Quantum state smoothing as an optimal Bayesian estimation problem with three different cost functions

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Quantum state smoothing is a technique to estimate an unknown true state of an open quantum system based on partial measurement information both prior and posterior to the time of interest. In this paper, we show that the smoothed quantum state is an optimal Bayesian state estimator, that is, it minimizes a Bayesian expected cost function. Specifically, we show that the smoothed quantum state is optimal with respect to two cost functions: the trace-square deviation from and the relative entropy to the unknown true state. However, when we consider a related cost function, the linear infidelity, we find, contrary to what one might expect, that the smoothed state is not optimal. For this case, we derive the optimal state estimator, which we call the lustrated smoothed state. It is a pure state, the eigenstate of the smoothed quantum state with the largest eigenvalue. We illustrate these estimates with a simple system, the driven, damped two-level atom.

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I. INTRODUCTION

Estimating the state of an open quantum system based on measurement information is an important task in quantum information science. Quantum trajectory theory, also referred to as quantum state filtering [1–3], utilizes a continuous-in-time past measurement record $\overleftarrow{O}$ to estimate the quantum state of a single system at time τ. One might think that one could obtain a more accurate estimate of the quantum state by conditioning on the *past-future* measurement record $\overleftrightarrow{O}$, that is, the measurement record both prior and posterior to the time of interest τ. However, due to the noncommutative nature of quantum states and operators, conditioning the estimate on future information is not straightforward in quantum mechanics, unlike in classical mechanics where the technique of smoothing is standard [4–10]. Consequently, utilizing past-future measurement information in quantum systems has attracted great interest and many smoothinglike formalisms have been proposed [11–20]. Here we are concerned with an approach that is guaranteed to yield a valid quantum state as its estimate, the quantum state smoothing theory [17,21–23].

The quantum state smoothing formalism is as follows. Consider an open quantum system coupled to two baths (which could represent sets of baths). An observer, say Alice, monitors one bath and constructs a measurement record O, called the observed record. The other bath, which is unobserved by Alice, is monitored by a second (perhaps hypothetical) observer, Bob, who also constructs a measurement record U, called the unobserved record. Now, if Alice were able to condition her estimate of the quantum state on both the past observed and unobserved measurement she would have effectively performed a perfect measurement on the system and Alice's estimated state would be the true state of the system $\rho_{\rm T} := \rho_{\overleftarrow{O}\overleftarrow{U}}$ which can be assumed to be pure (it will be as long as the initial state is pure). Note that it is not a necessary requirement for Alice's and Bob's records together to constitute a perfect monitoring of the quantum system or for the true state to be pure, but merely a convenient assumption to make when introducing the idea of quantum state smoothing. However, since Alice does not have access to the unobserved measurement record she cannot know the true quantum state. Nevertheless, Alice can calculate a Bayesian estimate of the true state ρ_{O_Ξ} conditioned on her observed measurement record O_Ξ by averaging over all possible unobserved measurement records with the appropriate probabilities, i.e.,

$$\rho_{O_\Xi} = \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\rho_{\rm T}\} \equiv \sum_{\overleftarrow{U}} \wp(\overleftarrow{U}|O_\Xi)\rho_{\overleftarrow{O}\overleftarrow{U}}, \tag{1}$$

where $\mathbb{E}\{\bullet\}$ denotes an ensemble average. For a filtered estimate of the quantum state, the past observed record is used ($O_\Xi = \overleftarrow{O}$) and $\rho_{\overleftarrow{O}} = \rho_{\rm F}$. For a smoothed estimate of the quantum state, the past-future observed record is used ($O_\Xi = \overleftrightarrow{O}$) and we define $\rho_{\overleftrightarrow{O}} = \rho_{\rm S}$. Unlike $\rho_{\rm F}$, $\rho_{\rm S}$ depends on how Bob chooses to measure his bath, although averaging over Alice's result removes this dependence, as one then obtains the same unconditioned state ρ in both cases.

Since its conception [17], various properties of and scenarios for the smoothed quantum state have been studied, providing considerable insight into the theory [21–25], including how different measurement choices by Bob can affect the purity improvement over the filtered state [22,23] and the differentiability of the smoothed quantum state [25]. However, these works did not address one question: Is the smoothed quantum state an optimal estimator of the true state and if so, in what sense? Here *optimal* means minimizing a expected cost function. The expected cost function is the appropriately conditioned expectation for a cost function $\mathcal{C}(\check{\rho}, \rho_{\rm T})$, a measure comparing the estimate state $\check{\rho}$ to the true state, that is,

$$\mathcal{B}_{O_\Xi}(\check{\rho}) = \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\mathcal{C}(\check{\rho}, \rho_{\rm T})\}. \tag{2}$$

Note that in Eq. (2) the expected cost function is defined in the Bayesian sense [26–29], where it is the unknown quantity which is averaged over with the observations remaining fixed. This is as opposed to the frequentist sense [26–29], which averages over the potential observations with the unknown quantity fixed, and one typically optimizes the estimator by considering the worst-case scenario. It is easy to see why we have adopted the Bayesian approach, as the standard quantum filtered state is a Bayesian estimator, conditioned on the past record, and the smoothed quantum state is a generalization of it, conditioned on the past-future record.

Returning to the Bayesian expected cost function, henceforth referred to as the expected cost function for simplicity, in Eq. (2), in this paper, we show that, for the trace-square deviation from the true state (Sec. II) and the relative entropy with the true state (Sec. III) as cost functions, the conditioned state (1) is the optimal estimator. Furthermore, for each cost function, we show that the expected cost function of the conditioned state ρ_{O_Ξ} reduces to simple measures involving only ρ_{O_Ξ}. When we consider a cost function closely related to the trace-square deviation, the linear infidelity with the true state (Sec. IV), we find, somewhat counterintuitively, that ρ_{O_Ξ} is not the optimal estimator. Rather, the *lustrated* conditioned state, a pure state corresponding to the largest eigenvalue of ρ_{O_Ξ}, is the optimal estimator. We derive upper and lower bounds on the expected cost function of the lustrated state for both the trace-square deviation and linear infidelity expected cost function.

II. COST FUNCTION: TRACE-SQUARE DEVIATION

In this section we focus on the trace-square deviation (TrSD) from the true state, a distance measure between two quantum states, as the cost function of interest, i.e., $\mathcal{C}(\check{\rho}, \rho_{\rm T}) = {\rm Tr}[(\check{\rho} - \rho_{\rm T})^2]$. This means that the expected cost function for a given estimate $\check{\rho}$ of the true quantum state is

$$\begin{aligned}\mathcal{B}^{\rm TrSD}_{O_\Xi}(\check{\rho}) &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{{\rm Tr}[(\check{\rho} - \rho_{\rm T})^2]\}\\ &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{P(\rho_{\rm T}) - 2L(\check{\rho}, \rho_{\rm T}) + P(\check{\rho})\},\end{aligned} \tag{3}$$

where the purity is defined as

$$P(\rho) = {\rm Tr}[\rho^2] \tag{4}$$

and the linear fidelity [30] is

$$L(\rho, \sigma) = {\rm Tr}[\rho\sigma]. \tag{5}$$

Note that here we have not restricted our discussion to smoothed estimates ($O_\Xi = \overleftrightarrow{O}$), but are also allowing for a filtered estimate ($O_\Xi = \overleftarrow{O}$).

To see that the conditioned state, Eq. (1), is the estimator that minimizes Eq. (3), we will show that any other estimator $\rho' = \rho_{O_\Xi} + \hat{\mathcal{O}}$ for any traceless Hermitian operator $\hat{\mathcal{O}} \neq 0$ is suboptimal, that is, the expected cost function for ρ' is strictly greater than the expected cost function for ρ_{O_Ξ}. By substituting ρ' into the expected cost function we obtain

$$\begin{aligned}\mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho') &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{{\rm Tr}[(\rho' - \rho_{\rm T})^2]\}\\ &= \mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho_{O_\Xi}) + 2\,{\rm Tr}[\hat{\mathcal{O}}\rho_{O_\Xi}]\\ &\quad - 2\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{{\rm Tr}[\hat{\mathcal{O}}\rho_{\rm T}]\} + {\rm Tr}[\hat{\mathcal{O}}^2]\\ &= \mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho_{O_\Xi}) + {\rm Tr}[\hat{\mathcal{O}}^2]\\ &> \mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho_{O_\Xi}).\end{aligned} \tag{6}$$

Here, to obtain the second line we have expanded the trace term and collected the terms that contribute to $\mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho_{O_\Xi})$, making use of the cyclic nature of the trace. To obtain the third line we have used the linear nature of the trace and Eq. (1). Finally, the inequality in the final line results from the fact that $\hat{\mathcal{O}}$ has real eigenvalues.

Interestingly, we can show that the expected linear fidelity between the conditioned state ρ_{O_Ξ} and the true state is equal to the purity of the conditioned state, that is,

$$\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{L(\rho_{O_\Xi}, \rho_{\rm T})\} = {\rm Tr}[\rho_{O_\Xi}\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\rho_{\rm T}\}] = P(\rho_{O_\Xi}). \tag{7}$$

This equality also holds for the Jozsa fidelity [31] $F(\rho_{O_\Xi}, \rho_{\rm T}) = ({\rm Tr}[\sqrt{\sqrt{\rho_{\rm T}}\rho_{O_\Xi}\sqrt{\rho_{\rm T}}}])^2$ provided the true state is pure as, in this case, the Jozsa fidelity is equal to the linear fidelity. Previously [17,22], Eq. (7) was proven only when also averaging both sides over the observed record O_Ξ. [Note that, in [22], the equality between the Jozsa fidelity and the purity is typeset incorrectly; in Eq. (9), the left-hand side should also average over the past unobserved measurement record.] The expected cost function, Eq. (3), for the optimal estimator $\check{\rho} = \rho_{O_\Xi}$, assuming a pure true state, thus reduces to the impurity of the conditioned state

$$\mathcal{B}^{\rm TrSD}_{O_\Xi}(\rho_{O_\Xi}) = 1 - P(\rho_{O_\Xi}). \tag{8}$$

To verify Eq. (7), we consider a physical example. The system is a two-level system, with a driving Hamiltonian $\hat{H} = (\Omega/2)\hat{\sigma}_x$ and radiative damping described by a Lindblad operator $\sqrt{\gamma}\hat{\sigma}_-$, where Ω is the Rabi frequency and γ is the damping rate. Alice monitors a fraction η of the output fluorescence using Y homodyne detection so that her Lindblad operator can be written $\hat{c}_\phi = \sqrt{\gamma\eta}e^{-i\pi/2}\hat{\sigma}_-$, defined so that her photocurrent is $dJ_\phi = {\rm Tr}[\hat{c}_\phi\rho + \rho\hat{c}_\phi^\dagger] + dW$, where dW is a Wiener increment satisfying

$$\mathbb{E}\{dW\} = 0, \quad \mathbb{E}\{dW^2\} = dt. \tag{9}$$

The remaining $1-\eta$ fraction of the output is monitored by Bob using photodetection, with the Lindblad operator $\hat{c}_N = \sqrt{\gamma(1-\eta)}\hat{\sigma}_-$ defined so that the average rate of his jumps is $\mathbb{E}\{dN\}/dt = {\rm Tr}[\hat{c}_N^\dagger\hat{c}_N\rho]$. Note that Alice and Bob's measurements collectively constitute a perfect measurement, so the true state is pure. We can describe the evolution of the true quantum state with the stochastic master equation [3]

$$\begin{aligned}d\rho_{\rm T} &= \mathcal{G}[\hat{c}_N]\rho dN - i[\hat{H}, \rho_{\rm T}]dt - \mathcal{H}\big[\tfrac{1}{2}\hat{c}_N^\dagger\hat{c}_N\big]\rho_{\rm T}dt\\ &\quad + \mathcal{D}[\hat{c}_\phi]\rho_{\rm T}dt + \mathcal{H}[\hat{c}_\phi]\rho_{\rm T}dW_{\rm T},\end{aligned} \tag{10}$$

with the initial condition chosen to be the ground state $\rho(t_0) = |0\rangle\langle 0|$. The superoperators are defined as $\mathcal{G}[\hat{c}]\rho = \hat{c}\rho\hat{c}^\dagger/{\rm Tr}[\hat{c}\rho\hat{c}^\dagger] - \rho$, $\mathcal{D}[\hat{c}]\rho = \hat{c}\rho\hat{c}^\dagger - \{\hat{c}^\dagger\hat{c}, \rho\}/2$, and $\mathcal{H}[\hat{c}]\rho = \hat{c}\rho + \rho\hat{c}^\dagger - {\rm Tr}[\hat{c}\rho + \rho\hat{c}^\dagger]\rho$. Here the vector of

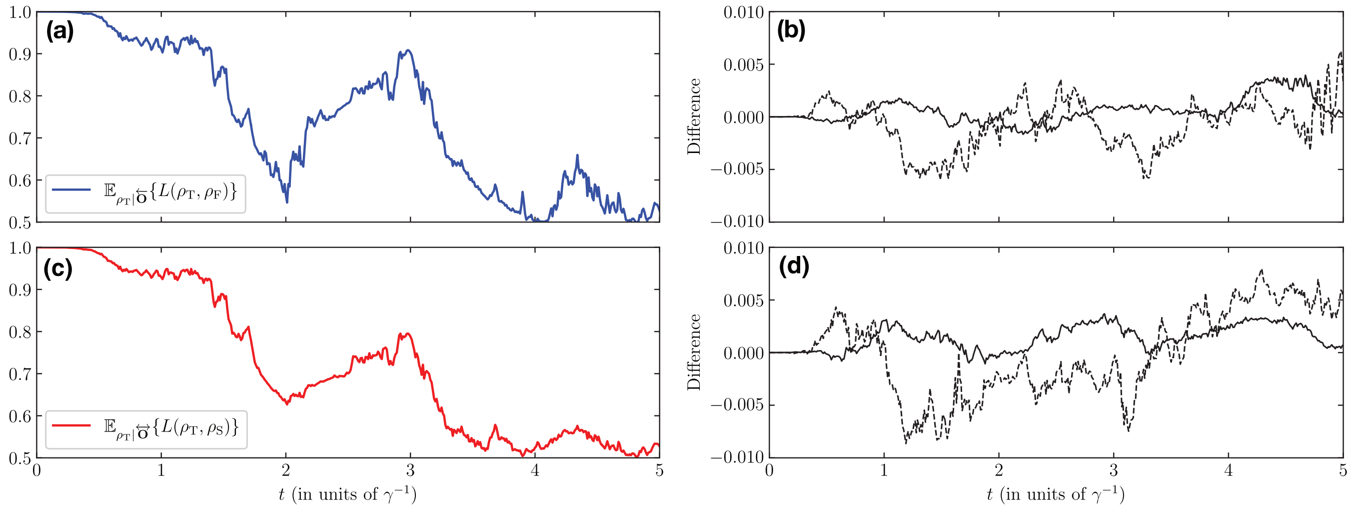

FIG. 1. Ensemble average of the linear fidelity between the conditioned state, (a) filtered and (c) smoothed, and the true state for the quantum system of Eq. (10), where Alice and Bob are monitoring the same channel with equal measurement efficiency ($\eta = 0.5$) using Y homodyne detection and photodetection, respectively. The number of trajectories M used to compute the ensemble average for the smoothed quantum state for this simulation is 2×10^4. The number of trajectories N used to compute the ensemble average for the linear fidelity between the true state and the smoothed quantum state is 3×10^4. Note that the observed record is fixed in all cases and the ensembles of unobserved records used to generate the smoothed state and to average the linear fidelity are generated independently. (b) and (d) In order to show the equality in Eq. (7), we have plotted the difference between the purity and the expected linear fidelity for $M = 2 \times 10^3$ and $N = 3 \times 10^3$ unobserved jump trajectories (dashed line) and for $M = 2 \times 10^4$ and $N = 3 \times 10^4$ (solid line). We can see that the difference between the purity and the average fidelity decreases as the number of trajectories used to compute the averages increases, supporting the equality in Eq. (7). Here $\hbar = 1$, $\Omega = 3\gamma$, and $\eta = 0.5$.

(true) innovations $dW_T = dJ_\phi - \text{Tr}[\hat{c}_\phi \rho_T + \rho_T \hat{c}_\phi^\dagger]dt$ satisfies properties similar to those in Eq. (9).

For the details of how the true state ρ_T, the filtered state ρ_F, the smoothed state ρ_S, and the ensemble averages $\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\bullet\}$ are computed, see the Appendix. In Fig. 1 we can see the convergence of $\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{F(\rho_{O_\Xi}, \rho_T)\}$ to the purity $P(\rho_{O_\Xi})$ as the number of unobserved trajectories averaged over increases for both the filtered state and the smoothed state. When a large number ensemble is used, the two are almost indistinguishable.

III. COST FUNCTION: RELATIVE ENTROPY

We now direct our attention to another cost function, the relative entropy with the true state. The consideration of this cost function stems from the work in Ref. [32], where the authors derived bounds on the average relative entropy between the state obtained by an omniscient observer (the true state) and either an ignorant (one who has no measurement information) or partially ignorant observer (one who has a fraction of the measurement output). The key difference here is that we are considering the optimal estimator that minimizes the (conditional) average of the relative entropy, as opposed to finding the upper and lower bounds. Furthermore, we consider a more general setting, allowing the omniscient observer to perform a different measurement (Bob's measurement in our formulation) on the remaining portion of the output to the partially ignorant observer (Alice in our formulation).

The relative entropy is defined as [30,33,34]

$$S(\rho||\sigma) = \text{Tr}[\rho \log \rho] - \text{Tr}[\rho \log \sigma]. \tag{11}$$

The relative entropy $S(\rho||\sigma)$ is a measure of state distinguishability between states ρ and σ, more specifically, it is akin to the likelihood that state σ will not be confused with state ρ, where a relative entropy of zero corresponds to completely indistinguishable states. Note that, unlike many other cost functions, like the TrSD, the relative entropy is not symmetric in its arguments. Consequently, the estimation task is to find the state that minimizes the expected cost function

$$\mathcal{B}^{\text{RE}}_{O_\Xi}(\check{\rho}) = \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{S(\rho_T||\check{\rho})\}. \tag{12}$$

Once again, the state that minimizes this expected cost function is the usual conditioned state. We can show the optimality of this state using the same method as before, that is, any state $\rho' \neq \rho_{O_\Xi}$ will be suboptimal. Substituting ρ' into the expected cost function, we get

$$\begin{aligned}\mathcal{B}^{\text{RE}}_{O_\Xi}(\rho') &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\text{Tr}[\rho_T \log \rho_T] - \text{Tr}[\rho_T \log \rho']\}\\ &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\text{Tr}[\rho_T \log \rho_T] - \text{Tr}[\rho_{O_\Xi} \log \rho_{O_\Xi}]\}\\ &\quad + \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\text{Tr}[\rho_{O_\Xi} \log \rho_{O_\Xi}] - \text{Tr}[\rho_T \log \rho']\}. \end{aligned} \tag{13}$$

Remembering that $\rho_{O_\Xi} = \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{\rho_T\}$, the expected cost function becomes

$$\begin{aligned}\mathcal{B}^{\text{RE}}_{O_\Xi}(\rho') &= \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{S(\rho_T||\rho_{O_\Xi})\} + S(\rho_{O_\Xi}||\rho')\\ &> \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{S(\rho_T||\rho_{O_\Xi})\},\end{aligned} \tag{14}$$

where the inequality results from the fact that the relative entropy is non-negative and is saturated only when $\rho' = \rho_{O_\Xi}$ [34].

In a similar vein to the TrSD case, when we restrict our analysis to pure true states, the expected cost function of the

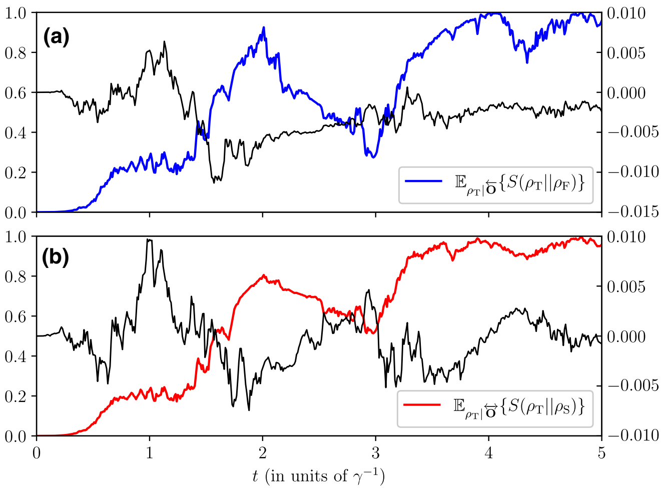

FIG. 2. Relative entropy expected cost function for the (a) filtered state and (b) smoothed state for the quantum system described in Eq. (10). We have also computed the difference (thin black line with the scale on the right y axis) between the expected cost function and the von Neumann entropy to illustrate that the equality in Eq. (12) holds, where $\log_2$ has been used for the entropy in both cases. Other details, including the randomly generated observed record $\overleftrightarrow{O}$, are as in Figs. 1(a) and 1(c).

conditioned state simplifies to the von Neumann entropy of ρ_{O_Ξ}, that is,

$$\mathcal{B}^{\mathrm{RE}}_{O_\Xi}(\rho_{O_\Xi}) = S(\rho_{O_\Xi}). \tag{15}$$

This follows from the fact that for a pure state ρ, $\mathrm{Tr}[\rho \log \rho] = 0$. Note that a similar equality was also derived in Ref. [32]; however, there only the average over both the observed and unobserved records was considered and not the conditional averages. We verify that this equality is correct using the previous physical example in Eq. (10), with good agreement seen in Fig. 2.

IV. COST FUNCTION: LINEAR INFIDELITY

Given the previous two cost functions, one might assume that the conditioned state ρ_{O_Ξ} would be the optimal estimator for any cost function that was a distance or distinguishability measure. However, this is not the case. The last measure we will consider as a cost function is the linear infidelity (LI) with the true state, i.e., $\mathcal{C}(\check{\rho}, \rho_{\mathrm{T}}) = 1 - L(\check{\rho}, \rho_{\mathrm{T}})$, where the linear fidelity is defined in Eq. (5). Note that one will obtain the same cost function for the Jozsa infidelity when assuming a pure true state, as discussed above. Once again, the task is to find the state that minimizes the expected cost function, here

$$\mathcal{B}^{\mathrm{LI}}_{O_\Xi}(\check{\rho}) = \mathbb{E}_{\overleftarrow{U}|O_\Xi}\{1 - L(\check{\rho}, \rho_{\mathrm{T}})\}. \tag{16}$$

By the linearity of the trace, we can immediately simplify this expected cost function to $\mathcal{B}^{\mathrm{LI}}_{O_\Xi}(\check{\rho}) = 1 - \mathrm{Tr}[\check{\rho}\rho_{O_\Xi}]$. Furthermore, we can reframe the optimization problem in this case to find the estimated state $\check{\rho}$ that maximize the linear fidelity $L(\check{\rho}, \rho_{O_\Xi}) = \mathrm{Tr}[\check{\rho}\rho_{O_\Xi}]$.

Now, using the fact that ρ_{O_Ξ} is Hermitian and hence diagonalizable with some unitary matrix U, the linear fidelity becomes

$$\begin{aligned}\mathrm{Tr}[\check{\rho}\rho_{O_\Xi}] &= \mathrm{Tr}[\check{\rho}U\Lambda_{O_\Xi}U^\dagger] = \sum_i \lambda_i (U^\dagger\check{\rho}U)_{ii} \\ &= \lambda^{\max}_{O_\Xi}\sum_i \frac{\lambda_i}{\lambda^{\max}_{O_\Xi}}(U^\dagger\check{\rho}U)_{ii},\end{aligned} \tag{17}$$

where Λ_{O_Ξ} is a diagonal matrix whose entries are the eigenvalues λ_i of ρ_{O_Ξ} and $\lambda^{\max}_{O_\Xi}$ is the largest eigenvalue of ρ_{O_Ξ}. Since ρ_{O_Ξ} is positive semidefinite, as it is a valid quantum state, $0 \leqslant \lambda_i/\lambda^{\max}_{O_\Xi} \leqslant 1$. Furthermore, we know $(U^\dagger\check{\rho}U)_{ii}$ is positive since $\check{\rho}$ must be a valid quantum state and hence is positive semidefinite. Thus we can take the upper bound on $\lambda_i/\lambda^{\max}_{O_\Xi}$, giving an upper bound on the linear infidelity

$$\mathrm{Tr}[\check{\rho}\rho_{O_\Xi}] \leqslant \lambda^{\max}_{O_\Xi}\sum_i (U^\dagger\check{\rho}U)_{ii} = \lambda^{\max}_{O_\Xi}\mathrm{Tr}[U^\dagger\check{\rho}U] = \lambda^{\max}_{O_\Xi}, \tag{18}$$

where the final equality is obtained using the cyclic property of the trace and $\mathrm{Tr}[\check{\rho}] = 1$. Importantly, this upper bound is independent of the particular choice of estimator, meaning that this is the maximum possible value any estimator can obtain for the linear fidelity. Thus, if we can find an estimator that will saturate this upper bound, it will minimize the linear infidelity expected cost function and will be an optimal estimator for this cost function. In fact, it is fairly easy to find such an estimator which saturates the upper bound in Eq. (18), that is, by choosing the estimated state to be

$$\rho^{\mathrm{L}}_{O_\Xi} = |\psi^{\max}_{O_\Xi}\rangle\langle\psi^{\max}_{O_\Xi}|, \tag{19}$$

$|\psi^{\max}_{O_\Xi}\rangle$ is the eigenstate corresponding to the largest eigenvalue $\lambda^{\max}_{O_\Xi}$ of the conditioned state. By choosing the eigenstate that corresponds to the largest eigenvalue of the conditioned state, we know that $\rho_{O_\Xi}|\psi^{\max}_{O_\Xi}\rangle = \lambda^{\max}_{O_\Xi}|\psi^{\max}_{O_\Xi}\rangle$, from which it is easy to see that this estimator will saturate the upper bound in Eq. (18). Since this state estimator is, in some sense, a purification of the conditioned state, we will call this state the lustrated conditioned state, hence the superscript L.

As was the case for the previous two cost functions, the expected cost function for the optimal estimator can be simplified to

$$\mathcal{B}^{\mathrm{LI}}_{O_\Xi}[\rho^{\mathrm{L}}_{O_\Xi}] = 1 - \lambda^{\max}_{O_\Xi}, \tag{20}$$

which follows trivially from

$$\mathbb{E}_{\overleftarrow{U}|O_\Xi}\{L(\rho^{\mathrm{L}}_{O_\Xi}, \rho_{\mathrm{T}})\} = \lambda^{\max}_{O_\Xi}. \tag{21}$$

In this case, contrary to the other distance measures, we notice that the expected cost function does not reduce to a simple measure of the optimal estimator. Instead, the expected cost function depends on the conditioned state. To verify Eq. (21), we will consider the (not-lustrated) physical system presented in Sec. II. In Fig. 3 we indeed see that the average fidelity of the lustrated conditioned state $\rho^{\mathrm{L}}_{O_\Xi}$ is equal to the largest eigenvalue of the conditioned state ρ_{O_Ξ}.

Due to the similarity between the LI and the TrSD cost functions, it is possible to derive upper and lower bounds on the expected cost function of the lustrated state for both a LI and a TrSD cost. To begin, since the lustrated conditioned state is the optimal estimator for the LI expected cost function,

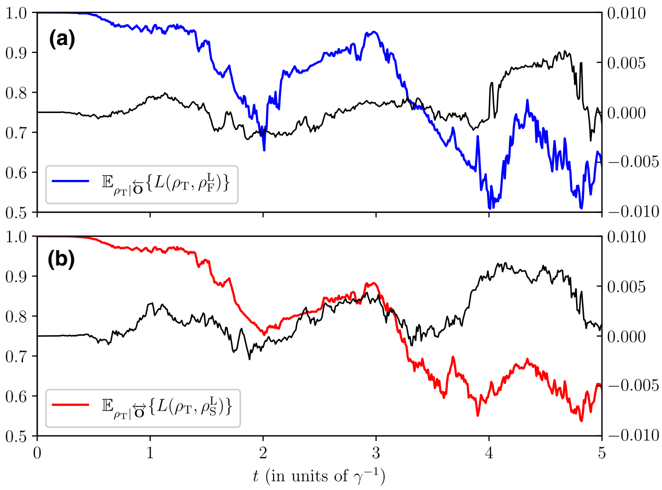

FIG. 3. Ensemble average of the linear fidelity between the true state and the lustrated conditioned state, (a) filtered and (b) smoothed, and its difference from the maximum eigenvalue of the conditioned state (thin black line with the scale on the right y axis) for the quantum system in Eq. (10) with Alice and Bob using Y homodyne detection and photodetection, respectively. Other details, including the randomly generated observed record $\overleftrightarrow{O}$, are as in Figs. 1(a) and 1(c).

we have the trivial bound

$$\mathcal{B}^{\rm LI}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}] \leqslant \mathcal{B}^{\rm LI}_{O_\Xi}[\rho_{O_\Xi}], \tag{22}$$

where an equality occurs when the conditioned state is pure. While this is trivial for a LI cost, this relationship places a nontrivial upper bound on the TrSD expected cost function for the lustrated conditioned state. Specifically, it is easy to show, using Eq. (7) and assuming a pure true state, that Eq. (22) implies that $\frac{1}{2}\mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}] \leqslant \mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho_{O_\Xi}]$. Similarly, we can derive a lower bound on the LI expected cost function for the lustrated conditioned state by considering $\mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho_{O_\Xi}] \leqslant \mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}]$. In this case, we obtain the lower bound $\frac{1}{2}\mathcal{B}^{\rm LI}_{O_\Xi}[\rho_{O_\Xi}] \leqslant \mathcal{B}^{\rm LI}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}]$. As a result, we obtain the following bounds on the expected cost functions of their respective optimal estimators:

$$\tfrac{1}{2}\mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}] \leqslant \mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho_{O_\Xi}] \leqslant \mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}], \tag{23}$$

$$\tfrac{1}{2}\mathcal{B}^{\rm LI}_{O_\Xi}[\rho_{O_\Xi}] \leqslant \mathcal{B}^{\rm LI}_{O_\Xi}[\rho^{\rm L}_{O_\Xi}] \leqslant \mathcal{B}^{\rm LI}_{O_\Xi}[\rho_{O_\Xi}]. \tag{24}$$

To verify these bounds on the expected cost functions, we will once again consider the physical example presented in Sec. II. In Fig. 4 we consider both the TrSD [Figs. 4(a) and 4(c)] and LI [Figs. 4(b) and 4(d)] expected cost functions for the filtered [Figs. 4(a) and 4(b)] and smoothed [Figs. 4(c) and 4(d)] states, observing the bounds in Eqs. (24) and (23). Note that one might think that the upper bound in Eq. (23) could become a trivial bound when $\mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho_{O_\Xi}] > 1$ as the maximum value of the TrSD expected cost function is 2. In fact, this is never trivial as, from Eq. (8), we can see that $\mathcal{B}^{\rm TrSD}_{O_\Xi}[\rho_{O_\Xi}] \leqslant 1$.

V. CONCLUSION

In this paper, we have shown that the smoothed quantum state is an optimal state estimator. Specifically, the smoothed quantum state simultaneously minimizes the expected cost function for a trace-square deviation and relative entropy cost function. Furthermore, we showed that, in both cases, the expected cost function of the smoothed state reduces to simple measures acting solely on the smoothed state, specifically, the impurity and von Neumann entropy of the smoothed state, respectively. However, when we considered the linear infidelity as a cost function we found, somewhat counterintuitively, that the smoothed quantum state was not optimal. Instead, the lustrated smoothed state, defined as the eigenstate corresponding to the maximum eigenvalue of the smoothed quantum state, is the optimal estimator for such a cost function.

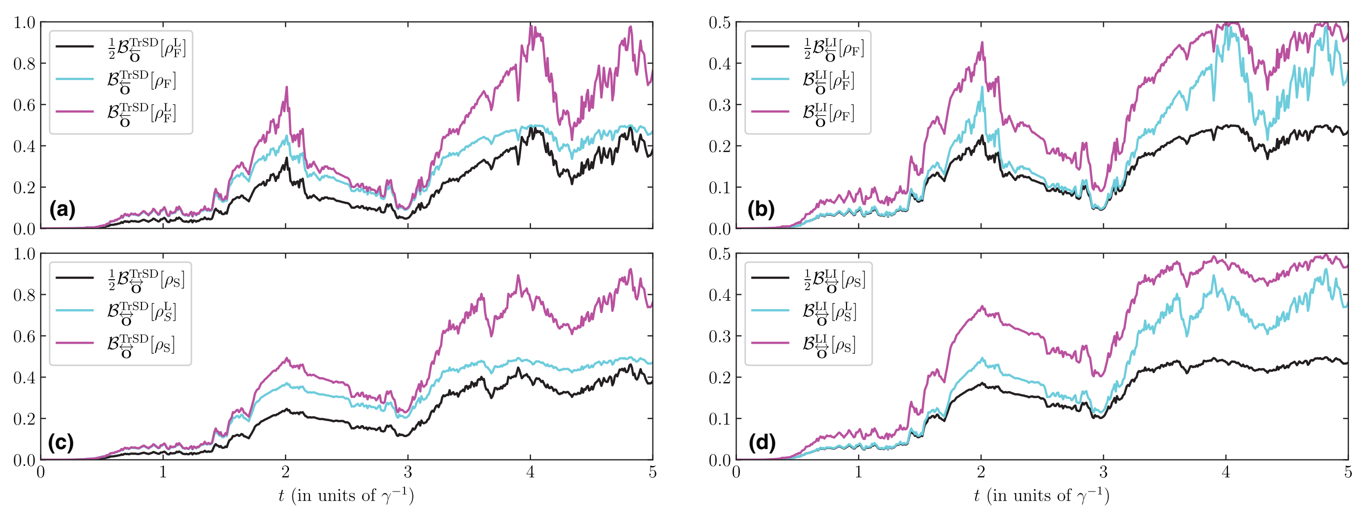

FIG. 4. The (a) and (c) TrSD and (b) and (d) LI expected cost functions for the conditioned and lustrated conditioned state, respectively, for the system in Eq. (10). We evaluate the TrSD expected cost function for the (a) filtered and (b) smoothed quantum states and evaluate the LI expected cost function for the (c) filtered lustrated and (c) smoothed lustrated states. The bounds on these expected cost functions are calculated for (a) and (c) from Eq. (23) and for (b) and (d) from Eq. (24). Other details, including the randomly generated observed record $\overleftrightarrow{O}$, are as in Figs. 1(a) and 1(c).

As was the case for the other cost functions, we showed that the linear infidelity expected cost function of the lustrated smoothed state reduces to a simple measure. However, in this case the measure does not depend on the lustrated smoothed state itself, rather it depends on the maximum eigenvalue of the smoothed quantum state. Finally, we calculated some upper and lower bounds on the expected cost function of the lustrated smoothed state for both the trace-square deviation and the linear infidelity.

Since the lustrated state is pure, an obvious question is whether it is related to the pure states in the most-likely-path approach of Refs. [15,35]. This and many other related questions are answered by the general cost function approach to quantum state estimation using past and future measurement records introduced in Ref. [36]. However, the most-likely path [15,35,36] is restricted to homodyne-like unknown records, whereas in this paper we have used an example where the unknown record is comprised of discrete photon counts. It remains an open question as to whether the most-likely-path cost functions of Ref. [36] can be generalized to such cases. Another interesting avenue for future work would be to consider the implications of using a frequentist approach as discussed in the Introduction.

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APPENDIX: NUMERICS

In this Appendix we present the methods used to compute the filtered, true, and smoothed quantum states and all the measures in this paper. To begin, a typical homodyne measurement current, which will remain fixed for all calculations, was generated in parallel with the associated filtered state and calculating the measurement current via

$$dJ_\phi(t) = \text{Tr}[\hat{c}_\phi \rho_{\text{F}}(t) + \rho_{\text{F}}(t)\hat{c}_\phi^\dagger]dt + dW_{\text{F}}(t), \tag{A1}$$

where dW_{F} is the filtered innovation generated from a Gaussian distribution with the moments in Eq. (9). The filtered state was computed using quantum maps [3], where the evolution of the filtered state in a finite time step δt is given by

$$\rho_{\text{F}}(t+\delta t) = \frac{\mathcal{M}_H \mathcal{M}_{dJ_\phi(t)} \mathcal{M}_u \rho_{\text{F}}(t)}{\text{Tr}[\mathcal{M}_H \mathcal{M}_{dJ_\phi(t)} \mathcal{M}_u \rho_{\text{F}}(t)]}, \tag{A2}$$

where the completely positive map $\mathcal{M}_A$ subscripts denote the particular type of evolution the system is undergoing: $\hat{H}$ denotes the Hamiltonian part, $dJ_\phi(t)$ denotes the homodyne part, and u denotes the remaining unconditioned dynamics. The Hamiltonian part is $\mathcal{M}_H \bullet = \exp(-iH\delta t) \bullet \exp(iH\delta t)$. The unconditioned map can be described by averaging over Bob's jump process to make a trace-preserving map,

$$\mathcal{M}_u \bullet = \sum_{dN(t)=0}^{1} \hat{M}_{dN(t)} \bullet \hat{M}_{dN(t)}^\dagger. \tag{A3}$$

The homodyne (conditioned) map is described by a single measurement operator $\mathcal{M}_{dJ_\phi(t)} \bullet = \hat{M}_{dJ_\phi(t)} \bullet \hat{M}_{dJ_\phi(t)}^\dagger$. In particular, we used completely positive quantum maps [37], where the $\hat{M}$ operators have been taken to a second order in δt to ensure the positivity of the quantum state to high accuracy. For details on the particular operators used for the homodyne measurement and the jump measurement see Ref. [37].

With this typical record, both the unnormalized filtered state $\tilde{\rho}_{\text{F}}$ and the unnormalized true state $\tilde{\rho}_{\text{T}}$ can be computed. The unnormalized filtered state is computed using Eq. (A2) without the trace term in the denominator, and the unnormalized true state evolves as

$$\tilde{\rho}_{\text{T}}(t+\delta t) = \mathcal{M}_H \mathcal{M}_{dJ_\phi(t)} \mathcal{M}_{dN(t)} \rho_{\text{T}}(t). \tag{A4}$$

The reason why the unnormalized versions of these states are computed, as opposed to the normalized version, is because their traces correspond to the ostensible probability distributions $\text{Tr}[\tilde{\rho}_{\text{F}}] = \wp_{\text{ost}}(\overleftarrow{O})$ and $\text{Tr}[\tilde{\rho}_{\text{T}}] = \wp_{\text{ost}}(\overleftarrow{O}, \overleftarrow{U})$. These distributions are needed for computing the ensemble average over $\overleftarrow{U}$ given $\overleftarrow{O}$ via $\wp_{\text{ost}}(\overleftarrow{U}|\overleftarrow{O}) = \wp_{\text{ost}}(\overleftarrow{O}, \overleftarrow{U})/\wp_{\text{ost}}(\overleftarrow{O})$. Specifically, the ensemble average using this conditional probability is

$$\mathbb{E}_{\overleftarrow{U}|\overleftarrow{O}}\{\bullet\} = \frac{1}{N_{\text{tot}}} \sum_{n=1}^{N_{\text{tot}}} \frac{\text{Tr}\big[\tilde{\rho}_{\text{T}}^{(n)}\big]}{\text{Tr}[\tilde{\rho}_{\text{F}}]} \bullet, \tag{A5}$$

where the superscript (n) labels the nth realization of the true state with N_{tot} being the total number of realizations computed.

For the smoothed quantum state, it is necessary to perform the ensemble average $\mathbb{E}_{\overleftarrow{U}|\overleftrightarrow{O}}$. Thus we require the ostensible distribution $\wp_{\text{ost}}(\overleftarrow{U}|\overleftrightarrow{O}) = \wp_{\text{ost}}(\overleftrightarrow{O}, \overleftarrow{U})/\wp_{\text{ost}}(\overleftrightarrow{O})$. Both the numerator and denominator are obtained by introducing the retrofiltered effect $\hat{E}_{\text{R}}$, a positive operator-valued measure. element that evolve backward-in-time from a final uninformative state $\hat{E}_{\text{R}}(T) = I$ conditioning on the measurement result back to the time τ. The retrofiltered effect is computed as the adjoint of the unnormalized filtered state, that is,

$$\hat{E}_{\text{R}}(t-\delta t) = \mathcal{M}_u^\dagger \mathcal{M}_{dJ_\phi(t)}^\dagger \mathcal{M}_H^\dagger \hat{E}_{\text{R}}(t), \tag{A6}$$

where $\text{Tr}[\hat{E}_{\text{R}}\rho] = \wp(\overrightarrow{O}|\rho)$. The ostensible distributions are then obtained by $\text{Tr}[\tilde{\rho}_{\text{T}}\hat{E}_{\text{R}}] = \wp_{\text{ost}}(\overleftrightarrow{O}, \overleftarrow{U})$ and $\text{Tr}[\tilde{\rho}_{\text{F}}\hat{E}_{\text{R}}] = \wp_{\text{ost}}(\overleftrightarrow{O})$. Thus the ensemble average conditioning on $\overleftrightarrow{O}$ is computed as

$$\mathbb{E}_{\overleftarrow{U}|\overleftrightarrow{O}}\{\bullet\} = \frac{1}{N_{\text{tot}}} \sum_{n=1}^{N_{\text{tot}}} \frac{\text{Tr}[\tilde{\rho}_{\text{T}}^{(n)}\hat{E}_{\text{R}}]}{\text{Tr}[\tilde{\rho}_{\text{F}}\hat{E}_{\text{R}}]} \bullet. \tag{A7}$$

Note that, for a fair comparison of the various equalities presented in this paper, the ensemble of true states (of size M) used to compute the ensemble average for the smoothed state was generated independently of the ensemble of true states (of size N) used to compute the ensemble averages of other quantities, like the linear fidelities and relative entropies.

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