Noninterfering and simultaneous Stern-Gerlach and Heisenberg microscope experiments to measure the full electron coordinate in an entangled state

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In two celebrated experiments of quantum mechanics, the Stern-Gerlach (SG) and Heisenberg microscope (HM) experiments, the electron spin s_z and spatial $\boldsymbol{r}$ coordinates are measured separately. In this paper, the combined SG + HM experiment is proposed to measure the full electron coordinate $\boldsymbol{x} = \{\boldsymbol{r}, s_z\}$. To this end, noninterfering and (virtually) simultaneous SG and HM experiments are proposed to apply to the individual fragments A and B of an entangled dissociating system A-B. The theoretical description of a spin-collapsed and partially spatially collapsed state of the SG + HM experiment is given for the prototype "perfectly" entangled system, the dissociating H_2 molecule.

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I. INTRODUCTION

The Stern-Gerlach (SG) experiment [1,2], to measure the spin of an electron, and the Heisenberg microscope (HM) thought experiment [3–6], to measure the electron spatial coordinate, are two famous experiments which helped to shape quantum mechanics [7,8]. These experiments use different types of interactions with electrons to measure their respective coordinates. In the SG-type experiment an inhomogeneous magnetic field along the chosen axis z is applied to a quantum system to measure the projection of the electron spin s_z on this axis [1,2]. In the HM experiment the scattering of a photon from the sample electron is applied to measure the electron spatial coordinate $\boldsymbol{r}$ with an x-ray microscope [3].

Since the corresponding operators $\hat{\sigma}_z$ and $\hat{\boldsymbol{r}}$ commute with each other, an accurate simultaneous measurement of $\boldsymbol{r}$ and s_z is, in principle, possible. Yet, a specific experiment to measure the full electron coordinate $\boldsymbol{x} = \{\boldsymbol{r}, s_z\}$ is lacking. In particular, a combination of the seminal SG and HM experiments, an apparent choice for the $\boldsymbol{x}$ measurement, needs to be considered. What makes the proper setup of the combined SG + HM experiment a nontrivial and worth pursuing development are two general requirements. The first requirement, coming from the foundations of quantum mechanics, is that the SG and HM experiments should be executed simultaneously. The second is a natural physical requirement that during their execution the SG and HM experiments should not interfere with each other.

In this paper, in order to fulfill these requirements, we propose to invoke yet another famous thought experiment of quantum mechanics by Einstein, Podolsky, and Rosen (EPR) [9], the essence of which is particle entanglement. In the EPR version by Bohm and Aharonov (BA) [10] the entanglement of two electrons with opposite spins on well-separated H atoms of the paradigmatic dissociating H_2 molecule causes an instant transfer of the β spin to the remote atom H_B, once the α spin is measured on another atom H_A.

In this paper, a setup of the combined SG + HM experiment is proposed to measure the full electron coordinate $\boldsymbol{x}$. Its original point is that the SG and HM experiments are carried out remotely on the different fragments of a singlet system with entangled electrons, with the dissociating H_2 molecule as the paradigmatic case. Because of entanglement, the SG measurement of the α spin on H_A is equivalent to the effective noninterfering measurement of the β spin of the electron on H_B, which can be synchronized with the measurement of its spatial coordinate $\boldsymbol{r}$, thus giving its full coordinate $\boldsymbol{x} = \{\boldsymbol{r}, \beta\}$. The mechanism of the SG + HM experiment is put forward, the original point of which is the formation of the spin-collapsed and partially spatial collapsed electron state on H_B. As a further development, the notion of the intrinsic accuracy of the $\boldsymbol{r}$ measurement is introduced and it is estimated for realistic conditions of the HM experiment.

II. COMBINED SG + HM MEASUREMENT OF x FOR EPR-BA ENTANGLED STATE

In this section, in order to measure the total coordinate $\boldsymbol{x}$ of an individual electron, we propose to carry out the remote SG and HM experiments on different fragments of a "perfectly" entangled essentially dissociated system AB. To provide "a proof of principle" of the proposed SG + HM experiment, we consider the paradigmatic perfectly entangled dissociating hydrogen molecule H_A-H_B.

Experiment No. 1: SG experiment on the fragment H_A. In the SG-type experiment carried out on the fragment H_A, the inhomogeneous magnetic field $B(z)$ along the chosen axis z is applied to measure the projection s_z of the spin of H_A on this axis. Due to the measurement, the H_A acquires the actualized spin, say, the α spin in the collapsed state.

Then, the crucial feature of the EPR-BA entanglement is that the entangled electron on the fragment H_B instantly acquires the opposite β spin. Thus, due to entanglement, the spin of the electron on the fragment H_B is actualized and it is known from the measurement of the spin of the entangled electron on the fragment H_A. In this sense, the SG measurement of the α spin on H_A is equivalent to the effective noninterfering measurement of the β spin of the electron on H_B. This justifies the use of entanglement in the proposed SG + HM experiment.

Experiment No. 2: HM experiment on the fragment H_B. The instant acquisition of the β spin by the H_B-fragment electron allows, in principle, to synchronize the HM measurement of the spatial position $\boldsymbol{r}$ of this electron with experiment No. 1. Strictly speaking, the HM experiment can be carried out the next instant after the spin-collapsed state of H_2 with spin polarization is set.

The position $\boldsymbol{r}$ is measured in the HM experiment via inelastic Compton scattering [4] of a photon from the electron on H_B. The Compton scattering is characterized with the energy ΔE^{C},

$$\Delta E^{\mathrm{C}} = E_i - E_s = h\nu_i - h\nu_s, \tag{1}$$

transferred from the photon to the electron. In (1), E_i is the energy of the incident photon, E_s is the energy of the scattered photon, ν_i and ν_s are the corresponding photon frequencies, and h is the Planck constant. The scattered photon is registered with an x-ray microscope. The accuracy of the HM measurement is restricted with the diffraction limit (DL). The latter is characterized with uncertainty Δz^{DL} of the measured electron coordinate z which, for the conventional settings, is given by Abbe's formula [11]

$$\Delta z^{\mathrm{DL}} = \frac{\lambda}{\sin(\phi)}. \tag{2}$$

Here, λ is the photon wavelength and ϕ is the angle of photon scattering.

As a result, in the considered case of the dissociating H_2, the remote SG and HM experiments measure the full electron coordinate $\boldsymbol{x}$ of the electron on the H_B atom. Due to the involvement of entanglement, this is achieved with noninterfering and (virtually) simultaneous measurements of its spatial $\boldsymbol{r}$ and spin β coordinates as described above. In the next section the description of the (partially) collapsed state of the combined SG + HM experiment will be given.

III. SPIN-COLLAPSED AND PARTIALLY SPATIALLY COLLAPSED STATE OF SG + HM EXPERIMENT

In this section we present the description of the resultant state of the SG + HM experiment $\Psi^{\mathrm{SG+HM}}$ for the considered case H_A-H_B. The initial entangled singlet state Ψ^{ent} is well represented with the Heitler-London (HL) wave function [12]

$$\Psi^{\mathrm{ent}} \approx \frac{\Psi_{\alpha\beta} + \Psi_{\beta\alpha}}{\sqrt{2\left(1 + S_{ab}^2\right)}}. \tag{3}$$

In (3) $\Psi_{\alpha\beta}$ and $\Psi_{\beta\alpha}$ are the spin-polarized components, in $\Psi_{\alpha\beta}$ the electron on H_A has α spin, and that on H_B has β spin,

$$\Psi_{\alpha\beta}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tfrac{1}{\sqrt{2}}[a(\boldsymbol{r}_1)\alpha(1)b(\boldsymbol{r}_2)\beta(2) - b(\boldsymbol{r}_1)\beta(1)a(\boldsymbol{r}_2)\alpha(2)], \tag{4}$$

while in $\Psi_{\beta\alpha}$ the opposite spin polarization takes place,

$$\Psi_{\beta\alpha}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tfrac{1}{\sqrt{2}}[b(\boldsymbol{r}_1)\alpha(1)a(\boldsymbol{r}_2)\beta(2) - a(\boldsymbol{r}_1)\beta(1)b(\boldsymbol{r}_2)\alpha(2)]. \tag{5}$$

In (4) and (5), $a(\boldsymbol{r})$ and $b(\boldsymbol{r})$ are the normalized atomic orbitals (AOs) centered on H_A and H_B, respectively, while in (3), S_{ab} is their overlap integral

$$S_{ab} = \int a^*(\boldsymbol{r})b(\boldsymbol{r})d\boldsymbol{r}. \tag{6}$$

Then, actualization of the spin α of the electron on H_A via the SG experiment on this fragment produces the spin-polarized component $\Psi_{\alpha\beta}$ of (4) as the spin-collapsed state. The instant transfer of the β spin to the electron on H_B is described with the factors $b(\boldsymbol{r}_2)\beta(2)$ and $b(\boldsymbol{r}_1)\beta(1)$ in the first and second terms, respectively, of (4).

Now, we turn to the HM measurement of the spatial position of the electron on H_B with the β spin acquired in the SG measurement. We start by stressing the fundamental difference between the SG and HM experiments. Indeed, while the former measures the expectation value of the operator $\hat{\sigma}_z$ with the discrete eigenspectrum, the latter encounters the operator $\hat{r}$ with the continuous eigenspectrum. One can suppose, in general, that in a hypothetical measurement of $\boldsymbol{r}$ the interaction of a measured quantum state with a macroscopic apparatus leads to a collapsed state with a pointlike particle accommodated in the apparatus [13].

At variance with this, in the HM experiment there is no direct quantum state–macroscopic apparatus interaction mentioned above. Instead, we have inelastic Compton scattering of the photon from the sample electron of a quantum state. In this case, the realization of the full collapse of the latter encounters problems such as non-normalizability or infinite energy of a microscopic collapsed state [14]. Then, instead of a full collapse, one can suggest a finite-energy partial spatial collapse as the instant result of the $\boldsymbol{r}$ measurement in the HM experiment.

A partial collapse occurs when the electron on the fragment H_B absorbs the incident photon of the energy E_i, which excites the electron from the spin-collapsed state $\Psi_{\alpha\beta}$ of (4) to the state $\Psi^{\mathrm{SG+HM}}(\epsilon)$,

$$\Psi^{\mathrm{SG+HM}}(\epsilon) = a(\boldsymbol{r}_1)\alpha(1)g^B(\epsilon)(\boldsymbol{r}_2)\beta(2). \tag{7}$$

Partial spatial collapse is represented in (7) with the floating Gaussian orbital (FGO) $g^B(\epsilon)$,

$$g^B(\epsilon)(\boldsymbol{r}) = \frac{e^{-|\boldsymbol{r}-\boldsymbol{r}^{\mathrm{act}}|^2/(2\epsilon)^2}}{(\sqrt{2\pi}\epsilon)^3}. \tag{8}$$

Here, r^{act} is the center of electron localization, and ϵ is the Gaussian exponential parameter, which characterizes the degree of localization. The function $g^B(\epsilon)(\boldsymbol{r})$ is denoted as "floating" since, unlike a conventional AO, such as the AO $a(\boldsymbol{r})$ of H_B of (4), it is not centered on a particular nucleus. Rather, $\boldsymbol{r}^{\mathrm{act}}$ is the center of the region, where the electron on H_B encounters a collision with a photon in the HM microscope.

Next, the scattered photon of the energy E_s is emitted from the state $\Psi^{\mathrm{SG+HM}}(\epsilon)$. This photon is registered with the HM and it carries the information on the actualization of the spatial coordinate. The latter is characterized with a finite accuracy of coordinate actualization $\Gamma(\epsilon)$,

$$\Delta w^{\mathrm{act}} = w^{\mathrm{act}} \pm \Gamma(\epsilon), \quad w = x, y, z, \tag{9}$$

where $\Gamma(\epsilon)$ is the half width of the Gaussian (8),

$$\Gamma(\epsilon) = \epsilon\sqrt{2\ln 2}. \tag{10}$$

The function $\Gamma(\epsilon)$ can be called the intrinsic accuracy of the $\boldsymbol{r}$ measurement, as opposed to the DL of Eq. (2), which characterizes the general optical limitation of the accuracy [11] due to the x-ray or electron microscope employed in the HM measurement. The probability $P(\boldsymbol{r}^{\mathrm{act}})$ of the coordinate actualization is given with the Born rule [15] applied to the spin-collapsed state $\Psi_{\alpha\beta}$ of (4),

$$P(\boldsymbol{r}^{\mathrm{act}}) \approx \int_{\boldsymbol{r}\in[\boldsymbol{r}^{\mathrm{act}}+\Delta\boldsymbol{r}^{\mathrm{act}}]} |b(\boldsymbol{r})|^2 d\boldsymbol{r}. \tag{11}$$

The intrinsic accuracy $\Gamma(\epsilon)$ of the $\boldsymbol{r}$ measurement can be related to the excitation energy E_i from the spin-collapsed state $\Psi_{\alpha\beta}$ of (4) to the state $\Psi^{\mathrm{SG+HM}}(\epsilon)$,

$$\begin{aligned} E_i \approx& \left[\langle g^B(\Gamma)|-\tfrac{1}{2}\nabla^2|g^B(\Gamma)\rangle - \langle b|-\tfrac{1}{2}\nabla^2|b\rangle\right] \\ &+\left[\langle g^B(\Gamma)|V_H^B|g^B(\Gamma)\rangle - \langle b|V_H^B|b\rangle\right]. \end{aligned} \tag{12}$$

The first square brackets of (12) contain the kinetic energy difference, while the second square brackets contain that for the energy of the electron attraction to the H_B nucleus, where

$$V_H^B = -\frac{1}{\left|\boldsymbol{r} - \boldsymbol{R}_H^B\right|} \tag{13}$$

is the electron-nucleus attraction potential.

As was pointed out in Ref. [4], in which the practical realization of the HM experiment was discussed, the incident photon energy E_i can reach 500 keV. This is much higher than typical values of the electron-nuclear attraction and electron-electron repulsion integrals for valence orbitals such as $b(\boldsymbol{r})$. Because of this, it is safe to approximately assume that the excitation energy is accumulated in (12) in the kinetic integral of the FGO (8),

$$E_i \approx \langle g^B(\Gamma)|-\tfrac{1}{2}\nabla^2|g^B(\Gamma)\rangle, \tag{14}$$

and the latter is expressed through $\Gamma(\epsilon)$ as follows [16],

$$\langle g^B(\Gamma)|-\frac{1}{2}\nabla^2|g^B(\Gamma)\rangle = \frac{3\ln 2}{\Gamma^2}. \tag{15}$$

Then, according to (14) and (15), absorption of the incident photon with an energy of 500 keV causes the actualization of the spatial electron coordinate with an accuracy $\Gamma(\epsilon) = 0.0075$ bohrs. Such a tight localization of a valence electron can be considered as its practical spatial collapse. Evidently, the limit of tight electron localization $\Gamma \to 0$ in $\Psi^{\mathrm{SG+HM}}(\epsilon)$ represents the full collapse.

With the energy ΔE^{C} of (1) being much larger than the binding energy of the sample electron, the latter, after emission of the scattered electron, is eventually ejected, leaving behind the bare proton. From the point of view of actualization and measurement of $\boldsymbol{r}$, this ejection is an accompanying process, so it will not be considered in the present paper.

To sum up, the proposed SG + HM experiment results in the actualization of the particular spin, say, β spin of the electron on the fragment H_B, the spatial coordinate $\boldsymbol{r}$ of which is instantly actualized with a finite accuracy $\Gamma(\epsilon)$. This spin-collapsed and partially spatially collapsed state is described with the decoherent product $\Psi^{\mathrm{SG+HM}}(\epsilon)$ with the FGO of (8). Our estimate indicates that at realistic energies of the photon beam a practically full spatial collapse is achievable.

IV. CONCLUSIONS

The setup of the combined SG + HM experiment to determine the full electron coordinate $\boldsymbol{x}$ and its mechanism proposed in this paper have the following original points:

(1) The proposed SG + HM experiment brings together three, arguably, of the most renowned experiments of quantum mechanics. These are the SG experiment to measure s_z, the HM experiment to measure $\boldsymbol{r}$, and the EPR experiment in its BA version to entangle remote electrons.

(2) Electron entanglement is proposed to be employed, in order to satisfy the requirements of noninterfering and simultaneous HG and HM measurements of $\boldsymbol{r}$ and s_z. Specifically, the SG measurement of the α spin on one fragment of an entangled system serves as an effective noninterfering measurement of the β spin of the electron on another fragment. Due to the instant transfer of the spin in the entangled system, it can be synchronized with the HM measurement of the spatial coordinate $\boldsymbol{r}$. As the result of the SG + HM experiment, the disentangled electron acquires the actualized coordinates $\boldsymbol{x} = \{\boldsymbol{r}, \beta\}$ in the collapsed state.

(3) In the proposed mechanism of the SG + HM experiment, the SG measurement of the expectation value of the spin operator with the discrete eigenspectrum is contrasted with the HM measurement for the position operator with the continuous eigenspectrum. According to this mechanism, the final SG + HM state combines spin collapse with partial spatial collapse.

(4) To characterize the partial spatial collapse of the HM part of the SG + HM experiment, the intrinsic accuracy of the $\boldsymbol{r}$ measurement is introduced. Its estimate emerges from the description of partial spatial collapse with the floating Gaussian orbital.

(5) The estimate of the intrinsic accuracy of the $\boldsymbol{r}$ measurement for the realistic condition indicates that it can be as precise as 0.01 bohrs. With this, the DL of (2) sets the apparent limit of $\boldsymbol{r}$ measurement with the x-ray microscope considered originally [11]. As was pointed out in Ref. [4], the use of an electron microscope might, in principle, increase the accuracy of the HM experiment.

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