Gravity effect on quantum resonance ratchet transport of cold atoms

Anatole Kenfack*

Theoretical and Applied Physics Department, African University of Science and Technology, Km 10 Airport Road, Galadimawa, Abuja, Nigeria
and Physikalische und Theoretische Chemie, Institut für Chemie und Biochemie, Freie Universität Berlin, Takustr. 3, DE-14195 Berlin, Germany

(Received 25 February 2021; accepted 16 September 2021; published 27 September 2021)

A quantum resonance model is prepared by periodically kicking cold atoms exposed to an optical ratchet potential in the direction of the gravitational field. Within tailored parameters, intriguing phenomena emerge including absolute negative mobility, in which the atom surprisingly moves against the bias (gravity). In its upward motion, the particle current can be less or greater than or equal to the zero-gravity case, for which current reversal may prevail. Contrary to its associated zero-gravity case, the particle can be fully trapped or can slowly move downward. These phenomena occur when the classical counterpart is chaotic. With the present-day optical-lattice setup, it will be interesting to observe all these phenomena and, in particular, to see how the atom flies uphill.

DOI: 10.1103/PhysRevA.104.033321

I. INTRODUCTION

Directed transport in periodic unbiased systems with broken spatiotemporal symmetries, or ratchet transport, has continued to draw interest since it was first proven as a good candidate for understanding molecular motors [1]. Early works focused on the role of noise [2], which was later replaced by deterministic chaotic dynamics with dissipation [3,4]. Ratchet models in quantum regimes with noise and dissipation have also been studied [5,6]. Purely Hamiltonian dynamics have demonstrated ratchet effects [7–11], including those exhibiting both regular and chaotic phase-space structures [12] and complete chaos [8,13], and a recent and topical review on quantum resonance ratchets with ultracold atoms [9].

A class of these quantum systems whose classical counterparts may be chaotic [13] has received great interest quite recently. The so-called quantum δ-kicked rotor or kicked rotor has ever since become an excellent testing ground for both theoretical and experimental studies of these systems. Their experimental study has gained new impetus through its realization using cold atoms exposed to a kicked optical-lattice potential from off-resonant standing light [14]. These systems have revealed a rich variety of effects including dynamical localization [15,16], quantum resonance [14,15,17–20], tunneling [5,21], and the quantum δ-kicked accelerator or kicked accelerator [22–26]. The latter can be realized by taking a kicked rotor and adding a linear potential along the direction of the standing wave [27,28]. This is in general characterized by a linear gain in momentum as the number of kicks increases, and the mechanism behind this phenomenon can be understood as a resonant rephasing effect of the system wave function that depends on the time interval between kicks [24,26,29]. In addition, the freshly introduced "quantum ratchet accelerator" was studied in a modified kicked rotor and in a Karper model [30].

In this paper, we consider a cold atom initially released from a magneto-optical trap and δ-kicked with a vertically oriented off-resonant optical-lattice potential of the ratchet type, recently achieved experimentally [31,32]. When the kicked potential is symmetric [14], the model constitutes the so-called kicked accelerator, equivalent to a kicked rotor with an additional linear potential due to gravity. Kicked accelerators have been used to investigate aspects of the transition to chaos in both classical and quantum regimes [22] and represent systems in which quantum accelerator modes (QAMs) have been observed [23,24]. However, the model considered here has demonstrated fascinating quantum phenomena when atoms are driven in the horizontal direction, for which gravity is negligible [7,8,10,11]. The question that naturally arises is how sensitive these transport characteristics are under gravity.

Our goal is to explore the influence of gravity on ratchet transport in this kicked accelerator system within quantum resonance regimes, that is, when the quantum accelerator mode, for which the initial velocity of the particle does not meet the condition of a momentum gain after each kick, is not possible. Solving the time-dependent Schrödinger equation of this system at quantum resonance, intriguing departures have been revealed. Compared to the zero-gravity case, the particle can unexpectedly be slowed down, fully trapped, or even accelerated upward. Interestingly, the mechanisms underlying these phenomena have been unraveled.

The paper is organized as follows. We describe the model and analyze phenomena likely to occur under gravity in Sec. II. In Sec. III, our results are presented and discussed within quantum resonance regime. Section IV is devoted to conclusions.

*kenfack@mpipks-dresden.mpg.de

II. DESCRIPTION OF THE QUANTUM RESONANCE RATCHET MODEL

The Hamiltonian of the system under consideration here, the kicked accelerator, can be written for an atom with mass m as

$$\hat{H}' = \frac{\hat{p}'^2}{2m} + mg\hat{x}' + V(\hat{x}')\sum_{n=1}^{N}\delta(t'-nT), \tag{1}$$

where t' is the time variable, $\hat{x}'$ the vertical position operator and $\hat{p}'$ its associated momentum operator, m the mass of the cold-atom particle, and g the earth's gravitational acceleration in the vertical direction of the standing wave represented by the ratchet potential, $V(\hat{x}') = V_0 v(\hat{x}') = V_0(\sin(2k_L\hat{x}') + \alpha\sin(4k_L\hat{x}'))$, of depth V_0. This potential is realized with a laser of wave number $k_L = 2\pi/\lambda_L$, that is, of wavelength λ_L, resulting from the superposition of two counterpropagating lasers of spatial periodicity $\lambda_L/2$ and $\lambda_L/4$. Such a dissipationless ratchet potential has been successfully engineered [31,32]. When $\alpha = 0$, the system becomes the standard kicked accelerator [14]. Note that the time dependency is due to the excitation of the potential by a train of δ kicks of period T, where n is an integer that counts the number of kicks.

The dynamics of the system is governed by the scaled time-dependent Schrödinger equation

$$i\tilde{\hbar}\frac{\partial\psi}{\partial t} = -\frac{\tilde{\hbar}^2}{2}\frac{\partial^2\psi}{\partial x^2} + \eta x\psi + Av(x)\sum_{n=1}^{N}\delta(t-n)\psi \tag{2}$$

with the dimensionless variables $\hat{x} = 2k_L\hat{x}'$, $t = t'/T$, where $T = 1/\omega$ is the kicking period, $\hat{p} = 2k_L\hat{p}'/m\omega$, $\hat{H} = (4k_L^2/m\omega^2)\hat{H}'$, and $A = P\tilde{\hbar}$, where $P = V_0/\hbar$ is the potential strength. The relation $[\hat{x},\hat{p}] = i\tilde{\hbar}$ defines the effective Planck constant $\tilde{\hbar}$, which is related to the recoil frequency of the atom $\omega_r = \hbar k_L^2/2m$ as $\tilde{\hbar} = 8\omega_r T$. Here, $\tilde{\hbar}$ changes if one adjusts T, and $\eta = G\tilde{\hbar}^2$ is the effective gravity constant, with $G = mg^2/(8k_L^3\hbar^2)$. It turns out from Eq. (2) that the quantum map attached to this δ-kicked model, for the evolution operator of the particle between two successive kicks, is governed by the Floquet operator [27] $\hat{U} = \exp(-i\tilde{\hbar}(\hat{k}^2/2 + G\hat{x}))\exp(-iPv(\hat{x}))$, where $\hat{x}$ and $\hat{k} = -i\frac{\partial}{\partial\hat{x}}$ are the position and the wave-number operators, respectively.

In general, if $\eta = 0$, Eq. (2) is formally similar to that of a kicked rotor whose motion is in a circle. This model differs from that of a kicked particle, which moves in a line instead. The link between the two models is established by the spatial periodicity of the kicking potential, with the associated evolution operator $\hat{U}$, which commutes with spatial translations by multiples of the spatial period. As is well known from the Bloch theory, this enforces conservation of quasimomentum β, which is the fractional part of the momentum p ($p = q + \beta$, q being the integer part). Along these lines, a family of fictitious rotors parametrized by $\beta \in [0, 1)$, with angle coordinate θ, can be introduced in order to define the state of β rotors. The dynamics of an atom thus corresponds to that of a whole bundle of these β rotors which evolve independently of one another, according to their respective Hamiltonians. The energy of such an atom is expected to grow linearly rather than quadratically in time, as reported in Refs. [24,33,34]. In particular, the particle dynamics corresponds to that of a single rotor only if this particle has a sharply defined quasimomentum, including the case of the standard rotor with $\beta = 0$. This picture can be experimentally realized with states involving a narrow distribution of quasimomenta for which the dynamics may indeed be rotorlike on some finite time scale.

In the case of nonzero gravity, however, the quasimomentum is no longer conserved because the spatial periodicity of the evolution operator $\hat{U}$ is destroyed, as is the case in this work, except when G is rational. Thus direct application of the Bloch theory is no longer possible, thereby preventing the reduction of the atomic dynamics to that of rotors, which is the keystone of the theory in the absence of gravity. This difficulty is circumvented by a gauge transformation, which amounts to measuring in a momentum in a freely falling frame. This leads to a Hamiltonian which is periodic in space though not in time. Hence the quasimomentum β is constant in time, allowing reduction to β-rotor dynamics. In this framework, the β-rotor dynamics at resonant values $\tilde{\hbar}$ is a nontrivial mathematical problem which may give rise to different types of quantum transport depending on the arithmetic type of G as clearly demonstrated in Ref. [24]. It turns out that quantum resonancelike motion is only possible for specific values of quasimomentum β and gravity G. Focusing here on main quantum resonances, our calculations are subsequently performed in the laboratory frame by solving Eq. (2) straighforwardly.

The current, defined as $\langle\hat{k}\rangle = \langle\psi(t)|\hat{k}|\psi(t)\rangle$, is numerically computed from the time-evolved wave function $\langle x|\psi(t)\rangle = \langle x|\hat{U}|\psi(t-1)\rangle$, implemented with the fast Fourier split operator method [35]. The initial state is taken as a homogeneous wave packet (with zero momentum; this is also a good approximation for a wave packet that extends over many lattice sites) and parameters of the cesium atom are used, thereby leading to the value of $G = 0.0217$. A comprehensive discussion of this initial state is given in Ref. [7], where an eigenstate of the static ratchet potential (inhomogeneous) could also be a good candidate. Two major effects are likely to occur here, namely, a QAM and quantum resonance. Recall that a QAM is produced at values of the driving frequency close to (but not at) those at which quantum resonance occurs in the δ-kicked rotor or ratchet [7,8]. A QAM is characterized by the asymmetric transfer of a fixed momentum impulse per kick which is about 20% of the initial ensemble of laser-cooled atoms [24,25]. Quantum resonance, however, occurs when the flashing period is commensurate with the recoil frequency and is related to the arithmetic nature of $\tilde{\hbar}$, which satisfies $\tilde{\hbar} = 4\pi r/s$, where (r, s) are mutual prime numbers. Here the particle is always kicked at the right time, thereby rephasing the system (revival); this leads to linear growth in the width of the momentum distribution [7,8]. Remarkably high-order quantum resonance ($s > 16$), which led to large currents and unexpected current reversals, has been demonstrated in Ref. [8]. The wave function here is reconstructed via fractional revivals. It was also shown that whenever such high-order quantum resonance occurs, the classical counterpart is always fully chaotic. In what follows our numerical experiment is precisely carried out within the quantum resonance regime. Furthermore, the potential is prepared and leaned in such a way to favor the typical ratchet motion, for which $\eta = 0$, downward ($x < 0$); any current reversal [8] that may occur here would thus lead to upward motion($x > 0$). If such a current reversal occurs in the presence of gravity, one will instead speak of absolute negative mobility (ANM), a counterintuitive phenomenon which has been attracting growing interest [36–44]. Here the particle subject to ANM surprisingly moves against the bias (gravity force). This phenomenon was orignally perceived as being solely due to quantum mechanics effects [36] and was later found in classical systems as a result, for instance, of noise or phase modulation of the driving force ([42] and references therein). For consistency of our numerical simulations, we checked that our calculations do not suffer from any artifact that may be due to the nonspatial periodicity of the quantum map. This is essentially attributable to the small value of $G \approx 0.0217$ and to the small number of kicks (100 kicks; a limit difficult to reach in present-day experiments).

III. RESULTS AND DISCUSSIONS

A. Classical mapping

Mechanisms of phenomena found here can be underpinned with the knowledge of the dynamics of the classical counterpart of the system, Eq. (2), in the laboratory frame. Such a dynamics can be described in discrete steps corresponding to successive application of kicks that lead to a kick-to-kick mapping. With the momentum variable p_n associated with position x_n, after the nth kick, the corresponding classical map reads $p_{n+1} = p_n - \eta - P(\cos(x_n) + 2\alpha\cos(2x_n))$ and $x_{n+1} = x_n + p_{n+1}$, where P stands for the classical stochasticity parameter. For $\eta = 0$, the standard map is recovered. It is found that the threshold value of P, $P_{\text{thr}} \approx 3/4\pi$, which separates the regime of regular islands in a chaotic sea and that of the full chaotic sea obtained in Ref. [8], is slightly displaced to the upper limits, and the Poincaré cross sections (not shown) are qualitatively similar. Specifically, $\bar{P}_{\text{thr}} \approx 0.8\pi > P_{\text{thr}}$, where $\bar{P}_{\text{thr}}$ is the threshold for $\eta \neq 0$.

B. Exploration of emerging quantum phenomena

Figure 1 illustrates how gravity actually affects the ratchet current $\langle k\rangle$ displayed as a function of the number of kicks, for specific values of P and $\tilde{\hbar}$ at quantum resonance. Each panel highlights a typical scenario likely to appear in this model for $\eta = 0$ (black curves) and for $\eta \neq 0$ (red curves). While one would expect the particle to accelerate downward due to gravity as in Fig. 1(a), it is not obvious to predict the scenarios depicted in the other panels, typically the trapping of the particle and, especially, its upward motion. Though the mechanism of current reversal in this model has hitherto not been theoretically investigated, its manifestation is crucial. Much like the effect of current reversal in [8], we have numerically checked that ANM also goes hand in hand with full chaos, with $\bar{P}_{\text{thr}}$ the corresponding threshold. Taking into account the gravity for a particle undergoing a current reversal, these phenomena may occur: (i) The particle is decelerated in its upward motion [Fig. 1(b)]. (ii) The particle continues to move upward as there is no gravity at all [Fig. 1(c)]. (iii) In Fig. 1(d), the current is stronger, with the gravity thereby accelerating the particle more upward; the gravity clearly enhances the uphill motion. This additional effect will be scrutinized further. Still, in this current reversal regime, the gravity effect can be strong in such a way that (iv) the particle completely reverses its motion [Fig. 1(e)], a sort of ANM rectifier, or (v) the particle can be simply trapped [Fig. 1(f)]. In the latter two cases, ANM is eradicated, thereby leaving room for the normal downward acceleration of the particle or its trapping. Finally, in cases where quantum resonance is not resolved, in general because of the weakness of the potential strength, the gravity affects the system as follows: (vi) upward motion is unexpectedly turned on [Fig. 1(g)], or (vii) downward motion is activated [Fig. 1(h)].

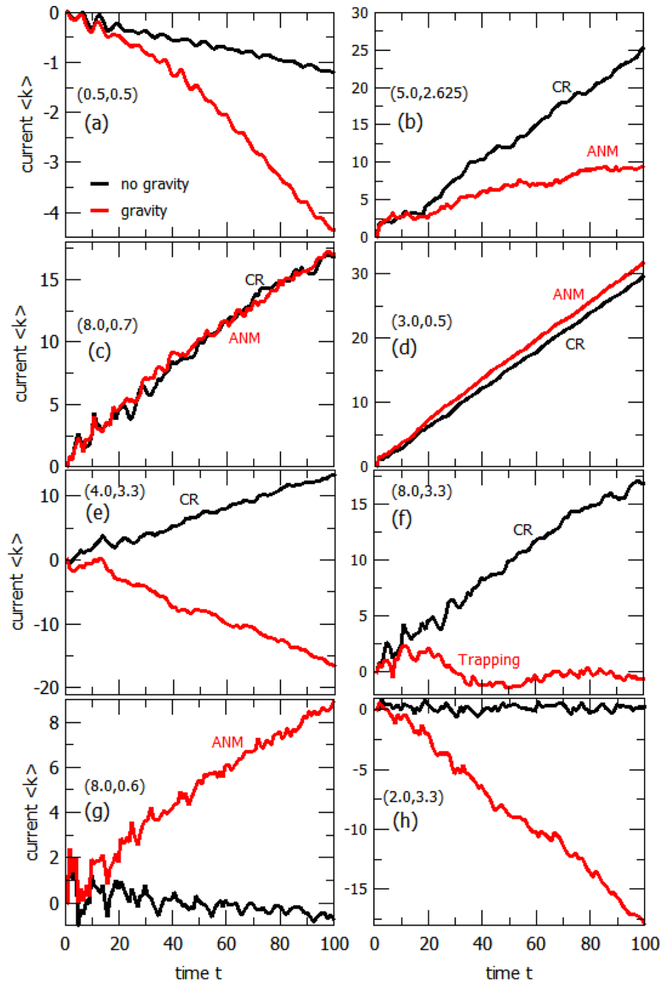

FIG. 1. Current as a function of time t, for given P and $\tilde{\hbar}$. Each panel, with $(P, \tilde{\hbar}/\pi)$ as indicated, represents a typical phenomenon emphasized in the absence (black curves) and in the presence (red curves) of gravity: (a) normal, (b) ANM, (c) ANM as there is no gravity, (d) enhanced ANM, (e) current reversal (CR) eradication, (f) trapping, (g) ANM activation, and (h) normal.

At this stage, it turns out that ANM alone is not enough to fully explain some of the phenomena observed here, in particular, when the particle is accelerated more upward compared to the zero-gravity case [Fig. 1(d)] or when gravity has no effect [Fig. 1(c)]. This can be understood by noting that the dynamics of the particle has now been performed in the tilted ratchet potential $\bar{v}(x) = G\tilde{\hbar}x + Pv(x)$, whose effect is to induce an additional momentum kick Δp that may contribute to reducing or increasing the particle current.

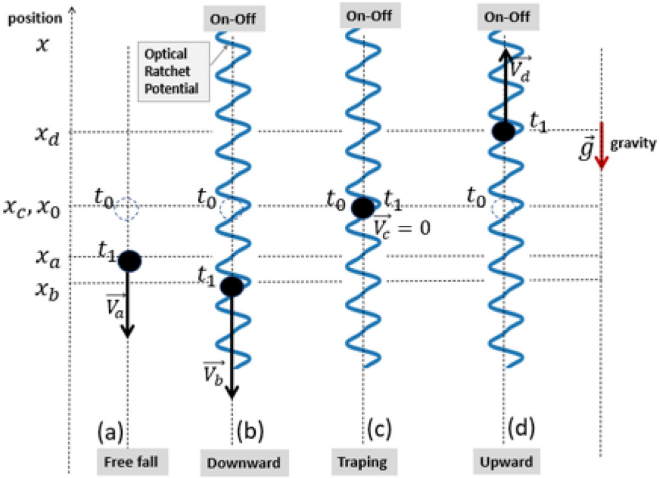

FIG. 2. Quantum resonance transport scenarios of an on-off optical ratchet particle (filled circle), subject to gravity, initially released at zero velocity at time t_0 from position x_0 and pictured at a later time, t_1, at positions (x_a)–(x_d), associated with the velocities $(\vec{V}_a)$–$(\vec{V}_d)$, respectively. (a) Free fall, (b) downward motion, (c) trapping, and (d) upward motion.

The key scenarios emerging from the dynamics of the particle, subject to gravity, are sketched in Fig. 2. While free fall [Fig. 2(a)] occurs at zero potential, downward motion [Fig. 2(b)], trapping [Fig. 2(c)], and upward motion [Fig. 2(d)] are induced by the on-off excitation of the potential. Here the particle is initially released from the magneto-optical trap at t_0 with zero velocity and is pictured at a later time t_1 with velocities $\vec{V}_a$, $\vec{V}_b$, $\vec{V}_c$, and $\vec{V}_d$, associated with different motions as indicated in Fig. 2. We note in passing that the dynamics of the kicked ratchet may be restored if the gravity effect is counteracted by appropriately shifting the potential as discussed in Ref. [27], where the decoherence due to noise can affect the phenomena expected here.

A global trend of such a rich transport would be crucial, especially as far as experiments are concerned. This picture can be obtained by computing, at a given quantum resonance, the acceleration rate $\Gamma(t)$, defined as

$$\Gamma(t) = \frac{\Delta\langle k\rangle(t)}{\Delta t} = \frac{\langle k\rangle(t) - \langle k\rangle(t_0)}{t - t_0}, \tag{3}$$

where $\langle k\rangle(t)$ is the current evaluated at time t. Figure 3 displays $\Gamma(t = t_1)$, after $t_1 = 100$ kicks, as a function of P and for some values of $\tilde{\hbar}$ satisfying quantum resonance. As in Fig. 1, the black and red curves stand for the zero-gravity and nonzero-gravity cases, respectively, and for these values of $\tilde{\hbar}$ at quantum resonance: 3.3π [Fig. 3(a)], 1.5π [Fig. 3(b)], 1.125π [Fig. 3(c)], and 1.55π [Fig. 3(d)]. These results are consistent with the phenomena found above for which the motion of the particle is downward ($\Gamma < 0.0$), upward ($\Gamma > 0.0$), or fully trapped ($\Gamma = 0.0$). Besides standard current reversal and ANM, other unexpected effects clearly occur: (i) Gravity has no effect and the gravity and no-gravity rates are identical; this is clearly shown in Fig. 3(b) for values of $P \in [4.25, 6.45]$. (ii) The particle moves faster or slower than that under zero gravity; examples of such cases are shown, for instance, in Fig. 3(c) for $P \in [7.0, 7.5]$ and in Fig. 3(d) for $P \in [6.0, 6.75]$. These plots are potentially good guides that could help in choosing parameters for a desired transport type. In addition, we have computed the momentum distribution wave function $|\langle k|\psi\rangle|^2$ after 100 kicks at quantum resonance for a few values of $(P, \tilde{\hbar}/\pi)$ taken from Fig. 3. The values of $(P, \tilde{\hbar}/\pi)$ indicated in each panel describe these cases: the current is zero for no gravity and nonzero for gravity (0.4,3.3) [Fig. 3(a)], the current is zero for no gravity and zero for gravity (2.9833,3.3) [Fig. 3(a)], the currents are nonzero and identical for both no gravity and gravity (5.6,1.5) [Fig. 3(b)], and the currents are nonzero and nonidentical (4.0,3.3) [Fig. 3(a)]. As is clearly shown, apart from the distributions in Fig. 4(a) (no gravity) and Fig. 4(b) (gravity), which are

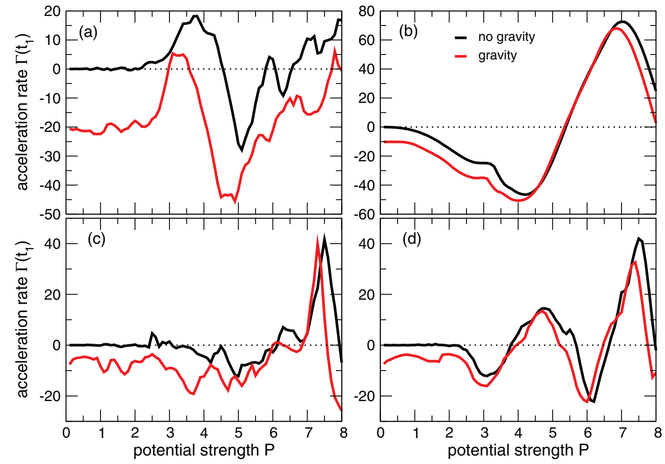

FIG. 3. Acceleration rate $\Gamma(t_1)$ as a function of P, evaluated after $t_1 = 100$ kicks and for given values of $\tilde{\hbar}$ at quantum resonance: (a) $\tilde{\hbar} = 3.3\pi$, (b) $\tilde{\hbar} = 1.5\pi$, (c) $\tilde{\hbar} = 1.125\pi$, and (d) $\tilde{\hbar} = 1.55\pi$. Throughout, various regimes of current reversal, ANM, enhanced ANM, trapping, and no effect of gravity are well resolved.

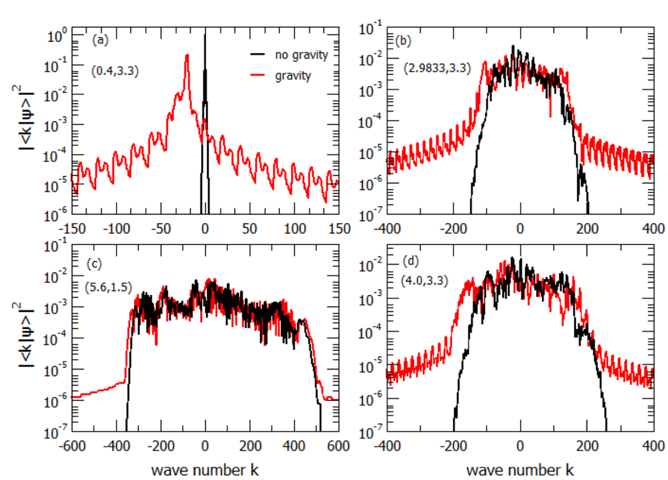

FIG. 4. Momentum distribution wave function $|\langle k|\psi\rangle|^2$ as a function of k computed after 100 kicks at quantum resonance for $(P, \tilde{\hbar}/\pi)$ indicated in each panel as in Fig. 1. As detailed in the text, symmetry and asymmetry distributions are fingerprints of Trapping [(a) black curve, (b) red curve] and directed transport (other curves), respectively.

symmetric as their associated currents are zero, the remaining cases are asymmetric, corresponding to the nonzero currents.

IV. CONCLUSIONS

To sum up, a transport phenomenon is identified in a system made by δ-kicking cold atoms exposed to an optical lattice ratchet potential prepared in the vertical direction and subject to gravity. Within tailored parameters and at quantum resonance, a number of interesting phenomena have emerged including current reversal (zero gravity), full trapping of the particle, and absolute negative mobility, in which the particle paradoxically flies uphill, that is, against the bias (gravity). Besides, compared to the zero-gravity case, it is also found that the particle in its uphill motion can be either faster or slower or identical, while the particle in its downward motion can be surprisingly slower. ANM as well as other fascinating ANM-related phenomena all go hand in hand with the classical counterpart chaos, thereby suggesting its theoretical analysis. Remarkably, these predictions should be readily tested in the present-day optical-lattice setup. In particular, the acceleration rate would be of great importance, especially where the choice of experimental parameters is concerned. One would also be interested to know how the ratchet potential influences QAMs which may occur, while the quasimomentum will be taken into account via the gauge transformation within the free-falling frame; this would be a great alternative to direct calculations in the laboratory frame. More other resonances due to QAMs may occur and influence the inherent current reversal and ANM found in this work. Finally, the introduction of noise in the present model may lead to decoherence, which may affect phenomena observed here.

ACKNOWLEDGMENTS

Useful comments and discussions with Kamal P. Singh are greatly appreciated. This work was supported by the African University of Science and Technology (AUST), Abuja, Nigeria, through African Development Bank funding.

[1] F. Jülicher *et al.*, Rev. Mod. Phys. **69**, 1269 (1997).

[2] P. Reimann, Phys. Rep. **361**, 57 (2002); R. D. Astumian and P. Hänggi, Phys. Today **55**, 33 (2002); P. Hänggi and F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009).

[3] P. Jung, J. G. Kissner, and P. Hänggi, Phys. Rev. Lett. **76**, 3436 (1996); M. Porto, M. Urbakh, and J. Klafter, *ibid.* **85**, 491 (2000); S. Flach, O. Yevtushenko, and Y. Zolotaryuk, *ibid.* **84**, 2358 (2000).

[4] J. L. Mateos, Phys. Rev. Lett. **84**, 258 (2000); U. E. Vincent, A. Kenfack, A. N. Njah, and O. Akinlade, Phys. Rev. E **72**, 056213 (2005); A. Kenfack, S. M. Sweetnam, and A. K. Pattanayak, *ibid.* **75**, 056215 (2007); U. E. Vincent, A. Kenfack, D. V. Senthilkumar, D. Mayer, and J. Kurths, *ibid.* **82**, 046208 (2010); B. S. Dandogbessi *et al.*, Phys. Scripta **90**, 055206 (2015); B. S. Dandogbessi, *ibid.* **90**, 069601 (2015).

[5] P. Reimann, M. Grifoni, and P. Hänggi, Phys. Rev. Lett. **79**, 10 (1997); P. Reimann *et al.*, Chaos **8**, 629 (1997).

[6] G. G. Carlo, Phys. Rev. Lett. **108**, 210605 (2012); L. Ermann and G. G. Carlo, Phys. Rev. E **91**, 010903(R) (2015).

[7] E. Lundh and M. Wallin, Phys. Rev. Lett. **94**, 110603 (2005).

[8] A. Kenfack, J. Gong, and A. K. Pattanayak, Phys. Rev. Lett. **100**, 044104 (2008).

[9] J. Ni, S. Dadras, W. K. Lam, R. K. Shrestha, M. Sadgrove, S. Wimberger, and G. S. Summy, Ann. Phys. **529**, 1600335 (2017).

[10] L. Chen *et al.*, Physica A **389**, 4071 (2010); L. Chen, C. Xiong, H.-C. Yuan, L.-H. Ding, *ibid.* **398**, 83 (2014).

[11] T. Salger, S. Kling, T. Hecking, C. Geckeler, L. Morales-Molina, and M. Weitz, Science **326**, 1241 (2009).

[12] T. Dittrich *et al.*, Ann. Phys. (Leipzig) **9**, 755 (2000); H. Schanz, M.-F. Otto, R. Ketzmerick, and T. Dittrich, Phys. Rev. Lett. **87**, 070601 (2001); T. Cheon *et al.*, J. Phys. Soc. Jpn. **72**, 1087 (2003); S. Denisov, J. Klafter, and M. Urbakh, Phys. Rev. E **66**, 046203 (2002); T. S. Monteiro, P. A. Dando, N. A. C. Hutchings, and M. R. Isherwood, Phys. Rev. Lett. **89**, 194102 (2002).

[13] G. Casati and B. Chirikov, *Quantum Chaos: Between Order and Disorder*, editor by G. Casati and B. Chirikov, (Cambridge University Press, Cambridge, UK, 1995).

[14] F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. **75**, 4598 (1995).

[15] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Phys. Rev. Lett. **73**, 2974 (1994).

[16] J. Ringot, P. Szriftgiser, J. C. Garreau, and D. Delande, Phys. Rev. Lett. **85**, 2741 (2000).

[17] F. M. Izrailev and D. L. Shepelyansky, Theor. Math. Phys. **43**, 553 (1980).

[18] C. Ryu, M. F. Andersen, A. Vaziri, M. B. d'Arcy, J. M. Grossman, K. Helmerson, and W. D. Phillips, Phys. Rev. Lett. **96**, 160403 (2006); F. M. Izrailev, Phys. Rep. **196**, 299 (1990); W. H. Oskay *et al.*, Opt. Commun. **179**, 137 (2000).

[19] A. J. Daley and A. S. Parkins, Phys. Rev. E **66**, 056210 (2002).

[20] G. J. Duffy, S. Parkins, T. Muller, M. Sadgrove, R. Leonhardt, and A. C. Wilson, Phys. Rev. E **70**, 056206 (2004).

[21] H. Linke *et al.*, Science **286**, 2314 (1999).

[22] L. E. Reichl, *The Transition to Chaos*, 2nd ed. (Springer, New York, 2004).

[23] M. K. Oberthaler, R. M. Godun, M. B. d'Arcy, G. S. Summy, and K. Burnett, Phys. Rev. Lett. **83**, 4447 (1999); G. Behinaein, V. Ramareddy, P. Ahmadi, and G. S. Summy, *ibid.* **97**, 244101 (2006); P. Ahmadi, G. Behinaein, V. Ramareddy, and G. S. Summy, Phys. Rev. A **80**, 053418 (2009); V. Ramareddy *et al.*, Europhys. Lett. **89**, 33001 (2010); R. K. Shrestha, S. Wimberger, J. Ni, W. K. Lam, and G. S. Summy, Phys. Rev. E **87**, 020902(R) (2013).

[24] S. Fishman, I. Guarneri, and L. Rebuzzini, Phys. Rev. Lett. **89**, 084101 (2002); S. Fishman, J. Stat. Phys. **110**, 911 (2003).

[25] S. Schlunk, M. B. d'Arcy, S. A. Gardiner, and G. S. Summy, Phys. Rev. Lett. **90**, 124102 (2003).

[26] R. M. Godun, M. B. d'Arcy, M. K. Oberthaler, G. S. Summy, and K. Burnett, Phys. Rev. A **62**, 013411 (2000).

[27] M. B. dArcy, R. M. Godun, M. K. Oberthaler, G. S. Summy, K. Burnett, and S. A. Gardiner, Phys. Rev. E **64**, 056233 (2001).

[28] P. H. Jones, M. Goonasekera, D. R. Meacher, T. Jonckheere, and T. S. Monteiro, Phys. Rev. Lett. **98**, 073002 (2007).

[29] Z.-Y. Ma, M. B. d'Arcy, and S. A. Gardiner, Phys. Rev. Lett. **93**, 164101 (2004).

[30] J. Gong and P. Brumer, Phys. Rev. E **70**, 016202 (2004); J. Gong, Phys. Rev. Lett. **97**, 240602 (2006).

[31] G. Ritt, C. Geckeler, T. Salger, G. Cennini, and M. Weitz, Phys. Rev. A **74**, 063622 (2006).

[32] T. Salger, C. Geckeler, S. Kling, and M. Weitz, Phys. Rev. Lett. **99**, 190405 (2007).

[33] Y. Sadgrove and S. Wimberger, Adv. At. Mol. Opt. Phys. **60**, 315 (2011).

[34] S. Wimberger, S. Guarneri, and S. Fishman, Nonlinearity **16**, 1381 (2003).

[35] M. D. Feit *et al.*, J. Comp. Phys. **47**, 412 (1982).

[36] B. J. Keay, S. Zeuner, S. J. Allen, Jr., K. D. Maranowski, A. C. Gossard, U. Bhattacharya, and M. J. W. Rodwell, Phys. Rev. Lett. **75**, 4102 (1995).

[37] R. Eichhorn, P. Reimann, and P. Hänggi, Phys. Rev. Lett. **88**, 190601 (2002).

[38] B. Cleuren and C. Van den Broeck, Phys. Rev. E **67**, 055101(R) (2003).

[39] A. Ros, R. Eichhorn, J. Regtmeier, T. T. Duong, P. Reimann, and D. Anselmetti, Nature (London) **436**, 928 (2005).

[40] L. Du and D. Mei, Phys. Rev. E **85**, 011148 (2012).

[41] C. Mulhern, Phys. Rev. E **88**, 022906 (2013).

[42] B. S. Dandogbessi and A. Kenfack, Phys. Rev. E **92**, 062903 (2015).

[43] A. Slapik, J. Luczka, P. Hänggi, and J. Spiechowicz, Phys. Rev. Lett. **122**, 070602 (2019).

[44] Y. Luo *et al.*, Chaos **30**, 053115 (2020).