Fundamental transfer matrix and dynamical formulation of stationary scattering in two and three dimensions

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We offer a consistent dynamical formulation of stationary scattering in two and three dimensions (2D and 3D) that is based on a suitable multidimensional generalization of the transfer matrix. This is a linear operator acting in an infinite-dimensional function space which we can represent as a 2×2 matrix with operator entries. This operator encodes the information about the scattering properties of the potential and enjoys an analog of the composition property of its one-dimensional ancestor. Our results improve an earlier attempt in this direction [Phys. Rev. A **93**, 042707 (2016)] by elucidating the role of the evanescent waves. We show that a proper formulation of this approach requires the introduction of a pair of intertwined transfer matrices, each related to the time-evolution operator for an effective nonunitary quantum system. We study the application of our findings in the treatment of the scattering problem for δ-function potentials in 2D and 3D and clarify its implicit regularization property which circumvents the singular terms appearing in the standard treatments of these potentials. We also discuss the utility of our approach in characterizing invisible (scattering-free) potentials and potentials for which the first Born approximation provides the exact expression for the scattering amplitude.

I. INTRODUCTION

Scattering of waves is a natural phenomenon of great importance. This was recognized quite early and has led to a comprehensive study of this phenomenon in the 19th and 20th centuries. The formulation of quantum scattering theory by Born [1], the introduction of the concept of the scattering (S) matrix by Wheeler [2], and the developments of the Green's function methods for treating scattering problems by Lippmann and Schwinger [3] are the milestones of our present understanding of the subject. The study and applications of the Born series, the S-matrix, and the Lippman-Schwinger equation constitute an integral part of the standard textbook treatments of scattering theory [4,5]. They have also provided the main impetus for developing rigorous mathematical theories of scattering [6–8]. There is essentially no conceptual difference between the utility of these concepts and theories in dealing with the scattering of waves in different dimensions. However, in one dimension (1D), one has the option of employing an alternative tool, called the *transfer matrix* [9–11]. This is an object which, like the S-matrix, stores the information about the scattering features of the system and enjoys a composition property that makes it an ideal tool for dealing with multilayer and locally periodic systems [12–15].

Consider a short-range potential in 1D $v : \mathbb{R} \to \mathbb{C}$, so that $|v(x)|$ tends to zero faster than $|x|^{-1}$ as $x \to \pm\infty$. Then every solution of the stationary Schrödinger equation

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x) \quad x \in \mathbb{R}, \tag{1}$$

satisfies

$$\psi(x) \to \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \to -\infty, \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \to +\infty, \end{cases}$$

where $k \in \mathbb{R}^+$ is a wave number, and $A_\pm$ and $B_\pm$ are complex coefficients. The transfer matrix of the potential v is a 2×2 matrix **M** that relates $A_\pm$ and $B_\pm$ according to

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}. \tag{2}$$

This equation determines **M** in a unique manner, provided that it is independent of A_- and B_- [16]. We can also identify the S-matrix for v by the 2×2 matrix **S** that satisfies [17,18]

$$\begin{bmatrix} A_+ \\ B_- \end{bmatrix} = \mathbf{S} \begin{bmatrix} A_- \\ B_+ \end{bmatrix}. \tag{3}$$

If we enforce (2) and (3) for the coefficients $A_\pm$ and $B_\pm$ of the left- and right-incident scattering solutions $\psi_{l/r}$ of (1), i.e., those fulfilling

$$\psi_l(x) \to \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for } x \to -\infty, \\ T^l\, e^{ikx} & \text{for } x \to +\infty, \end{cases}$$

$$\psi_r(x) \to \begin{cases} T^r\, e^{-ikx} & \text{for } x \to -\infty, \\ e^{-ikx} + R^r e^{ikx} & \text{for } x \to +\infty, \end{cases} \tag{4}$$

we can express the left/right reflection and transmission amplitudes $R^{l/r}$ and $T^{l/r}$ of the potential in terms of the entries of

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M and **S** [15,17,18]:

$$R^l = -\frac{M_{21}}{M_{22}}, \quad R^r = \frac{M_{12}}{M_{22}}, \quad T^{l/r} = \frac{1}{M_{22}}, \quad (5)$$

$$R^l = S_{21}, \quad R^r = S_{12}, \quad T^{l/r} = S_{11} = S_{22}. \quad (6)$$

These relations show that both **M** and **S** contain the information about the scattering properties of the potential. Therefore, their determination is equivalent to the solution of the scattering problem.

The advantage of the transfer matrix over the S-matrix is its composition property. If we divide $\mathbb{R}$ into n adjacent intervals of the form

$$I_1 := (-\infty, a_1), \qquad I_2 := [a_1, a_2),$$
$$I_3 := [a_2, a_3), \qquad \cdots \qquad I_{n-1} := [a_{n-2}, a_{n-1}),$$
$$I_n := [a_{n-1}, \infty),$$

where $a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ such that $a_1 < a_2 < \cdots < a_{n-1}$, let $v_j : \mathbb{R} \to \mathbb{C}$ be the truncation of v given by

$$v_j(x) := \begin{cases} v(x) & \text{for } x \in I_j, \\ 0 & \text{for } x \notin I_j, \end{cases}$$

and $\mathbf{M}_j$ be the transfer matrix of v_j, then the following composition rule holds [15]:

$$\mathbf{M} = \mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_1. \quad (7)$$

This relation is the main reason for the wide range of applications of the transfer matrix [12,19–33]. It has provided the main guideline for the development of the multichannel [34–38] and higher-dimensional [39–45] generalizations of the transfer matrix (2). The latter generalizations involve discretization of the configuration or momentum space variables along the normal direction(s) to the scattering/propagation axis of the wave and yield large numerical transfer matrices whose treatment requires appropriate numerical schemes. Therefore, unlike the S-matrix, they do not arise from a fundamental concept with a universal definition.

Reference [46] pursues a different route and arrives at a higher-dimensional notion of transfer matrix whose definition does not involve any discretization or approximation scheme. This is a linear operator acting in an infinite-dimensional function space. It admits a suitable realization in terms of a 2×2 matrix with operator entries that has the same structure as the transfer matrix in 1D. It also shares the basic properties of the latter; namely, it carries the information about the scattering features of the potential and possesses a similar composition property. Another appealing feature of this transfer matrix is that it admits a Dyson series expansion. This follows from a higher-dimensional generalization of a dynamical formulation of stationary scattering in 1D [47], where the standard transfer matrix is identified with the S-matrix of an effective nonunitary two-level quantum system [48]. This observation has interesting applications in constructing tunable unidirectionally invisible potentials [15], extending the notion of the transfer matrix to long-range potentials [49] and performing low-energy scattering calculations in 1D [50]. See also Ref. [51].

The application of the transfer matrix of Ref. [46] to δ-function potentials in two and three dimensions (2D and 3D)

turns out not to involve the unwanted singularities of the standard treatments of these potentials [52–60]. Furthermore, this approach to scattering theory opens up a line of attack on the problem of identifying invisible (scattering-free) complex potentials [61–64] and admits a useful electromagnetic counterpart [65].

A careful examination of the analysis of Ref. [46] reveals certain underlying assumptions and implicit rules of calculation that require further justification. These shortcomings may be traced back to an inadequate treatment of the role of the evanescent waves. These waves do not enter the general expression for the transfer matrix in an explicit manner, but their contribution to the scattering phenomenon cannot be ignored in general. They play an important role in the identification of the correct form of the composition property of the transfer matrix which makes it into a powerful practical tool. The purpose of this paper is to offer a comprehensive refinement of the concept of a fundamental transfer matrix in 2D and 3D that eliminates the shortcomings of Ref. [46] and yields a consistent alternative to the standard treatment of stationary scattering.

The organization of this paper is as follows. In Sec. II, we introduce a variant of the transfer matrix of Ref. [46] in 2D, called the *fundamental transfer matrix*, which considers the contribution of the evanescent waves. In Sec. III, we identify a Hamiltonian operator whose evolution operator yields the fundamental transfer matrix. In Sec. IV, we derive the composition rule for the fundamental transfer matrix. This turns out to require the introduction of an alternative (auxiliary) transfer matrix which is also related to the evolution operator of an effective quantum system. In Sec. V, we examine the utility of the auxiliary transfer matrix in the solution of the scattering problem and derive a basic formula describing its relationship to the fundamental transfer matrix. In Sec. VI, we present the application of our general results to δ-function potentials in 2D. Here, we elucidate the conceptual basis for the implicit regularization property of the fundamental transfer matrix that makes it avoid the singularities of the standard treatments of these potentials. In Sec. VII, we discuss the application of the fundamental transfer matrix in constructing potentials that display perfect broadband omnidirectional invisibility in 2D. Here, we also present a characterization theorem for potentials for which the first Born approximation gives the exact expression for the scattering amplitude. In Sec. VIII, we generalize the results of Secs. II–VII to 3D. Section IX provides a summary of our findings, and the Appendixes include the derivation of some of the results we present in the text.

II. FUNDAMENTAL TRANSFER MATRIX IN 2D

Consider the stationary Schrödinger equation in 2D:

$$\left[-\partial_x^2 - \partial_y^2 + v(x, y) \right] \psi(x, y) = k^2 \psi(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (8)$$

for a short-range potential $v : \mathbb{R}^2 \to \mathbb{C}$ and a wave number $k \in \mathbb{R}^+$. This equation has a unique solution possessing the
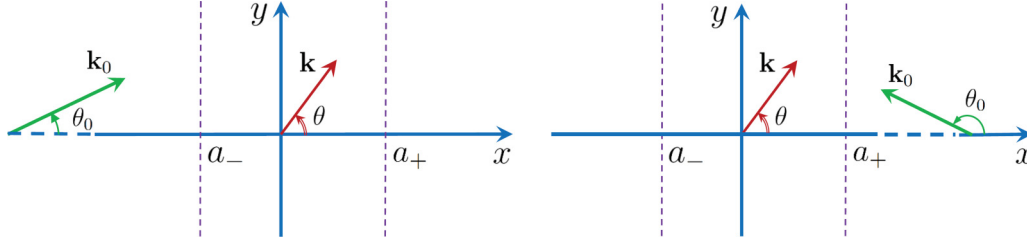
FIG. 1. Schematic view of the scattering setup for a left-incident wave (on the left) and a right-incident wave (on the right). $\mathbf{k}_0$ and $\mathbf{k}$ are, respectively, the incident and scattered wave vectors. For the left- and right-incident waves, the incidence angle θ_0 takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$. Dashed lines represent $x = a_\pm$.

following asymptotic expression [8]:

$$\psi(\mathbf{r}) = \frac{1}{2\pi} \left[e^{i\mathbf{k}_0 \cdot \mathbf{r}} + \sqrt{\frac{i}{kr}} \, e^{ikr} \mathfrak{f}(\theta) \right] + o(r^{-1/2}) \quad \text{for } r \to \infty, \tag{9}$$

where $\mathbf{k}_0 \in \mathbb{R}^2$ is the incident wave vector, $\mathbf{r} := x \, \mathbf{e}_x + y \, \mathbf{e}_y$, $\mathbf{e}_j$ is the unit vector along the j axis for $j \in \{x, y\}$, (r, θ) are the polar coordinates of $\mathbf{r}$, $\mathbf{k}_0 \cdot \mathbf{r} := kr \cos(\theta - \theta_0)$, θ_0 is the incidence angle which determines $\mathbf{k}_0/k$ according to $\mathbf{k}_0/k = \cos\theta_0 \, \mathbf{e}_x + \sin\theta_0 \, \mathbf{e}_y$, $\mathfrak{f}$ is a complex-valued function called the *scattering amplitude*, and $o(r^{-1/2})$ stands for a function of r and θ such that

$$\lim_{r \to \infty} r^{1/2} o(r^{-1/2}) = 0.$$

By solving the scattering problem for the potential v, we mean determining $\mathfrak{f}(\theta)$ for every wave number k and incidence angle θ_0.

Let us adopt a coordinate system in which the source of the incident wave and the detectors used to observe the scattered wave lie on the planes $x = \pm\infty$. If the source of the incident wave lies at $x = -\infty$, $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and we speak of a left-incident wave. If the source lies at $x = +\infty$, $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and we have a right-incident wave. See Fig. 1. We use $\mathfrak{f}^l$ and $\mathfrak{f}^r$ to label the scattering amplitudes for the left- and right-incident waves, respectively:

$$\mathfrak{f}(\theta) = \begin{cases} \mathfrak{f}^l(\theta) & \text{for } \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \mathfrak{f}^r(\theta) & \text{for } \theta_0 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases} \tag{10}$$

Let $\mathcal{F}_{y,p}$ and $\mathcal{F}_{p,y}^{-1}$ denote the Fourier transformation of a function of y and its inverse, i.e.,

$$\mathcal{F}_{y,p}\{f(y)\} := \tilde{f}(p) := \int_{-\infty}^{\infty} dy \, e^{-ipy} f(y), \tag{11}$$

$$\mathcal{F}_{p,y}^{-1}\{g(p)\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{ipy} g(p), \tag{12}$$

where $f, g : \mathbb{R} \to \mathbb{C}$ are functions.[1] Performing the Fourier transform of both sides of (8) with respect to y, we find

$$-\tilde{\psi}''(x, p) + [\widehat{\mathscr{V}}(x)\tilde{\psi}](x, p)$$
$$= \varpi(p)^2 \, \tilde{\psi}(x, p), \quad (x, p) \in \mathbb{R}^2, \tag{13}$$

where a prime stands for differentiation with respect to x, $\tilde{\psi}(x, p) := \mathcal{F}_{y,p}\{\psi(x, y)\}$,

$$[\widehat{\mathscr{V}}(x)\tilde{f}](p) := \mathcal{F}_{y,p}\{v(x, y)f(y)\}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \, \tilde{v}(x, p - q)\tilde{f}(q), \tag{14}$$

and

$$\varpi(p) := \begin{cases} \sqrt{k^2 - p^2} & \text{for } |p| < k, \\ i\sqrt{p^2 - k^2} & \text{for } |p| \geqslant k. \end{cases} \tag{15}$$

An important feature of potential scattering in 2D (and higher dimensions), which has no counterpart in 1D, is the presence of evanescent waves. To elucidate their role, in the following, we confine our attention to the class of potentials whose supports lie between a pair of lines that are parallel to the y axis. In other words, we suppose that there are real numbers $a_\pm$ such that $a_- < a_+$ and

$$v(x, y) = 0 \quad \text{for } x \notin [a_-, a_+]. \tag{16}$$

Then $\tilde{v}(x, p) = 0$ for $x \notin [a_-, a_+]$, and (13) gives

$$\left[\partial_x^2 + \varpi(p)^2\right]\tilde{\psi}(x, p) = 0 \quad \text{for } x \notin [a_-, a_+].$$

Solving this equation and performing the inverse Fourier transform with respect to p, we can write ψ in the form

$$\psi = \psi_{\text{os}} + \psi_{\text{ev}}, \tag{17}$$

where $\psi_{\text{os}}, \psi_{\text{ev}} : \mathbb{R}^2 \to \mathbb{C}$ are, respectively, functions representing the oscillating and evanescent waves outside the region defined by $a_- < x < a_+$ in $\mathbb{R}^2$. This means that

$$\psi_{\text{os}}(x, y) = \frac{1}{2\pi} \int_{-k}^{k} dp \left[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x} \right] e^{ipy}$$
$$\text{for } \pm x \geqslant \pm a_\pm, \tag{18}$$

$$\psi_{\text{ev}}(x, y) = \frac{1}{2\pi} \int_{|p| \geqslant k} dp \, C_\pm(p)e^{\mp|\varpi(p)|x} e^{ipy} \quad \text{for } \pm x \geqslant \pm a_\pm, \tag{19}$$

where $A_\pm, B_\pm, C_\pm : \mathbb{R} \to \mathbb{C}$ are functions such that[2]

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for } |p| \geqslant k,$$
$$C_\pm(p) = 0 \quad \text{for } |p| < k. \tag{20}$$

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[1]Throughout this paper, we use the term *function* to mean a classical function or a tempered distribution [66].

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[2]More precisely, $A_\pm$ and $B_\pm$ (respectively, $C_\pm$) are tempered distributions supported in the open interval $(-k, k)$ [respectively, $\mathbb{R} \setminus (-k, k)$].

In the following, we use $\mathscr{F}_k$ to denote the vector space of functions ϕ such that $\phi(p) = 0$ for $|p| \geqslant k$. Then (20) states that $A_\pm, B_\pm \in \mathscr{F}_k$.

We also introduce

$$\mathscr{B}_-(p) := B_-(p) + C_-(p) = \begin{cases} B_-(p) & \text{for } |p| < k, \\ C_-(p) & \text{for } |p| \geqslant k, \end{cases} \quad (21)$$

$$\mathscr{A}_+(p) := A_+(p) + C_+(p) = \begin{cases} A_+(p) & \text{for } |p| < k, \\ C_+(p) & \text{for } |p| \geqslant k, \end{cases} \quad (22)$$

for future use and employ (11) and (17)–(22) to conclude that

$$\tilde{\psi}(x, p) = \begin{cases} A_-(p)e^{i\varpi(p)x} + \mathscr{B}_-(p)e^{-i\varpi(p)x} & \text{for } x \leqslant a_-, \\ \mathscr{A}_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x} & \text{for } x \geqslant a_+. \end{cases}$$
$$(23)$$

According to (19), $\psi_{\text{ev}}(x, y) \to 0$ as $x \to \pm\infty$. In light of (9) and (17), this means that the scattering properties of the potential are encoded in the functions $A_\pm$ and $B_\pm$. To see this, we examine the scattering of the left- and right-incident waves separately, use $A_\pm^{l/r}$ and $B_\pm^{l/r}$ to identify the functions $A_\pm$ and $B_\pm$ associated with left/right-incident waves, and set

$$p := k \sin\theta, \quad p_0 := k \sin\theta_0. \quad (24)$$

For a left-indecent wave, $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and as we show in Ref. [46, Appendix A],

$$A_-^l(p) = 2\pi\delta(p - p_0) = \frac{2\pi\,\delta(\sin\theta - \sin\theta_0)}{k} = \frac{2\pi\,\delta(\theta - \theta_0)}{k|\cos\theta_0|}, \quad B_+^l(p) = 0, \quad (25)$$

$$\mathfrak{f}^l(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} k|\cos\theta|A_+^l(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ k|\cos\theta|B_-^l(k\sin\theta) & \text{for } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases} \quad (26)$$

Similarly, for a right-incident wave, where $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, we have [61]

$$A_-^r(p) = 0, \quad B_+^r(p) = 2\pi\delta(p - p_0) = \frac{2\pi\,\delta(\theta - \theta_0)}{k|\cos\theta_0|}, \quad (27)$$

$$\mathfrak{f}^r(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} k|\cos\theta|A_+^r(k\sin\theta) & \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ k|\cos\theta|B_-^r(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases} \quad (28)$$

In view of Eqs. (26) and (28) and the fact that $k|\cos\theta| = \varpi(p)$, we can express the scattering amplitudes $\mathfrak{f}^{l/r}$ in terms of the dimensionless coefficient functions:

$$\check{A}_\pm := \varpi A_\pm, \quad \check{B}_\pm := \varpi B_\pm, \quad (29)$$

which also belong to $\mathscr{F}_k$. This gives

$$\check{B}_+^l(p) = \check{A}_-^r(p) = 0, \quad \check{A}_-^l(p) = \check{B}_+^r(p) = 2\pi\varpi(p_0)\,\delta(p - p_0), \quad (30)$$

$$\mathfrak{f}^l(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} \check{A}_+^l(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \check{B}_-^l(k\sin\theta) & \text{for } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \end{cases} \quad (31)$$

$$\mathfrak{f}^r(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} \check{A}_+^r(k\sin\theta) & \text{for } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \check{B}_-^r(k\sin\theta) - 2\pi\delta(\theta - \theta_0) & \text{for } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases} \quad (32)$$

Notice that $\mathfrak{f}^{l/r}(\theta)$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ respectively correspond to the scattering amplitudes measured by detectors placed at $x = +\infty$ and $x = -\infty$. Equations (31) and (32) identify these with certain linear combinations of the scaled amplitudes $\check{A}_\pm$ and $\check{B}_\pm$ and the δ function. This is the main reason for our use of these amplitudes.

As seen from (30)–(32), the scattering problem for the potential has a solution if there is a procedure to determine $\check{A}_+$ and $\check{B}_-$ for the $\check{A}_-$ and $\check{B}_+$ given by (30). The scattering operator, more commonly known as the S-matrix, achieves this goal. We can represent it by a 2×2 matrix $\widehat{\mathbf{S}}$ that fulfills

$$\widehat{\mathbf{S}} \begin{bmatrix} \check{A}_- \\ \check{B}_+ \end{bmatrix} = \begin{bmatrix} \check{A}_+ \\ \check{B}_- \end{bmatrix}. \quad (33)$$

This is the 2D analog of the definition of the S-matrix in 1D, namely, (3). Notice however that, unlike the latter, $\widehat{\mathbf{S}}$ is

a linear operator acting in the infinite-dimensional space of two-component functions:

$$\mathscr{F}_k^{2\times 1} := \mathbb{C}^{2\times 1} \otimes \mathscr{F}_k = \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \middle| \phi_\pm \in \mathscr{F}_k \right\},$$

where $\mathbb{C}^{2\times 1}$ stands for the vector space of 2×1 complex matrices.

Reference [46] uses a direct extension of the definition of the transfer matrix in 1D, namely, (2), to arrive at a multidimensional notion of the transfer matrix. In 2D, this is done by identifying the transfer matrix with a 2×2 matrix $\widehat{\mathbf{M}}$ with operator entries $\widehat{M}_{ij} : \mathscr{F}_k \to \mathscr{F}_k$ such that

$$\widehat{\mathbf{M}} \begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix}. \quad (34)$$

The utility of this transfer matrix in potential scattering relies on the following relations which, in conjunction with (26) and

(28), allow for the determination of the scattering amplitudes $f^{l/r}$:

$$A_+^l = 2\pi \widehat{M}_{11}\, \delta_{p_0} + \widehat{M}_{12} B_-^l, \tag{35}$$

$$\widehat{M}_{22} B_-^l = -2\pi \widehat{M}_{21}\, \delta_{p_0}, \tag{36}$$

$$A_+^r = \widehat{M}_{12} B_-^r, \tag{37}$$

$$\widehat{M}_{22} B_-^r = 2\pi\, \delta_{p_0}. \tag{38}$$

where δ_{p_0} stands for the Dirac δ function centered at p_0, i.e., $\delta_{p_0}(p) := \delta(p - p_0)$, and $|p_0| < k$. Equations (35)–(38) follow from (34) once we substitute (25) and (27) for $A_\pm$ and $B_\pm$ [46,61]. Notice that (35) and (37) reduce the determination of $A_+^{l/r}$ to that of $B_-^{l/r}$, while (36) and (38) provide linear integral equations for the latter.[3]

The above procedure for determining the scattering amplitudes using the entries of the transfer matrix admits a slightly simpler variation if we employ the dimensionless coefficient functions $\breve{A}_\pm$ and $\breve{B}_\pm$ instead of $A_\pm$ and $B_\pm$ and make use of (30)–(32). In light of (29), the role of $\widehat{\mathbf{M}}$ is now played by

$$\widehat{\overset{\smile}{\mathbf{M}}} := \varpi(\hat{p}) \widehat{\mathbf{M}} \varpi(\hat{p})^{-1}, \tag{39}$$

where for every pair of functions $f, g : \mathbb{R} \to \mathbb{C}$, we define the function $f(\hat{p})g$ by

$$[f(\hat{p})g](p) := f(p)g(p).$$

It is easy to see that $\widehat{\overset{\smile}{\mathbf{M}}}$ is a transfer matrix (with operator entries $\overset{\smile}{\widehat{M}}_{jl}$ acting in $\mathscr{F}_k$) that satisfies

$$\widehat{\overset{\smile}{\mathbf{M}}} \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix} = \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix}. \tag{40}$$

With the help of this equation, we can express (35)–(38) as

$$\breve{A}_+^l = 2\pi \varpi(p_0) \overset{\smile}{\widehat{M}}_{11}\, \delta_{p_0} + \overset{\smile}{\widehat{M}}_{12} \breve{B}_-^l, \tag{41}$$

$$\overset{\smile}{\widehat{M}}_{22} \breve{B}_-^l = -2\pi \varpi(p_0) \overset{\smile}{\widehat{M}}_{21}\, \delta_{p_0}, \tag{42}$$

$$\breve{A}_+^r = \overset{\smile}{\widehat{M}}_{12} \breve{B}_-^r, \tag{43}$$

$$\overset{\smile}{\widehat{M}}_{22} \breve{B}_-^r = 2\pi \varpi(p_0)\, \delta_{p_0}. \tag{44}$$

Again, (42) and (44) are linear integral equations for $\breve{B}_-^{l/r}$. Solving them and using the result in Eqs. (41) and (43), we can determine $\breve{A}_+^{l/r}$. This yields a solution for the scattering problem by virtue of (31) and (32).

It is important to note that, unlike its predecessors [39–45], the definition of the transfer matrix $\overset{\smile}{\mathbf{M}}$ does not involve a slicing or discretization of the configuration or momentum space variables. The same holds for the transfer matrix $\widehat{\mathbf{M}}$.

However, $\overset{\smile}{\mathbf{M}}$ turns out to be more convenient to use.[4] For this reason, we will refer to it as the fundamental transfer matrix.

The introduction of the fundamental transfer matrix offers an alternative to the traditional methods of solving scattering problems. Its utility for this purpose involves the following steps:

(1) Find the fundamental transfer matrix $\overset{\smile}{\mathbf{M}}$.
(2) Solve (42) and (44) for $\breve{B}_-^{l/r}$.
(3) Substitute $\breve{B}_-^{l/r}$ in Eqs. (41) and (43) to find $\breve{A}_+^{l/r}$.
(4) Substitute $\breve{B}_-^{l/r}$ and $\breve{A}_+^{l/r}$ in Eqs. (31) and (32) to determine the scattering amplitudes $f^{l/r}$.

In Sec. III, we derive a dynamical equation that yields a Dyson series expansion of $\overset{\smile}{\mathbf{M}}$. The solution of (42) and (44) requires the knowledge of the potential. These equations have a unique solution if $\overset{\smile}{\widetilde{M}}_{22}$ has a trivial kernel (null space). The operator $\overset{\smile}{\widetilde{M}}_{22}$ is the 2D counterpart of the M_{22} entry of the transfer matrix $\mathbf{M}$ in 1D. Therefore, the condition that $\overset{\smile}{\widetilde{M}}_{22}$ has a nontrivial kernel is the 2D counterpart of $M_{22} = 0$. This equation is of particular interest because the wave numbers k for which it holds correspond to the spectral singularities of the potential [67]. This suggests a characterization of spectral singularities in 2D in terms of the nontriviality of the kernel of $\overset{\smile}{\widetilde{M}}_{22}$.

It is also worth mentioning that (41)–(44) reproduce their 1D counterparts, namely, (5), if we replace $2\pi \varpi(p_0)\delta_{p_0}$ with 1 and identify $\breve{A}_+^l$, $\breve{B}_+^l$, $\breve{A}_+^r$, and $\breve{B}_+^r$, respectively, with the left transmission amplitude T^l, left reflection amplitude R^l, right reflection amplitude R^r, and right transmission amplitude T^r.

III. DYNAMICAL EQUATION FOR FUNDAMENTAL TRANSFER MATRIX

The formulation of the stationary scattering in terms of the fundamental transfer matrix $\overset{\smile}{\mathbf{M}}$ is useful, provided that we can evaluate it and solve the integral equations (42) and (44) for $\breve{B}_-^{l/r}$. Because $\overset{\smile}{\mathbf{M}}$ and $\widehat{\mathbf{M}}$ are related by a similarity transformation, the determination of any of these transfer matrices will yield the other. Reference [46] shows that, like its 1D analog [47,48], we can relate $\widehat{\mathbf{M}}$ to the time-evolution operator for an effective nonunitary quantum system and derive a Dyson series expansion for it. In what follows, we pursue a similar route that considers the role of the evanescent waves. This leads to a slightly different dynamical equation and Dyson series expansion for $\widehat{\mathbf{M}}$ than those obtained in Ref. [46]. As we show in the next section, an implicit assumption employed in Ref. [46] corrects the discrepancy, and the prescription proposed in Ref. [46] for calculating the scattering amplitudes $f^{l/r}$ produces the correct result for the applications considered therein.

Let ψ be the general bounded solution ψ of the stationary Schrödinger equation (8) that we consider in Sec. II, and for

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[3]In general, we can associate integral kernels $M_{ij}(p, q)$ to $\widehat{M}_{ij}$ and express (36) and (38) as the integral equations $\int_{-k}^{k} dq\, M_{22}(p, q) B_-^l(q) = -2\pi M_{21}(p, p_0)$ and $\int_{-k}^{k} dq\, M_{22}(p, q) B_-^r(q) = 2\pi \delta(p - p_0)$, respectively. This follows from Eqs. (59) and (60) below and the fact that $\widehat{\mathscr{V}}(x)$ is the integral operator given by (14).

[4]This is because the entries of $\overset{\smile}{\mathbf{M}}$ enter the equations for scaled amplitudes $\breve{A}_\pm$ and $\breve{B}_\pm$ and that the scattering amplitudes $f^{l/r}$ have simpler expressions in terms of these amplitudes than $A_\pm$ and $B_\pm$ (which are related by $\widehat{\mathbf{M}}$).

each $x \in \mathbb{R}$, let $\Psi_\pm(x) : \mathbb{R} \to \mathbb{C}$ and $\boldsymbol{\Psi}(x) : \mathbb{R} \to \mathbb{C}^{2\times1}$ be the functions defined by

$$[\Psi_\pm(x)](p) := \frac{1}{2} \exp[\pm i\varpi_r(p)x][\varpi(p)\tilde{\psi}(x, p) \pm i\, \tilde{\psi}'(x, p)],$$

$$\boldsymbol{\Psi}(x) := \begin{bmatrix} \Psi_-(x) \\ \Psi_+(x) \end{bmatrix}, \tag{45}$$

where

$$\varpi_r(p) := \mathrm{Re}[\varpi(p)] = \begin{cases} \sqrt{k^2 - p^2} & \text{for } |p| < k \\ 0 & \text{for } |p| \geqslant k \end{cases}$$

$$= \begin{cases} \varpi(p) & \text{for } |p| < k, \\ 0 & \text{for } |p| \geqslant k. \end{cases}$$

Then in view of (21)–(23) and (29),

$$[\boldsymbol{\Psi}(x)](p) = \begin{bmatrix} \check{A}_-(p) \\ \check{B}_-(p) + \check{C}_-(p)e^{|\varpi(p)|x} \end{bmatrix} \quad \text{for } x \leqslant a_-, \tag{46}$$

$$[\boldsymbol{\Psi}(x)](p) = \begin{bmatrix} \check{A}_+(p) + \check{C}_+(p)e^{-|\varpi(p)|x} \\ \check{B}_+(p) \end{bmatrix} \quad \text{for } x \geqslant a_+, \tag{47}$$

where

$$\check{C}_\pm := \varpi\, C_\pm. \tag{48}$$

Equations (46) and (47) imply

$$\lim_{x \to -\infty} \boldsymbol{\Psi}(x) = \begin{bmatrix} \check{A}_- \\ \check{B}_- \end{bmatrix}, \quad \lim_{x \to +\infty} \boldsymbol{\Psi}(x) = \begin{bmatrix} \check{A}_+ \\ \check{B}_+ \end{bmatrix}. \tag{49}$$

We can also use (20), (46), and (47) to show that, for $|p| < k$,

$$[\boldsymbol{\Psi}(x)](p) = \begin{bmatrix} \check{A}_-(p) \\ \check{B}_-(p) \end{bmatrix} \quad \text{for } x \leqslant a_-,$$

$$[\boldsymbol{\Psi}(x)](p) = \begin{bmatrix} \check{A}_+(p) \\ \check{B}_+(p) \end{bmatrix} \quad \text{for } x \geqslant a_+. \tag{50}$$

Next, we observe that (13) is equivalent to

$$i\boldsymbol{\Psi}'(x) = \widehat{\breve{\mathbf{H}}}(x)\boldsymbol{\Psi}(x), \quad x \in \mathbb{R}, \tag{51}$$

where

$$\widehat{\breve{\mathbf{H}}}(x) := \frac{1}{2} \exp[-i\varpi_r(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\mathscr{V}}(x)\mathcal{K} \exp[i\varpi_r(\hat{p})x\boldsymbol{\sigma}_3]\varpi(\hat{p})^{-1}$$

$$- i\varpi_i(\hat{p})\boldsymbol{\sigma}_3, \tag{52}$$

$$\mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \tag{53}$$

where $\boldsymbol{\sigma}_3$ is the diagonal Pauli matrix, and

$$\varpi_i(p) := \mathrm{Im}[\varpi(p)] = \begin{cases} 0 & \text{for } |p| < k \\ \sqrt{p^2 - k^2} & \text{for } |p| \geqslant k \end{cases}$$

$$= \begin{cases} 0 & \text{for } |p| < k, \\ |\varpi(p)| & \text{for } |p| \geqslant k. \end{cases}$$

We may view (51) as the time-dependent Schrödinger equation for a nonstationary quantum system with the Hamiltonian operator (52) and x playing the role of time.

Let $\widehat{\breve{\mathbf{U}}}(x, x_0)$ denote the evolution operator for the Hamiltonian (52) and an initial time x_0. This is the operator satisfying

$$i\partial_x\widehat{\breve{\mathbf{U}}}(x, x_0) = \widehat{\breve{\mathbf{H}}}(x)\widehat{\breve{\mathbf{U}}}(x, x_0), \quad \widehat{\breve{\mathbf{U}}}(x_0, x_0) = \widehat{\mathbf{I}}, \tag{54}$$

$$\boldsymbol{\Psi}(x) = \widehat{\breve{\mathbf{U}}}(x, x_0)\boldsymbol{\Psi}(x_0), \tag{55}$$

where $\widehat{\mathbf{I}}$ is the identity operator for the space $\mathscr{F}^{2\times1} := \mathbb{C}^{2\times1} \otimes \mathscr{F}$ of two-component state vectors, where $\mathscr{F}$ is the space of complex-valued functions of p.[5] In view of (34), (49), and (55),

$$\widehat{\breve{\mathbf{M}}} = \lim_{x_\pm \to \pm\infty} \widehat{\breve{\mathbf{U}}}(x_+, x_-). \tag{56}$$

We can express (54) in the form of the following Dyson series [4]:

$$\widehat{\breve{\mathbf{U}}}(x, x_0) = \widehat{\mathbf{I}} + \sum_{n=1}^{\infty}(-i)^n \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1}$$

$$\cdots \int_{x_0}^{x_2} dx_1 \widehat{\breve{\mathbf{H}}}(x_n)\widehat{\breve{\mathbf{H}}}(x_{n-1})\cdots\widehat{\breve{\mathbf{H}}}(x_1)$$

$$=: \mathscr{T} \exp\left[-i \int_{x_0}^{x} dx' \widehat{\breve{\mathbf{H}}}(x')\right], \tag{57}$$

where $\mathscr{T}$ denotes the time-ordering operation with x playing the role of time. Substituting (57) in Eq. (56) yields the Dyson series expansion of the fundamental transfer matrix:

$$\widehat{\breve{\mathbf{M}}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\infty}(-i)^n \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{x_n} dx_{n-1}$$

$$\cdots \int_{-\infty}^{x_2} dx_1 \widehat{\breve{\mathbf{H}}}(x_n)\widehat{\breve{\mathbf{H}}}(x_{n-1})\cdots\widehat{\breve{\mathbf{H}}}(x_1)$$

$$=: \mathscr{T} \exp\left[-i \int_{-\infty}^{\infty} dx \widehat{\breve{\mathbf{H}}}(x)\right]. \tag{58}$$

In view of (39), we can use (58) to derive the following Dyson series expansion for the transfer matrix $\widehat{\mathbf{M}}$:

$$\widehat{\mathbf{M}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\infty}(-i)^n \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{x_n} dx_{n-1}$$

$$\cdots \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{H}}(x_n)\widehat{\mathbf{H}}(x_{n-1})\cdots\widehat{\mathbf{H}}(x_1), \tag{59}$$

where

$$\widehat{\mathbf{H}}(x) := \varpi(\hat{p})^{-1} \widehat{\breve{\mathbf{H}}}(x)\,\varpi(\hat{p})$$

$$= \tfrac{1}{2} \varpi(\hat{p})^{-1} \exp[-i\varpi_r(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\mathscr{V}}(x)\mathcal{K}$$

$$\times \exp[i\varpi_r(\hat{p})x\boldsymbol{\sigma}_3] - i\varpi_i(\hat{p})\boldsymbol{\sigma}_3. \tag{60}$$

IV. COMPOSITION RULE FOR FUNDAMENTAL TRANSFER MATRIX

In the absence of the potential, i.e., $v(x, y) = 0$, we have $\widehat{\mathscr{V}}(x) = \hat{\mathbf{0}}$, where $\hat{0}$ is the zero operator acting in $\mathscr{F}$, and

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[5]We can write $\widehat{\mathbf{I}}$ as the product of the 2×2 identity matrix $\mathbf{I}$ and the identity operator $\hat{1}$ for $\mathscr{F}$.

$\widehat{\vec{\mathbf{H}}}(x) = \widehat{\mathbf{H}}_0 := -i\varpi_i(\hat{p})\boldsymbol{\sigma}_3$. We identify the latter with the free Hamiltonian and examine the dynamics of the system in the interaction picture [4], where the evolving state vectors are given by

$$\boldsymbol{\Phi}(x) := \exp(i\widehat{\mathbf{H}}_0 x)\boldsymbol{\Psi}(x) = \exp[\varpi_i(\hat{p})x\boldsymbol{\sigma}_3]\boldsymbol{\Psi}(x). \quad (61)$$

These satisfy the time-dependent Schrödinger equation $i\boldsymbol{\Phi}'(x) = \widehat{\vec{\mathcal{H}}}(x)\boldsymbol{\Phi}(x)$ for the interaction-picture Hamiltonian:

$$\begin{aligned}\widehat{\vec{\mathcal{H}}}(x) &:= \exp(i\widehat{\mathbf{H}}_0 x)\widehat{\vec{\mathbf{H}}}(x)\exp(-i\widehat{\mathbf{H}}_0 x) - \widehat{\mathbf{H}}_0 \\ &= \tfrac{1}{2}\exp[-i\varpi(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\vec{\mathcal{V}}}(x)\mathcal{K}\exp[i\varpi(\hat{p})x\boldsymbol{\sigma}_3]\varpi(\hat{p})^{-1}.\end{aligned}$$
$$(62)$$

We can use the interaction-picture evolution operator $\widehat{\vec{\mathcal{U}}}(x, x_0)$, which is given by

$$\begin{aligned}\widehat{\vec{\mathcal{U}}}(x, x_0) &:= \exp(i\widehat{\mathbf{H}}_0 x)\widehat{\vec{\mathbf{U}}}(x, x_0)\exp(-i\widehat{\mathbf{H}}_0 x_0) \\ &= \exp[\varpi_i(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\vec{\mathbf{U}}}(x, x_0)\exp[-\varpi_i(\hat{p})x_0\boldsymbol{\sigma}_3], \quad (63)\end{aligned}$$

to define the operator:

$$\widehat{\vec{\mathfrak{M}}} := \lim_{x_\pm \to \pm\infty}\widehat{\vec{\mathcal{U}}}(x_+, x_-) = \mathcal{T}\exp\left[-i\int_{-\infty}^{\infty}dx\,\widehat{\vec{\mathcal{H}}}(x)\right].$$
$$(64)$$

We call this the *auxiliary transfer matrix*. It is related to the transfer matrix $\widehat{\mathfrak{M}}$ constructed in Ref. [46] according to

$$\widehat{\mathfrak{M}} := \varpi(\hat{p})^{-1}\widehat{\vec{\mathfrak{M}}}\varpi(\hat{p}) = \mathcal{T}\exp\left[-i\int_{-\infty}^{\infty}dx\,\widehat{\mathcal{H}}(x)\right],$$
$$(65)$$

where

$$\begin{aligned}\widehat{\mathcal{H}}(x) &:= \varpi(\hat{p})^{-1}\widehat{\vec{\mathcal{H}}}\varpi(\hat{p}) \\ &= \tfrac{1}{2}\varpi(\hat{p})^{-1}\exp[-i\varpi(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\mathcal{V}}(x)\mathcal{K}\exp[i\varpi(\hat{p})x\boldsymbol{\sigma}_3].\end{aligned}$$
$$(66)$$

Note however that $\widehat{\mathfrak{M}} \neq \widehat{\mathbf{M}}$.

Next, consider slicing the space along a finite set of lines that are parallel to the y axis. Let ℓ be a positive integer and $a_0, a_1, a_2, \cdots, a_\ell$ and $x_\pm$ are arbitrary real numbers such that

$$x_- \leqslant a_- = a_0 < a_1 < a_2 < \cdots < a_{\ell-1} < a_\ell = a_+ \leqslant x_+.$$
$$(67)$$

Then the interaction-picture evolution operator has the following semigroup property:

$$\begin{aligned}\widehat{\vec{\mathcal{U}}}(x_+, x_-) &= \widehat{\vec{\mathcal{U}}}(x_+, a_\ell)\,\widehat{\vec{\mathcal{U}}}(a_\ell, a_{\ell-1})\,\widehat{\vec{\mathcal{U}}}(a_{\ell-1}, a_{\ell-2}) \\ &\quad \cdots \widehat{\vec{\mathcal{U}}}(a_1, a_0)\,\widehat{\vec{\mathcal{U}}}(a_0, x_-).\end{aligned}$$
$$(68)$$

If $v(x, y)$ vanishes for a range of values of x, $\widehat{\mathcal{V}}(x) = \hat{0}$ and $\widehat{\vec{\mathcal{H}}}(x) = \widehat{\mathbf{0}}$, where $\widehat{\mathbf{0}}$ is the zero operator acting in $\mathscr{F}^{2\times 1}$. This feature of $\widehat{\vec{\mathcal{H}}}(x)$ together with (68) is responsible for the composition property of the auxiliary transfer matrix [46]. To see this, we let $v_j : \mathbb{R} \to \mathbb{C}$ be truncations of the potential v given

by

$$v_1(x, y) := \begin{cases} v(x, y) & \text{for } x \in [a_0, a_1], \\ 0 & \text{for } x \notin [a_0, a_1], \end{cases}$$

$$v_{m+1}(x, y) := \begin{cases} v(x, y) & \text{for } x \in (a_m, a_{m+1}], \\ 0 & \text{for } x \notin (a_m, a_{m+1}], \end{cases} \quad (69)$$

with $m \in \{1, 2, \cdots, \ell - 1\}$, so that $v_1 + v_2 + \cdots + v_\ell = v$. Let $\widehat{\mathcal{V}}_j(x)$, $\widehat{\vec{\mathbf{H}}}_j(x)$, $\widehat{\vec{\mathbf{U}}}_j(x, x_0)$, $\widehat{\vec{\mathcal{H}}}_j(x)$, $\widehat{\vec{\mathcal{U}}}_j(x, x_0)$, and $\widehat{\vec{\mathfrak{M}}}_j$ be, respectively, the analogs of $\widehat{\mathcal{V}}(x)$, $\widehat{\vec{\mathbf{H}}}(x)$, $\widehat{\vec{\mathbf{U}}}(x, x_0)$, $\widehat{\vec{\mathcal{H}}}(x)$, $\widehat{\vec{\mathcal{U}}}(x, x_0)$, and $\widehat{\vec{\mathfrak{M}}}$ for the potentials v_j. Then

$$\widehat{\vec{\mathcal{H}}}_j(x) = \begin{cases} \widehat{\vec{\mathcal{H}}}(x) & \text{for } x \in [a_{j-1}, a_j], \\ \widehat{\mathbf{0}} & \text{for } x \notin [a_{j-1}, a_j], \end{cases} \quad (70)$$

$$\widehat{\vec{\mathcal{U}}}_j(x, x_0) = \begin{cases} \widehat{\mathbf{I}} & \text{for } x_0 \leqslant x \leqslant a_{j-1}, \\ \widehat{\vec{\mathcal{U}}}(x, x_0) & \text{for } a_{j-1} \leqslant x_0 \leqslant x \leqslant a_j, \\ \widehat{\mathbf{I}} & \text{for } a_j \leqslant x_0 \leqslant x. \end{cases} \quad (71)$$

This together with $\widehat{\vec{\mathfrak{M}}}_j := \lim_{x_\pm \to \pm\infty}\widehat{\vec{\mathcal{U}}}_j(x_+, x_-)$ implies

$$\widehat{\vec{\mathfrak{M}}}_j = \widehat{\vec{\mathcal{U}}}(a_j, a_{j-1}). \quad (72)$$

Furthermore, because $\widehat{\vec{\mathcal{U}}}(x_+, a_\ell) = \widehat{\vec{\mathcal{U}}}(a_0, x_-) = \widehat{\mathbf{I}}$, we can express (68) as

$$\begin{aligned}\widehat{\vec{\mathcal{U}}}(x_+, x_-) &= \widehat{\vec{\mathcal{U}}}_\ell(a_\ell, a_{\ell-1})\,\widehat{\vec{\mathcal{U}}}_{\ell-1}(a_{\ell-1}, a_{\ell-2}) \\ &\quad \cdots \widehat{\vec{\mathcal{U}}}_1(a_1, a_0) \quad \text{for } \pm x_\pm \geqslant \pm a_\pm. \quad (73)\end{aligned}$$

Equations (64), (67), (72), and (73) lead to

$$\widehat{\vec{\mathfrak{M}}} = \widehat{\vec{\mathfrak{M}}}_\ell \widehat{\vec{\mathfrak{M}}}_{\ell-1} \cdots \widehat{\vec{\mathfrak{M}}}_1. \quad (74)$$

Next, we use the analog of (63) for v_j, namely,

$$\widehat{\vec{\mathcal{U}}}_j(x, x_0) = \exp[\varpi_i(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\vec{\mathbf{U}}}_j(x, x_0)\exp[-\varpi_i(\hat{p})x_0\boldsymbol{\sigma}_3],$$

and (56), (63), (72), and (74) to establish the following composition property of the fundamental transfer matrix:

$$\begin{aligned}\widehat{\vec{\mathbf{M}}} &= \lim_{x_\pm \to \pm\infty}\exp[-\varpi_i(\hat{p})x_+\boldsymbol{\sigma}_3]\widehat{\vec{\mathfrak{M}}}_\ell\,\widehat{\vec{\mathfrak{M}}}_{\ell-1} \\ &\quad \cdots \widehat{\vec{\mathfrak{M}}}_1\exp[\varpi_i(\hat{p})x_-\boldsymbol{\sigma}_3].\end{aligned}$$
$$(75)$$

In view of (74), this equation implies

$$\widehat{\vec{\mathbf{M}}} = \lim_{x_\pm \to \pm\infty}\exp[-\varpi_i(\hat{p})x_+\boldsymbol{\sigma}_3]\widehat{\vec{\mathfrak{M}}}\exp[\varpi_i(\hat{p})x_-\boldsymbol{\sigma}_3]. \quad (76)$$

Notice also that we can use (56) and the semigroup property of the evolution operator $\widehat{\vec{\mathbf{U}}}(x, x_0)$ to express the fundamental transfer matrix in the form

$$\begin{aligned}\widehat{\vec{\mathbf{M}}} &= \lim_{x_\pm \to \pm\infty}\widehat{\vec{\mathbf{U}}}(x_+, a_\ell)\,\widehat{\vec{\mathbf{U}}}_\ell(a_\ell, a_{\ell-1})\,\widehat{\vec{\mathbf{U}}}_{\ell-1}(a_{\ell-1}, a_{\ell-2}) \\ &\quad \cdots \widehat{\vec{\mathbf{U}}}_1(a_1, a_0)\,\widehat{\vec{\mathbf{U}}}(a_0, x_-), \\ &= \lim_{x_\pm \to \pm\infty}\exp[-\varpi_i(\hat{p})x_+\boldsymbol{\sigma}_3]\widehat{\vec{\mathbf{U}}}_\ell(a_\ell, a_{\ell-1}) \\ &\quad \cdots \widehat{\vec{\mathbf{U}}}_1(a_1, a_0)\exp[\varpi_i(\hat{p})x_-\boldsymbol{\sigma}_3], \quad (77)\end{aligned}$$

where we have made use of $\widehat{\mathbf{U}}(x_+, a_\ell) = \exp[-\varpi_i(\hat{p})(x_+ - a_\ell)\boldsymbol{\sigma}_3]$ and $\widehat{\mathbf{U}}(a_0, x_-) = \exp[\varpi_i(\hat{p})(x_- - a_0)\boldsymbol{\sigma}_3]$. These follow from the fact that, for $x < a_0$ and $x > a_\ell$, $\mathscr{V}(x) = \hat{0}$, and $\widehat{\mathbf{H}}(x) = \widehat{\mathbf{H}}_0 = -i\varpi_i(\hat{p})\boldsymbol{\sigma}_3$.

The composition rule (77) reduces the problem of obtaining the fundamental transfer matrix for the potential v to the calculation of $\widehat{\mathbf{U}}_j(x_+, x_-)$. Alternatively, one may compute $\widehat{\mathfrak{M}}_j$ and utilize (75).

V. SOLUTION OF THE SCATTERING PROBLEM USING AUXILIARY TRANSFER MATRIX

Because of the presence of the second term on the right-hand side of (52), the Dyson series expansion for the Hamiltonian $\widehat{\mathcal{H}}(x)$ has a simpler structure than that for $\widehat{\mathbf{H}}(x)$. This in turn suggests that it should be easier to compute $\widehat{\mathfrak{M}}$ and use it to determine the fundamental transfer matrix $\widehat{\mathbf{M}}$ using (76). Alternatively, we may try to express (35)–(38) in terms of the entries of $\widehat{\mathfrak{M}}$ and obtain a solution for the scattering problem by solving these equations. In the following, we examine the latter approach.

First, consider an arbitrary function $\phi : \mathbb{R} \to \mathbb{C}$ such that $|\phi(p)|$ does not diverge as $p \to \pm\infty$ faster than a polynomial, i.e., there is some $n \in \mathbb{Z}^+$ such that $\lim_{p\to\pm\infty} |\phi(p)|/(1 + |p|^n) = 0$.[6] Let $\phi_{\mathrm{os}}, \phi_{\mathrm{ev}} : \mathbb{R} \to \mathbb{C}$ be defined by

$$\phi_{\mathrm{os}}(p) := \begin{cases} \phi(p) & \text{for } |p| < k, \\ 0 & \text{for } |p| \geqslant k, \end{cases}$$

$$\phi_{\mathrm{ev}}(p) = \phi(p) - \phi_{\mathrm{os}}(p).$$

Then by virtue of (15),

$$\lim_{x\to\infty} \exp[-\varpi_i(\hat{p})x]\phi = \phi_{\mathrm{os}},$$

$$\lim_{x\to\infty} \exp[-\varpi_i(\hat{p})x]\phi_{\mathrm{ev}} = 0. \tag{78}$$

These equations identify the operator:

$$\widehat{\Pi}_k := \lim_{x\to\infty} \exp[-\varpi_i(\hat{p})x] = \lim_{x\to-\infty} \exp[\varpi_i(\hat{p})x], \tag{79}$$

with the projection operator that acts in $\mathscr{F}$ according to

$$\widehat{\Pi}_k\phi = \phi_{\mathrm{os}}. \tag{80}$$

In view of (20), (78), and (79), we have

$$\widehat{\Pi}_k A_\pm = A_\pm, \quad \widehat{\Pi}_k B_\pm = B_\pm, \quad \widehat{\Pi}_k C_\pm = 0. \tag{81}$$

These in turn imply

$$\widehat{\Pi}_k \breve{\mathscr{B}}_- = \breve{B}_-, \quad \widehat{\Pi}_k \breve{\mathscr{A}}_+ = \breve{A}_+, \quad \widehat{\Pi}_k \breve{\mathscr{C}}_\pm = 0, \tag{82}$$

where

$$\breve{\mathscr{B}}_- := \varpi(\hat{p})\mathscr{B}_-, \quad \breve{\mathscr{A}}_+ := \varpi(\hat{p})\mathscr{A}_+, \tag{83}$$

and we have made use of (21), (22), (29), and (48).

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[6]This allows for identifying ϕ and its Fourier transform with tempered distributions.

Next, we use (46), (47), and (61) to show that

$$\lim_{x\to-\infty} \boldsymbol{\Phi}(x) = \Phi(a_-) = \begin{bmatrix} \breve{A}_- \\ \breve{\mathscr{B}}_- \end{bmatrix},$$

$$\lim_{x\to+\infty} \boldsymbol{\Phi}(x) = \Phi(a_+) = \begin{bmatrix} \breve{\mathscr{A}}_+ \\ \breve{B}_+ \end{bmatrix}. \tag{84}$$

In view of (64), these imply

$$\widehat{\mathfrak{M}} \begin{bmatrix} \breve{A}_- \\ \breve{\mathscr{B}}_- \end{bmatrix} = \begin{bmatrix} \breve{\mathscr{A}}_+ \\ \breve{B}_+ \end{bmatrix}. \tag{85}$$

This relation together with (31) and (32) provides a route to the solution of the scattering problem. If we replace the role of $\breve{B}_-, \breve{A}_+$, and $\widehat{\mathbf{M}}$ in the derivation of (41)–(44) with $\breve{\mathscr{B}}_-, \breve{\mathscr{A}}_+$, and $\widehat{\mathfrak{M}}$ and employ (85), we are led to

$$\breve{\mathscr{A}}_+^l = 2\pi \varpi(p_0)\widehat{\mathfrak{M}}_{11} \delta_{p_0} + \widehat{\mathfrak{M}}_{12}\breve{\mathscr{B}}_-^l, \tag{86}$$

$$\widehat{\mathfrak{M}}_{22} \breve{\mathscr{B}}_-^l = -2\pi \varpi(p_0)\widehat{\mathfrak{M}}_{21} \delta_{p_0}, \tag{87}$$

$$\breve{\mathscr{A}}_+^r = \widehat{\mathfrak{M}}_{12}\breve{\mathscr{B}}_-^r, \tag{88}$$

$$\widehat{\mathfrak{M}}_{22} \breve{\mathscr{B}}_-^r = 2\pi \varpi(p_0) \delta_{p_0}. \tag{89}$$

Determination of $\breve{\mathscr{B}}_-^{l/r}$ requires the solution of (87) and (89). Solving these equations and substituting the result in (86) and (88) yield $\breve{\mathscr{A}}_+^{l/r}$. We can obtain the coefficient functions $\breve{A}_+^{l/r}$ and $\breve{B}_-^{l/r}$ entering the formulas (26) and (28) for the scattering amplitudes $\mathfrak{f}^{l/r}$ by affecting the projection operator $\widehat{\Pi}_k$ on $\breve{\mathscr{B}}_-^{l/r}$ and $\breve{\mathscr{A}}_+^{l/r}$, respectively:

$$\breve{B}_-^{l/r} = \widehat{\Pi}_k\breve{\mathscr{B}}_-^{l/r}, \quad \breve{A}_+^{l/r} = \widehat{\Pi}_k\breve{\mathscr{A}}_+^{l/r}. \tag{90}$$

In practice, this means that we can obtain $\breve{B}_-^{l/r}(p)$ and $\breve{A}_+^{l/r}(p)$ by evaluating $\breve{\mathscr{B}}_-^{l/r}(p)$ and $\breve{\mathscr{A}}_+^{l/r}(p)$ at the values of p such that $|p| < k$. This is simply because

$$\breve{B}_-^{l/r}(p) = \begin{cases} \breve{\mathscr{B}}_-^{l/r}(p) & \text{for } |p| < k, \\ 0 & \text{for } |p| \geqslant k, \end{cases}$$

$$\breve{A}_+^{l/r}(p) = \begin{cases} \breve{\mathscr{A}}_+^{l/r}(p) & \text{for } |p| < k, \\ 0 & \text{for } |p| \geqslant k. \end{cases} \tag{91}$$

Next, we pursue an alternative approach for using $\widehat{\mathfrak{M}}$ as a tool for solving the scattering problem for the potential v. With the help of (34), (76), (79), (82), and (85), we can show that

$$\mathbf{0} = \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix} - \widehat{\mathbf{M}} \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix}$$

$$= \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix} - \lim_{x\to\infty} \exp[-\varpi_i(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\mathfrak{M}} \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix}$$

$$= \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix} - \lim_{x\to\infty} \exp[-\varpi_i(\hat{p})x\boldsymbol{\sigma}_3]\left(\widehat{\mathfrak{M}} \begin{bmatrix} \breve{A}_- \\ \breve{\mathscr{B}}_- \end{bmatrix} - \widehat{\mathfrak{M}} \begin{bmatrix} 0 \\ \breve{C}_- \end{bmatrix}\right)$$

$$= \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix} - \lim_{x \to \infty} \exp\left[-\varpi_i(\hat{p})x\sigma_3\right] \left(\begin{bmatrix} \breve{\mathscr{A}}_+ \\ \breve{B}_+ \end{bmatrix} - \widehat{\mathfrak{M}} \begin{bmatrix} 0 \\ \breve{C}_- \end{bmatrix} \right)$$

$$= \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix} - \begin{bmatrix} \widehat{\Pi}_k \breve{\mathscr{A}}_+ \\ \breve{B}_+ \end{bmatrix} + \lim_{x \to \infty} \exp\left[-\varpi_i(\hat{p})x\sigma_3\right] \widehat{\widetilde{\mathfrak{M}}} \begin{bmatrix} 0 \\ \breve{C}_- \end{bmatrix}$$

$$= \lim_{x \to \infty} \exp\left[-\varpi_i(\hat{p})x\sigma_3\right] \widehat{\widetilde{\mathfrak{M}}} \begin{bmatrix} 0 \\ \breve{C}_- \end{bmatrix}.$$

According to the latter equation,

$$\lim_{x \to \infty} \exp\left[-\varpi_i(\hat{p})x\right] \widehat{\widetilde{\mathfrak{M}}}_{12} \breve{C}_- = 0,$$

$$\lim_{x \to \infty} \exp\left[\varpi_i(\hat{p})x\right] \widehat{\widetilde{\mathfrak{M}}}_{22} \breve{C}_- = 0, \qquad (92)$$

which mean that

$$\widehat{\Pi}_k \widehat{\widetilde{\mathfrak{M}}}_{12} \breve{C}_- = 0, \quad \widehat{\widetilde{\mathfrak{M}}}_{22} \breve{C}_- = 0. \qquad (93)$$

In view of the latter relation, we can write (85) as

$$\widehat{\widetilde{\mathfrak{M}}} \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix} = \begin{bmatrix} \breve{\mathscr{A}}_+ - \widehat{\widetilde{\mathfrak{M}}}_{12} \breve{C}_- \\ \breve{B}_+ \end{bmatrix}. \qquad (94)$$

If we apply

$$\widehat{\boldsymbol{\Pi}}_k := \widehat{\Pi}_k \mathbf{I}, \qquad (95)$$

to both sides of this equation and use (82) and (93), we can show that

$$\widehat{\boldsymbol{\Pi}}_k \widehat{\widetilde{\mathfrak{M}}} \, \widehat{\boldsymbol{\Pi}}_k \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix} = \widehat{\boldsymbol{\Pi}}_k \widehat{\widetilde{\mathfrak{M}}} \begin{bmatrix} \breve{A}_- \\ \breve{B}_- \end{bmatrix} = \begin{bmatrix} \breve{A}_+ \\ \breve{B}_+ \end{bmatrix}. \qquad (96)$$

Equations (34) and (96) suggest

$$\widehat{\widetilde{\mathbf{M}}} = \widehat{\boldsymbol{\Pi}}_k \widehat{\widetilde{\mathfrak{M}}} \, \widehat{\boldsymbol{\Pi}}_k, \qquad (97)$$

provided that we identify $\widehat{\widetilde{\mathbf{M}}}$ as an operator acting in $\mathscr{F}_k^{2 \times 1}$. With the help of this identification and (94), we can also check the consistency of (76) and (97). Furthermore, because $\widehat{\Pi}_k$ and $\varpi(\hat{p})$ commute, we can use (39), (65), and (97) to infer

$$\widehat{\mathbf{M}} = \widehat{\boldsymbol{\Pi}}_k \widehat{\mathfrak{M}} \, \widehat{\boldsymbol{\Pi}}_k. \qquad (98)$$

Reference [46] advocates the utility of the transfer matrix $\widehat{\mathbf{M}}$ as a tool for solving scattering problems but fails to recognize its difference with $\widehat{\mathfrak{M}}$. For this reason, it proposes to solve the scattering problem by employing (86)–(89) with $\mathscr{A}_+^{l/r}$ and $\mathscr{B}_-^{l/r}$ changed to $A_+^{l/r}$ and $B_-^{l/r}$. This discrepancy has escaped various analytical and numerical tests performed on the specific applications considered in Refs. [46,61,62,64,68] because the final result of the calculation of the scattering amplitudes turns out to be correct. A careful analysis shows that, for all of these applications, the expression for $\widehat{\mathbf{M}}$ can be recovered from that of $\widehat{\mathfrak{M}}$ by replacing $\widehat{\mathscr{V}}(x)$ with the operator $\widehat{\Pi}_k \widehat{\mathscr{V}}(x) \widehat{\Pi}_k$. In view of (98), it is not difficult to see that letting $\widehat{\Pi}_k \widehat{\mathscr{V}}(x) \widehat{\Pi}_k$ play the role of $\widehat{\mathscr{V}}(x)$ in the Dyson series for $\widehat{\mathfrak{M}}$ will produce $\widehat{\mathbf{M}}$ if this series terminates after its second term, i.e.,

$$\widehat{\mathfrak{M}} = \widehat{\mathbf{I}} - i \int_{-\infty}^{\infty} dx \, \widehat{\mathcal{H}}(x). \qquad (99)$$

As we show in Sec. VI, this happens for the δ-function potential in 2D. In general, (99) does not hold, and one cannot obtain $\widehat{\mathbf{M}}$ from the expression for $\widehat{\mathfrak{M}}$ by replacing $\widehat{\mathscr{V}}(x)$ with $\widehat{\Pi}_k \widehat{\mathscr{V}}(x) \widehat{\Pi}_k$.

For $v = 0$, $\widehat{\mathfrak{M}} = \mathbf{I}$, and (93) implies $C_- = 0$. At first sight, this seems unjustified because there is no reason why one should not be able to construct a solution ψ of the free Schrödinger equation (8) such that $\psi(x, y) = C_-(p) \exp[k\varpi(k)x + ipy]$ for $x < a_-$, where p is a real number such that $|p| > k$, and C_- is a nonzero function. Note however that, because $v = 0$, this solution satisfies $\psi(x, y) = C_-(k) \exp[k\varpi(k)x + ipy]$ for all $x \in \mathbb{R}$. It diverges exponentially as $x \to +\infty$. The fact that (93) reduces to $C_- = 0$ for $v = 0$ is a direct consequence of the requirement that ψ is a bounded solution of (8).

The application of the auxiliary and fundamental transfer matrices in solving the scattering problem for short-range potentials satisfying (16) does not seem to encounter any serious problems when we deal with short-range potentials violating (16) or potentials v with an infinite range such that, for each $y \in \mathbb{R}$, $v_y(x) := v(x, y)$ is a short-range potential in 1D. A typical example is an infinite-range potential of the form:

$$v(x, y) = \begin{cases} \mathfrak{z} \, e^{i\alpha y} & \text{for } x \in [a_-, a_+], \\ 0 & \text{for } x \notin [a_-, a_+], \end{cases}$$

where α is a positive real parameter, and $\mathfrak{z}$ is a real or complex coupling constant [63,69]. This is an exactly solvable potential that fails to satisfy (99). A careful examination of the solution of its scattering problem shows that one cannot use $\widehat{\Pi}_k \widehat{\mathscr{V}}(x) \widehat{\Pi}_k$ in place of $\widehat{\mathscr{V}}(x)$ to compute its scattering amplitude.

VI. IMPLICIT REGULARIZATION OF δ-FUNCTION POTENTIAL IN 2D

Consider the potentials of the form:

$$v(x, y) = \delta(x)\mathfrak{g}(y), \qquad (100)$$

where $\mathfrak{g} : \mathbb{R} \to \mathbb{C}$ is a function with Fourier transform $\tilde{\mathfrak{g}}$ [68]. Then (14) and (62) give

$$\widehat{\mathscr{V}}(x) = \delta(x)\widehat{\mathscr{G}}, \quad \widehat{\mathcal{H}}(x) = \tfrac{1}{2} \delta(x)\widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} \mathcal{K}, \qquad (101)$$

where

$$(\widehat{\mathscr{G}}\phi)(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \, \tilde{\mathfrak{g}}(p - q)\phi(q), \qquad (102)$$

and $\phi \in \mathscr{F}$.[7] Because $\mathcal{K}^2 = \mathbf{0}$, (101) implies $\widehat{\mathcal{H}}(x_2)\widehat{\mathcal{H}}(x_1) = \mathbf{0}$, the Dyson series (65) for $\widehat{\mathfrak{M}}$ terminates, and we find

$$\widehat{\mathfrak{M}} = \mathbf{I} - \frac{i}{2} \widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} \mathcal{K}$$

$$= \begin{bmatrix} \hat{1} - \frac{i}{2} \widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} & -\frac{i}{2} \widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} \\ \frac{i}{2} \widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} & \hat{1} + \frac{i}{2} \widehat{\mathscr{G}} \, \varpi(\hat{p})^{-1} \end{bmatrix}. \qquad (103)$$

---

[7]$\widehat{\mathscr{G}}\phi$ gives the convolution of $\tilde{g}$ and ϕ.

Substituting this equation in Eq. (97), we have

$$\widehat{\widetilde{\mathbf{M}}} = \widehat{\mathbf{\Pi}}_k - \frac{i}{2}\, \widehat{\mathbf{\Pi}}_k \widehat{\mathscr{G}}\, \varpi(\hat{p})^{-1} \widehat{\mathbf{\Pi}}_k\, \mathcal{K}$$

$$= \widehat{\mathbf{\Pi}}_k - \frac{i}{2}\, \widehat{\mathbf{\Pi}}_k \widehat{\mathscr{G}}\, \widehat{\mathbf{\Pi}}_k \varpi(\hat{p})^{-1}\, \mathcal{K}. \qquad (104)$$

Because $\widehat{\widetilde{\mathbf{M}}}$ acts in $\mathscr{F}_k^{2\times 1}$, we can replace the first $\widehat{\mathbf{\Pi}}_k$ on the right-hand side of this equation by $\widehat{\mathbf{I}}$ and observe that we can obtain it from (103) by replacing $\widehat{\mathscr{G}}$ with $\widehat{\mathbf{\Pi}}_k \widehat{\mathscr{G}}\, \widehat{\mathbf{\Pi}}_k$. In view of (101) and (103), this shows that we can obtain $\widehat{\widetilde{\mathbf{M}}}$ by letting $\widehat{\mathbf{\Pi}}_k \widehat{\mathscr{V}}(x)\, \widehat{\mathbf{\Pi}}_k$ play the role of $\widehat{\mathscr{V}}(x)$ in the calculation of $\widehat{\widetilde{\mathfrak{M}}}$. Similarly, we can determine $\widehat{\mathbf{M}}$ from the expression for $\widehat{\mathfrak{M}}$. This is the procedure used in Ref. [68] to obtain $\widehat{\mathbf{M}}$ for the class of potentials given by (100).

For a variety of different choices for the function $\mathfrak{g}$, it is possible to find analytic closed-form expressions for the solutions of (36) and (38). These together with (26), (28), (35), and (37) lead to an exact solution of the scattering problem for these potentials. Among these are the multi-δ-function potentials:

$$v_N(x, y) = \delta(x) \sum_{n=1}^{N} \mathfrak{z}_n \delta(y - a_n), \qquad (105)$$

for arbitrary $N \in \mathbb{Z}^+$, $\mathfrak{z}_n \in \mathbb{C}$, and $a_n \in \mathbb{R}$, as well as the Dirac comb potential:

$$v_{\mathrm{DC}}(x, y) = \mathfrak{z}\, \delta(x) \sum_{n=-\infty}^{\infty} \delta(y - na), \qquad$$

where $\mathfrak{z} \in \mathbb{C}$ and $a \in \mathbb{R}^+$ [68].

The standard treatment of the scattering problem for the δ-function potentials (105) leads to divergent terms, and there are well-known regularization and renormalization procedures to derive a physically sensible expression for the scattering problem for these potentials [52–56,58–60]. The application of the fundamental transfer matrix to these potentials does not involve any divergent terms and produces the expression obtained by the standard approach. In the remainder of this section, we examine the implicit regularization property of the fundamental transfer matrix. For simplicity of presentation, we confine our attention to the δ-function potentials:

$$v(x, y) = \mathfrak{z}\, \delta(x)\delta(y - a), \qquad (106)$$

with $\mathfrak{z} \in \mathbb{C}$ and $a \in \mathbb{R}$, which corresponds to setting $\mathfrak{g}(y) = \mathfrak{z}\, \delta(y - a)$.

For this choice of $\mathfrak{g}$, we have $\tilde{\mathfrak{g}}(p) := \mathcal{F}_{y,p}\{g(y)\} = \mathfrak{z}\, e^{-iap}$. Substituting this equation in Eq. (102) and making use of (12), we have

$$(\widehat{\mathscr{G}}\phi)(p) := \mathfrak{z}\, e^{-iap} \mathcal{F}_{q,a}^{-1}\{\phi(q)\}. \qquad (107)$$

In view of (103) and (107), the entries of the auxiliary transfer matrix satisfy

$$(\widehat{\widetilde{\mathfrak{M}}}_{jl}\phi)(p) = \phi(p)\, \delta_{jl} + \frac{(-1)^j\, i\mathfrak{z}\, f_\phi(a)\, e^{-iap}}{2}, \qquad (108)$$

where δ_{jl} is the Kronecker δ symbol, and

$$f_\phi(a) := \mathcal{F}_{q,a}^{-1}\{\varpi(q)^{-1}\phi(q)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq\, \frac{e^{iaq}\phi(q)}{\varpi(q)}. \qquad (109)$$

Because the δ-function potential (106) is invariant under the reflection about the y axis, its left and right scattering amplitudes coincide: $\mathfrak{f}^l = \mathfrak{f}^r$. This allows us to confine our attention to the study of its scattering problem for a right-incident wave. Having determined auxiliary transfer matrix $\widehat{\widetilde{\mathfrak{M}}}$, we first try to use it to calculate the right scattering amplitude $\mathfrak{f}^r$. This requires the solution of (89), which in view of (108) takes the following explicit form:

$$\breve{\mathscr{B}}_-^r(p) = 2\pi\varpi(p_0)\delta(p - p_0) - \frac{i\mathfrak{z}}{2}\, f(a)\, e^{-iap}, \qquad (110)$$

where

$$f(a) := f_{\breve{\mathscr{B}}_-^r}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq\, \frac{e^{iaq}\breve{\mathscr{B}}_-^r(q)}{\varpi(q)}. \qquad (111)$$

We can try to determine $f(a)$ by substituting (110) in Eq. (111). This gives

$$f(a) = \frac{e^{iap_0}}{1 + \frac{i\mathfrak{z}}{4\pi}\mathcal{F}_{q,0}\{\varpi(q)^{-1}\}}. \qquad (112)$$

However,

$$\mathcal{F}_{q,0}\{\varpi(q)^{-1}\} = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{k^2 - q^2}} = \infty. \qquad (113)$$

This is the same logarithmic singularity that arises in the standard treatment of the δ-function potentials in 2D [52–56,58–60]. We can remove it by a regularization of the integral in Eq. (113) followed by a renormalization of the coupling constant $\mathfrak{z}$.

The above calculation shows that, at least for the δ-function potential (106), the application of the auxiliary transfer matrix faces the same complications as the standard methods based on the Lippmann-Schwinger equation.

Next, we examine the application of the fundamental transfer matrix (104) in addressing the scattering problem for the δ-function potential (106). To do this, first, we use (80) and (107) to show that

$$(\widehat{\mathbf{\Pi}}_k \widehat{\mathscr{G}}\, \widehat{\mathbf{\Pi}}_k \phi)(p) := \frac{\chi_k(p)e^{-iap}}{2\pi} \int_{-k}^{k} dq\, e^{iaq}\phi(q), \qquad (114)$$

where

$$\chi_k(p) := \begin{cases} 1 & \text{for } |p| < k, \\ 0 & \text{for } |p| \geqslant k. \end{cases} \qquad (115)$$

Employing (114) in Eq. (104) and assuming that $|p| < k$, we obtain

$$(\widehat{\widetilde{M}}_{jl}\phi)(p) = \chi_k(p)\left[\phi(p)\delta_{jl} + \frac{(-1)^j\, i\mathfrak{z}\, g_\phi\, e^{-iap}}{2}\right], \qquad (116)$$

where

$$g_\phi := \frac{1}{2\pi} \int_{-k}^{k} dq\, \frac{e^{iaq}\phi(q)}{\varpi(q)}.$$

Now we are in a position to solve (44) for $\breve{B}_-^r$. In view of (116), this equation reads

$$\breve{B}_-^r(p) = \chi_k(p)\left[2\pi\varpi(p_0)\delta(p-p_0) - \frac{i\mathfrak{z}}{2}\, g_{\breve{\mathscr{B}}_-^r}\, e^{-iap}\right],$$
(117)

where

$$g_{\breve{\mathscr{B}}_-^r} := \frac{1}{2\pi}\int_{-k}^{k} dq\, \frac{e^{iaq}\breve{B}_-^l(q)}{\varpi(q)}.$$
(118)

Again, we substitute (117) in Eq. (118) to determine $g_{\breve{\mathscr{B}}_-^l}$. Because $\int_{-k}^{k} dq/\sqrt{k^2-q^2} = \pi$, this gives

$$g_{\breve{\mathscr{B}}_-^r} = \frac{4\, e^{iap_0}}{4+i\mathfrak{z}}.$$
(119)

Inserting this equation in Eq. (117) and making use of (32), (43), (116), and (119), we find

$$\breve{B}_-^r(p) = \chi_k(p)\bigg\{2\pi\varpi(p_0)\delta(p-p_0)$$
$$- \frac{2i\mathfrak{z}\,\exp[-ia(p-p_0)]}{4+i\mathfrak{z}}\bigg\},$$

$$\breve{A}_+^r(p) = -\frac{i\mathfrak{z}}{2}\,\chi_k(p)\,g(a)e^{-iap}$$
$$= -\frac{2i\mathfrak{z}\,\chi_k(p)\exp[-ia(p-p_0)]}{4+i\mathfrak{z}},$$

$$\mathfrak{f}^r(\theta) = -\sqrt{\frac{2}{\pi}}\,\frac{\mathfrak{z}}{4+i\mathfrak{z}}\,\exp[-iak(\sin\theta-\sin\theta_0)].$$

This expression for the scattering amplitude agrees with the one obtained in Refs. [46,68]. The standard coupling constant normalization scheme also produces the same result, albeit with $\mathfrak{z}$ replaced with the renormalized coupling constant [60].

Comparing the application of the auxiliary and fundamental transfer matrices for the treatment of the δ-function potential (106), we can explain the source of the implicit regularization feature of the latter. The presence of the projection operator $\widehat{\Pi}_k$ in Eq. (97) has the effect of imposing a cutoff on p, namely, $|p| < k$. Therefore, the application of the fundamental transfer matrix involves an implicit cutoff regularization of the δ-function potentials (106). The same holds for the multi-δ-function potentials (105).

VII. PERFECT BROADBAND INVISIBILITY IN 2D

References [61,62,64] use the transfer-matrix formulation of potential scattering proposed in Ref. [46] to construct complex potentials displaying perfect broadband omnidirectional and unidirectional invisibility. In this section, we discuss the application of the fundamental transfer matrix for the same purpose.

We have treated the scattering amplitudes $\mathfrak{f}$ and $\mathfrak{f}^{l/r}$ as functions of the scattering angle θ, but they also depend on the incidence angle θ_0 and the wave number k. Making the θ_0 and k dependence of the scattering amplitudes explicit, we can express (10) as

$$\mathfrak{f}(\theta;\theta_0,k) = \begin{cases} \mathfrak{f}^l(\theta;\theta_0,k) & \text{for } -\frac{\pi}{2}<\theta_0<\frac{\pi}{2}, \\ \mathfrak{f}^r(\theta;\theta_0,k) & \text{for } \frac{\pi}{2}<\theta_0<\frac{3\pi}{2}. \end{cases}$$
(120)

If $\mathfrak{f}(\theta;\theta_0,k)=0$ for all $\theta_0 \in [0,2\pi)$, we say that the potential has *omnidirectional invisibility* for wave number k. Similarly, we speak of *unidirectional invisibility* for a wave number k when one and only one of the following conditions holds [61]:

$$\mathfrak{f}^l(\theta;\theta_0,k)=0 \quad \text{for all } \theta_0 \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right),$$

$$\mathfrak{f}^r(\theta;\theta_0,k)=0 \quad \text{for all } \theta_0 \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right).$$

We refer to these cases as *left* and *right* unidirectional invisibility at k, respectively. We qualify these invisibility properties as being *broadband* if they hold for a continuous range of values of k. We call them *perfect* if we can realize them without invoking any approximation scheme. For example, if there is some $\alpha_\pm \in \mathbb{R}^+$ such that $\alpha_- < \alpha_+$, for all $k \in [\alpha_-,\alpha_+]$, $\mathfrak{f}^r(\theta;\theta_0,k)=0$ for all $\theta_0 \in (\frac{\pi}{2},\frac{3\pi}{2})$, and $\mathfrak{f}^l(\theta;\theta_0,k) \neq 0$ for some $\theta_0 \in (\frac{\pi}{2},\frac{3\pi}{2})$, and we can verify these conditions without relying on an approximation scheme, we say that v has perfect broadband unidirectional invisibility from the right [61,64].

The fundamental transfer matrix provides a convenient characterization of invisible potentials. For example, according to (31), (32), and (41)–(44), omnidirectional invisibility for a wave number k corresponds to the requirement that

$$\widehat{\mathbf{M}} = \widehat{\mathbf{I}},$$
(121)

for this particular value of k.[8] Here, and in what follows, we view $\widehat{\mathbf{M}}$ as an operator acting in $\mathscr{F}_k^{2\times1}$ and interpret $\mathbf{I}$ as the identity operator for $\mathscr{F}_k^{2\times1}$.

In view of (64) and (97), (121) holds if, for all $n \in \mathbb{Z}^+$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$\widehat{\Pi}_k\widehat{\widetilde{\mathcal{H}}}(x_n)\widehat{\widetilde{\mathcal{H}}}(x_{n-1})\cdots\widehat{\widetilde{\mathcal{H}}}(x_1)\widehat{\Pi}_k = \widehat{\mathbf{0}}.$$
(122)

According to (62), the entries of $\widehat{\widetilde{\mathcal{H}}}(x)$ are given by

$$\widehat{\widetilde{\mathcal{H}}}_{jl}(x) = \frac{(-1)^{j+1}}{2}\,\exp[i(-1)^j x\varpi(\hat{p})]\widehat{\mathscr{V}}(x)$$
$$\times \exp[i(-1)^{l+1}x\varpi(\hat{p})]\omega(\hat{p})^{-1}.$$

This together with (95) and the fact that functions of $\hat{p}$ commute with the projection operator $\widehat{\Pi}_k$ shows that (122) holds if, for arbitrary complex-valued functions $f_1, f_2, \ldots, f_{n-1}$ of p,

$$\widehat{\Pi}_k\widehat{\mathscr{V}}(x_n)f_{n-1}(\hat{p})\widehat{\mathscr{V}}(x_{n-1})f_{n-2}(\hat{p})\widehat{\mathscr{V}}(x_{n-2})$$
$$\cdots\cdots f_1(\hat{p})\widehat{\mathscr{V}}(x_1)\widehat{\Pi}_k = \widehat{\mathbf{0}}.$$
(123)

For $n=1$, this is equivalent to

$$\widehat{\Pi}_k\,\widehat{\mathscr{V}}(x)\,\widehat{\Pi}_k = \hat{\mathbf{0}}.$$
(124)

We can use (14) and (79) to express (124) as

$$\int_{-k}^{k} dq\,\tilde{v}(x,p-q)\phi(q)=0 \quad \text{for } |p|<k \quad \text{and} \quad \phi \in \mathscr{F}_k.$$
(125)

---

[8] $\widehat{\mathbf{M}}$ and $\widehat{\mathfrak{M}}$ depend on k but not on θ_0.

In light of the identity:

$$\int_{-k}^{k} dq \, \tilde{v}(x, p - q)\phi(q) = \int_{p-k}^{p+k} d\mathfrak{K} \, \tilde{v}(x, \mathfrak{K})\phi(p - \mathfrak{K}),$$
(126)

and the fact that $|p| < k$ and $p - k \leqslant \mathfrak{K} \leqslant p + k$ imply $|\mathfrak{K}| < 2k$, we can satisfy (125) by demanding that $\tilde{v}(x, \mathfrak{K}) = 0$ for $|\mathfrak{K}| < 2k$. If there is some $\alpha \in \mathbb{R}^{+}$ such that

$$\tilde{v}(x, \mathfrak{K}) = 0 \quad \text{for } \mathfrak{K} \leqslant 2\alpha,$$
(127)

(125) and consequently (124) hold for all wave numbers k in the range $(0, \alpha]$ [62].

A less obvious consequence of (127) is that it implies (123). This follows from a more general result. As we show in Appendix A, if there are positive real numbers α and β such that $k \leqslant \alpha$ and

$$\tilde{v}(x, \mathfrak{K}) = 0 \quad \text{for } \mathfrak{K} \leqslant \beta,$$
(128)

then

$$\widehat{\mathcal{V}}(x_n)f_{n-1}(\hat{p})\widehat{\mathcal{V}}(x_{n-1})f_{n-2}(\hat{p})\widehat{\mathcal{V}}(x_{n-2})\cdots f_1(\hat{p})\widehat{\mathcal{V}}(x_1)\widehat{\Pi}_k = 0$$

$$\text{for } n \geqslant \frac{k + \alpha}{\beta}.$$
(129)

Because $\alpha \geqslant k$, (123) and consequently (122) hold for $n \geqslant 2\alpha/\beta$. This together with (64) and (97) proves the following theorem:

Theorem 1: Let $\alpha, \beta \in \mathbb{R}^{+}$, and v be a short-range potential such that $\tilde{v}(x, \mathfrak{K}) = 0$ for $\mathfrak{K} \leqslant \beta$. Then the following assertions hold for $k \in (0, \alpha]$:

(a) For $\beta \geqslant 2\alpha$, $\widehat{\mathbf{M}} = \widehat{\mathbf{I}}$.

(b) For $0 < \beta < 2\alpha$,

$$\widehat{\mathbf{M}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\lceil 2\alpha/\beta - 1\rceil}(-i)^n \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1}$$

$$\cdots \int_{x_0}^{x_2} dx_1 \, \widehat{\mathbf{\Pi}}_k \widehat{\bar{\mathcal{H}}}(x_n)\widehat{\bar{\mathcal{H}}}(x_{n-1})\cdots\widehat{\bar{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k,$$
(130)

where $\widehat{\mathbf{I}}$ is to be interpreted as the identity operator acting in $\mathscr{F}_k^{2\times 1}$, and $\lceil x \rceil$ stands for the smallest integer not smaller than x.

For a potential satisfying (127), $\beta = 2\alpha$, and Theorem 1 implies that $\widehat{\mathbf{M}} = \widehat{\mathbf{I}}$ for $k \in (0, \alpha]$. This proves the following theorem on perfect broadband omnidirectional invisibility [62]:

Theorem 2: Let α be a positive real number and v be a short-range potential such that $\tilde{v}(x, \mathfrak{K}) = 0$ for $\mathfrak{K} \leqslant 2\alpha$. Then v is omnidirectionally invisible for every wave number k that does not exceed α.

Reference [62] provides concrete examples of potentials satisfying the hypothesis of Theorem 2 and discusses their optical realizations in effectively 2D isotropic media. This provides the first examples of complex permittivity profiles that display perfect broadband omnidirectional invisibility for transverse electric or transverse magnetic incident waves. Reference [65] reports a 3D extension of this result that allows for the construction of isotropic media possessing perfect broadband omnidirectional invisibility for incident waves of arbitrary polarization.

Next, consider a potential satisfying (128) for $\beta = \alpha$, i.e.,

$$\tilde{v}(x, \mathfrak{K}) = 0 \quad \text{for } \mathfrak{K} \leqslant \alpha,$$
(131)

and suppose that $k \in (0, \alpha]$. Then (129) implies

$$\widehat{\mathcal{V}}(x_2)f_1(\hat{p})\widehat{\mathcal{V}}(x_1)\widehat{\Pi}_k = \hat{0},$$
(132)

and Theorem 1 gives the following expression for the fundamental transfer matrix:

$$\widehat{\mathbf{M}} = \widehat{\mathbf{I}} - i\int_{-\infty}^{\infty} dx \, \widehat{\mathbf{\Pi}} \, \widehat{\bar{\mathcal{H}}}(x)\,\widehat{\mathbf{\Pi}}$$

$$= \widehat{\mathbf{I}} - \frac{i}{2}\int_{-\infty}^{\infty} dx \, \exp[-i\varpi(\hat{p})x\boldsymbol{\sigma}_3]\widehat{\Pi}\widehat{\mathcal{V}}(x)\widehat{\Pi}\mathcal{K}$$

$$\times \exp[i\varpi(\hat{p})x\boldsymbol{\sigma}_3]\varpi(\hat{p})^{-1}.$$
(133)

This relation justifies the use of $\widehat{\Pi}\widehat{\mathcal{V}}(x)\widehat{\Pi}$ instead of $\widehat{\mathcal{V}}(x)$ in Ref. [64].

In view of (132) and (133), for all $\phi \in \mathscr{F}_k$, and $i_1, i_2, j_1, j_2 \in \{1, 2\}$,

$$\left(\widehat{M}_{i_1 j_1} - \delta_{i_1 j_1}\hat{1}\right)\left(\widehat{M}_{i_2 j_2} - \delta_{i_2 j_2}\hat{1}\right)\phi = 0.$$
(134)

As noted in Ref. [64], we can use (134) to obtain a closed-form expression for the scattering amplitudes $\mathfrak{f}^{l/r}$. For completeness, we summarize the derivation of this expression. First, we write (42) and (44) as

$$\breve{B}_-^l = -(\widehat{M}_{22} - \hat{1})\breve{B}_-^l - 2\pi\varpi(p_0)\widehat{M}_{21}\,\delta_{p_0},$$
(135)

$$\breve{B}_-^r = -(\widehat{M}_{22} - \hat{1})\breve{B}_-^r + 2\pi\varpi(p_0)\,\delta_{p_0}.$$
(136)

We can then use (134) and the fact that $\breve{B}_-^{l/r}$, δ_{p_0}, and $\widehat{M}_{21}\,\delta_{p_0}$ belong to $\mathscr{F}_k$ to check that

$$\breve{B}_-^l = -2\pi\varpi(p_0)\widehat{M}_{21}\,\delta_{p_0},$$

$$\breve{B}_-^r = -2\pi\varpi(p_0)(\widehat{M}_{22} - 2\hat{1})\,\delta_{p_0},$$
(137)

to solve (135) and (136). Substituting (137) in Eqs. (41) and (43) and employing (134), we obtain

$$\breve{A}_+^l = 2\pi\varpi(p_0)\widehat{M}_{11}\,\delta_{p_0},$$

$$\breve{A}_+^r = 2\pi\varpi(p_0)\widehat{M}_{12}\,\delta_{p_0}.$$
(138)

If we insert (137) and (138) in Eqs. (31) and (32) and use (133) to evaluate the $\widehat{M}_{jl}\,\delta_{p_0}$ that appear in the resulting expressions for the scattering amplitudes, we are led to the following most remarkable formula [64]:

$$\mathfrak{f}^{l/r}(\theta; \theta_0, k) = -\frac{\tilde{\tilde{v}}[k(\cos\theta - \cos\theta_0), k(\sin\theta - \sin\theta_0)]}{2\sqrt{2\pi}},$$
(139)

where $\tilde{\tilde{v}}(\mathfrak{K}_x, \mathfrak{K}_y)$ stands for the 2D Fourier transform of $v(x, y)$, i.e.,

$$\tilde{\tilde{v}}(\mathfrak{K}_x, \mathfrak{K}_y) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp[-i(\mathfrak{K}_x x + \mathfrak{K}_y y)]v(x, y).$$

Because the right-hand side of (139) is precisely the formula one obtains for the scattering amplitude in the first Born approximation, the above calculation proves the following theorem that was originally reported in Ref. [64]:

Theorem 3: Let α be a positive real number and v be a short-range potential such that $\tilde{v}(x, \mathfrak{K}) = 0$ for $\mathfrak{K} \leqslant \alpha$. Then the first Born approximation gives the exact expression for the scattering amplitude of v for wave numbers k not exceeding α.

Reference [64] provides concrete examples of potentials fulfilling the hypothesis of this theorem and characterizes particular classes of such potentials that display perfect broadband unidirectional invisibility.

VIII. GENERALIZATION TO 3D

The developments we report in Secs. II–V admit a straightforward 3D generalization, where the scattering phenomenon is defined through the stationary Schrödinger equation:

$$[-\nabla^2 + v(\mathbf{r})]\psi(\mathbf{r}) = k^2 \psi(\mathbf{r}), \tag{140}$$

where ∇^2 stands for the 3D Laplacian, $v : \mathbb{R}^3 \to \mathbb{C}$ is a short-range potential, $\mathbf{r} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z$ is the position vector, $\mathbf{e}_j$ is the unit vector along the j axis, and $j \in \{x, y, z\}$. Following the standard convention, we consider situations where the source of the incident wave is located on either of the planes $z = -\infty$ or $z = +\infty$. These respectively correspond to the scattering of left- and right-incident waves.

The standard formulation of stationary scattering [8] relies on the existence of the so-called scattering solutions of (140).

By definition, these satisfy

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_0 \cdot \mathbf{r}} + \frac{\mathfrak{f}(\vartheta, \varphi)\, e^{ikr}}{r} + o(r^{-1}) \quad \text{for } r \to \infty,$$

where $\mathbf{k}_0$ is the incident wave vector, (r, ϑ, φ) are the spherical coordinates of $\mathbf{r}$, and $\mathfrak{f}(\vartheta, \varphi)$ is the scattering amplitude. We can quantify the direction of $\mathbf{k}_0$ by the spherical angles ϑ_0 and φ_0. Because $|\mathbf{k}_0| = k$, $(k, \vartheta_0, \varphi_0)$ are the spherical coordinates of $\mathbf{k}_0$. For left- and right-incident waves, ϑ_0 takes values in the intervals $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$, and we label the corresponding scattering amplitudes by $\mathfrak{f}^l$ and $\mathfrak{f}^r$, respectively. In other words,

$$\mathfrak{f}(\vartheta, \varphi) = \begin{cases} \mathfrak{f}^l(\vartheta, \varphi) & \text{for } \vartheta_0 \in [0, \frac{\pi}{2}), \\ \mathfrak{f}^r(\vartheta, \varphi) & \text{for } \vartheta_0 \in (\frac{\pi}{2}, \pi]. \end{cases} \tag{141}$$

Given a vector $\mathbf{u} \in \mathbb{R}^3$ with components u_x, u_y, and u_z, let $\vec{u}$ denote the projection of $\mathbf{u}$ onto the x-y plane, i.e., $\vec{u} := u_x \mathbf{e}_x + u_y \mathbf{e}_y$. In the following, we identify $\vec{u}$ and $\mathbf{u}$, respectively, with (u_x, u_y) and $(\vec{u}, u_z)$, e.g.,

$$\vec{r} = (x, y), \quad \mathbf{r} = (\vec{r}, z).$$

Suppose that, for $z \to \pm\infty$, $|v(\vec{r}, z)|$ tends to zero with such a rate that every bounded solution of (140) satisfies

$$\psi(\vec{r}, z) \to \int_{\mathscr{D}_k} \frac{d^2\vec{p}}{4\pi^2 \varpi(\vec{p})} \, e^{i\vec{p} \cdot \vec{r}} [\breve{A}_\pm(\vec{p}) e^{i\varpi(\vec{p})z} + \breve{B}_\pm(\vec{p}) e^{-i\varpi(\vec{p})z}] \quad \text{for } z \to \pm\infty, \tag{142}$$

where

$$\mathscr{D}_k := \{\vec{p} \in \mathbb{R}^2 \,|\, |\vec{p}| < k\}, \quad \varpi(\vec{p}) := \begin{cases} \sqrt{k^2 - |\vec{p}|^2} & \text{for } |\vec{p}| < k, \\ i\sqrt{|\vec{p}|^2 - k^2} & \text{for } |\vec{p}| \geqslant k, \end{cases}$$

and $\breve{A}_\pm$ and $\breve{B}_\pm$ are functions of $\vec{p} \in \mathbb{R}^2$ that vanish for $|\vec{p}| \geqslant k$. In analogy to 2D, we use $\mathscr{F}$ to denote the set of functions of $\vec{p}$, and let $\mathscr{F}_k := \{\phi \in \mathscr{F} \,|\, \phi(\vec{p}) = 0 \text{ for } |\vec{p}| \geqslant k\}$. Then $\breve{A}_\pm, \breve{B}_\pm \in \mathscr{F}_k$, and we identify the fundamental transfer matrix $\widehat{\mathbf{M}}$ with the linear operator acting in $\mathscr{F}_k^{2\times1} := \mathbb{C}^{2\times1} \otimes \mathscr{F}_k$ that relates $\breve{A}_\pm$ and $\breve{B}_\pm$ via (40). This equation leads to the following analogs of (41)–(44):

$$\breve{A}_+^l = 4\pi^2 \varpi(\vec{p}_0) \widehat{M}_{11}\, \delta_{\vec{p}_0} + \widehat{M}_{12} \breve{B}_-^l, \tag{143}$$

$$\widehat{M}_{22} \breve{B}_-^l = -4\pi^2 \varpi(\vec{p}_0) \widehat{M}_{21}\, \delta_{\vec{p}_0}, \tag{144}$$

$$\breve{A}_+^r = \widehat{M}_{12} \breve{B}_-^r, \tag{145}$$

$$\widehat{M}_{22} \breve{B}_-^r = 4\pi^2 \varpi(\vec{p}_0)\, \delta_{\vec{p}_0}, \tag{146}$$

where

$$\vec{p}_0 := k \sin\vartheta_0 (\cos\varphi_0\, \mathbf{e}_x + \sin\varphi_0\, \mathbf{e}_y), \tag{147}$$

and $\delta_{\vec{p}_0}$ stands for the δ function centered at $\vec{p}_0$ in 2D, i.e., $\delta_{\vec{p}_0}(\vec{p}) := \delta(\vec{p} - \vec{p}_0)$.

The link between the fundamental transfer matrix and the left/right scattering amplitudes is provided by Eqs. (143)–(146) and the following identity which follows from an argument given in Ref. [46, Appendix F]:

$$\mathfrak{f}^l(\vartheta, \varphi) = -\frac{i}{2\pi} \times \begin{cases} \breve{A}_+^l(\vec{p}) - 4\pi^2 \varpi(\vec{p}_0)\delta(\vec{p} - \vec{p}_0) & \text{for } \vartheta \in [0, \frac{\pi}{2}), \\ \breve{B}_-^l(\vec{p}) & \text{for } \vartheta \in (\frac{\pi}{2}, \pi], \end{cases} \tag{148}$$

$$\mathfrak{f}^r(\vartheta, \varphi) = -\frac{i}{2\pi} \times \begin{cases} \breve{A}_+^r(\vec{p}) & \text{for } \vartheta \in [0, \frac{\pi}{2}), \\ \breve{B}_-^r(\vec{p}) - 4\pi^2 \varpi(\vec{p}_0)\delta(\vec{p} - \vec{p}_0) & \text{for } \vartheta \in (\frac{\pi}{2}, \pi], \end{cases} \tag{149}$$

where $\vec{p}_0$ and $\vec{p}$ are respectively given by (147) and

$$\vec{p} := k \sin\vartheta (\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y). \tag{150}$$

In practice, it is easier to determine the auxiliary transfer matrix $\widehat{\mathfrak{M}}$ and obtain the fundamental transfer matrix $\widehat{\mathbf{M}}$ using $\widehat{\mathbf{M}} = \widehat{\mathbf{\Pi}}_k\,\widehat{\mathfrak{M}}\,\widehat{\mathbf{\Pi}}_k$, where $\widehat{\mathbf{\Pi}}_k := \widehat{\Pi}_k\mathbf{I}$ and $\widehat{\Pi}_k$ is the projection operator mapping $\mathscr{F}$ onto $\mathscr{F}_k$ via

$$(\widehat{\Pi}_k f)(p) := \begin{cases} f(p) & \text{for } |\vec{p}| < k, \\ 0 & \text{for } |\vec{p}| \geqslant k. \end{cases}$$

We identify the auxiliary transfer matrix with the operator:

$$\widehat{\mathfrak{M}} := \lim_{z_\pm \to \pm\infty} \widehat{\mathcal{U}}(z_+, z_-) = \mathscr{T} \exp\left[-i \int_{-\infty}^{\infty} dz\, \widehat{\mathcal{H}}(z)\right], \tag{151}$$

where $\widehat{\mathcal{U}}(z, z_0)$ is the evolution operator for a quantum system with Hamiltonian operator:

$$\begin{aligned} \widehat{\mathcal{H}}(z) := &\tfrac{1}{2} \exp[-i\varpi(\widehat{\vec{p}})z\boldsymbol{\sigma}_3]\widehat{\mathcal{V}}(z)\mathcal{K} \\ &\times \exp[i\varpi(\widehat{\vec{p}})z\boldsymbol{\sigma}_3]\varpi(\widehat{\vec{p}})^{-1}, \end{aligned} \tag{152}$$

where z plays the role of time, z_0 is an initial value of z, for all $f, g \in \mathscr{F}$, we have $[f(\widehat{\vec{p}})g](\vec{p}) := f(\vec{p})g(\vec{p})$,

$$[\widehat{\mathcal{V}}(z)f](\vec{p}) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} d^2\vec{q}\, \tilde{\tilde{v}}(\vec{p} - \vec{q}, z)f(\vec{q}), \tag{153}$$

and $\tilde{\tilde{v}}(\vec{p}, z) := \int_{\mathbb{R}^2} d^2\vec{r}\, e^{i\vec{p}\cdot\vec{r}}v(\vec{r}, z)$ is the 2D Fourier transform of $v(\vec{r}, z)$ over $\vec{r}$.

The application of the fundamental transfer matrix for the solution of the scattering problem for the δ-function potential in 3D:

$$v(\mathbf{r}) = \mathfrak{z}\,\delta(\mathbf{r}), \tag{154}$$

avoids the singularities arising in the standard treatment of this potential and produces the same result for the scattering amplitudes $\mathfrak{f}^{l/r}$ as the one obtained by a regularization of the singularities and the renormalization of the coupling constant in the standard treatment. To see this, we consider the following 3D analog of (100):

$$v(x, y, z) = \mathfrak{z}\,\mathfrak{g}(\vec{r})\,\delta(z), \tag{155}$$

where $\mathfrak{g} : \mathbb{R}^2 \to \mathbb{C}$ is a function with (2D) Fourier transform $\tilde{\tilde{\mathfrak{g}}}$.

Pursuing the approach of Sec. VI, we can verify that the fundamental transfer matrix for the potential (155) is given by (104), where

$$(\widehat{\Pi}_k\widehat{\mathscr{G}}\,\widehat{\Pi}_k\phi)(\vec{p}) = \frac{\chi_k(\vec{p})}{4\pi^2} \int_{\mathscr{D}_k} d^2\vec{q}\, \tilde{\tilde{\mathfrak{g}}}(\vec{p} - \vec{q})\phi(\vec{q}), \tag{156}$$

and

$$\chi_k(\vec{p}) := \begin{cases} 1 & \text{for } |\vec{p}| < k, \\ 0 & \text{for } |\vec{p}| \geqslant k. \end{cases}$$

For $\mathfrak{g}(\vec{r}) = \delta(\vec{r})$, (155) reduces to (154), and (156) becomes $(\widehat{\Pi}_k\widehat{\mathscr{G}}\,\widehat{\Pi}_k\phi)(\vec{p}) = (4\pi^2)^{-1}\chi_k(\vec{p}) \int_{\mathscr{D}_k} d^2\vec{q}\,\phi(\vec{q})$. Substituting this equation in Eq. (104), we find the following expression for the entries of the fundamental transfer matrix:

$$(\widehat{M}_{jl}\phi)(\vec{p}) = \chi_k(\vec{p})\left[\delta_{jl}\phi(\vec{p}) + \frac{(-1)^j\,i\mathfrak{z}\,h_\phi}{2}\right], \tag{157}$$

where

$$h_\phi := \frac{1}{4\pi^2} \int_{\mathscr{D}_k} d^2\vec{q}\, \frac{\phi(\vec{q})}{\varpi(\vec{q})}. \tag{158}$$

Next, we use (157) to express (146) as

$$\check{B}_-^r(\vec{p}) = 4\pi^2\varpi(\vec{p}_0)\delta(\vec{p} - \vec{p}_0) - \frac{i\mathfrak{z}\,h_{\check{B}_-^r}}{2}. \tag{159}$$

We can compute $h_{\check{B}_-^r}$ by substituting this equation in Eq. (158). The result is

$$h_{\check{B}_-^r} = \frac{4\pi}{4\pi + ik\,\mathfrak{z}}, \tag{160}$$

where we have used the identity $\int_{\mathscr{D}_k} d^2\vec{q}/\sqrt{k^2 - q^2} = 2\pi k$. Inserting (160) in Eq. (159) and using (145), (149), (157), (160), and $\mathfrak{f}^l = \mathfrak{f}^r$, which follows from the invariance of the potential (154) under the transformation $z \to -z$, we have

$$\begin{aligned} \check{B}_-^r(\vec{p}) &= \chi_k(\vec{p})\left[4\pi^2\varpi(\vec{p}_0)\delta(\vec{p} - \vec{p}_0) - \frac{2\pi i\,\mathfrak{z}}{4\pi + ik\,\mathfrak{z}}\right], \\ \check{A}_+^r(\vec{p}) &= -\frac{2\pi i\,\mathfrak{z}\,\chi_k(\vec{p})}{4\pi + ik\,\mathfrak{z}}, \\ \mathfrak{f}^l(\vartheta, \varphi) &= \mathfrak{f}^r(\vartheta, \varphi) = -\frac{\mathfrak{z}}{4\pi + ik\,\mathfrak{z}}. \end{aligned}$$

The standard treatment of the δ-function potential (154), which involves a regularization of a polynomial singularity and the renormalization of the coupling constant, gives the same expression for $\mathfrak{f}^{l/r}$ with $\mathfrak{z}$ changed to the renormalized coupling constant. For the cases where $\mathfrak{z}$ is real, there is an argument due to Rajeev which leads to the same result [57]. Again our treatment avoids singular terms, for the solution of the scattering problem using the fundamental transfer matrix has a built-in cutoff regularization property.

We conclude this section by pointing out that we can easily generalize the results of Sec. VII to 3D and establish the following 3D analogs of Theorems 2 and 3:

Theorem 4: Let α be a positive real number, $\vec{e}$ be a unit vector lying in the x-y plane, and $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential such that $\tilde{\tilde{v}}(\vec{\mathfrak{K}}, z) = 0$ for $\vec{\mathfrak{K}} \cdot \vec{e} \leqslant 2\alpha$. Then v is omnidirectionally invisible for every wave number k that does not exceed α.

Theorem 5: Let α be a positive real number, $\vec{e}$ be a unit vector lying in the x-y plane, and v be a short-range potential such that $\tilde{\tilde{v}}(\vec{\mathfrak{K}}, z) = 0$ for $\vec{\mathfrak{K}} \cdot \vec{e} \leqslant \alpha$. Then the first Born approximation gives the exact expression for the scattering amplitude of v for wave numbers $k \leqslant \alpha$.

We outline the proofs of these theorems in Appendix B.

IX. CONCLUDING REMARKS

The idea of making use of an analog of the transfer matrix to describe the propagation and scattering of waves in dimensions higher than one dates back to the 1980s [39]. The transfer matrices used for this purpose are obtained by slicing the space along one direction, discretizing the spatial or momentum variable(s) along the normal directions, assigning a transfer matrix for each slice, and multiplying the transfer matrices for the slices according to an analog of the composition rule for the transfer matrix in 1D [45]. The outcome is a

large numerical transfer matrix whose manipulation requires appropriate numerical schemes. This is in sharp contrast to the notion of the S-matrix which is identified with a linear operator mapping between infinite-dimensional function spaces [5] and leads to a rigorous mathematical theory of scattering [6,8]. We proposed a different notion of the transfer matrix in 2D and 3D, which shared this feature of the S-matrix [46]. The main theoretical input leading to the introduction of this notion is the idea of expressing it in terms of the evolution operator for a nonunitary effective quantum system [47], hence the name *dynamical formulation of stationary scattering*.

A more detailed examination of the developments reported in Ref. [46] revealed an ambiguity whose resolution required a more careful treatment of the evanescent waves. In this paper, we propose a refinement of the conceptual framework offered in Ref. [46] that addresses this issue. It turns out that a proper implementation of this approach to stationary scattering entails the introduction of two different transfer matrices, which we call auxiliary and fundamental transfer matrices. Like the S-matrix, these are linear operators mapping between infinite-dimensional function spaces. We have derived their basic properties and explored their utility in solving scattering problems. We have established and elucidated the origin of an implicit regularization property of the fundamental transfer matrix for δ-function potentials in 2D and 3D. We have also offered proper derivations of a couple of basic results on achieving perfect broadband omnidirectional invisibility (Theorem 2) and constructing potentials for which the first Born approximation yields the exact expression for the scattering amplitude (Theorem 3) in 2D. The results we have obtained for 2D admit straightforward extensions to 3D. We have obtained 3D analogs of Theorems 2 and 3.

APPENDIX A: PROOF OF EQ. (129)

First, we introduce the function spaces

$$\mathcal{S}_\varsigma := \{f \in \mathscr{F} | f(p) = 0 \text{ for } p \leqslant \varsigma\}, \quad \varsigma \in \mathbb{R},$$

and prove a couple of lemmas.

Lemma 1: Let $\beta, \gamma \in \mathbb{R}$, v be a short-range potential satisfying

$$\tilde{v}(x, \mathfrak{K}) = 0 \quad \text{for } \mathfrak{K} \leqslant \beta, \tag{A1}$$

and $g \in \mathscr{F}$ be an arbitrary function such that

$$g(p) = 0 \quad \text{for } p \leqslant \gamma, \tag{A2}$$

i.e., for all $x \in \mathbb{R}$, $\tilde{v}(x, \cdot) \in \mathcal{S}_\beta$ and $g \in \mathcal{S}_\gamma$. Then $\widehat{\mathscr{V}}(x)g \in \mathcal{S}_{\beta+\gamma}$.

Proof: According to (14) and (A2),

$$[\widehat{\mathscr{V}}(x)g](p) = \frac{1}{2\pi} \int_\gamma^\infty dq\, \tilde{v}(x, p - q)g(q)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{p-\gamma} d\mathfrak{K}\, \tilde{v}(x, \mathfrak{K})g(p - \mathfrak{K}). \tag{A3}$$

For $p \leqslant \beta + \gamma$, $p - \gamma \leqslant \beta$, and (A1) implies that the integrand on the right-hand side of (A3) vanishes. Therefore, $\widehat{\mathscr{V}}(x)g \in \mathcal{S}_{\beta+\gamma}$. ∎

Lemma 2: Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $k \in (0, \alpha]$, $\phi \in \mathscr{F}_k$, v be a short-range potential satisfying (A1), $n \in \mathbb{Z}^+$, $x_1, x_2, \cdots, x_n \in \mathbb{R}$, $f_1, f_2, \cdots, f_n$ be functions of p, and

$$\phi_n := f_n(\hat{p})\widehat{\mathscr{V}}(x_n)f_{n-1}(\hat{p})\widehat{\mathscr{V}}(x_{n-1})f_{n-2}(\hat{p})\widehat{\mathscr{V}}(x_{n-2})$$

$$\cdots f_1(\hat{p})\widehat{\mathscr{V}}(x_1)\phi. \tag{A4}$$

Then $\phi_n(p) = 0$ for $p \leqslant n\beta - \alpha$, i.e.,

$$\phi_n \in \mathcal{S}_{n\beta-\alpha}. \tag{A5}$$

Proof (by induction on n): Because $\phi \in \mathscr{F}_k$ and $-\alpha \leqslant -k$, $\phi(p) = 0$ for $p \leqslant -k$. This implies $\phi(p) = 0$ for $p \leqslant -\alpha$. Hence, $\phi \in \mathcal{S}_{-\alpha}$. In view of this observation and (A1), which means $\tilde{v}(x, .) \in \mathcal{S}_\beta$, we can use Lemma 1 with $g = \phi$ and $\gamma = -\alpha$ to conclude that $\widehat{\mathscr{V}}(x)\phi \in \mathcal{S}_{\beta-\alpha}$. This together with the identity:

$$\phi_1(p) = [f_1(\hat{p})\widehat{\mathscr{V}}(x_1)\phi](p) = f_1(p)[\widehat{\mathscr{V}}(x_1)\phi](p),$$

imply $\phi_1 \in \mathcal{S}_{\beta-\alpha}$. Therefore, (A5) holds for $n = 1$. Next, we suppose that there is a positive integer m such that (A5) holds for $n = m$ and prove it for $n = m + 1$. Clearly, $\phi_{m+1} = f_{m+1}(\hat{p})\widehat{\mathscr{V}}(x_{m+1})\phi_m$. According to (A1), $\tilde{v}(x, \cdot) \in \mathcal{S}_\beta$, and induction hypothesis states that $\phi_m \in \mathcal{S}_{m\beta-\alpha}$. By virtue of Lemma 1, these imply $\widehat{\mathscr{V}}(x_{m+1})\phi_m \in \mathcal{S}_{(m+1)\beta-\alpha}$. Combining this relation with $\phi_{m+1}(p) = f_{m+1}(p)[\widehat{\mathscr{V}}(x_{m+1})\phi_m](p)$, we are led to $\phi_{m+1} \in \mathcal{S}_{(m+1)\beta-\alpha}$. This proves (A5) for $n = m + 1$. ∎

Now suppose that (A1) holds for some $\beta > 0$, $k \in (0, \alpha]$, and $\xi \in \mathscr{F}$ is arbitrary. Then applying Lemma 2 for $\phi := \widehat{\Pi}_k \xi$ and $f_n(p) = 1$, we find

$$[\widehat{\mathscr{V}}(x_n)f_{n-1}(\hat{p})\widehat{\mathscr{V}}(x_{n-1})f_{n-2}(\hat{p})\widehat{\mathscr{V}}(x_{n-2})$$

$$\cdots f_1(\hat{p})\widehat{\mathscr{V}}(x_1)\widehat{\Pi}_k\xi](p) = 0 \quad \text{for } p \leqslant n\beta - \alpha.$$

This relation together with the fact that $(\Pi_k\xi)(p) = 0$ for $p > k$ implies

$$\widehat{\mathscr{V}}(x_n)f_{n-1}(\hat{p})\widehat{\mathscr{V}}(x_{n-1})f_{n-2}(\hat{p})\widehat{\mathscr{V}}(x_{n-2})$$

$$\cdots f_1(\hat{p})\widehat{\mathscr{V}}(x_1)\widehat{\Pi}_k\xi = 0 \quad \text{for } n \geqslant \frac{k + \alpha}{\beta}.$$

This establishes (129) because ξ is an arbitrary element of $\mathscr{F}$.

APPENDIX B: PROOFS OF THEOREMS 4 AND 5

Let $\mathbf{e}_1$ be a unit vector belonging to the x-y plane, and $\mathbf{e}_2 := \mathbf{e}_z \times \mathbf{e}_1$, where $\times$ stands for the cross product of vectors belonging to $\mathbb{R}^3$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms an orthonormal basis of the x-y plane, and we can use it to introduce a Cartesian coordinate system for this plane. Let x' and y' denote the coordinates along the axes defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

Clearly, there is a rotation angle $\varphi \in [0, 2\pi)$ such that

$$x' = \cos\varphi\, x - \sin\varphi\, y, \quad y' = \sin\varphi\, x + \cos\varphi\, y.$$

Similarly, if $\vec{p} = p_x\, \mathbf{e}_x + p_y\, \mathbf{e}_y$ for some $(p_x, p_y) \in \mathbb{R}^2$, we can express the components of $\vec{p}$ in the coordinate frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ as

$$p_1 = \cos\varphi\, p_x - \sin\varphi\, p_y, \quad p_2 = \sin\varphi\, p_x + \cos\varphi\, p_y.$$

Next, for each $\varphi \in [0, 2\pi)$ and $\varsigma \in \mathbb{R}$, we use $\mathscr{R}_{\varphi,\varsigma}$ to denote the subset of the x-y plane defined by $x' \leqslant \varsigma$, i.e.,

$$\mathscr{R}_{\varphi,\varsigma} := \{(x, y) \in \mathbb{R}^2 \,|\, \cos\varphi\, x - \sin\varphi\, y \leqslant \varsigma\},$$

and

$$\mathcal{S}_{\varphi,\varsigma} := \{f \in \mathscr{F} \,|\, f(\vec{p}) = 0 \quad \text{for } \vec{p} \in \mathscr{R}_{\varphi,\varsigma}\},$$

where $\mathscr{F}$ denotes the set of functions of $\vec{p}$. We are now in a position to state 3D analogs of Lemmas 1 and 2 of Appendix A and Theorem 1.

Lemma 3: Let $\varphi \in [0, 2\pi)$, $\beta, \gamma \in \mathbb{R}$, $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential such that, for all $z \in \mathbb{R}$, $\tilde{v}(\cdot, z) \in \mathcal{S}_{\varphi,\beta}$, and $g \in \mathcal{S}_{\varphi,\gamma}$. Then $\widehat{\mathcal{V}}(z)g \in \mathcal{S}_{\varphi,\beta+\gamma}$.

Proof: Using $g \in \mathcal{S}_{\varphi,\gamma}$ in Eq. (153), we have

$$[\widehat{\mathcal{V}}(z)g](\vec{p})$$
$$= \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus \mathscr{R}_{\varphi,\gamma}} d^2\vec{q}\; \tilde{v}(\vec{p} - \vec{q})g(\vec{q})$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq_2 \int_{\gamma}^{\infty} dq_1 \tilde{v}(p_1 - q_1, p_2 - q_2, z)g(q_1, q_2)$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{p_1 - \gamma} d\mathfrak{K}_1 \tilde{v}(\mathfrak{K}_1, p_2 - q_2, z)$$
$$\times g(q_1 - \mathfrak{K}_1, q_2).$$

The integrand on the right-hand side of this equation vanishes for $p_1 \leqslant \beta + \gamma$ because $\tilde{v}(\cdot, z) \in \mathcal{S}_{\varphi,\beta}$ and $\mathfrak{K}_1 \leqslant p_1 - \gamma \leqslant \beta$. ∎

Lemma 4: Let $\varphi \in [0, 2\pi)$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $k \in (0, \alpha]$, $\phi \in \mathscr{F}_k$, $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential such that, for all $z \in \mathbb{R}$, $\tilde{v}(\cdot, z) \in \mathcal{S}_{\varphi,\beta}$, $n \in \mathbb{Z}^+$, $z_1, z_2, \ldots, z_n \in \mathbb{R}$, $f_1, f_2, \ldots, f_n \in \mathcal{F}$, and

$$\phi_n := f_n(\widehat{\vec{p}})\widehat{\mathcal{V}}(z_n)f_{n-1}(\widehat{\vec{p}})\widehat{\mathcal{V}}(z_{n-1})f_{n-2}(\widehat{\vec{p}})\widehat{\mathcal{V}}(z_{n-2})$$
$$\cdots f_1(\widehat{\vec{p}})\widehat{\mathcal{V}}(z_1)\phi.$$

Then $\phi_n \in \mathcal{S}_{\varphi,n\beta-\alpha}$.

Proof: This follows from the inductive argument used in the proof of Lemma 2 and the fact that $k \in (0, \alpha]$ and $\phi \in \mathscr{F}_k$ imply $\phi \in \mathcal{S}_{\varphi,-\alpha}$. ∎

The following 3D analog of Theorem 1 is an immediate consequence of Lemmas 3 and 4:

Theorem 6: Let $\varphi \in [0, 2\pi)$, $\alpha, \beta \in \mathbb{R}^+$, $k \in (0, \alpha]$, and $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential such that, for all $z \in \mathbb{R}$, $\widetilde{\widetilde{v}}(\vec{\mathfrak{K}}, z) = 0$ for $\mathfrak{K}_1 \leqslant \beta$. Then $\widehat{\mathbf{M}} = \widehat{\mathbf{I}}$ for $\beta \geqslant 2\alpha$, and

$$\widehat{\mathbf{M}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\lceil 2\alpha/\beta - 1 \rceil} (-i)^n \int_{z_0}^{z} dz_n \int_{z_0}^{z_n} dz_{n-1}$$
$$\cdots \int_{z_0}^{z_2} dz_1\, \widehat{\mathbf{\Pi}}_k \widehat{\widetilde{\mathcal{H}}}(z_n)\widehat{\widetilde{\mathcal{H}}}(z_{n-1})\cdots\widehat{\widetilde{\mathcal{H}}}(z_1)\widehat{\mathbf{\Pi}}_k, \quad \text{(B1)}$$

for $0 < \beta < 2\alpha$.

Theorems 4 and 5 follow as corollaries of Theorem 6; their hypotheses imply that of Theorem 6 with $\beta = 2\alpha$ and $\beta = \alpha$, respectively. This shows that

(1) if $\widetilde{\widetilde{v}}(\vec{\mathfrak{K}}, z) = 0$ for $\mathfrak{K}_1 \leqslant 2\alpha$, $\widehat{\mathbf{M}} = \widehat{\mathbf{I}}$ and the potential is omnidirectionally invisible for $k \leqslant \alpha$;

(2) if $\widetilde{\widetilde{v}}(\vec{\mathfrak{K}}, z) = 0$ for $\mathfrak{K}_1 \leqslant \alpha$, we have $\beta = \alpha$, $\lceil 2\alpha/\beta - 1 \rceil = 1$, and only the first two terms on the right-hand side of (B1) survive for $k \leqslant \alpha$.

In the latter case, we have a closed-form expression for $\widehat{\mathbf{M}}$ which, like 2D, allows for the determination of $\breve{B}_{-}^{l/r}$, $\breve{A}_{+}^{l/r}$, and $\mathfrak{f}^{l/r}$. Specifically, we obtain $\breve{B}_{-}^{l/r}$ and $\breve{A}_{+}^{l/r}$ by multiplying the right-hand sides of (137) and (138) by 2π and replacing p_0 with $\vec{p}_0$. Substituting these in Eqs. (148) and (149) and making use of (141), we arrive at

$$\mathfrak{f}(\vartheta, \varphi) = -\frac{\widetilde{\widetilde{v}}[\mathbf{k}(\vartheta, \varphi) - \mathbf{k}_0]}{4\pi}, \quad \text{(B2)}$$

where $\widetilde{\widetilde{v}}$ stands for the 3D Fourier transform of v, i.e., $\widetilde{\widetilde{v}}(\mathfrak{K}) = \int_{\mathbb{R}^3} d^3\mathbf{r}\, e^{-i\mathfrak{K}\cdot\mathbf{r}}v(\mathbf{r})$, and $\mathbf{k}$ and $\mathbf{k}_0$ are, respectively, the scattered and incident wave vectors, i.e.,

$$\mathbf{k}(\vartheta, \varphi) = k(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta),$$
$$\mathbf{k}_0 = k(\sin\vartheta_0\cos\varphi_0, \sin\vartheta_0\sin\varphi_0, \cos\vartheta_0).$$

This completes the proof of Theorem 5 because (B2), which provides an exact expression for the scattering amplitude of the potential, coincides with the outcome of applying the first Born approximation [4, §7.2].

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