Alternative approach to the quantization of the damped harmonic oscillator

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In this paper, an alternative approach for constructing Lagrangians for driven and undriven linearly damped systems is proposed, by introducing a redefined time coordinate and an associated coordinate transformation to ensure that the resulting Lagrangian satisfies the Helmholtz conditions. The approach is applied to canonically quantize the damped harmonic oscillator and although it predicts an energy spectrum that decays at the same rate to previous models, unlike those approaches it recovers the classical critical damping condition, which determines transitions between energy eigenstates, and is therefore consistent with the correspondence principle. It is also demonstrated how to apply the procedure to a driven damped harmonic oscillator.

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I. INTRODUCTION

Harmonic oscillators experiencing linear and nonlinear damping, both with and without a driving force, arise in a range of physical contexts including modeling superconducting qubits using Josephson junctions [1–3], heavy ion scattering [4–6], and nuclear fission [7–9]. There is consensus in the literature that quantization techniques for the linearly damped, undriven harmonic oscillator,

$$\ddot{q} + 2\alpha\dot{q} + \omega^2 q = 0, \tag{1}$$

can be generalized to the above contexts [10,11]. Here, q, α, and ω are the position coordinate, the coefficient of linear damping, and the oscillator frequency, respectively.

Existing canonical quantization procedures, however, do not work for nonconservative systems; i.e., it has not been possible with approaches previously reported in the literature to write a Lagrangian which produces Eq. (1) as its Euler-Lagrange equation, as it does not satisfy the Helmholtz conditions [12]. The attempts to overcome this problem have resulted in a wide range of proposals over the last century, the most prominent being that of Bateman [13], Caldirola [14], and Kanai [15] (hereafter referred to as the BCK approach). In the BCK approach, one quantizes the Bateman Lagrangian $\mathcal{L}_B$,

$$\mathcal{L}_B = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)e^{2\alpha t}, \tag{2}$$

from which the Euler-Lagrange equations produces Eq. (1) multiplied by the integrating factor $e^{2\alpha t}$. The canonical momentum associated with q is $p = e^{2\alpha t}\dot{q}$, and can be used to write the corresponding Hamiltonian H_{BCK}:

$$H_{\mathrm{BCK}} = \frac{1}{2}\left(e^{-2\alpha t}p^2 + \omega^2 e^{2\alpha t}q^2\right). \tag{3}$$

Many other proposed approaches, such as those employing separation of variables or redefinition of the position variable q, are equivalent to this BCK approach [10,16–20]. Another class of approaches involve coupling an undamped oscillator to a loss mechanism [11,21–23], including through a spin-boson model [24,25] or a Lindbladian master equation formalism [26–28]. While these approaches have proved successful, they require a model of how the dissipation occurs. However, such an approach is not often practical, for example when characterizing superconducting quantum circuits, which rely on experimentally determined estimations of the damping conditions [29–31]. By providing a closed description of the system, the BCK approach therefore remains a highly favored approach to quantize the damped harmonic oscillator—as evidenced by continued analysis of the approach [19,20,32–36]—and solutions to H_{BCK} have been thoroughly investigated [10,37,38].

There are two features of the BCK approach which have recently caused debate in the literature. First, it was proved that a square-integrable vacuum cannot be found for the BCK Hamiltonian [32], a result which has withstood some debate [33,34,36]. Second, in Ref. [35] the authors claim that Eq. (2) describes a doubled system of both a damped harmonic oscillator and an amplified harmonic oscillator [Eq. (1) with $\alpha < 0$]. In that paper, they propose a modified Bateman Lagrangian, by introducing additional real dynamical variables and obtain a ladder of energies,

$$E_n(t) = \hbar\omega e^{-2\alpha t}\left(n + \frac{1}{2}\right), \tag{4}$$

and find a critical damping condition for the quantized system, $\alpha = \omega(\sqrt{5} - 1)/2$, which differs from the classical condition $\alpha = \omega$. Their approach, however, does not satisfy the correspondence principle and relies upon modifying an existing Lagrangian with multiple dynamical variables.

In this work, we propose an alternative approach for quantizing driven and undriven linearly damped systems of

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the form

$$\ddot{q} + 2\alpha\dot{q} + \sum_i g_i(q,t) = 0, \tag{5}$$

where $g_i(q,t)$ is an arbitrary continuous function of q and t. Our approach only requires a single dynamical variable, does not require modification of an existing Lagrangian, and is therefore easily generalized. We do so by introducing a new time coordinate in Eq. (5), for which we find an exact Lagrangian and Hamiltonian which is then quantized. Importantly, while we predict the energy spectrum given by Eq. (4), we recover the classical critical damping condition $\alpha = \omega$. Therefore, our quantization satisfies the correspondence principle. Finally, we demonstrate how to apply our approach to driven linearly damped systems, by considering an example from superconducting quantum computing.

II. DERIVING A HAMILTONIAN VIA THE HELMHOLTZ CONDITIONS

For a set of coordinates q_i and their derivatives with respect to time t, the Helmholtz conditions state that a Lagrangian can be written for a system of differential equations $E_i(t, q_i, \dot{q}_i, \ddot{q}_i)$ if they satisfy [12]

$$\frac{\partial E_i}{\partial \ddot{q}_k} - \frac{\partial E_k}{\partial \ddot{q}_i} = 0, \tag{6a}$$

$$\frac{\partial E_i}{\partial \dot{q}_k} + \frac{\partial E_k}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial E_i}{\partial \ddot{q}_k} + \frac{\partial E_k}{\partial \ddot{q}_i}\right) = 0, \tag{6b}$$

$$\frac{\partial E_i}{\partial q_k} - \frac{\partial E_k}{\partial q_i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial E_i}{\partial \dot{q}_k} - \frac{\partial E_k}{\partial \dot{q}_i}\right) = 0. \tag{6c}$$

A single differential equation trivially satisfies Eqs. (6a) and (6c); however, Eq. (5) [and thus Eq. (1)] does not satisfy Eq. (6b). Therefore, in this coordinate system, it is not possible to write a Lagrangian which produces Eq. (1) as its Euler-Lagrange equation.

We propose considering a new time coordinate $\tau = f(t)$, where $f(t)$ is a continuous and differentiable function for all $t \geqslant 0$, such that the Helmholtz conditions are satisfied. Equation (5) becomes

$$\left(\frac{d\tau}{dt}\right)^2 \ddot{q}(\tau) + \left[\left(\frac{d^2\tau}{dt^2}\right) + 2\alpha\left(\frac{d\tau}{dt}\right)\right]\dot{q}(\tau) + \sum_i f_i(q(\tau),\tau) = 0, \tag{7}$$

assuming that $d\tau/dt \neq 0$ for all times t of interest, that α and ω are time independent, and derivatives are with respect to τ. Since Eq. (7) is a differential equation $E(\tau, q, \dot{q}, \ddot{q})$, it may be substituted into Eq. (6b) to ensure that the Helmholtz conditions are satisfied:

$$\left(\frac{d^2\tau}{dt^2}\right) + 2\alpha\frac{d\tau}{dt} = \frac{d}{d\tau}\left[\left(\frac{d\tau}{dt}\right)^2\right]. \tag{8}$$

This condition is trivially satisfied for $\alpha = 0$ (the undamped oscillator), and we consider $\alpha \neq 0$ for the remainder of this work. In that case, Eq. (8) is generally satisfied by

$$\tau = Ke^{2\alpha t} - \tau_0, \tag{9}$$

where K and τ_0 are constants of integration. K must be nonzero for the transform to be defined, but τ_0 corresponds only to a time translation of our system $\tau \to \tau + \tau_0$. Since this has no physical implications, we set $\tau_0 = 0$ for the remainder of this work.

Observe that Eq. (8) is independent of $g_i(q,t)$, so it is possible to write a Lagrangian for any linearly damped system. However, we start by considering the undriven damped harmonic oscillator, for which the equation of motion in τ is

$$4\alpha^2\tau^2\frac{d^2q}{d\tau^2} + 8\alpha^2\tau\frac{dq}{d\tau} + \omega^2 q = 0. \tag{10}$$

It is straightforward to show that Eq. (10) is the Euler-Lagrange equation corresponding to the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[4\alpha^2\tau^2\dot{q}^2 - \omega^2 q^2\right]. \tag{11}$$

The canonical momentum p for $\mathcal{L}$ is

$$p = \frac{\partial L}{\partial \dot{q}} = 4\alpha^2\tau^2\dot{q}, \tag{12}$$

and thus the classical Hamiltonian H for our system is

$$H = \frac{1}{2}\left[\frac{1}{4\alpha^2\tau^2}p^2 + \omega^2 q^2\right]. \tag{13}$$

It follows from Eq. (9) that $\tau > 0$ for $t > -\infty$, so H is defined for all time and $\alpha > 0$. So the resulting classical Hamiltonian, Eq. (13), produces Eq. (1) as its equation of motion. To the best of our knowledge, Eq. (13) has not been found in the earlier literature on the quantization of the damped harmonic oscillator. Nor has this procedure for deriving a Hamiltonian for a dissipative system been proposed in the earlier literature on the quantization of dissipative systems.

III. CANONICAL QUANTIZATION

Canonical quantization is now carried out in the usual way by promoting q and p to their corresponding self-adjoint operators satisfying $[\hat{q}, \hat{p}] = i\hbar$. We now define, in the Schrödinger picture, the annihilation operator

$$\hat{a}(\tau) = \frac{1}{\sqrt{2\hbar}}\left[\sqrt{2\omega\alpha\tau}\,\hat{q} + i\frac{1}{\sqrt{2\omega\alpha\tau}}\hat{p}\right], \tag{14}$$

which satisfies $[\hat{a}(\tau), \hat{a}^\dagger(\tau)] = 1$ for all τ, where $\hat{a}^\dagger(\tau)$ is the corresponding creation operator. Our quantized Hamiltonian may thus be written

$$\hat{H}(\tau) = \frac{\hbar\omega}{2\alpha\tau}\left[\hat{a}^\dagger(\tau)\hat{a}(\tau) + \frac{1}{2}\right]. \tag{15}$$

Using the ground-state vector $|0,\tau\rangle$ which satisfies $\hat{a}(\tau)|0,\tau\rangle = 0$, the Fock basis vectors are constructed as $|n,\tau\rangle = (1/\sqrt{n!})(\hat{a}^\dagger(\tau))^n|0,\tau\rangle$ for positive integer n. The Fock basis vectors are eigenstates of the Hamiltonian $\hat{H}(\tau)|n,\tau\rangle = E_n(\tau)|n,\tau\rangle$. The expectation value of the Hamiltonian, $\bar{E}_n(\tau)$, in state $|n,\tau\rangle$ is then

$$\bar{E}_n(\tau) = \langle n,\tau|\,\hat{H}\,|n,\tau\rangle = \frac{\hbar\omega}{2\alpha\tau}\left(n + \frac{1}{2}\right). \tag{16}$$

Thus, in the original time coordinate t, the expectation values of the energy spectrum of the damped harmonic oscillator are

$$\bar{E}_n(\tau) = \frac{\hbar\omega}{2\alpha K} e^{-2\alpha t}\left(n + \frac{1}{2}\right). \tag{17}$$

Consistent with Eq. (4), the energy eigenvalues constitute an equally spaced ladder, and their expectation values decrease exponentially with time. Note that a choice of $K = 1/2\alpha$ ensures that $\bar{E}_n$ reproduces exactly the harmonic oscillator spectrum at $t = 0$. The eigenvalues decrease as $\exp(-2\alpha t)$, which is the same rate as that expected for a corresponding classical oscillator.

IV. POSITION-SPACE ENERGY EIGENFUNCTIONS

We define the Fock state $|n, \tau\rangle$ in the position basis $|x\rangle$ as

$$|n, \tau\rangle = \int dx \psi_n(x, \tau)\, |x\rangle\,, \tag{18}$$

where the $\psi_n(x, \tau)$ are the position-space wavefunctions. Having defined the annihilation operator $\hat{a}(\tau)$, following a standard approach [39] the eigenfunctions

$$\psi_n(x, \tau) = \left(\frac{2\omega\alpha\tau}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{2\omega\alpha\tau}{\hbar}}\, x\right) \times \exp\left(-\frac{\omega\alpha\tau}{\hbar} x^2\right) \tag{19}$$

are obtained, where H_n is the nth Hermite polynomial. Writing these wavefunctions in the original time coordinate then gives

$$\psi_n(x, t) = \left(\frac{2\omega\alpha K}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{2\omega\alpha K}{\hbar}} \exp(\alpha t) x\right) \times \exp\left(\frac{\alpha t}{2} - \frac{\omega\alpha K}{\hbar} x^2 \exp(2\alpha t)\right). \tag{20}$$

Observe that, up to a phase factor, these are the same wavefunctions as found in Ref. [35] for $K = m/2\alpha$ (where m is the mass of the oscillator). In Fig. 1(a), we plot $|\psi_n(x, t)|^2$ for Eq. (20) at $t = 0$ for the $n = 0$ and $n = 1$ cases, which correspond to the initial position-space wavefunctions of the ground state and first excited state, for $\omega = 10$, $\alpha = 0.005$, $\hbar = 1$, and $K = 1/2\alpha$.

In Fig. 1(b) we plot the $n = 0$ and $n = 1$ wavefunctions at $t = 250$ for Eq. (20) with the same choice of constants. As t increases, observe that $|\psi_n(x, t)|^2$ becomes increasingly localized around the origin $x = 0$, as the oscillator's motion is damped.

V. SOLUTIONS TO THE SCHRÖDINGER EQUATION

We now calculate the evolution of a state obeying the time-dependent Hamiltonian equation (15) in the Schrödinger picture. In particular, our time-dependent wavefunctions obey the eigenvalue equation,

$$\hat{H}(\tau)\psi_n(x, \tau) = E_n(\tau)\psi_n(x, \tau). \tag{21}$$

As the energy levels of our Hamiltonian are nondegenerate, the general solutions $\Psi(x, \tau)$ for the time-dependent Schrödinger equation are found in the standard way to give,

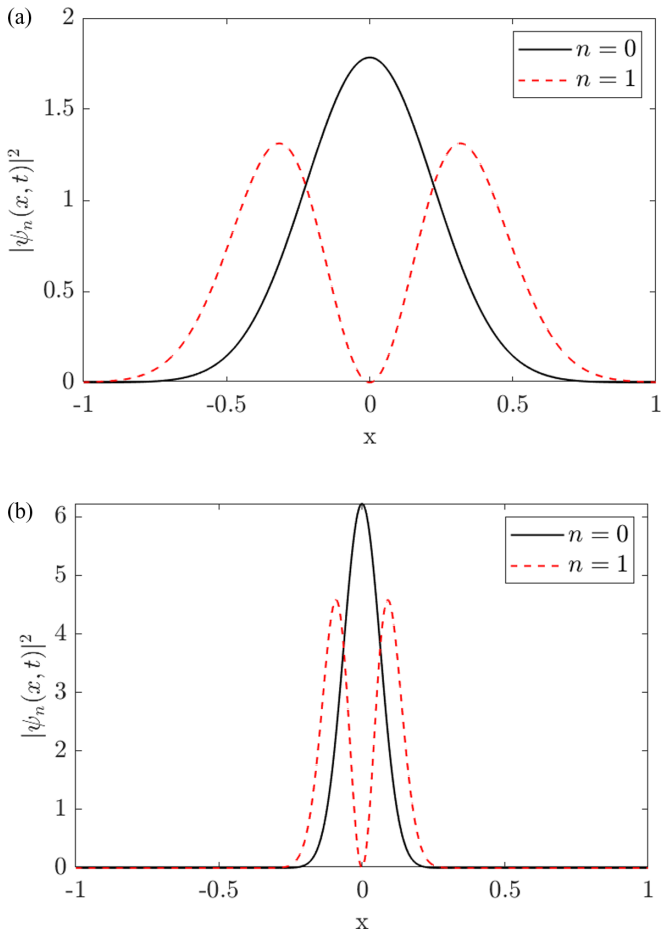

FIG. 1. $|\psi_n(x, t)|^2$ for Eq. (20) for $n = 0, 1$ at (a) $t = 0$ and (b) $t = 250$. All quantities are in natural units with $\hbar = 1$.

in terms of τ,

$$\Psi(x, \tau) = \sum_n c_n(\tau)\psi_n(x, \tau)e^{i\theta_n(\tau)}, \tag{22}$$

where $|c_n(\tau)|^2$ is the probability of being in the nth energy level at time τ, and

$$\theta_n(\tau) = -\frac{1}{\hbar}\int_K^\tau E_n(\tau')\, d\tau'. \tag{23}$$

Here, the c_m obey

$$\frac{dc_m}{d\tau} = -c_m \langle m, \tau| \frac{d}{d\tau} |m, \tau\rangle - \sum_{n\neq m} c_n \frac{\langle m, \tau| \frac{dH}{d\tau} |n, \tau\rangle}{E_n - E_m} e^{i(\theta_n - \theta_m)}, \tag{24}$$

where $|m, \tau\rangle$ are the Fock basis states defined before. The integral in Eq. (23) is integrated from K, corresponding to $\tau(t = 0)$. Then, for the Hamiltonian given by Eq. (13), Eq. (24) becomes

$$\frac{dc_m}{d\tau} = \frac{1}{4\tau}\left[c_{m-2}\sqrt{m(m-1)}\left(\frac{\tau}{K}\right)^{i\omega/\alpha} - c_{m+2}\sqrt{(m+2)(m+1)}\left(\frac{\tau}{K}\right)^{-i\omega/\alpha}\right]. \tag{25}$$

Transforming back to the original time coordinate t gives

$$\frac{dc_m}{dt} = \frac{\alpha}{2}\left[c_{m-2}\sqrt{m(m-1)}e^{i2\omega t} - c_{m+2}\sqrt{(m+2)(m+1)}e^{-i2\omega t}\right], \tag{26}$$

for the time rate of change of the coefficients. Equation (26) may be solved as described in Ref. [35], and the references therein, for initial condition $c_m(0) = \delta_{m,n}$, where $|n, 0\rangle$ is the initial state of our system. This is achieved by solving

$$\frac{\partial G}{\partial t} = -\left[\frac{\alpha}{2}\left(\frac{\partial^2}{\partial q^2} - q^2\right) + i\omega q\frac{\partial}{\partial q}\right]G, \tag{27}$$

for $G(q,t) \triangleq \sum_j q^j e^{-ij\omega t/2} c_j(t)/\sqrt{j!}$ with initial conditions $G(q,0) = q^n/\sqrt{n!}$ and $G(0,t) = c_0(t)$. The $c_m(\tau)$ can then be used to find the probability of the system being found in any eigenstate, after initially being in the nth energy level. This is of particular interest in the study of superconducting quantum circuits, where knowledge of the transition rates is necessary for the control of the quantum system. We consider here the cases $n = 0$ and $n = 2$ in particular. Observe that interestingly Eq. (26) only couples modes of the same parity, so it is sufficient to consider only the even-order modes.

A. Case $n = 0$

In the case $n = 0$, where the system in initially in the ground state, $c_m(t)$ may be written

$$c_m(t) = \begin{cases} \frac{(m-1)!!}{\sqrt{m!}}\sqrt{\xi}e^{i(m+1/2)\omega t} \\ \quad \times \frac{(\sinh(\xi\alpha t))^{m/2}}{(\cosh(\zeta+\xi\alpha t))^{(m+1)/2}} & \text{for } m = 2k, \\ 0 & \text{for } m = 2k+1, \end{cases} \tag{28}$$

where $\sum_m |c_m(t)|^2 = 1$, $k \in \mathbb{Z}$, and

$$\xi = \sqrt{1 - \omega^2/\alpha^2}, \tag{29a}$$

$$e^{\pm\zeta} = \xi \pm i\omega/\alpha. \tag{29b}$$

Critical damping occurs when $\xi = 0$ which, from Eq. (29a), gives the condition for critical damping as $\alpha = \omega$, consistent with the result for an equivalent classical system. Note that this condition differs from the result of Ref. [35] ($\alpha = \omega(\sqrt{5}-1)/2$).

In Figs. 2(a)–2(c) we show $|c_m(t)|^2$ with $m = 0, 2, 4, 6$ for the underdamped case ($\alpha < \omega$), the case of critical damping ($\alpha = \omega$), and the overdamped case ($\alpha > \omega$), with $\omega = 1$ and α defined accordingly.

In Fig. 2(a), ξ is imaginary and $c_m(t)$ is oscillatory. In Fig. 2(a), we show the underdamped case, with $\alpha = 0.75 < \omega$. The transition probabilities oscillate with period $\pi/|\alpha\xi| = \pi/\sqrt{\omega^2 - \alpha^2}$.

In Fig. 2(b), $\xi = 0$. Expanding about $\xi = 0$, we then obtain

$$c_m(t) = \frac{(m-1)!!}{\sqrt{m!}}e^{i(m+1/2)\omega t}\frac{(\omega t)^{n/2}}{(1+i\omega t)^{(n+1)/2}}. \tag{30}$$

In Fig. 2(b), we show this critically damped case, with $\alpha = \omega = 1$. The transition probability $|c_m(t)|^2$ decreases monotonically for the ground state $m = 0$, while for $m \geqslant 2$ it increases to a local maximum before also decreasing monotonically.

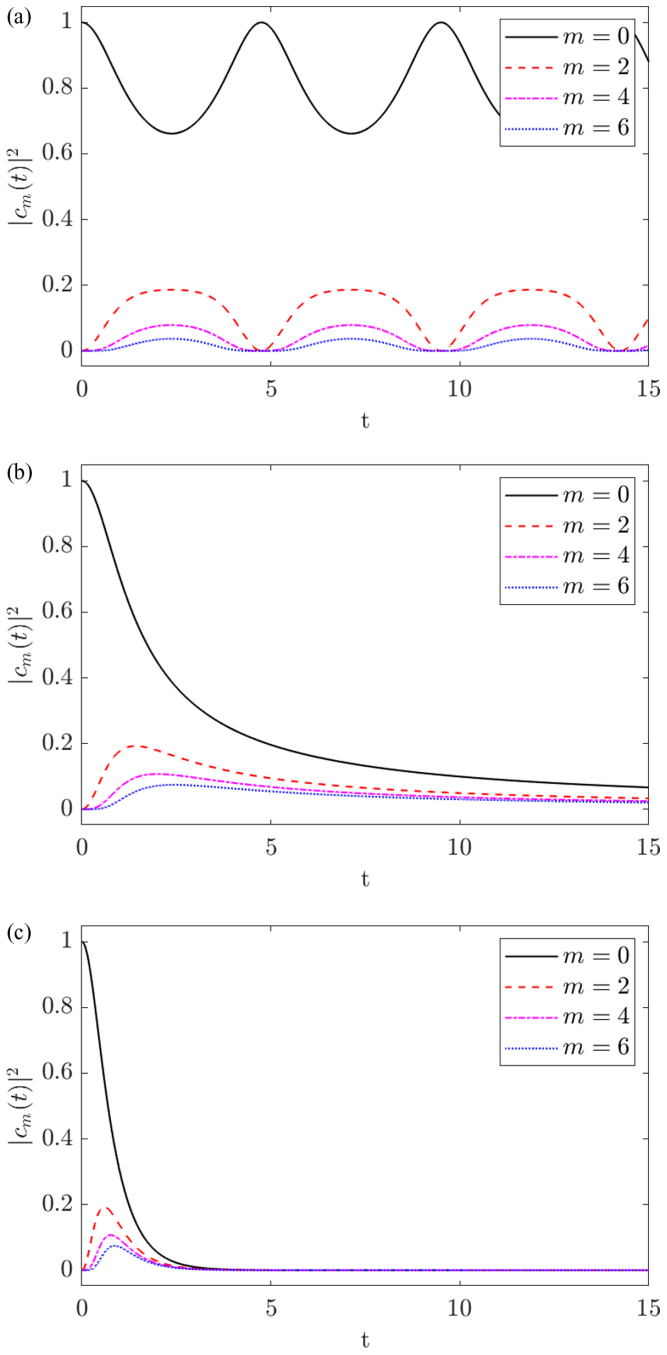

FIG. 2. For an initial state of $|0, 0\rangle$, we plot $|c_m(t)|^2$ for $m = 0, 2, 4, 6$ in (a) the underdamped case, (b) the critically damped case, and (c) the overdamped case. All quantities are in natural units with $\hbar = 1$.

In Fig. 2(c) we show the overdamped case, $\alpha = 2 > \omega$. Now $\xi \neq 0$ and has only real components, so we observe similar behavior to the case of critical damping except for the expected faster rate of decay.

B. Case $n = 2$

When the initial state is the second excited state of the system, $n = 2$, the transition amplitudes are

$$c_m(t) = \begin{cases} \frac{(m-1)!!}{\sqrt{2m!}}\sqrt{\xi}e^{i(m+1/2)\omega t} \\ \quad \times \frac{(\sinh(\xi\alpha t))^{m/2}}{(\cosh(\zeta+\xi\alpha t))^{(m+3)/2}} \\ \quad \times\left(\frac{m\xi^2}{\sinh(\xi\alpha t)} - \sinh(\xi\alpha t)\right) & \text{for } m = 2k, \\ 0 & \text{for } m = 2k+1, \end{cases} \tag{31}$$

which again satisfies the normalization condition $\sum_m |c_m(t)|^2 = 1$, with the parameters ξ and ζ defined as before.

In Figs. 3(a)–3(c), we plot $|c_m(t)|^2$ with $m = 0, 2, 4, 6$ for the underdamped case ($\alpha < \omega$), the case of critical damping ($\alpha = \omega$), and the overdamped case ($\alpha > \omega$), with $\omega = 1$ and α defined accordingly.

Figure 3(a) shows $|c_m(t)|^2$ for $\alpha = 0.75 < \omega$. As before, we observe oscillations in the transition probabilities with period $\pi/|\alpha\xi| = \pi/\sqrt{\omega^2 - \alpha^2}$. However, we observe "beating" between α and ω in the modes $m > n$. This beating is not present in the $n = 0$ case considered earlier.

As before, in the critically damped case $\xi = 0$, so following Ref. [35] we expand around $\xi = 0$ to obtain

$$c_m(t) = \frac{(m-1)!!}{\sqrt{2m!}}e^{i(m+1/2)\omega t}\left(\frac{m}{\omega t} - \omega t\right) \times \frac{(\omega t)^{m/2}}{(1+i\omega t)^{(m+3)/2}}. \tag{32}$$

In Fig. 3(b) we show the critically damped case with $\alpha = \omega = 1$. In this case it is the initially occupied $m = 2$ mode which decreases monotonically, while the $m = 0$ mode increases before decreasing.

In Fig. 3(c), we show the overdamped case $\alpha = 2 > \omega$. As before, we observe qualitatively similar behavior to the critical damping case, except for the expected greater rate of decay.

VI. EXAMPLES OF DRIVEN DAMPED HARMONIC OSCILLATORS

As foreshadowed in the Introduction, this quantization of the damped harmonic oscillator has many potential applications. In what follows, the quantization procedure is applied to two of the simplest superconducting qubits, the phase qubit and flux qubit, and provides an example of the application of the quantization procedure in the case of a driven damped harmonic oscillator. The coordinate of interest here is δ, the phase of the current through the Josephson junction, and, as is well known, the equations of motion for these qubits in terms of this dynamical variable are determined through the use of the resistively capacitance shunted junction model model and Kirchhoff's circuital laws and may be written

$$\frac{\hbar C}{2e}\frac{\partial^2\delta}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial\delta}{\partial t} + (I_0\sin\delta - I) = 0, \tag{33a}$$

$$\frac{\hbar C}{2e}\frac{\partial^2\delta}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial\delta}{\partial t} + I_0\sin\delta + \frac{\hbar}{2eL}(\delta - \delta_X) = 0. \tag{33b}$$

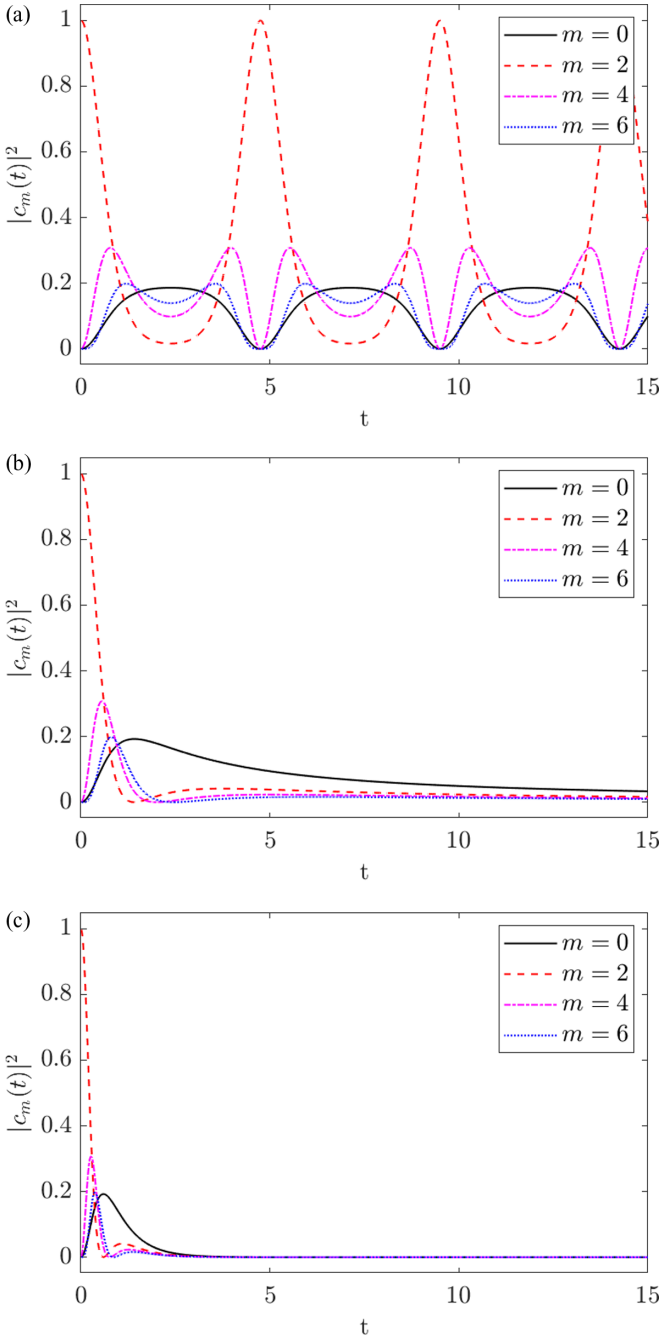

FIG. 3. For an initial state of $|2, 0\rangle$, we plot $|c_m(t)|^2$ for $m = 0, 2, 4, 6$ in (a) the underdamped case, (b) the critically damped case, and (c) the overdamped case. All quantities are in natural units with $\hbar = 1$.

Here, C and R are the capacitance and resistance of the circuit, respectively, and I_0 is the critical current of the Josephson junction. I is the current bias in the phase qubit, and L and δ_X respectively the inductance and a dimensionless parameter for the flux through the flux qubit. Neither Eq. (33a) nor Eq. (33b) satisfies the second Helmholtz condition [Eq. (6b)]. Thus, in both cases, we introduce the dimensionless time coordinate

$$\tau = K\exp\left(\frac{t}{CR}\right), \tag{34}$$

where K is a constant of integration [40], obtaining

$$\left(\frac{\tau}{CR}\right)^2\frac{d^2\delta}{d\tau^2}+\frac{2\tau}{C^2R^2}\frac{d\delta}{d\tau}+\frac{2e}{\hbar C}(I_0\sin\delta-I)=0, \tag{35a}$$

$$\left(\frac{\tau}{CR}\right)^2\frac{d^2\delta}{d\tau^2}+\frac{2\tau}{C^2R^2}\frac{d\delta}{d\tau}+\frac{2e}{\hbar C}I_0\sin\delta+\frac{1}{LC}(\delta-\delta_X)=0, \tag{35b}$$

for the phase and flux qubits, respectively. We can define the same canonical momentum π for each case,

$$\pi=\left(\frac{\tau}{CR}\right)^2\dot{\delta}, \tag{36}$$

and obtain Hamiltonians for each system. These are H_P and H_F,

$$H_P=\frac{1}{2}\left[\left(\frac{CR}{\tau}\right)^2\pi^2-\frac{4e}{\hbar C}(I_0\cos\delta+I\delta)\right], \tag{37a}$$

$$H_F=\frac{1}{2}\left[\left(\frac{CR}{\tau}\right)^2\pi^2-\frac{4e}{\hbar C}I_0\cos\delta+\frac{1}{LC}\left(\delta^2-2\delta_X\delta\right)\right], \tag{37b}$$

for the phase and flux qubit, respectively. Quantization is now achieved through the standard method of promoting δ and π to their corresponding self-adjoint operators satisfying the commutator $[\hat{\delta},\hat{\pi}]=i\hbar$. In the regime where $\hat{\delta}$ is small, we use the series expansion

$$\cos\hat{\delta}\approx 1-\frac{1}{2}\hat{\delta}^2 \tag{38}$$

to write

$$\hat{H}_P=\frac{1}{2}\left[\frac{1}{\alpha_P^2\tau^2}\hat{\pi}^2+\Omega_P^2\left(\hat{\delta}^2-2\frac{I}{I_0}\hat{\delta}-2\right)\right], \tag{39a}$$

$$\hat{H}_F=\frac{1}{2}\left[\frac{1}{\alpha_F^2\tau^2}\hat{\pi}^2+\Omega_F^2\left(\hat{\delta}^2-\frac{2\delta_X}{LC\Omega_F^2}\hat{\delta}-2\frac{\Omega_P^2}{\Omega_F^2}\right)\right], \tag{39b}$$

where we have defined

$$\alpha_P=\alpha_F=\frac{1}{CR}, \tag{40a}$$

$$\Omega_P=\sqrt{\frac{2eI_0}{\hbar C}}, \tag{40b}$$

$$\Omega_F=\sqrt{\Omega_P^2+\frac{1}{LC}}. \tag{40c}$$

The translated operators

$$\hat{\delta}_P=\hat{\delta}-\frac{I}{I_0}, \tag{41a}$$

$$\hat{\delta}_F=\hat{\delta}-\frac{\delta_X}{LC\Omega_F^2}, \tag{41b}$$

obey the commutation relations $[\hat{\delta}_P,\hat{\pi}]=[\hat{\delta}_F,\hat{\pi}]=i\hbar$. In terms of these new canonically conjugate pairs, we obtain the Hamiltonians

$$\hat{H}_P=\frac{1}{2}\left[\frac{1}{\alpha_P^2\tau^2}\hat{\pi}^2+\Omega_P^2\hat{\delta}_P^2\right]-\left(\Omega_P^2+\left(\frac{I}{I_0}\right)^2\right), \tag{42a}$$

$$\hat{H}_F=\frac{1}{2}\left[\frac{1}{\alpha_P^2\tau^2}\hat{\pi}^2+\Omega_F^2\hat{\delta}_F^2\right]-\left(\Omega_P^2+\left(\frac{\delta_X}{LC\Omega_F^2}\right)^2\right). \tag{42b}$$

From our earlier results, the expectation values of the energy spectra $\bar{E}_n^{(P)}(t)$ and $\bar{E}_n^{(F)}(t)$ for each of the qubits are

$$\bar{E}_n^{(P)}(t)=\frac{\hbar\Omega_P}{\alpha_P K}e^{-\alpha_P t}\left(n+\frac{1}{2}\right)-\left(\Omega_P^2+\left(\frac{I}{I_0}\right)^2\right), \tag{43a}$$

$$\bar{E}_n^{(F)}(t)=\frac{\hbar\Omega_F}{\alpha_F K}e^{-\alpha_F t}\left(n+\frac{1}{2}\right)-\left(\Omega_P^2+\left(\frac{\delta_X}{LC\Omega_F^2}\right)^2\right). \tag{43b}$$

The associated conditions for critical damping are

$$R=\sqrt{\frac{\hbar}{8eI_0C}}, \tag{44a}$$

$$R=\sqrt{\frac{\hbar L}{4C(2eI_0L+\hbar)}}, \tag{44b}$$

for the phase and flux qubits, respectively.

VII. CONCLUSIONS

We have proposed an alternative approach to the quantization of driven and undriven linearly damped harmonic oscillators via a change of time coordinate to produce classical equations of motion that satisfy the Helmholtz conditions. The resulting quantum model is qualitatively similar to previous approaches, such as Refs. [19,35], since we predict the expectation values of the energy eigenvalues to obey an equally spaced ladder decaying at a rate given by $\exp(-2\alpha t)$. However, unlike Ref. [35], the result of our quantization predicts a critical damping parameter $\alpha=\omega$, which is the same as for an equivalent classical oscillator. Thus, our quantization is unique in that it recovers the correspondence principle in this respect.

We have shown that a system initially in a single eigenstate has a nonzero probability to transition to a different eigenstate of the same parity. Additionally, for systems initially in the second excited state, the quantization predicts different dynamics and the presence of beating when $\alpha<\omega$, vanishing at the point of critical damping. These transition rates are of particular relevance to modeling the control of superconducting quantum circuits. Indeed, the expectation values of the energy eigenvalues and the critical damping points have been found for the cases of the simplest phase and flux qubits.

Importantly, this new quantization method for the damped harmonic oscillator requires only a single classical dynamical variable and is therefore easily generalized. In particular, this approach should also be applicable to systems with time-dependent damping $\alpha(t)$, in which case the constraint

equation for τ becomes

$$\left(\frac{d^2\tau}{dt^2}\right) + 2\alpha(t)\frac{d\tau}{dt} = \frac{d}{d\tau}\left[\left(\frac{d\tau}{dt}\right)^2\right], \tag{45}$$

which will admit different solutions depending upon the functional form of α.

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