k}|}, \frac{k_2}{|\vec{k}|}, \frac{k_3}{|\vec{k}|})$. Let (ξ,η) be the stereographic coordinates of the point $\vec{k}'$ for the stereographic projection of S^2 from the north pole (0,0,1) on the projection 2-plane $k_3 = 0$. Finally, to the original point of the Cartesian coordinates (k_1,k_2,k_3) we assign the coordinates (ξ,η,ζ), where $\zeta := |\vec{k}| > 0$. Thus we construct a new coordinate system on $\widetilde{\mathbb{R}}^3$. From the well-known theory of stereographic projection we easily infer the relations between the Cartesian coordinates (k_1,k_2,k_3) and the coordinates (ξ,η,ζ),

$$\xi = \frac{k_1}{|\vec{k}| - k_3}, \quad \eta = \frac{k_2}{|\vec{k}| - k_3}, \quad \zeta = |\vec{k}|, \tag{2.39a}$$

$$k_1 = \zeta\frac{2\xi}{\xi^2+\eta^2+1}, \quad k_2 = \zeta\frac{2\eta}{\xi^2+\eta^2+1}, \quad k_3 = \zeta\frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1} \tag{2.39b}$$

for $(\xi,\eta) \in \mathbb{R}^2$ and $\zeta > 0$. The natural basis of the vector fields on $\widetilde{\mathbb{R}}^3$ defined by the coordinates (ξ,η,ζ) is given by

$$\begin{aligned}
\frac{\partial}{\partial\xi} &= |\vec{k}|^{-1}\left(\left[|\vec{k}|(|\vec{k}|-k_3) - k_1^2\right]\frac{\partial}{\partial k_1} - k_1k_2\frac{\partial}{\partial k_2} + k_1(|\vec{k}|-k_3)\frac{\partial}{\partial k_3}\right)\\
&= \frac{2\zeta}{(\xi^2+\eta^2+1)^2}\left((\eta^2-\xi^2+1)\frac{\partial}{\partial k_1} - 2\xi\eta\frac{\partial}{\partial k_2} + 2\xi\frac{\partial}{\partial k_3}\right)\\
&= 2|\vec{k}|\sin^2\frac{\theta}{2}\left[\left(1-2\cos^2\frac{\theta}{2}\cos^2\varphi\right)\frac{\partial}{\partial k_1} - \cos^2\frac{\theta}{2}\sin 2\varphi\frac{\partial}{\partial k_2} + \sin\theta\cos\varphi\frac{\partial}{\partial k_3}\right],
\end{aligned} \tag{2.40a}$$

$$\begin{aligned}
\frac{\partial}{\partial\eta} &= |\vec{k}|^{-1}\left(-k_1k_2\frac{\partial}{\partial k_1} + \left[|\vec{k}|(|\vec{k}|-k_3) - k_2^2\right]\frac{\partial}{\partial k_2} + k_2(|\vec{k}|-k_3)\frac{\partial}{\partial k_3}\right)\\
&= \frac{2\zeta}{(\xi^2+\eta^2+1)^2}\left(-2\xi\eta\frac{\partial}{\partial k_1} + (\xi^2-\eta^2+1)\frac{\partial}{\partial k_2} + 2\eta\frac{\partial}{\partial k_3}\right)\\
&= 2|\vec{k}|\sin^2\frac{\theta}{2}\left[-\cos^2\frac{\theta}{2}\sin 2\varphi\frac{\partial}{\partial k_1} + \left(1-2\cos^2\frac{\theta}{2}\sin^2\varphi\right)\frac{\partial}{\partial k_2} + \sin\theta\sin\varphi\frac{\partial}{\partial k_3}\right],
\end{aligned} \tag{2.40b}$$

$$\begin{aligned}
\frac{\partial}{\partial\zeta} &= |\vec{k}|^{-1}\left(k_1\frac{\partial}{\partial k_1} + k_2\frac{\partial}{\partial k_2} + k_3\frac{\partial}{\partial k_3}\right)\\
&= \frac{1}{(\xi^2+\eta^2+1)}\left(2\xi\frac{\partial}{\partial k_1} + 2\eta\frac{\partial}{\partial k_2} + (\xi^2+\eta^2-1)\frac{\partial}{\partial k_3}\right)\\
&= \sin\theta\cos\varphi\frac{\partial}{\partial k_1} + \sin\theta\sin\varphi\frac{\partial}{\partial k_2} + \cos\theta\frac{\partial}{\partial k_3},
\end{aligned} \tag{2.40c}$$

where (θ,φ) $(0 < \theta \leqslant \pi$ and $0 \leqslant \varphi < 2\pi)$ are the spherical coordinates of the point (k_1,k_2,k_3) or, equivalently, of the point $(\frac{k_1}{|\vec{k}|}, \frac{k_2}{|\vec{k}|}, \frac{k_3}{|\vec{k}|})$. [Note that for $\theta = \pi$ the coordinate φ is undefined, but since $\sin\pi = 0$, the formulas (2.40) hold true also for $\theta=\pi$.] We quickly find that the vectors $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta}$, and $\frac{\partial}{\partial\zeta}$ are mutually orthogonal. Hence the coordinate system (ξ,η,ζ) is orthogonal. However, the most important advantage of this new system is the fact that

$$\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial\eta}\in\Pi^{\perp}(\vec{k}) \tag{2.41}$$

for every $\vec{k}=(k_1,k_2,k_3)\in\widetilde{\mathbb{R}}^3$. Consequently, we easily construct an orthonormal triad of vector fields $(\vec{E}_1(\vec{k}),\vec{E}_2(\vec{k}),\vec{E}_3(\vec{k}))$ on $\widetilde{\mathbb{R}}^3$ satisfying the conditions (2.29). It reads

$$\vec{E}_1(\vec{k}):=-(|\vec{k}|-k_3)^{-1}\frac{\partial}{\partial\xi}=-\frac{\xi^2+\eta^2+1}{2\zeta}\frac{\partial}{\partial\xi},$$
$$\vec{E}_2(\vec{k}):=(|\vec{k}|-k_3)^{-1}\frac{\partial}{\partial\eta}=\frac{\xi^2+\eta^2+1}{2\zeta}\frac{\partial}{\partial\eta},\quad \vec{E}_3(\vec{k}):=\frac{\partial}{\partial\zeta}. \tag{2.42}$$

[The system of coordinates (ξ,η,ζ) has the opposite orientation to the Cartesian system (k_1,k_2,k_3), hence the minus sign in the formula defining $\vec{E}_1(\vec{k})$.] Inserting (2.42) into the definition of $\mathbf{A}_l$ given by (2.33), after performing straightforward calculations we get

$$i\mathbf{A}_l=\frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2}+\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)}\mathbf{\Sigma}, \tag{2.43}$$

where $\vec{\mathbf{S}}=(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3)$, with

$$\mathbf{S}_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \mathbf{S}_2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},$$
$$\mathbf{S}_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \tag{2.44}$$

the spin-1 matrices, and

$$\mathbf{\Sigma}=\frac{\vec{k}\cdot\vec{\mathbf{S}}}{|\vec{k}|}=i|\vec{k}|^{-1}\begin{pmatrix}0&-k_3&k_2\\k_3&0&-k_1\\-k_2&k_1&0\end{pmatrix} \tag{2.45}$$

is the helicity operator. Thus the photon position operator (2.33) now takes the form

$$\hat{X}_l=i\mathbf{1}\partial_l-i\frac{k_l}{2|\vec{k}|^2}\mathbf{1}+\frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2}+\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)}\mathbf{\Sigma} \tag{2.46}$$

and for general s [see (2.37)] we have

$$\hat{X}_l=i\mathbf{1}\partial_l-is\frac{k_l}{|\vec{k}|^2}\mathbf{1}+\frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2}+\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)}\mathbf{\Sigma}. \tag{2.47}$$

A result analogous to (2.47) was found in [11] for the case when instead of $\mathbb{R}^3\setminus\{(0,0,k_3)\in\mathbb{R}^3:k_3\geqslant 0\}$ one takes $\mathbb{R}^3\setminus\{(0,0,k_3)\in\mathbb{R}^3:k_3\leqslant 0\}$. In [11] this result was obtained from the photon position operator for the orthonormal triad defined by the spherical coordinates, with the use of the suitable rotation of this triad around the vector $\vec{k}$ (see the example in the present section).

Now we are able to get the more general formula for the position operator of the photon. First, note that the general real orthonormal triad of vector fields on $\Omega\subset\widetilde{\mathbb{R}}^3$ fulfilling the conditions (2.29) has the form

$$\vec{E}'_1=a\vec{E}_1-b\vec{E}_2,\quad \vec{E}'_2=b\vec{E}_1+a\vec{E}_2,\quad \vec{E}'_3=\vec{E}_3 \tag{2.48}$$

for $a=a(\vec{k})$, $b=b(\vec{k})$, $a^2+b^2=1$, and $\vec{k}\in\Omega$. From (2.48), employing (2.43) and (2.42) with (2.40) making simple manipulations, we find

$$i\mathbf{A}'_l=i\mathbf{E}'\partial_l\mathbf{E}'^{-1}=\frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2}+\left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)}+(a\partial_l b-b\partial_l a)\right)\mathbf{\Sigma}. \tag{2.49}$$

Hence, the more general form of the photon position operator reads

$$\hat{X}'_l=i\mathbf{1}\partial_l-is\frac{k_l}{|\vec{k}|^2}\mathbf{1}+\frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2}+\left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)}+(a\partial_l b-b\partial_l a)\right)\mathbf{\Sigma},$$
$$a=a(\vec{k}),\quad b=b(\vec{k}),\quad a^2+b^2=1,\quad \vec{k}\in\Omega\subset\widetilde{\mathbb{R}}^3 \tag{2.50}$$

In the case of the Białynicki-Birula scalar product (2.9) we set $s=\frac{1}{2}$. Note that the first three terms on the right-hand side of (2.50) with $s=\frac{1}{2}$ define the position operator of the photon proposed by Pryce in his pioneering work [5]. However, the components of Pryce's position operator do not commute.

Remark 1. To emphasize the geometric meaning of our construction we have decided to follow the path of real Riemannian geometry. For this reason the connection $\hat{\tilde{D}}$ is originally defined as the connection in the tangent bundle $T(\widetilde{\mathbb{R}}^3)$ and then extended to its complexification, yielding in turn the photon position operator $\hat{\tilde{X}}$. It should be noted, however, that in the general approach the connection coefficients Γ_{ijk} may be complex valued. This can be taken into account by allowing unitary (instead of orthogonal) transformations of the triad $(\vec{E}_1,\vec{E}_2,\vec{E}_3)$ in (2.48). To this end consider

$$\mathbf{E}'=\mathbf{E}\mathbf{U}, \tag{2.51}$$

$$\mathbf{U}=\begin{pmatrix}\mathbf{U}_\perp&&0\\&&0\\0&0&1\end{pmatrix}, \tag{2.52}$$

with some unitary 2×2 matrix $\mathbf{U}_\perp(\vec{k})$. After straightforward calculations we obtain

$$\mathbf{A}'_l=\mathbf{E}'\partial_l\mathbf{E}'^{-1}=\mathbf{A}_l+\mathbf{E}\mathbf{U}(\partial_l\mathbf{U}^\dagger)\mathbf{E}^T=\mathbf{A}_l+\mathbf{E}_\perp\mathbf{U}_\perp(\partial_l\mathbf{U}_\perp^\dagger)\mathbf{E}_\perp^T, \tag{2.53}$$

where $\mathbf{E}_\perp$ is a 3×2 matrix determined by vectors $\vec{E}_1$ and $\vec{E}_2$,

$$(\mathbf{E}_\perp)_{j\mu}=E_{\mu j},\quad j=1,2,3;\quad \mu=1,2. \tag{2.54}$$

Writing the general $\vec{k}$-dependent unitary 2×2 matrix as

$$\mathbf{U}_\perp = e^{i\beta}\begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi}\end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta}\end{pmatrix}, \tag{2.55}$$

with real functions $\beta = \beta(\vec{k})$, $\alpha = \alpha(\vec{k})$, $\psi = \psi(\vec{k})$, and $\Delta = \Delta(\vec{k})$, the general form of the anti-Hermitian matrix $\mathbf{U}_\perp \partial_l \mathbf{U}_\perp^\dagger$ can be calculated as

$$\mathbf{U}_\perp \partial_l \mathbf{U}_\perp^\dagger = i\begin{pmatrix} -\cos(2\alpha)\partial_l\Delta - \partial_l\beta - \partial_l\psi & e^{2i\psi}[\sin(2\alpha)\partial_l\Delta + i\partial_l\alpha] \\ e^{-2i\psi}[\sin(2\alpha)\partial_l\Delta - i\partial_l\alpha] & \cos(2\alpha)\partial_l\Delta - \partial_l\beta + \partial_l\psi\end{pmatrix}. \tag{2.56}$$

The relation (2.56) expanded in terms of Pauli matrices $\boldsymbol{\sigma}_j$ and substituted into (2.53) produces the formula for the photon position operator

$$\begin{aligned}\hat{X}'_l = {}& i\mathbf{1}\partial_l - is\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2} + (\partial_l\beta)\mathbf{R}_0 - [\cos(2\psi)\sin(2\alpha)\partial_l\Delta - \sin(2\psi)\partial_l\alpha]\mathbf{R}_1 \\ &+ \left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}| - k_3)} + \sin(2\psi)\sin(2\alpha)\partial_l\Delta + \cos(2\psi)\partial_l\alpha\right)\mathbf{R}_2 + [\cos(2\alpha)\partial_l\Delta + \partial_l\psi]\mathbf{R}_3,\end{aligned} \tag{2.57}$$

where

$$\mathbf{R}_j := \mathbf{E}_\perp\boldsymbol{\sigma}_j\mathbf{E}_\perp^T, \quad j = 0, 1, 2, 3. \tag{2.58}$$

(We use the fact that $\mathbf{R}_2 = \boldsymbol{\Sigma}$.) Clearly, the formula (2.50) can be obtained from (2.57) by choosing $\beta = \psi = \Delta \equiv 0$ and setting $a = \cos\alpha$ and $b = \sin\alpha$.

Remark 2. Using the matrix notation of (2.33), (2.51), and (2.52), it can be easily observed that all versions of the photon position operator (2.57), including those given by the formula (2.50), are unitarily equivalent. Indeed, since the $\mathbf{E}$ related to the triad (2.42) is an orthogonal matrix, we can rewrite $\mathbf{E}' = \mathbf{E}\mathbf{U} = \mathbf{V}\mathbf{E}$ for the unitary matrix $\mathbf{V} = \mathbf{E}\mathbf{U}\mathbf{E}^{-1}$, and consequently

$$\mathbf{A}'_l = \mathbf{E}'\partial_l\mathbf{E}'^{-1} = \mathbf{V}\mathbf{A}_l\mathbf{V}^{-1} + \mathbf{V}\partial_l\mathbf{V}^{-1}. \tag{2.59}$$

Thus, the action of the photon position operator (2.57) can be rephrased as

$$\begin{aligned}\hat{X}'_l\widetilde{\boldsymbol{\Psi}} &= i\left(\mathbf{1}\partial_l - s\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \mathbf{A}'_l\right)\widetilde{\boldsymbol{\Psi}} = i\left(\mathbf{V}\mathbf{V}^{-1}\partial_l - s\frac{k_l}{|\vec{k}|^2}\mathbf{V}\mathbf{V}^{-1} + \mathbf{V}\mathbf{A}_l\mathbf{V}^{-1} + \mathbf{V}\partial_l\mathbf{V}^{-1}\right)\widetilde{\boldsymbol{\Psi}} \\ &= i\mathbf{V}\left(\mathbf{1}\partial_l - s\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \mathbf{A}_l\right)\mathbf{V}^{-1}\widetilde{\boldsymbol{\Psi}} = \mathbf{V}\hat{X}_l\mathbf{V}^{-1}\widetilde{\boldsymbol{\Psi}},\end{aligned} \tag{2.60}$$

showing that the unitary transformation given by $\mathbf{V}$ converts $\hat{X}_l$ to $\hat{X}'_l$.

As an example of the real transformation (2.48) we investigate the case when the orthonormal triad $(\vec{E}'_1, \vec{E}'_2, \vec{E}'_3)$ is determined in a natural way by the spherical system of coordinates $(\theta, \varphi, |\vec{k}|)$ $(0 < \theta < \pi$ and $0 \leqslant \varphi < 2\pi)$,

$$\begin{aligned}\vec{E}'_1 &= \cos\theta\cos\varphi\frac{\partial}{\partial k_1} + \cos\theta\sin\varphi\frac{\partial}{\partial k_2} - \sin\theta\frac{\partial}{\partial k_3}, \\ \vec{E}'_2 &= -\sin\varphi\frac{\partial}{\partial k_1} + \cos\varphi\frac{\partial}{\partial k_2}, \quad \vec{E}'_3 = \sin\theta\cos\varphi\frac{\partial}{\partial k_1} + \sin\theta\sin\varphi\frac{\partial}{\partial k_2} + \cos\theta\frac{\partial}{\partial k_3}.\end{aligned} \tag{2.61}$$

From (2.40) and (2.42) we get

$$\begin{aligned}\vec{E}_1 &= \left(2\cos^2\frac{\theta}{2}\cos^2\varphi - 1\right)\frac{\partial}{\partial k_1} + \cos^2\frac{\theta}{2}\sin 2\varphi\frac{\partial}{\partial k_2} - \sin\theta\cos\varphi\frac{\partial}{\partial k_3}, \\ \vec{E}_2 &= -\cos^2\frac{\theta}{2}\sin 2\varphi\frac{\partial}{\partial k_1} + \left(1 - 2\cos^2\frac{\theta}{2}\sin^2\varphi\right)\frac{\partial}{\partial k_2} + \sin\theta\sin\varphi\frac{\partial}{\partial k_3}, \\ \vec{E}_3 &= \sin\theta\cos\varphi\frac{\partial}{\partial k_1} + \sin\theta\sin\varphi\frac{\partial}{\partial k_2} + \cos\theta\frac{\partial}{\partial k_3}.\end{aligned} \tag{2.62}$$

Substituting (2.61) and (2.62) into (2.48), we quickly conclude that

$$a = \cos\varphi = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}, \quad b = \sin\varphi = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}. \tag{2.63}$$

Then Eq. (2.50) gives (compare with [11])

$$\hat{X}'_l = i\mathbf{1}\partial_l - is\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2} + \left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|-k_3)} + \partial_l\varphi\right)\mathbf{\Sigma} \tag{2.64}$$

or after simple direct calculations we get

$$\hat{X}'_l = i\mathbf{1}\partial_l - is\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2} + \epsilon_{lm3}\frac{k_m k_3}{|\vec{k}|(k_1^2+k_2^2)}\mathbf{\Sigma}. \tag{2.65}$$

This is just the photon position operator found by Hawton in [6]. The domain $\Omega \subset \widetilde{\mathbb{R}}^3$ where the right-hand side of (2.65) is nonsingular is defined as $\Omega = \mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \in \mathbb{R}^1\}$. As the next example consider the transformation

$$a = \cos 2\varphi = \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}, \quad b = \sin 2\varphi = \frac{2k_1k_2}{k_1^2+k_2^2} \tag{2.66}$$

defined on the same $\Omega = \mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \in \mathbb{R}^1\}$. Inserting this into (2.50), we quickly get (see also [11])

$$\hat{X}'_l = i\mathbf{1}\partial_l - is\frac{k_l}{|\vec{k}|^2}\mathbf{1} + \frac{(\vec{k}\times\vec{\mathbf{S}})_l}{|\vec{k}|^2} - \epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}|+k_3)}\mathbf{\Sigma}. \tag{2.67}$$

Observe that $\hat{X}_l$ given by (2.67) is nonsingular on the open submanifold of $\mathbb{R}^3$ defined as $\mathbb{R}^3 \setminus \{(0,0,k_3) \in \mathbb{R}^3 : k_3 \leqslant 0\}$. Straightforward calculations show that we can arrive at (2.67) by using the stereographic projection of the unit 2-sphere from the south pole $(0,0,-1)$ and not from the north pole (0,0,1) as it has been done in the case of the formula (2.47). On the other hand, this example demonstrates the physical irrelevance of the particular choice of half-line excluded from $\mathbb{R}^3$ and it can be generalized in the following way. Starting from $\mathbf{E}(\vec{k})$ given by (2.42), we can construct $\mathbf{E}'(\vec{k}) = \mathbf{M}\mathbf{E}(\mathbf{M}^{-1}\vec{k})$ for an arbitrary rotation matrix $\mathbf{M}$. (Here $\vec{k}' = \mathbf{M}^{-1}\vec{k}$ defines obvious action of a matrix on $\vec{k}$, i.e., $k'_i = M^{-1}_{ij}k_j$.) Simple but rather long calculations confirm that the triad $\vec{E}'_\mu$ is of the form (2.48) and that the singularity half-line of photon position operator is rotated from the direction defined by vector $\vec{k}_{z_+} = (0,0,1)$ to the direction given by $\mathbf{M}\vec{k}_{z_+}$. Thus, since the reasoning of Remark 2 can be applied, all possible choices of singularity half-line are unitarily equivalent and its actual variant is physically irrelevant as long as quantum-mechanical predictions are formulated in terms of the Białynicki-Birula scalar product (2.9).

We can furthermore analyze the question of torsion of connection $\hat{\hat{D}}$. The connection coefficients corresponding to the formula (2.50) read

$$\Gamma_{jml} = -\frac{1}{|\vec{k}|^2}\left(s\delta_{jm}k_l - \delta_{ml}k_j + \delta_{jl}k_m + \epsilon_{rjm}\epsilon_{lp3}\frac{k_pk_r}{|\vec{k}|-k_3} + |\vec{k}|\epsilon_{rjm}k_r\partial_l\alpha\right) \tag{2.68}$$

for $a(\vec{k}) = \cos\alpha(\vec{k})$ and $b(\vec{k}) = \sin\alpha(\vec{k})$. From this expression it can be easily observed that no choice of $\alpha(\vec{k})$ can make torsion $Q_{jml} = \Gamma_{jml} - \Gamma_{jlm}$ vanishing. Indeed, the condition $Q_{112} = 0$ gives

$$\partial_1\alpha = \frac{(s-1)k_2}{|\vec{k}|k_3} - \frac{k_2}{k(|\vec{k}|-k_3)}, \tag{2.69}$$

while $Q_{113} = 0$ yields

$$\partial_1\alpha = \frac{(1-s)k_3}{|\vec{k}|k_2} - \frac{k_2}{k(|\vec{k}|-k_3)}. \tag{2.70}$$

These equations are immediately inconsistent for $s \neq 1$. For $s = 1$ it is enough to consider $Q_{221} = 0$ producing

$$\partial_2\alpha = \frac{(1-s)k_1}{|\vec{k}|k_3} + \frac{k_1}{k(|\vec{k}|-k_3)}. \tag{2.71}$$

As it can be verified by direct calculation, Eqs. (2.69) and (2.71) do not satisfy the basic integrability condition $\partial_1\partial_2\alpha = \partial_2\partial_1\alpha$. Thus, we conclude that the connection defining the photon position operator (2.50) must have a nonvanishing torsion tensor.

It can be interesting and informative to find the phase-space image of the position operator $\hat{\hat{X}}$. To this end we use extensively the formalism developed in [19]. In line with that formalism, the photon phase space is given as

$$\{(\vec{p},\vec{x},\phi_m,n)\} = \mathbb{R}^3 \times \mathbb{R}^3 \times \Gamma^3, \tag{2.72}$$

where Γ^3 is the 3×3 grid, $\Gamma^3 = \{(\phi_m,n)\}$, with $m,n = 0,1,2$ and $\phi_m = \frac{2\pi}{3}m$. We assume that the kernels $\mathcal{P}(\frac{\hbar\vec{\lambda}\cdot\vec{\mu}}{2})$, for $\vec{\lambda},\vec{\mu} \in \mathbb{R}^3$, and $\mathcal{K}(\frac{\pi kl}{3})$, for $k,l = 0,1,2$, determining the Stratonovich-Weyl quantizer (the Fano operators) are taken as

$$\mathcal{P}\left(\frac{\hbar\vec{\lambda}\cdot\vec{\mu}}{2}\right) = 1, \quad \mathcal{K}\left(\frac{\pi kl}{3}\right) = (-1)^{kl} \tag{2.73}$$

[see Eqs. (5.8) and (6.7) of Ref. [19]]. Then the phase-space image of the position operator $\hat{\hat{X}}$ is given by the function [see Eq. (5.16) of Ref. [19]]

$$\vec{X}(\vec{p},\vec{x},\phi_m,n) = \mathrm{Tr}\{\hat{\hat{X}}\hat{H}^{1/2}\hat{\Omega}(\vec{p},\vec{x},\phi_m,n)\hat{H}^{-1/2}\}, \tag{2.74}$$

where

$$\hat{H} = c|\hat{\vec{p}}| = c\hbar|\hat{\vec{k}}| \tag{2.75}$$

is the Hamiltonian operator and $\hat{\Omega}(\vec{p},\vec{x},\phi_m,n)$ is the Stratonovich-Weyl quantizer for the kernels given by (2.73). This quantizer is defined as

$$\hat{\Omega}(\vec{p},\vec{x},\phi_m,n) = \left(\frac{\hbar}{2\pi}\right)^3\frac{1}{3}\sum_{k,l=0}^{2}\int d^3\lambda\, d^3\mu(-1)^{kl}\exp[-i(\vec{\lambda}\cdot\vec{p}+\vec{\mu}\cdot\vec{x})]\exp[-i(k\phi_m+\phi_l n)]\hat{\mathcal{U}}(\vec{\lambda},\vec{\mu})\otimes\hat{\mathcal{D}}(k,l), \tag{2.76}$$

with

$$\hat{\mathcal{U}}(\vec{\lambda},\vec{\mu}) = \exp[i(\vec{\lambda}\cdot\hat{\vec{p}}+\vec{\mu}\cdot\hat{\vec{x}})] = \int d^3x \exp(i\vec{\mu}\cdot\vec{x})\left|\vec{x}-\frac{\hbar\vec{\lambda}}{2}\right\rangle\!\left\langle\vec{x}+\frac{\hbar\vec{\lambda}}{2}\right|$$
$$= \int d^3p \exp(i\vec{\lambda}\cdot\vec{p})\left|\vec{p}+\frac{\hbar\vec{\mu}}{2}\right\rangle\!\left\langle\vec{p}-\frac{\hbar\vec{\mu}}{2}\right|, \quad (2.77a)$$

$$\hat{\mathcal{D}}(k,l) = \exp\left(-i\frac{\pi kl}{3}\right)\exp(ik\hat{\phi})\exp\left(i\frac{2\pi}{3}l\hat{n}\right) = \exp\left(i\frac{\pi kl}{3}\right)\exp\left(i\frac{2\pi}{3}l\hat{n}\right)\exp(ik\hat{\phi})$$
$$= \exp\left(i\frac{\pi kl}{3}\right)\sum_{m=0}^{2}\exp\left(i\frac{2\pi km}{3}\right)|\phi_{(m+l)\mathrm{mod}3}\rangle\langle\phi_m|$$
$$= \exp\left(i\frac{\pi kl}{3}\right)\sum_{n=0}^{2}\exp\left(i\frac{2\pi nl}{3}\right)|n\rangle\langle(n+k)\mathrm{mod}3|,$$
$$\hat{n} = \sum_{n=0}^{2} n\,|n\rangle\langle n|, \quad \hat{\phi} = \sum_{m=0}^{2}\phi_m|\phi_m\rangle\langle\phi_m|, \quad |\phi_m\rangle = \frac{1}{\sqrt{3}}\sum_{n=0}^{2}\exp(in\phi_m)\,|n\rangle \quad (2.77b)$$

[see Eqs. (5.3), (5.4), and (5.8) of [19]; see also [23]]. We choose the basis $\{|n\rangle\}_{n=0,1,2}$ so that the representation of the spin-1 operator $\hat{\vec{\mathcal{S}}}$ with respect to this basis is given by (2.44), i.e., $(\mathcal{S}_j)_{rl} = -i\epsilon_{jrl}$, with $j,r,l = 1,2,3$. Inserting $\hat{\Omega}(\vec{p},\vec{x},\phi_m,n)$ given by (2.76) with (2.77) and $\hat{\vec{X}}$ defined by (2.50) with $s=\frac{1}{2}$ into (2.74) and performing straightforward but tedious manipulations, we find the components of vector function $\vec{X}(\vec{p},\vec{x},\phi_m,n)$ as

$$X_l(\vec{p},\vec{x},\phi_m,n) = x_l + 2\hbar\Bigg[\frac{p_{(n+2)\mathrm{mod}3+1}}{|\vec{p}|^2}\delta_{l,(n+1)\mathrm{mod}3+1} - \frac{p_{(n+1)\mathrm{mod}3+1}}{|\vec{p}|^2}\delta_{l,(n+2)\mathrm{mod}3+1}$$
$$+\left(\frac{p_j}{|\vec{p}|(|\vec{p}|-p_3)}\epsilon_{lj3} + a\frac{\partial b}{\partial p_l} - b\frac{\partial a}{\partial p_l}\right)\frac{p_r}{|\vec{p}|}\epsilon_{r,(n+1)\mathrm{mod}3+1,(n+2)\mathrm{mod}3+1}\Bigg]\sin\phi_m \quad (2.78)$$

for $l=1,2,3$; $m,n=0,1,2$; $\phi_m = \frac{2\pi}{3}m$; and summation over $j,r=1,2,3$. Observe that the phase-space image $\vec{X}(\vec{p},\vec{x},\phi_m,n)$ of the position operator $\hat{\vec{X}}$ depends not only on $\vec{x}$ but also on $\vec{p}$ and on the grid Γ^3 coordinates (ϕ_m,n). Namely, it is equal to $\vec{x}$ plus a term linear in $\hbar$ dependent on $(\vec{p},\phi_m,n)$.

III. EIGENFUNCTIONS OF $\hat{\vec{X}}$

Now we are going to show how the geometrical interpretation of the Hawton position operator for the photon enables us to find in an easy way the respective eigenfunctions. These eigenfunctions were introduced in [10,11,14,15].

Employing our results, we infer from (2.23) and (2.28) that

$$\hat{D}_l(|\vec{k}|^{1/2}\vec{E}_\mu) = 0, \quad \mu = 1,2,3. \quad (3.1)$$

Hence

$$\hat{D}_l[\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2}\vec{E}_\mu] = -iX_l\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2}\vec{E}_\mu. \quad (3.2)$$

Finally, as $\hat{X}_l = i\hat{D}_l$ we get

$$\hat{X}_l[\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2}\vec{E}_\mu] = X_l\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2}\vec{E}_\mu \quad (3.3)$$

for $\mu=1,2,3$. Since for any photon state the condition (2.2) must be fulfilled, the position eigenfunction of the photon $\widetilde{\mathbf{\Psi}}_{\vec{X}}(\vec{k})$ has the form

$$\widetilde{\mathbf{\Psi}}_{\vec{X}}(\vec{k}) = [c_1\mathbf{E}_1(\vec{k}) + c_2\mathbf{E}_2(\vec{k})]\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2}, \quad (3.4)$$

where $c_1, c_2 \in \mathbb{C}^1$, while $\mathbf{E}_1(\vec{k})$ and $\mathbf{E}_2(\vec{k})$ are the one-column matrices representing the vectors $\vec{E}_1(\vec{k})$ and $\vec{E}_2(\vec{k})$, respectively. Then the Białynicki-Birula scalar product (2.9) of the position eigenfunctions is normalized to the Dirac δ if and only if $|c_1|^2+|c_2|^2=1$,

$$|c_1|^2+|c_2|^2 = 1 \Longleftrightarrow \int\frac{d^3k}{(2\pi)^3|\vec{k}|}\widetilde{\mathbf{\Psi}}^\dagger_{\vec{X}}(\vec{k})\widetilde{\mathbf{\Psi}}_{\vec{X}'}(\vec{k}) = \delta(\vec{X}-\vec{X}'). \quad (3.5)$$

In particular, taking $c_1 = \frac{1}{\sqrt{2}}$ and $c_2 = \pm\frac{i}{\sqrt{2}}$, we have

$$\widetilde{\mathbf{\Psi}}_{\vec{X},\pm1}(\vec{k}) = \frac{1}{\sqrt{2}}[\mathbf{E}_1(\vec{k}) \pm i\mathbf{E}_2(\vec{k})]\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^{1/2},$$
$$\hat{\vec{X}}\widetilde{\mathbf{\Psi}}_{\vec{X},\pm1} = \vec{X}\widetilde{\mathbf{\Psi}}_{\vec{X},\pm1}, \quad \mathbf{\Sigma}\widetilde{\mathbf{\Psi}}_{\vec{X},\pm1} = \pm\widetilde{\mathbf{\Psi}}_{\vec{X},\pm1}. \quad (3.6)$$

Analogously, when the scalar product (2.34) applies we employ (2.36) and consequently

$$\widetilde{\mathbf{\Psi}}_{\vec{X}}(\vec{k}) = [c_1\mathbf{E}_1(\vec{k}) + c_2\mathbf{E}_2(\vec{k})]\exp(-i\vec{k}\cdot\vec{X})|\vec{k}|^s \quad (3.7)$$

for $c_1, c_2 \in \mathbb{C}^1$ (see [10,11,14,15]).

Given $\widetilde{\boldsymbol{\Psi}}_{\vec{X}}(\vec{k})$ by (3.4), using the results of [16,17,19], we find the eigenfunction in the $\vec{x}$ representation $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$ as the Fourier transform of $\widetilde{\boldsymbol{\Psi}}_{\vec{X}}(\vec{k})$,

$$\begin{aligned}\boldsymbol{\Psi}_{\vec{X}}(\vec{x}) &= \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^3}\widetilde{\boldsymbol{\Psi}}_{\vec{X}}(\vec{k})\exp(i\vec{k}\cdot\vec{x}) \\ &= \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^3}[c_1\mathbf{E}_1(\vec{k})+c_2\mathbf{E}_2(\vec{k})]|\vec{k}|^{1/2} \\ &\quad\times \exp[i\vec{k}\cdot(\vec{x}-\vec{X})]. \end{aligned}\tag{3.8}$$

In the general case when $\widetilde{\boldsymbol{\Psi}}_{\vec{X}}(\vec{k})$ is given by (3.7) we get

$$\boldsymbol{\Psi}_{\vec{X}}(\vec{x}) \sim \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^3}[c_1\mathbf{E}_1(\vec{k})+c_2\mathbf{E}_2(\vec{k})]|\vec{k}|^s \times \exp[i\vec{k}\cdot(\vec{x}-\vec{X})]. \tag{3.9}$$

Example 1. Assume that $\mathbf{E}_1(\vec{k})$ and $\mathbf{E}_2(\vec{k})$ are given by (2.62). Then (3.8) in the matrix form reads

$$\begin{aligned}\boldsymbol{\Psi}_{\vec{X}}(\vec{x}) &= \frac{\sqrt{\hbar c}}{(2\pi)^3}\int_0^\infty dk\, k^{5/2}\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \\ &\times \begin{pmatrix} c_1\big(2\cos^2\frac{\theta}{2}\cos^2\varphi - 1\big) + c_2\big(-\cos^2\frac{\theta}{2}\sin 2\varphi\big) \\ c_1\big(\cos^2\frac{\theta}{2}\sin 2\varphi\big) + c_2\big(1-2\cos^2\frac{\theta}{2}\sin^2\varphi\big) \\ c_1(-\sin\theta\cos\varphi) + c_2(\sin\theta\sin\varphi)\end{pmatrix} \\ &\times \exp[ik\sin\theta\cos\varphi(x_1-X_1) + ik \\ &\times \sin\theta\sin\varphi(x_2-X_2) + ik\cos\theta(x_3-X_3)]. \end{aligned}\tag{3.10}$$

We quickly recognize that the integrand in (3.10) goes to infinity for $k\to\infty$. To avoid this problem we proceed in a standard way. Namely, we multiply the integrand by $\exp(-\varepsilon k)$, $\varepsilon > 0$, and after performing integration we take the limit $\varepsilon \to 0^+$. Note that the limit should be calculated in the sense of distribution theory. To do it all one can apply the Wolfram *Mathematica*. Then, without going into details, we arrive at the following results. The wave function $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$ can be written as

$$\boldsymbol{\Psi}_{\vec{X}}(\vec{x}) = \begin{pmatrix} c_1\mathcal{F}_I + c_2\mathcal{F}_{II} \\ -c_1\mathcal{F}_{II} + c_2(\mathcal{F}_I+\mathcal{F}_{III}) \\ c_1\mathcal{F}_{IV} + c_2\mathcal{F}_V\end{pmatrix}, \tag{3.11}$$

where $\mathcal{F}_I = \mathcal{F}_I(\vec{x}),\dots,\mathcal{F}_V = \mathcal{F}_V(\vec{x})$ are distributions which (in the sense of distribution theory) are equal to the following functions on respective domains:

$$\begin{aligned}\mathcal{F}_I &= \frac{\sqrt{\hbar c}}{8\sqrt{2}\pi^{3/2}X^{7/2}}\bigg(\frac{2i\cos 2\varphi_1\cos\theta_1}{\sin^2\theta_1} - \frac{2\cos 2\varphi_1}{\sin^2\theta_1} + \frac{2\cos 2\varphi_1[1-i\,\mathrm{sgn}(\cos\theta_1)]}{|\cos\theta_1|^{3/2}\sin^2\theta_1} \\ &\quad + 3\sin^2\varphi_1 - 5i\cos\theta_1\sin^2\varphi_1\bigg) \quad \text{for}\quad X\neq 0,\quad \theta_1\neq\frac{\pi}{2}, \\ \mathcal{F}_{II} &= \frac{\sqrt{\hbar c}\sin 2\varphi_1}{4\sqrt{2}\pi^{3/2}X^{7/2}\sin^2\theta_1}\bigg(\frac{11}{8} - \frac{21\cos\theta_1}{16} - \frac{3\cos 2\theta_1}{8} + \frac{5i\cos 3\theta_1}{16} - \frac{1-i\,\mathrm{sgn}(\cos\theta_1)}{|\cos\theta_1|^{3/2}}\bigg)\quad \text{for}\quad X\neq 0,\quad \theta_1\neq\frac{\pi}{2}, \\ \mathcal{F}_{III} &= \frac{\sqrt{\hbar c}(-3-5i\cos\theta_1)}{8\sqrt{2}\pi^{3/2}X^{7/2}}\quad \text{for}\quad X\neq 0,\quad \mathcal{F}_{IV} = \frac{-5i\sqrt{\hbar c}\sin\theta_1\cos\varphi_1}{8\sqrt{2}\pi^{3/2}X^{7/2}}\quad \text{for } X\neq 0, \\ \mathcal{F}_V &= \frac{5i\sqrt{\hbar c}\sin\theta_1\sin\varphi_1}{8\sqrt{2}\pi^{3/2}X^{7/2}}\quad \text{for}\quad X\neq 0. \end{aligned}\tag{3.12}$$

Here we use the abbreviation $X := |\vec{x}-\vec{X}|$ and θ_1 and φ_1 are the angles defining the direction of the vector $\vec{x}-\vec{X}$,

$$\frac{\vec{x}-\vec{X}}{|\vec{x}-\vec{X}|} = (\sin\theta_1\cos\varphi_1, \sin\theta_1\sin\varphi_1, \cos\theta_1). \tag{3.13}$$

It is evident that the wave function $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$, $\vec{x}\in\mathbb{R}^3$, is a one-column matrix with elements being distributions given by regularization of the functions (3.12) [24,25]. We have not been able yet to find these distributions in a clear compact form. The systematic and careful use of the theory of distributions should be helpful here. This is not completely surprising: In the case of the standard position operator of a nonrelativistic particle the eigenfunctions (the Dirac δ functions) are also nonregular distributions. However, from the partial result given by Eq. (3.12) we can draw an interesting conclusion. From the Białynicki-Birula [16,17] and Sipe [18] interpretation of the photon wave function $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$, the quantity $\boldsymbol{\Psi}^\dagger_{\vec{X}}(\vec{x})\boldsymbol{\Psi}_{\vec{X}}(\vec{x})d^3x$ (if it exists) is proportional to the probability that the energy of the photon is localized in the domain d^3x. Using this interpretation, we can state that the formula (3.11) under (3.12) shows that the energy of a photon in the quantum state $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$ is localized in a small region where $|\vec{x}-\vec{X}|\to 0$ and $\theta_1\to\frac{\pi}{2}$. This behavior of $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$ is illustrated in Fig. 1.

Finally, it could be noted that this interpretation of $\boldsymbol{\Psi}_{\vec{X}}(\vec{x})$ is rather nonstandard. The Fourier transform (3.8) corresponds to the Białynicki-Birula scalar product (2.9) of $\widetilde{\boldsymbol{\Psi}}_{\vec{X}}(\vec{k})$ with the functions

$$\widetilde{\boldsymbol{\Phi}}_{1,\vec{x}}(\vec{k}) \sim \begin{pmatrix}|\vec{k}|e^{-i\vec{k}\cdot\vec{x}} \\ 0 \\ 0\end{pmatrix},\quad \widetilde{\boldsymbol{\Phi}}_{2,\vec{x}}(\vec{k}) \sim \begin{pmatrix}0 \\ |\vec{k}|e^{-i\vec{k}\cdot\vec{x}} \\ 0\end{pmatrix},$$

$$\widetilde{\boldsymbol{\Phi}}_{3,\vec{x}}(\vec{k}) \sim \begin{pmatrix}0 \\ 0 \\ |\vec{k}|e^{-i\vec{k}\cdot\vec{x}}\end{pmatrix}. \tag{3.14}$$

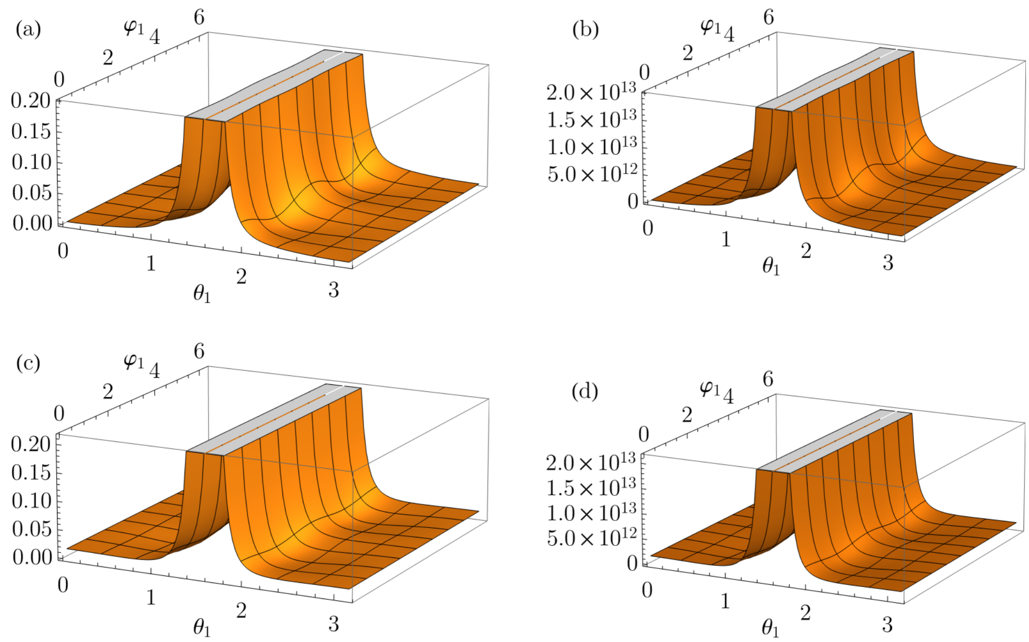

FIG. 1. Distribution of probability for photon energy divided by $\hbar c$, $\frac{1}{\hbar c}\mathbf{\Psi}^{\dagger}_{\vec{X}}(\vec{x})\mathbf{\Psi}_{\vec{X}}(\vec{x})$ (in units of $1/\mathrm{m}^4$), as a function of θ_1 (rad) and φ_1 (rad) for specific values of X (m) and c_i ($\mathrm{m}^{3/2}$): (a) $c_1 = 1$, $c_2 = 0$, and $X = 1$; (b) $c_1 = 1$, $c_2 = 0$, and $X = 0.01$; (c) $c_1 = 0$, $c_2 = 1$, and $X = 1$; and (d) $c_1 = 0$, $c_2 = 1$, and $X = 0.01$.

However, these functions do not satisfy the condition (2.2) and they are not orthogonal in the sense of Białynicki-Birula scalar product. In turn, the orthodox quantum-mechanical requirement of relating observables to self-adjoint operators has been abandoned here.

IV. FROM THE PHOTON POSITION OPERATOR TO BERRY'S POTENTIAL

It has been noted by Hawton [6] and then investigated further in [11] that the last term on the right-hand side of Eq. (2.65) defines some Berry potential leading to a Berry phase predicted by Chiao and Wu [7] and confirmed experimentally by Tomita and Chiao [8]. A deep group-theoretic and geometrical interpretation of this Berry potential was given by Białynicki-Birula and Białynicka-Birula [9]. Here we will briefly repeat the problem by employing the general formula (2.50).

Let us define a new covariant derivative (a connection) $\hat{\vec{D}}' = (\hat{D}'_1, \hat{D}'_2, \hat{D}'_3)$ on some domain $\Omega \subset \widetilde{\mathbb{R}}^3$ [6,9,11,17,26–28];

$$\hat{D}'_l := \partial_l + i\left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}| - k_3)} + (a\partial_l b - b\partial_l a)\right)\mathbf{\Sigma},$$
$$a = a(\vec{k}), \quad b = b(\vec{k}), \quad a^2 + b^2 = 1, \quad \vec{k} \in \Omega \subset \widetilde{\mathbb{R}}^3. \tag{4.1}$$

The curvature of $\hat{\vec{D}}'$ is given as

$$\begin{aligned}[\hat{D}'_l, \hat{D}'_m] &= i\epsilon_{lmr}\frac{k_r}{|\vec{k}|^3}\mathbf{\Sigma} + i\left(\epsilon_{mr3}\frac{k_r}{|\vec{k}|(|\vec{k}| - k_3)} + a\partial_m b - b\partial_m a\right)\\ &\times \frac{\delta_{lj} - \frac{k_l k_j}{|\vec{k}|^2}}{|\vec{k}|}\mathbf{S}_j - i\left(\epsilon_{lr3}\frac{k_r}{|\vec{k}|(|\vec{k}| - k_3)} + a\partial_l b - b\partial_l a\right)\\ &\times \frac{\delta_{mj} - \frac{k_m k_j}{|\vec{k}|^2}}{|\vec{k}|}\mathbf{S}_j.\end{aligned} \tag{4.2}$$

To extract a Berry potential from the connection (4.1) consider the photon with the helicity $\lambda = \pm 1$ moving so that the photon wave function $\widetilde{\mathbf{\Psi}}(\vec{k})$ is propagated parallel with respect to $\hat{\vec{D}}'$. The momentum of photon changes along the curve

$$C : \vec{k} = \vec{k}(\tau) = (k_1(\tau), k_2(\tau), k_3(\tau)), \quad \tau_0 \leqslant \tau \leqslant \tau_1. \tag{4.3}$$

Hence

$$\begin{aligned}&\frac{dk_l}{d\tau}\hat{D}'_l\widetilde{\mathbf{\Psi}}(\vec{k}(\tau)) = 0\\ &\quad\Longrightarrow \frac{d\widetilde{\mathbf{\Psi}}}{d\tau} + i\left(\epsilon_{lm3}\frac{k_m\frac{dk_l}{d\tau}}{|\vec{k}|(|\vec{k}| - k_3)} + a\frac{db}{d\tau} - b\frac{da}{d\tau}\right)\mathbf{\Sigma}\widetilde{\mathbf{\Psi}}\\ &\quad= 0.\end{aligned} \tag{4.4}$$

Then, since

$$\mathbf{\Sigma}\widetilde{\mathbf{\Psi}} = \lambda\widetilde{\mathbf{\Psi}}, \quad \lambda = \pm 1, \tag{4.5}$$

we arrive at the ordinary differential equation

$$\frac{d\widetilde{\mathbf{\Psi}}}{d\tau} + i\lambda\left(\frac{k_2\frac{dk_1}{d\tau} - k_1\frac{dk_2}{d\tau}}{|\vec{k}|(|\vec{k}| - k_3)} + a\frac{db}{d\tau} - b\frac{da}{d\tau}\right)\widetilde{\mathbf{\Psi}} = 0. \tag{4.6}$$

The solution of Eq. (4.6) reads

$$\widetilde{\mathbf{\Psi}}(\vec{k}(\tau)) = \exp\left(i\int_C \mathcal{A}_l dk_l\right)\widetilde{\mathbf{\Psi}}_0(\vec{k}), \quad \widetilde{\mathbf{\Psi}}_0(\vec{k}) := \widetilde{\mathbf{\Psi}}(\vec{k}(\tau_0)), \tag{4.7}$$

where

$$\vec{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3),$$
$$\mathcal{A}_l := -\lambda\left(\epsilon_{lm3}\frac{k_m}{|\vec{k}|(|\vec{k}| - k_3)} + a\partial_l b - b\partial_l a\right) \tag{4.8}$$

is the Berry potential.

If C is a closed loop so that $\vec{k}(\tau_1) = \vec{k}(\tau_0) = \vec{k}$, we have, from (4.7),

$$\widetilde{\boldsymbol{\Psi}}(\vec{k}) = \exp(i\gamma[C])\widetilde{\boldsymbol{\Psi}}_0(\vec{k}), \quad \gamma[C] := \oint_C \mathcal{A}_l dk_l \tag{4.9}$$

and $\gamma[C]$ is the Berry phase.

We consider now some examples.

Example 2. Assume that $\Omega = \widetilde{\mathbb{R}}^3$ and $a = a(\vec{k})$ and $b = b(\vec{k})$ are arbitrary differentiable functions on $\widetilde{\mathbb{R}}^3$ satisfying the condition $a^2 + b^2 = 1$ [see (2.50)]. From (4.8) and (4.9) we get

$$\gamma_1[C] = \lambda \oint_C \left(\frac{k_1 dk_2 - k_2 dk_1}{|\vec{k}|(|\vec{k}| - k_3)} + a\, db - b\, da\right). \tag{4.10}$$

Then, since

$$a^2 + b^2 = 1 \Longrightarrow a\, da + b\, db = 0 \Longrightarrow da \wedge db = 0, \tag{4.11}$$

we quickly obtain

$$d(a\, db - b\, db) = 2da \wedge db = 0. \tag{4.12}$$

Consequently, as the domain $\widetilde{\mathbb{R}}^3$ is simply connected, the Stokes theorem gives

$$\oint_C a\, db - b\, da = \int_S d(a\, db - b\, da) = 0, \tag{4.13}$$

where S is a 2-surface such that the loop C is the boundary of S, $C = \partial S$. Finally, $\gamma_1[C]$ is independent of a and b and it reads

$$\gamma_1[C] = \lambda \oint_C \frac{k_1 dk_2 - k_2 dk_1}{|\vec{k}|(|\vec{k}| - k_3)}. \tag{4.14}$$

Using the spherical coordinates

$$k_1 = |\vec{k}| \sin\theta \cos\varphi, \quad k_2 = |\vec{k}| \sin\theta \sin\varphi, \quad k_3 = |\vec{k}| \cos\theta \tag{4.15}$$

for $|\vec{k}| > 0$, $0 < \theta \leqslant \pi$, and $0 \leqslant \varphi < 2\pi$, we get

$$\gamma_1[C] = \lambda \oint_C 2\cos^2\frac{\theta}{2} d\varphi. \tag{4.16}$$

In the special case when $\theta = \text{const}$ and φ changes from 0 to 2π the Berry phase (4.16) reads

$$\gamma_1[C] = \lambda 4\pi \cos^2\frac{\theta}{2} = \lambda 2\pi(\cos\theta + 1). \tag{4.17}$$

Example 3. Here we assume that $\Omega = \widetilde{\mathbb{R}}^3 \setminus \{(0, 0, k_3) \in \mathbb{R}^3 : k_3 < 0\} = \mathbb{R}^3 \setminus \{(0, 0, k_3) \in \mathbb{R}^3 : k_3 \in \mathbb{R}^1\}$, and $a = a(\vec{k})$ and $b = b(\vec{k})$ are given by (2.63). The Berry phase is now

$$\gamma_2[C] = \lambda \oint_C \left(\frac{k_1 dk_2 - k_2 dk_1}{|\vec{k}|(|\vec{k}| - k_3)} - d\varphi\right)$$
$$= \lambda \oint_C \left(2\cos^2\frac{\theta}{2} - 1\right) d\varphi$$
$$= \lambda \oint_C \cos\theta\, d\varphi = \gamma_1[C] - \lambda \oint_C d\varphi, \tag{4.18}$$

where $\gamma_1[C]$ is given by (4.16). Therefore,

$$\exp(i\gamma_1[C]) = \exp(i\gamma_2[C]). \tag{4.19}$$

If $\theta = \text{const}$ and φ changes from 0 to 2π, the Berry phase (4.18) is (see [6,9])

$$\gamma_2[C] = \lambda 2\pi \cos\theta. \tag{4.20}$$

Example 4. The domain Ω is as in Example 3; the functions a and b are given by (2.66). Now we quickly get

$$\gamma_3[C] = \lambda \oint_C 2\left(\cos^2\frac{\theta}{2} - 1\right) d\varphi = \gamma_2[C] - \lambda \oint_C d\varphi. \tag{4.21}$$

An important conclusion is that the phase factor $\exp(i\gamma[C])$ is the same in all three examples:

$$\exp(i\gamma_1[C]) = \exp(i\gamma_2[C]) = \exp(i\gamma_3[C]). \tag{4.22}$$

Finally, when $\theta = \text{const}$ and φ changes from 0 to 2π (i.e., the closed loop goes around the k_3 axis) we have (see [11])

$$\gamma_3[C] = \lambda 2\pi(\cos\theta - 1) = -\lambda 4\pi \sin^2\frac{\theta}{2}. \tag{4.23}$$

V. SUMMARY

In the paper we have shown that the Hawton position operator for the photon with commuting components can be easily derived from assumptions which are formulated in a natural manner within differential geometry language. We were able to find the general photon position operator satisfying those assumptions. We do not claim that the operator given in this paper is the only acceptable photon position operator. One can argue that it is more reasonable to assume that the components of the photon position operator do not commute. However, the operator considered in the present work has a simple geometrical interpretation, it satisfies canonical commutation rules, and it commutes with helicity operator. Our approach enables one to find the eigenfunctions of the photon position operator in an easy way. These eigenfunctions in the $\vec{x}$ representation are not spherically symmetric, which is in accordance with [1]. Of course, the spherical asymmetry is evident from the fact that the Hawton photon position operator depends on the spin operator. Moreover, the obtained eigenfunctions have an interpretation compatible with the interpretation given by Białynicki-Birula [17] and Sipe [18]. We still cannot understand in full detail the properties of the eigenfunctions found in our paper. Another problem also seems interesting and is worth considering: In previous work [19] devoted to the Weyl-Wigner-Moyal formalism of the photon we concluded that in the phase-space formulation for any quantum relativistic particle "...the problems with interpretation of the vector $\vec{x}$ are to be expected since for relativistic particles the operator $\hat{\vec{x}}$ does not represent the position observable ....” Therefore, an interesting question is if one can reformulate the Weyl-Wigner-Moyal formalism for the photon in such a way that instead of the operators $(\hat{\vec{p}}, \hat{\vec{x}})$ the operators $(\hat{\vec{p}}, \hat{\vec{X}})$ are applied. Consideration of this question is left for future work.

Note added in proof. It is proved in our paper that the different options for selecting the 3-axis are unitarily equivalent. We are convinced that this observation makes the particular choice of this axis physically irrelevant. However, a criticism of the Hawton position operator has been made by one of the referees, who claims that as a result of these transformations, the expectation value of the position operator for a given photon state can assume different values for different chosen distinguished directions, causing an ambiguity.

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