Solvable class of non-Markovian quantum multipartite dynamics

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We study a class of multipartite open quantum dynamics for systems with an arbitrary number of qubits. The non-Markovian quantum master equation can involve arbitrary single or multipartite and time-dependent dissipative coupling mechanisms, expressed in terms of strings of Pauli operators. We formulate the general constraints that guarantee the complete positivity of this dynamics. We characterize in detail the underlying mechanisms that lead to memory effects, together with properties of the dynamics encoded in the associated system rates. We specifically derive multipartite "eternal" non-Markovian master equations that we term hyperbolic and trigonometric due to the time dependence of their rates. For these models we identify a transition between positive and periodically divergent rates. We also study non-Markovian effects through an operational (measurement-based) memory witness approach.

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I. INTRODUCTION

In the theory of open quantum systems, the formulation of quantum Markovian master equations is completely determined by the theory of quantum semigroups [1]. In contrast, the study of non-Markovian memory effects presents two problems. The first one is that the most general structure of a quantum master equation that captures memory effects and at the same time is consistent with the completely positive (CP) condition of the solution map [2–4] is not known. The second one is that different inequivalent memory witnesses can be used to define and measure non-Markovian effects [5,6].

The first problem has been known for many years. In fact, arbitrary non-Markovian quantum master equations may lead to unphysical solutions [7–10] in which the average state (the density matrix) is not positive definite. For tackling this issue a broad class of phenomenological and theoretical approaches has been formulated [3], dealing with both time-convoluted and convolutionless master equations [11]. Examples include the dynamics induced by stochastic Hamiltonians defined by nonwhite noises [12], phenomenological single-memory kernels [13–16], interaction with incoherent degrees of freedom [17–22] and arbitrary ancilla systems [23,24], related quantum collisional models [25–32], quantum generalizations of semi-Markov processes [33,34], and random (convex) superpositions of unitary and unital maps [35–37], together with some exact derivations from underlying (microscopic or effective) unitary dynamics [38–46].

Despite these advances [7–46] most studies of non-Markovian evolutions are restricted in general to single or bipartite systems. In fact, in general checking the CP condition of the dynamics is a nontrivial task, whose difficulty in turn increases with the system's Hilbert space dimension. However, quantum information intrinsically requires multipartite processing, and as a consequence the formulation of multipartite non-Markovian dynamics is of interest from both theoretical and practical points of view.

Our main goal in this paper is to formulate and study a class of solvable multipartite non-Markovian master equations. The class of systems we consider is defined in terms of an arbitrary number N of qubits, whose interaction with the environment can be taken into account through arbitrary Pauli channels. The evolution of the system's density matrix ρ_t is given by the time-local master equation $(d/dt)\rho_t = \mathcal{L}[\rho_t]$, where the generator of the evolution has the general structure

$$\begin{aligned}\mathcal{L}[\bullet] = &\sum_{\substack{i=1,\cdots N\\ \alpha=x,y,z}} \Gamma_i^{\alpha}(t)\big(\sigma_i^{\alpha}\bullet\sigma_i^{\alpha}-\bullet\big)\\ &+\sum_{\substack{i=1,\cdots N\\ \alpha,\beta=x,y,z}} \Gamma_i^{\alpha\beta}(t)\big(\sigma_i^{\alpha}\sigma_{i+1}^{\beta}\bullet\sigma_{i+1}^{\beta}\sigma_i^{\alpha}-\bullet\big)\\ &+\sum_{\substack{i=1,\cdots N\\ \alpha,\beta,\gamma=x,y,z}} \Gamma_i^{\alpha\beta\gamma}(t)\big(\sigma_i^{\alpha}\sigma_{i+1}^{\beta}\sigma_{i+2}^{\gamma}\bullet\sigma_{i+2}^{\gamma}\sigma_{i+1}^{\beta}\sigma_i^{\alpha}-\bullet\big)\\ &+\cdots. \end{aligned} \tag{1}$$

Here, $[\bullet]$ denotes an arbitrary state of the system. Furthermore, σ_i^{α} is the αth Pauli operator ($\alpha = x, y, z$) acting on qubit i, and $\Gamma_i^{\alpha\cdots\beta}(t)$ define local and multipartite time-dependent (coupling) rates. In general, these rate functions may take both positive and negative values. The problem is to characterize which constraints must be fulfilled by them in order to obtain physically valid solutions. Interestingly, the resolution of this issue leads us to consider all possible multipartite interaction terms, that is, decoherence channels that involve coupling between an arbitrary number of qubits. We also explore which rates emerge when the memory effects arise from different underlying mechanisms based on coupling with incoherent degrees of freedom [20,21]. The explicit formulation of an operational (measurement based) memory witness [47–50] further provides an alternative characterization of non-Markovian effects.

As a specific example we study a family of "hyperbolic" and "trigonometric" eternal multipartite non-Markovian master equations in which some rates are negative or develop divergences at all times. These cases provide a nontrivial extension and generalization of previous results valid for single systems [51].

This paper is structured as follows. In Sec. II we present the general class of multipartite dynamics we consider and characterize the solution of the master equation, resolving as a consequence the constraints that guarantee the CP condition of the map. General properties are derived for this class of models. In Sec. III the eternal multipartite dynamics are characterized. In Sec. IV we study memory effects through an operational memory witness. In Sec. V we provide our conclusions. The Appendixes give the details of derivations and also provide the rates associated with different underlying memory mechanisms.

II. MULTIPARTITE DYNAMICS

The system of interest consists of an arbitrary number N of qubits. For notational convenience we define a set of Pauli strings $S_{\mathbf{a}} \equiv \sigma_{a_1} \otimes \sigma_{a_2} \otimes \sigma_{a_N}$, each one associated with the vector $\mathbf{a} = (a_1, a_2, \dots, a_N)$. Each component a_k $(k = 1, 2, \dots, N)$ assumes the values $a_k = (0, 1, 2, 3) \leftrightarrow (\mathrm{I}, \sigma_x, \sigma_y, \sigma_z)$, with each one being associated with the (two-dimensional) identity matrix and the standard three Pauli matrices.

The evolution of the system's density matrix ρ_t is written in a local-in-time way. Arbitrary multipartite decoherence channels are considered,

$$\frac{d}{dt}\rho_t = \mathcal{L}[\rho_t] = \sum_{\mathbf{a}\neq\mathbf{0}} \gamma_t^{\mathbf{a}} (S_{\mathbf{a}}\rho_t S_{\mathbf{a}} - \rho_t). \quad (2)$$

The set of functions $\{\gamma_t^{\mathbf{a}}\}$ defines the rates associated with the multipartite Pauli channels. In general, there are $4^N - 1$ different rate functions, as the vector $\mathbf{0} = (0, 0, \dots, 0)$ is associated with the identity operator in the full Hilbert space. Our goal is to characterize the different aspects of this general evolution. A time-convoluted formulation of the above dynamics is provided in Appendix A.

A. Subsystem dynamics

Given the evolution above, we ask about the dynamics of any particular subsystem. Introducing the splitting $\mathbf{a} = (\mathbf{a_s}, \mathbf{a_e})$, where $\mathbf{a_s}$ corresponds to the set of local operators that define the marginal Pauli string of the subsystem of interest and $\mathbf{a_e}$ corresponds to that of the rest of qubits (now considered part of the environment), from Eq. (2) the subsystem density matrix $\rho_t^{\mathbf{s}} = \mathrm{Tr}_{\mathbf{e}}[\rho_t]$ (where $\mathrm{Tr}[\bullet]$ is the trace operation) reads

$$\frac{d}{dt}\rho_t^{\mathbf{s}} = \sum_{\mathbf{a_s}} \gamma_t^{\mathbf{a_s}} \big(S_{\mathbf{a_s}}\rho_t^{\mathbf{s}} S_{\mathbf{a_s}} - \rho_t^{\mathbf{s}}\big), \quad \gamma_t^{\mathbf{a_s}} \equiv \sum_{\mathbf{a_e}} \gamma_t^{\mathbf{a_s},\mathbf{a_e}}. \quad (3)$$

From this equation we conclude that any subsystem, even when, in general, it is correlated with the complementary part, has an independent self-evolution. In addition, the structure of this evolution belongs to the same class as that of the full system [Eq. (2)]. Consequently, the following results can be particularized for any subsystem of arbitrary size.

B. Solution map and completely positive condition

We now show that by using the method of damping bases or spectral decomposition [52], the solution map $\rho_0 \to \rho_t$ corresponding to Eq. (2) can be obtained in an exact way. In order the apply this technique, first, we establish a set of relations fulfilled by the (two-dimensional) Pauli operators. Maintaining the notation $(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \leftrightarrow (I, \sigma_x, \sigma_y, \sigma_z)$, it is easy to check that

$$\sigma_a \mathrm{Tr}[\sigma_a \bullet] = \frac{1}{2}\sum_b H_{ab}(\sigma_b \bullet \sigma_b), \quad (4)$$

where here the input $[\bullet]$ is an arbitrary two-dimensional operator and $b = 0, 1, 2, 3$. The inverse relation reads

$$\sigma_a \bullet \sigma_a = \frac{1}{2}\sum_b H_{ab}\, \sigma_b \mathrm{Tr}[\sigma_b \bullet]. \quad (5)$$

In these expressions, the coefficients $\{H_{ab}\}$ define a four-dimensional Hadamard matrix H, which reads

$$H \equiv \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (6)$$

In deriving Eq. (5), we used that its inverse reads $H^{-1} = H/4$. Also notice that $H = H^T$.

Now, we introduce an extra rate $\gamma_t^{\mathbf{0}}$, which is associated with the identity string in the full Hilbert space

$$\gamma_t^{\mathbf{0}} \equiv -\sum_{\mathbf{a}\neq\mathbf{0}} \gamma_t^{\mathbf{a}}. \quad (7)$$

With this definition, the Lindbladian-like structure of Eq. (2) can straightforwardly be written as

$$\mathcal{L}[\bullet] = \sum_{\mathbf{a}} \gamma_t^{\mathbf{a}} (S_{\mathbf{a}} \bullet S_{\mathbf{a}}), \quad (8)$$

where the sum now includes the (identity) string $\mathbf{a} = \mathbf{0}$. Written in this way, applying the "vectorial extension" of Eq. (5) to the Hilbert space of N qubits, it follows that

$$\mathcal{L}[\bullet] = \frac{1}{2^N}\sum_{\mathbf{a}} S_{\mathbf{a}} \mathrm{Tr}[S_{\mathbf{a}}\bullet] \sum_{\mathbf{b}} H_{\mathbf{ab}}\gamma_t^{\mathbf{b}}, \quad (9)$$

where $H_{\mathbf{ab}} \equiv H_{a_1 b_1} H_{a_2 b_2} \cdots H_{a_N b_N}$ can be read as the matrix elements of the external product of N single Hadamard matrices [see Eq. (6)]. From this last expression, by using

$\mathrm{Tr}[S_{\mathbf{a}}S_{\mathbf{b}}]=2^N\delta_{\mathbf{a},\mathbf{b}}$, it is straightforward to determine the eigenvalues and eigenoperators of $\mathcal{L}[\bullet]$. They read

$$\mathcal{L}[S_{\mathbf{a}}]=\mu_t^{\mathbf{a}}S_{\mathbf{a}},\quad \mu_t^{\mathbf{a}}=\sum_{\mathbf{b}}H_{\mathbf{ab}}\gamma_t^{\mathbf{b}}. \tag{10}$$

Consequently, any Pauli string $S_{\mathbf{a}}$ is a right eigenoperator with eigenvalue $\mu_t^{\mathbf{a}}$. Given that $\mathcal{L}[\bullet]$ also defines the adjoint evolution (as the "jump operators" are Hermitian) [52], $S_{\mathbf{a}}$ is also a left eigenoperator. Notice also that by using the inverse of the Hadamard matrix, the inverse relation $\gamma_t^{\mathbf{a}}=\sum_{\mathbf{b}}H_{\mathbf{ab}}\mu_t^{\mathbf{b}}/4^N$ follows.

From the method of damping bases [52], Eq. (10) allows us to write the solution of Eq. (2) as

$$\rho_t=\frac{1}{2^N}\sum_{\mathbf{a}}\exp\left[\int_0^t dt'\mu_{t'}^{\mathbf{a}}\right]S_{\mathbf{a}}\mathrm{Tr}[S_{\mathbf{a}}\rho_0]. \tag{11}$$

One can see that the conditions $\mathrm{Tr}[\rho_t]=\mathrm{Tr}[\rho_0]=1$ are satisfied after noting that $\mathrm{Tr}[S_{\mathbf{a}}]=2^N\delta_{\mathbf{a},\mathbf{0}}$ and $\mu_{t'}^{\mathbf{0}}=0$. This last equality follows from Eqs. (7) and (10) jointly with the property $H_{\mathbf{0b}}=1\ \forall\mathbf{b}$. By using the vectorial extension of Eq. (4), we get the density matrix written in a Kraus representation [1,2],

$$\rho_t=\sum_{\mathbf{a}}p_t^{\mathbf{a}}(S_{\mathbf{a}}\rho_0S_{\mathbf{a}}). \tag{12}$$

The weights are $p_t^{\mathbf{a}}=4^{-N}\sum_{\mathbf{b}}H_{\mathbf{ab}}\exp[\int_0^t dt'\mu_{t'}^{\mathbf{b}}]$, which from Eq. (10) can explicitly be written in terms of the time-dependent rates as

$$p_t^{\mathbf{a}}=\frac{1}{4^N}\sum_{\mathbf{b}}H_{\mathbf{ab}}\exp\left[\sum_{\mathbf{c}}H_{\mathbf{bc}}\int_0^t dt'\gamma_{t'}^{\mathbf{c}}\right]. \tag{13}$$

Expressions (12) and (13) are the main results of this section. They completely characterize the solution map in terms of the set of rates $\{\gamma_t^{\mathbf{a}}\}$ and the initial condition ρ_0. In addition, they naturally provide a constraint that the rates must fulfill in order to obtain a CP map, that is, one that gives physical solution. In fact, the Kraus representation theorem [1,2] implies the conditions $0\leqslant p_t^{\mathbf{a}}\leqslant 1$, which means that $\{p_t^{\mathbf{a}}\}$ are a set of normalized probabilities. In the single-qubit case ($N=1$), previously obtained constraints are recovered [35]. In the general case, 4^N inequalities must be fulfilled. We notice that a sufficient, but not necessary, condition is $\int_0^t dt'\gamma_{t'}^{\mathbf{a}}\geqslant 0\ \forall\,\mathbf{a}\neq\mathbf{0}$. In fact, this constraint implies that all eigenvalues [see Eq. (10)] satisfy $\mu_t^{\mathbf{a}}\leqslant 0$ ($\mathbf{a}\neq\mathbf{0}$). Consequently, taking an arbitrary, but *fixed*, time t, the solution (11) of the non-Markovian dynamics, via the association $\int_0^t dt'\mu_{t'}^{\mathbf{a}}=t\mu_M^{\mathbf{a}}$, is equivalent to the solution of a (well-behaved) Markovian dynamics generated by a Lindbladian with eigenvalues $\{\mu_M^{\mathbf{a}}\}$.

C. Non-Markovianity and time-dependent rates

Different (inequivalent) memory witnesses based only on the system propagator can be used to define non-Markovianity [5,6] such as the trace distance between two different initial conditions [53] or those based on the k positivity of the solution map [54]. Here, as the dynamics is written naturally in a canonical form [51], memory effects can also be defined by the negativity of the time-dependent rates $\{\gamma_t^{\mathbf{a}}\}$. In this way, it is of interest to determine these elements for any well-behaved solution defined by the probabilities $\{p_t^{\mathbf{a}}\}$ in Eq. (12).

We can invert Eq. (13),

$$\mu_t^{\mathbf{a}}=\frac{d}{dt}\ln\left[\sum_{\mathbf{b}}H_{\mathbf{ab}}p_t^{\mathbf{b}}\right], \tag{14}$$

and using Eq. (10), we get explicit expressions for the set of rates $\{\gamma_t^{\mathbf{a}}\}$ in terms of the normalized time-dependent weights $0\leqslant p_t^{\mathbf{c}}\leqslant 1$,

$$\gamma_t^{\mathbf{a}}=\frac{1}{4^N}\sum_{\mathbf{b}}H_{\mathbf{ab}}\frac{d}{dt}\ln\left[\sum_{\mathbf{c}}H_{\mathbf{bc}}p_t^{\mathbf{c}}\right]. \tag{15}$$

The signs of $\{\gamma_t^{\mathbf{a}}\}$ can be taken as a signature of departure from a Markovian regime [51]. Alternatively, in Sec. V we study operational measures for non-Markovianity. We notice that Eqs. (13) and (15) provide a multipartite generalization of the case $N=1$ studied in Ref. [35].

D. Additivity of non-Markovian master equations

Given two sets of (arbitrary) normalized probabilities $\{p_t^{\mathbf{a}}\}$ and $\{\tilde{p}_t^{\mathbf{a}}\}$, the relation (15) allows us to obtain the corresponding sets of rates $\{\gamma_t^{\mathbf{a}}\}$ and $\{\tilde{\gamma}_t^{\mathbf{a}}\}$. From these we can obtain a new master equation defined by Eq. (2) with rates $\{\gamma_t^{\mathbf{a}}+\tilde{\gamma}_t^{\mathbf{a}}\}$. In fact, it is always possible to associate a set of probabilities $\{q_t^{\mathbf{a}}\}$ with these added rates, that is,

$$\{p_t^{\mathbf{a}}\}\leftrightarrow\{\gamma_t^{\mathbf{a}}\},\quad \{\tilde{p}_t^{\mathbf{a}}\}\leftrightarrow\{\tilde{\gamma}_t^{\mathbf{a}}\},\quad\Rightarrow\ \exists\{q_t^{\mathbf{a}}\}\leftrightarrow\{\gamma_t^{\mathbf{a}}+\tilde{\gamma}_t^{\mathbf{a}}\}. \tag{16}$$

Consequently, as occurs with Markovian Lindblad equations [2], for our class of models arbitrary well-behaved evolutions (defined by a given set of rates) can be added in an arbitrary way. The validity of this result follows from the commutation of two arbitrary propagators, Eq. (12), a property supported by the relation

$$S_{\mathbf{a}}S_{\mathbf{b}}\bullet S_{\mathbf{b}}S_{\mathbf{a}}=S_{\mathbf{b}}S_{\mathbf{a}}\bullet S_{\mathbf{a}}S_{\mathbf{b}}=S_{\mathbf{c}}\bullet S_{\mathbf{c}}^{\dagger}, \tag{17}$$

which is valid for arbitrary Pauli strings $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$, where $S_{\mathbf{c}}=S_{\mathbf{a}}S_{\mathbf{b}}$ or, equivalently, $S_{\mathbf{c}}=S_{\mathbf{b}}S_{\mathbf{a}}$. Equation (17) can be straightforwardly demonstrated from Eq. (5).

E. Coupling with incoherent degrees of freedom

Memory effects are induced whenever extra degrees of freedom are traced out. Here, we consider a general coupling with incoherent degrees of freedom, which define the environment. Based on Ref. [17], the more general case can always be described by writing the system density matrix ρ_t and the probabilities of the incoherent system $\{q_t^{\mathbf{h}}\}$ [$\sum_{\mathbf{h}}q_t^{\mathbf{h}}=1$] as

$$\rho_t=\sum_{\mathbf{h}}\rho_t^{\mathbf{h}},\quad q_t^{\mathbf{h}}=\mathrm{Tr}\big[\rho_t^{\mathbf{h}}\big], \tag{18}$$

where the auxiliary states $\{\rho_t^{\mathbf{h}}\}$ correspond to the system state given that the extra (hidden) incoherent degrees of freedom are in the particular state $\mathbf{h}$. The evolution of the states $\{\rho_t^{\mathbf{h}}\}$ may involve coupling between all of them. Assuming separable initial conditions, the general coupling structure defined in Ref. [17] guarantees the CP condition of the solution map $\rho_0\to\rho_t$.

Given the structure Eq. (2), each auxiliary state $\rho_t^{\mathbf{h}}$ must assume the form

$$\rho_t^{\mathbf{h}} = \sum_{\alpha} g_\alpha^{\mathbf{h}}(t)(S_\alpha \rho_0 S_\alpha), \quad (19)$$

where the parameter α runs over a set of Pauli strings that depends on each specific problem. The functions $g_\alpha^{\mathbf{h}}(t)$ in turn obey a classical master equation whose structure also depends on each specific model.

The (separable) initial conditions read $\rho_0^{\mathbf{h}} = \rho_0 q_0^{\mathbf{h}}$, where ρ_0 is the initial system state and $q_0^{\mathbf{h}}$ is the initial probability of the incoherent degrees of freedom. In fact, at time t, $q_t^{\mathbf{h}} = \sum_\alpha g_\alpha^{\mathbf{h}}(t)$. On the other hand, the system density matrix evolution [Eq. (12)] is defined by the probabilities $p_t^\alpha = \sum_{\mathbf{h}} g_\alpha^{\mathbf{h}}(t)$.

A general treatment for getting ρ_t is not possible, nor is it possible to predict the specific memory properties of the solution map for a given underlying coupling. Relevant examples are worked out in Appendix B such as a mapping with a classical Markovian master equation, stochastic Hamiltonians, and statistical mixtures of Markovian evolutions. In all cases, explicit expressions for the rates [Eq. (15)] can be obtained. A representative class of dynamics is studied in the next section.

III. MULTIPARTITE ETERNAL NON-MARKOVIANITY

For a single qubit, $N = 1$, the system density-matrix evolution, Eq. (2), may involve rates that are negative at all times. This property is called "eternal non-Markovianity" [20,51]. The results of Appendix B [see Eqs. (B8), (B13), and (B19)] and Appendix C [see Eqs. (C2) and (C4)] guarantee that this property also emerges in multipartite dynamics, $N > 1$, which have $4^N - 1$ rates.

In order to provide simple (multipartite) examples, here, we restrict ourselves to the case where the evolution is

$$\mathcal{L}[\bullet] = \{\gamma_t^{\mathbf{a}}(S_{\underline{\mathbf{a}}} \bullet S_{\underline{\mathbf{a}}} - \bullet) + \gamma_t^{\mathbf{b}}(S_{\underline{\mathbf{b}}} \bullet S_{\underline{\mathbf{b}}} - \bullet) + \gamma_t^{\mathbf{c}}(S_{\underline{\mathbf{c}}} \bullet S_{\underline{\mathbf{c}}}^\dagger - \bullet)\}, \quad (20)$$

where $S_{\underline{\mathbf{a}}}$ and $S_{\underline{\mathbf{b}}}$ are two arbitrary multipartite Pauli strings and $S_{\underline{\mathbf{c}}} = S_{\underline{\mathbf{a}}} S_{\underline{\mathbf{b}}}$. Depending on the time dependence of the rates, we define what we term "hyperbolic" and "trigonometric" cases of eternal non-Markovianity. Interestingly, these cases emerge by considering that the environment has only two possible states.

A. Hyperbolic eternal non-Markovianity

The system density matrix is written as the addition of two auxiliary states $\rho_t = \rho_t^{(1)} + \rho_t^{(2)}$ [Eq. (18)], whose evolution reads

$$\frac{d\rho_t^{(1)}}{dt} = -\gamma\rho_t^{(1)} + \gamma S_{\underline{\mathbf{a}}} \rho_t^{(1)} S_{\underline{\mathbf{a}}}, \quad (21a)$$

$$\frac{d\rho_t^{(2)}}{dt} = -\varphi\rho_t^{(2)} + \varphi S_{\underline{\mathbf{b}}} \rho_t^{(2)} S_{\underline{\mathbf{b}}}. \quad (21b)$$

The initial conditions for the auxiliary states are taken to be $\rho_0^{(1)} = \rho_0^{(2)} = \rho_0/2$. Given that the auxiliary states do not couple between them, from Eq. (21) it is simple to notice that this property is inherited by the incoherent degrees of freedom, which in turn do not evolve in time, $q_t^{(i)} = \mathrm{Tr}[\rho_t^{(i)}] = (1/2)$ $(i = 1, 2)$. Thus, the system dynamic is defined by a statistical superposition (with equal weights) of two different uncoupled Lindblad evolutions (with rates γ and φ).

The rates of the non-Markovian evolution follow from Eq. (15) with probabilities $p_t^{\mathbf{a}} = [p_1^{\mathbf{a}}(t) + p_2^{\mathbf{a}}(t)]/2$, with the sets $\{p_1^{\mathbf{a}}(t)\}$ and $\{p_2^{\mathbf{a}}(t)\}$, via Eq. (13), associated with $\rho_t^{(1)}$ and $\rho_t^{(2)}$, respectively. Taking $\varphi = \gamma$, we get [see also the derivation of Eq. (B19) in Appendix C]

$$\gamma_t^{\mathbf{a}} = \gamma_t^{\mathbf{b}} = \tfrac{1}{2}\gamma, \quad \gamma_t^{\mathbf{c}} = -\tfrac{1}{2}\gamma \tanh(\gamma t). \quad (22)$$

This result provides a multipartite generalization, $(N > 1)$, of the single-qubit case $(N = 1)$ studied in Ref. [51]. We notice that similar to the results of Ref. [20], in this particular case, alternative dynamics such as the mapping to a classical master equation [see Eq. (B8)] and stochastic Hamiltonians [see Eq. (B13)] also lead to the same rates.

B. Trigonometric eternal non-Markovianity

Based on Eq. (18), instead of the evolution (21), here, we consider

$$\frac{d\rho_t^{(1)}}{dt} = -\gamma\rho_t^{(1)} + \varphi S_{\underline{\mathbf{b}}} \rho_t^{(2)} S_{\underline{\mathbf{b}}}, \quad (23a)$$

$$\frac{d\rho_t^{(2)}}{dt} = -\varphi\rho_t^{(2)} + \gamma S_{\underline{\mathbf{a}}} \rho_t^{(1)} S_{\underline{\mathbf{a}}}. \quad (23b)$$

Both auxiliary states are intrinsically coupled. In this case, it is simple to check that the probabilities $q_t^{(i)} = \mathrm{Tr}[\rho_t^{(i)}]$ $(i = 1, 2)$ of the incoherent degrees of freedom obey a classical master equation. Notice that the incoherent transitions $(1) \leftrightarrow (2)$ imply the system transformations $\rho \to S_{\underline{\mathbf{a}}/\underline{\mathbf{b}}} \rho S_{\underline{\mathbf{a}}/\underline{\mathbf{b}}}$.

The initial conditions are taken as $\rho_0^{(1)} = [\varphi/(\varphi + \gamma)]\rho_0$ and $\rho_0^{(2)} = [\gamma/(\varphi + \gamma)]\rho_0$, where ρ_0 is the system initial state and the weights correspond to the stationary solution of the (environment) classical master equation. Thus, $q_t^{(1)} = \varphi/(\varphi + \gamma)$, and $q_t^{(2)} = \gamma/(\varphi + \gamma)$.

Taking into account Eq. (19), in order to solve Eq. (23) each auxiliary state is written as $(h = 1, 2)$

$$\rho_t^{(h)} = g_{\mathbf{0}}^{(h)}\rho_0 + g_{\underline{\mathbf{a}}}^{(h)} S_{\underline{\mathbf{a}}}\rho_0 S_{\underline{\mathbf{a}}} + g_{\underline{\mathbf{b}}}^{(h)} S_{\underline{\mathbf{b}}}\rho_0 S_{\underline{\mathbf{b}}} + g_{\underline{\mathbf{c}}}^{(h)} S_{\underline{\mathbf{c}}}\rho_0 S_{\underline{\mathbf{c}}}^\dagger, \quad (24)$$

where as before $S_{\underline{\mathbf{c}}} = S_{\underline{\mathbf{a}}} S_{\underline{\mathbf{b}}}$ and $g_\alpha^{(h)}$ are time-dependent functions. Using Eq. (17), it is possible to derive a classical master equation for the (eight) g functions, which involves coupling between pairs of them. The corresponding solutions allow us to obtain the probabilities $p_t^{\mathbf{a}} = \sum_h g_{\mathbf{a}}^{(h)}(t)$. Finally, the rates associated with the non-Markovian evolution follow from Eq. (15),

$$\gamma_t^{\mathbf{a}} = \gamma_t^{\mathbf{b}} = \frac{\varphi\gamma(\varphi + \gamma)}{e^{t(\varphi+\gamma)}(\varphi - \gamma)^2 + 4\varphi\gamma}. \quad (25)$$

Furthermore,

$$\begin{aligned}\gamma_t^{\mathbf{c}} = {} & \varphi\gamma(\delta_+ \Upsilon^2(1 - e^{t\Upsilon}) - \delta_-^2\{\Upsilon(1 + e^{t\Upsilon}) \\ & + e^{t\delta_+}[(\delta_+ - \Upsilon) - e^{t\Upsilon}(\delta_+ + \Upsilon)]\}) \\ & \times \{(e^{t\delta_+}\delta_-^2 + 4\varphi\gamma)[(1 + e^{t\Upsilon})\Upsilon\delta_+ \\ & - (1 - e^{t\Upsilon})\delta_-^2]\}^{-1},\end{aligned} \quad (26)$$

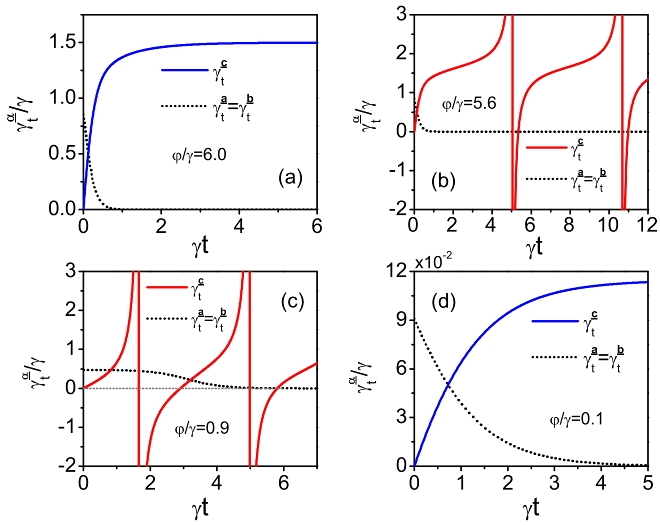

FIG. 1. Scaled time-dependent rates [Eqs. (25) and (26)] corresponding to the multipartite trigonometric eternal non-Markovian evolution [Eq. (23)] for different values of the rate ratio φ/γ.

where the coefficients are

$$\Upsilon \equiv (\varphi^2 - 6\varphi\gamma + \gamma^2)^{1/2}, \quad \delta_\pm \equiv \varphi \pm \gamma. \tag{27}$$

Depending on the ratio φ/γ, different characteristic behaviors are obtained. In Fig. 1 we plot both rates. Consistent with Eq. (25), $\gamma_t^{\underline{\mathbf{a}}}$ and $\gamma_t^{\underline{\mathbf{b}}}$ are always positive functions. However, this is not the case for $\gamma_t^{\underline{\mathbf{c}}}$, Eq. (26), which, depending on φ/γ, develops a transition between *positivity* [Figs. 1(a) and 1(d)] and a *periodic divergent behavior* [Figs. 1(b) and 1(c)]. From Eq. (27) we deduce that this change occurs in the boundaries of the interval $3-\sqrt{8} < (\varphi/\gamma) < 3+\sqrt{8}$, with $\gamma_t^{\underline{\mathbf{c}}}$ developing divergences in this interval while being positive outside it. Notice that this interval defines the regime where the system dynamics is CP indivisible.

From the plots it is also evident that $\gamma_t^{\underline{\mathbf{a}}}$ and $\gamma_t^{\underline{\mathbf{b}}}$ approach a constant when $\varphi \approx \gamma$. In fact, when $\varphi = \gamma$, the previous expressions reduce to

$$\gamma_t^{\underline{\mathbf{a}}} = \gamma_t^{\underline{\mathbf{b}}} = \tfrac{1}{2}\gamma, \quad \gamma_t^{\underline{\mathbf{c}}} = \tfrac{1}{2}\gamma \tan(\gamma t). \tag{28}$$

Based on Eq. (22), we name this case a *trigonometric eternal non-Markovian*. The probabilities $\{p_t^{\underline{\mathbf{a}}}\}$ [Eq. (12)] also assume a simple form,

$$p_t^{\underline{\mathbf{0}}} = \frac{1}{2}e^{-\gamma t}[\cosh(\gamma t) + \cos(\gamma t)], \tag{29a}$$

$$p_t^{\underline{\mathbf{a}}} = p_t^{\underline{\mathbf{b}}} = \frac{1}{4}[1 - e^{-2\gamma t}], \tag{29b}$$

$$p_t^{\underline{\mathbf{c}}} = \frac{1}{2}e^{-\gamma t}[\cosh(\gamma t) - \cos(\gamma t)]. \tag{29c}$$

These solutions apply to arbitrary multipartite Pauli strings $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

C. Adding non-Markovian evolutions

Added to the previous examples (see also Appendix C), the possibility of adding arbitrary (well-defined) rates [Eq. (16)] gives us a procedure for constructing a large family of well-behaved dynamics. Consistent with the goal of studying master equations with the structure defined by Eq. (1), as an example we write

$$\begin{aligned}\mathcal{L}[\bullet] = \sum_{i=1}^{N} \frac{\gamma_i}{2}\big\{&\big(\sigma_i^x\sigma_{i+1}^x \bullet \sigma_{i+1}^x\sigma_i^x - \bullet\big)\\ &+\big(\sigma_i^y\sigma_{i+1}^y \bullet \sigma_{i+1}^y\sigma_i^y - \bullet\big)\\ &+f_i(t)\big(\sigma_i^z\sigma_{i+1}^z \bullet \sigma_{i+1}^z\sigma_i^z - \bullet\big)\big\}.\end{aligned} \tag{30}$$

In this translational-invariant generator (say, with periodic boundaries in one dimension), we may choose $f_i(t) = -\tanh(\gamma_i t)$ or, alternatively, $f_i(t) = \tan(\gamma_i t)$ [see Eqs. (22) and (28), respectively].

One interesting aspect of using additivity for constructing multipartite evolutions is that, even when the underlying evolutions have a clear memory mechanism (see also Appendix B), the resulting dynamics does not necessarily have one. For example, while our approach guarantees that Eq. (30) leads to a completely positive dynamics [with a solution defined by Eqs. (12) and (13)], it is not evident which underlying processes may lead to this master equation. In addition, in general there may be subsystems that are coupled between them, one part being Markovian and the other being non-Markovian. For example, take $f_i(t) = \gamma_i/2$ for $i \leqslant N_0$ and $f_i(t) = \tan(\gamma_i t)$ for $i > N_0$.

IV. OPERATIONAL MEMORY WITNESS

An alternative and deeper characterization of quantum non-Markovianity can be obtained by defining memory effects via measurement-based approaches [47–49]. Here, we study a conditional past-future (CPF) correlation [48]. This object relies on performing three successive measurements of arbitrary system observables and calculating the correlation between the last (future) and first (past) outcomes conditioned to a given intermediate (present) outcome. For Markovian dynamics it vanishes identically, while memory effects lead to a non-null CPF correlation. The method was experimentally implemented in different quantum optical arrangements [50].

The measurements, denoted in successive order by $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$, and $\underline{\mathbf{z}}$, correspond to observations of three Hermitian operators $S_{\underline{\mathbf{m}}}$ with eigenvectors $\{|m\rangle\}$ and eigenvalues $\{m\}$,

$$S_{\underline{\mathbf{m}}}|m\rangle = m|m\rangle, \quad \underline{\mathbf{m}} = \underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{z}}. \tag{31}$$

The CPF correlation then reads [48]

$$C_{pf}(t,\tau)|_y = \sum_{z,x} zx[P(z,x|y) - P(z|y)P(x|y)], \tag{32}$$

where $\{x\}$, $\{y\}$, and $\{z\}$ denote the three sets of successive outcomes (operators eigenvalues, assumed to be dimensionless) and t and τ are the (first and second) time intervals between the successive measurements. With $P(u|v)$ we denote the conditional probability of u given v.

All probabilities appearing in Eq. (32) can be determine from the (outcome) joint probability $P(z,y,x) \leftrightarrow P(z,t+\tau, y,t; x,0)$, which in turn can be calculated after knowing the underlying system-environment dynamics. In Appendix D we show that $P(z,y,x)$ and $C_{pf}(t,\tau)|_y$ can be calculated exactly assuming that memory effects emerge due to the coupling with incoherent degrees of freedom [Eqs. (18) and (19)].

Each specific model [see examples (21) and (23)] is completely defined by the set of functions $\{g_\alpha^{\mathbf{h}}(t)\}$ [Eq. (19)]. Given that they obey a (linear) classical master equation, they can be written as

$$g_\alpha^{\mathbf{h}}(t) = \sum_{\mathbf{h}'} f_\alpha^{\mathbf{h}\mathbf{h}'}(t) q_0^{\mathbf{h}'} \equiv (\mathbf{h}|\mathbb{F}_\alpha(t)|q_0), \tag{33}$$

where the set of functions $\{f_\alpha^{\mathbf{h}\mathbf{h}'}(t)\}$ is independent of the initial conditions $\{q_0^{\mathbf{h}}\}$. Furthermore, for notational simplicity, we introduced a vectorial orthogonal base $\{|\mathbf{h})\}$ for the incoherent degrees of freedom, such that $f_\alpha^{\mathbf{h}\mathbf{h}'}(t) \leftrightarrow (\mathbf{h}|\mathbb{F}_\alpha(t)|\mathbf{h}')$ and $q_0^{\mathbf{h}} \leftrightarrow (\mathbf{h}|q_0)$.

The observables $S_{\underline{\mathbf{m}}}$ [Eq. (31)] may, in principle, be defined by arbitrary linear combinations of Pauli strings $\{S_{\mathbf{a}}\}$. Here, for simplicity they are defined by a unique Pauli string. In this case, the general expression for the CPF correlation [Eq. (D5)] reduces to (see Appendix D)

$$\begin{aligned} C_{pf}(t,\tau)|_y &= \delta_{\underline{\mathbf{z}},\mathbf{y}}\delta_{\underline{\mathbf{y}},\underline{\mathbf{x}}}\frac{(1-\langle x\rangle^2)}{[2^N P(y)]^2}\sum_{\alpha,\beta} H_{\underline{\mathbf{y}}\alpha}H_{\underline{\mathbf{y}}\beta}[(1|\mathbb{F}_\alpha(\tau)\mathbb{F}_\beta(t)|q_0) \\ &\quad - (1|\mathbb{F}_\alpha(\tau)|q_t)(1|\mathbb{F}_\beta(t)|q_0)]. \end{aligned} \tag{34}$$

Here, $|q_t) = \sum_\alpha \mathbb{F}_\alpha(t)|q_0)$ define the probabilities of the incoherent degrees of freedom at time t, while $(1| \equiv \sum_{\mathbf{h}}(\mathbf{h}|$. Furthermore, $\langle x\rangle \equiv \sum_x xP(x)$, where $P(x) = \langle x|\rho_0|x\rangle$. Finally, $P(y)$ is the probability of the outcomes of the second measurement. It is

$$P(y) = \frac{1}{2^N}\left[1 + y\langle x\rangle\delta_{\underline{\mathbf{y}},\underline{\mathbf{x}}}\sum_\alpha H_{\underline{\mathbf{y}}\alpha}(1|\mathbb{F}_\alpha(t)|q_0)\right]. \tag{35}$$

The term $\delta_{\underline{\mathbf{z}},\mathbf{y}}\delta_{\underline{\mathbf{y}},\underline{\mathbf{x}}}$ in Eq. (34) implies that, for observables defined by unique Pauli strings, memory effects are detected only when the three observables are the same $S_{\underline{\mathbf{x}}} = S_{\underline{\mathbf{y}}} = S_{\underline{\mathbf{z}}}$. This constraint does not emerge when the observables correspond to other bases of operators (see, for example, Ref. [49]).

The general solution (34) can be specified for the trigonometric eternal model [Eq. (23)]. Stationary initial conditions are assumed, $|q_t) = |q_0)$, with $q_0^{(1)} = \varphi/(\varphi+\gamma)$ and $q_0^{(2)} = \gamma/(\varphi+\gamma)$. For simplicity, first, we consider the case $N=1$. When the three measurements are performed in direction $\underline{\mathbf{a}}$ or $\underline{\mathbf{b}}$, we get

$$\begin{aligned} C_{pf}(t,\tau)|_y &= -\frac{(1-\langle x\rangle^2)}{[2^N P(y)]^2}\exp[-(t+\tau)(\gamma+\varphi)/2] \\ &\quad \times \frac{4^2\gamma^2\varphi^2}{(\gamma+\varphi)^2\Upsilon^2}\sinh\left(\frac{\Upsilon t}{2}\right)\sinh\left(\frac{\Upsilon\tau}{2}\right). \end{aligned} \tag{36}$$

When the three measurements are performed in direction $\underline{\mathbf{c}}$, we get

$$\begin{aligned} C_{pf}(t,\tau)|_y &= \frac{(1-\langle x\rangle^2)}{[2^N P(y)]^2}\,\frac{4\gamma\varphi(\gamma-\varphi)^2}{(\gamma+\varphi)^4} \\ &\quad \times[1-e^{-\tau(\gamma+\varphi)}][1-e^{-t(\gamma+\varphi)}]. \end{aligned} \tag{37}$$

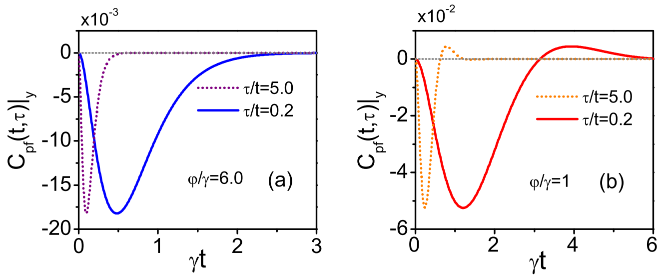

FIG. 2. (Dimensionless) CPF correlation [Eq. (36)] corresponding to the eternal non-Markovian trigonometric model [Eq. (23)] for different values of φ/γ and measurement time-interval relations τ/t. In all cases, the system initial condition is such that $\langle x\rangle = 0$.

These results allow us to analyze the transition to divergent rates [Eq. (26)] in a complementary way. In Fig. 2 we plot the CPF correlation (36). We observe that when the rate $\gamma_t^{\underline{\mathbf{c}}}$ does not develop divergences and is positive [Fig. 2(a)], the CPF correlation is negative for any value of the time intervals t and τ. Thus, even when criteria based on CP divisibility lead us to consider the dynamics to be "Markovian," the CPF method detects the presence of memory effects. In fact, they are induced by the underlying fluctuations of the incoherent degrees of freedom. On the other hand, in the interval $3-\sqrt{8} < (\varphi/\gamma) < 3+\sqrt{8}$, where the rate $\gamma_t^{\underline{\mathbf{c}}}$ develops divergences [Fig. 2(b)], the CPF correlation presents (finite) oscillations between positive and negative values. In this case, both memory criteria (negative rates and a nonvanishing CPF correlation) coincide.

For the model (23), the generalization to $N>1$, independent of the chosen observables, always leads to Eq. (36) or Eq. (37). This results follows by noting that in Eq. (34) the coefficients α and β assume only the four values $\alpha,\beta = (\mathbf{0},\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}})$ [see Eq. (24)]. Furthermore, using $H_{\underline{\mathbf{y}}\alpha}H_{\underline{\mathbf{y}}\beta} = H_{\underline{\mathbf{y}}\gamma}$, where γ corresponds to the string $S_\gamma = S_\alpha S_\beta$, for a fixed $\underline{\mathbf{y}}$ ($\underline{\mathbf{y}}=\underline{\mathbf{a}}$, $\underline{\mathbf{y}}=\underline{\mathbf{b}}$, or $\underline{\mathbf{y}}=\underline{\mathbf{c}}$) the four matrix elements $H_{\underline{\mathbf{y}}\gamma}$, similar to the case with $N=1$, can assume only the values (± 1), which always lead to Eq. (36) or Eq. (37). On the other hand, for $N>1$, accidentally, it may also happen that the CPF correlation vanishes. This occurs because we assumed that the incoherent degrees of freedom are stationary, which implies $\sum_{\alpha,\beta}[(1|\mathbb{F}_\alpha(\tau)\mathbb{F}_\beta(t)|q_0) - (1|\mathbb{F}_\alpha(\tau)|q_t)(1|\mathbb{F}_\beta(t)|q_0)] = 0$. Thus, when $H_{\underline{\mathbf{y}}\gamma} = 1$, it follows that $C_{pf}(t,\tau)|_y = 0$ [see Eq. (34)]. These accidental cases can always be surpassed by considering arbitrary measurement operators written as linear combinations of the Pauli strings.

As an example, we consider a bipartite case where $S_{\underline{\mathbf{a}}} = \sigma_1^x\sigma_2^x$, $S_{\underline{\mathbf{b}}} = \sigma_1^y\sigma_2^y$, and $S_{\underline{\mathbf{c}}} = \sigma_1^z\sigma_2^z$. The CPF correlation (36) is obtained when the three measurement are defined by any of the bipartite operators $S_{\underline{\mathbf{m}}} = (\sigma_1^x\sigma_2^z)$, $(\sigma_1^y\sigma_2^z)$, $(\sigma_1^z\sigma_2^x)$, $(\sigma_1^z\sigma_2^y)$; Eq. (37) is obtained when $S_{\underline{\mathbf{m}}} = (\sigma_1^x\sigma_2^y), (\sigma_1^y\sigma_2^x)$, while $C_{pf}(t,\tau)|_y = 0$ when $S_{\underline{\mathbf{m}}} = (\sigma_1^x\sigma_2^x)$, $(\sigma_1^y\sigma_2^y)$, $(\sigma_1^z\sigma_2^z)$.

V. SUMMARY AND CONCLUSIONS

We studied a class of solvable multipartite non-Markovian master equations in which the system consists of an arbitrary number of qubits and whose structure is written in terms of arbitrary multipartite Pauli coupling terms. Starting from a local-in-time representation of the evolution, we found the explicit solution for the system density matrix, which in turn allowed us to formulate the constraints that time-dependent rates must obey in order to guarantee the completely positive condition of the solution map.

We also found explicit analytical expressions for the time-dependent rates associated with a given evolution. Their sign (positive or negative) can be used as an indicator of non-Markovianity. Memory effects were also characterized by operational methods, where a CPF correlation defined by a set of three consecutive system measurements becomes a memory witness. We showed that this quantity can be obtained in an exact way for arbitrary measurement processes and arbitrary interaction with incoherent degrees of freedom.

As an application of the previous results, we presented simple underlying dynamics that lead to the phenomenon of eternal non-Markovianity, that is, multipartite dynamics in which some rates depart at all times from that of a Markovian regime. Both hyperbolic and trigonometric cases were established, characterized by a rate that is negative at all times or that develops periodical divergences. Even when these features develop, the CPF correlation is always a smooth function.

In the Appendixes we find the rates associated with different underlying memory mechanisms such as a mapping with a classical master equation, stochastic Hamiltonians, and statistical superpositions of Markovian dynamics. We show that under particular conditions different mechanisms may lead to the same time-dependent rates. Nevertheless, these accidental degeneracies do not occur in general. We also find that the phenomenon of eternal non-Markovianity becomes quite common in multipartite dynamics.

The class of models we studied here provides a useful solvable framework for studying quantum non-Markovianity in multipartite settings. This allows us to formulate a wide range of well-behaved multipartite non-Markovian master equations. The study of diverse memory witnesses can be tackled starting from here. Our results also lead to interesting questions such as determining which kind of underlying dynamics can be associated with an arbitrary non-Markovian multipartite Pauli evolution. Finally, our approach could be the starting point to studying other system operator algebras with more complex commutation relations.

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APPENDIX A: TIME-CONVOLUTED APPROACH

Instead of the local-in-time formulation defined by Eq. (2), alternatively, one may start with a time-convoluted evolution

$$\frac{d}{dt}\rho_t = \mathcal{L}[\rho_t] = \sum_{\substack{\mathbf{a} \\ \mathbf{a}\neq\mathbf{0}}} \int_0^t dt' k_{\mathbf{a}}(t-t')(S_{\mathbf{a}}\rho_{t'}S_{\mathbf{a}} - \rho_{t'}), \quad \text{(A1)}$$

where the set of time-dependent kernels $\{k_{\mathbf{a}}(t)\}$ must be constrained such that the solution map is CP. Like in Sec. II, by defining the kernel $k_{\mathbf{0}}(t) \equiv -\sum_{\mathbf{a}\,(\mathbf{a}\neq\mathbf{0})} k_{\mathbf{a}}(t)$, here, the weights of solution (12) can be written as

$$p_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}}\lambda_{\mathbf{b}}(t), \quad \text{(A2)}$$

where the coefficients $\lambda_{\mathbf{b}}(t)$ obey the evolution

$$\frac{d}{dt}\lambda_{\mathbf{b}}(t) = \int_0^t dt' k_{\mathbf{b}}(t-t')\lambda_{\mathbf{b}}(t'). \quad \text{(A3)}$$

The inverse relations for determining the kernels $\{k_{\mathbf{a}}(t)\}$ as a function of probabilities $\{p_{\mathbf{a}}(t)\}$ can be written in a Laplace domain $[f(z) = \int_0^\infty dt e^{-zt} f(t)]$ as

$$k_{\mathbf{a}}(z) = \frac{z\lambda_{\mathbf{a}}(z) - 1}{\lambda_{\mathbf{a}}(z)}, \quad \lambda_{\mathbf{a}}(z) = \sum_{\mathbf{b}} H_{\mathbf{ab}} p_{\mathbf{b}}(z). \quad \text{(A4)}$$

APPENDIX B: NON-MARKOVIAN UNDERLYING MECHANISMS

Here, we consider different mechanisms that lead to memory effects. The present analysis provides nontrivial multipartite extensions of some results developed in Ref. [20] for the case $N = 1$.

1. Mapping with a classical Markovian master equation

The solution map [Eq. (12)] is defined by a set of normalized probabilities $\{p_t^{\mathbf{a}}\}$. It is possible to formulate an underlying mechanism such that $\{p_t^{\mathbf{a}}\}$ correspond to the solution of an arbitrary Markovian classical master equation with 4^N different states.

We assume that the system density matrix interacts with an incoherent system whose states, in contrast to Eq. (18), can be put in one-to-one correspondence with the Pauli string vectors $\{\mathbf{a}\}$. Therefore, the system density matrix ρ_t can be written in terms of a set of auxiliary states $\{\rho_t^{\mathbf{a}}\}$ [17] such that

$$\rho_t = \sum_{\mathbf{a}} \rho_t^{\mathbf{a}}. \quad \text{(B1)}$$

The evolution of the auxiliary states is Markovian and involves coupling between all of them. We write

$$\frac{d}{dt}\rho_t^{\mathbf{a}} = -\sum_{\substack{\mathbf{b} \\ \mathbf{b}\neq\mathbf{a}}} \phi_{\mathbf{ba}}\rho_t^{\mathbf{a}} + \sum_{\substack{\mathbf{b} \\ \mathbf{b}\neq\mathbf{a}}} \phi_{\mathbf{ab}} S_{\mathbf{a}}S_{\mathbf{b}}\rho_t^{\mathbf{b}}S_{\mathbf{b}}S_{\mathbf{a}}. \quad \text{(B2)}$$

Here, $\{\phi_{\mathbf{ba}}\}$ are arbitrary rates. The stochastic interpretation of this equation is quite simple. Whenever the incoherent system undergoes the transition $\mathbf{b} \to \mathbf{a}$, the quantum system undergoes the transformation $\rho \to S_{\mathbf{a}}S_{\mathbf{b}}\rho S_{\mathbf{b}}S_{\mathbf{a}}$. Between transitions the system is frozen. The average system dynamics is given by Eq. (B2), where $\rho_t^{\mathbf{a}}$ corresponds to the conditional system state given that the incoherent one is in the state associated with $\mathbf{a}$.

It is simple to check that the solutions $\{\rho_t^{\mathbf{a}}\}$ of Eq. (B2) can be written as

$$\rho_t^{\mathbf{a}} = p_t^{\mathbf{a}}(S_{\mathbf{a}}\rho_0 S_{\mathbf{a}}), \quad \text{(B3)}$$

where the weights $p_t^{\mathbf{a}}$ must fulfill the classical master equation

$$\frac{d}{dt}p_t^{\mathbf{a}} = -\sum_{\substack{\mathbf{b}\\ \mathbf{b}\neq\mathbf{a}}}\phi_{\mathbf{ba}}p_t^{\mathbf{a}} + \sum_{\substack{\mathbf{b}\\ \mathbf{b}\neq\mathbf{a}}}\phi_{\mathbf{ab}}p_t^{\mathbf{b}}. \tag{B4}$$

Consequently, from Eqs. (B1) and (B3) we recover the solution Eq. (12) [$\rho_t = \sum_{\mathbf{a}} p_t^{\mathbf{a}}(S_{\mathbf{a}}\rho_0 S_{\mathbf{a}})$], where the probabilities $\{p_t^{\mathbf{a}}\}$ fulfill a the classical master equation (B4). For consistence, its initial condition must be $p_0^{\mathbf{a}} = \delta_{\mathbf{a},\mathbf{0}}$.

Particular case. Given that Eq. (B4) is arbitrary, it is not possible to find a general expression for the rates $\{\gamma_t^{\mathbf{a}}\}$ [Eq. (15)] in terms of the underlying ones $\{\phi_{\mathbf{ba}}\}$. Nevertheless, this mapping can be performed, for example, when Eq. (B4) assumes the form

$$\frac{d}{dt}p_t^{\mathbf{0}} = -\phi p_t^{\mathbf{0}} + \varphi\sum_{\substack{\mathbf{a}\\ \mathbf{a}\neq\mathbf{0}}}p_t^{\mathbf{a}}, \quad \frac{d}{dt}p_t^{\mathbf{a}} = -\varphi p_t^{\mathbf{a}} + x_{\mathbf{a}}\phi p_t^{\mathbf{0}}, \tag{B5}$$

where ϕ and φ are arbitrary rates and the weights $\{x_{\mathbf{a}}\}$ satisfy $\sum_{\mathbf{a}(\mathbf{a}\neq\mathbf{0})} x_{\mathbf{a}} = 1$. The probabilities, with initial condition $p_0^{\mathbf{a}} = \delta_{\mathbf{a},\mathbf{0}}$, can be written as

$$p_t^{\mathbf{a}} = p_\infty^{\mathbf{a}}[1-\exp(-\Phi t)] + \delta_{\mathbf{a},\mathbf{0}}\exp(-\Phi t), \tag{B6}$$

where $\Phi \equiv (\phi+\varphi)$ and the stationary values are

$$p_\infty^{\mathbf{0}} = \frac{\varphi}{\phi+\varphi}, \quad p_\infty^{\mathbf{a}} = \frac{x_{\mathbf{a}}\phi}{\phi+\varphi}. \tag{B7}$$

From the solutions (B6), the general expression (15), after some calculations steps [55], leads to

$$\gamma_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}}\frac{\Phi}{2}H_{\mathbf{ab}}\left[\tanh\left(\frac{t\Phi}{2}+\zeta_{\mathbf{b}}\right)-1\right], \tag{B8}$$

where the parameters are

$$\zeta_{\mathbf{b}} \equiv \frac{1}{2}\ln\left(\frac{h_\infty^{\mathbf{b}}}{1-h_\infty^{\mathbf{b}}}\right), \quad h_\infty^{\mathbf{b}} \equiv \sum_{\mathbf{c}}H_{\mathbf{bc}}p_\infty^{\mathbf{c}}. \tag{B9}$$

It is simple to check that, due to probability normalization, $h_\infty^{\mathbf{0}} = 1$. Hence, in Eq. (B8) the term with $\mathbf{b} = \mathbf{0}$ cancels out. Furthermore, if $h_\infty^{\mathbf{b}} = 0$, it follows that $\tanh(t\Phi/2+\zeta_{\mathbf{b}}) \to -1$. In general, the time dependence of the rate $\gamma_t^{\mathbf{a}}$ arises from a linear combination of hyperbolic tangent functions with coefficients that are ± 1. Thus, in general some rates can be negative at any time.

2. Stochastic Hamiltonians

We consider a stochastic evolution where the system wave vector $|\psi_t\rangle$ is driven by a stochastic Hamiltonian,

$$\frac{d|\psi_t\rangle}{dt} = -iH_{st}|\psi_t\rangle = -i\frac{1}{2}\xi_t^\alpha S_\alpha|\psi_t\rangle. \tag{B10}$$

The Hamiltonian H_{st} is characterized by a noise with an arbitrary statistics but null average $\langle\langle\xi_t^\alpha\rangle\rangle = 0$. The index $\alpha \leftrightarrow \alpha_t$ run overs all possible Pauli strings. Its time variation is very slow such that over a single realization it can be considered a frozen parameter. Thus, the average state $\rho_t^\alpha = \langle\langle|\psi_t\rangle\langle\psi_t|\rangle\rangle$ for a given α reads $\rho_t^\alpha = (1/2)[1+G_t^\alpha]\rho_0 + (1/2)[1-G_t^\alpha](S_\alpha\rho_0 S_\alpha)$, where

$$G_t^\alpha \equiv \left\langle\!\!\left\langle\exp\left(i\int_0^t dt'\xi_{t'}^\alpha\right)\right\rangle\!\!\right\rangle, \tag{B11}$$

is the characteristic noise function for a given α. After averaging this parameter, the system state can be written as $\rho_t = \sum_{\alpha,(\alpha\neq\mathbf{0})} x_\alpha\rho_t^\alpha$, where $\sum_{\alpha,(\alpha\neq\mathbf{0})} x_\alpha = 1$. The parameters $\{x_\alpha\}$ correspond to the statistical weight of each Pauli string during the variation of the coefficient α. It is straightforward to check that $\rho_t = \sum_{\mathbf{a}} p_t^{\mathbf{a}}(S_{\mathbf{a}}\rho_0 S_{\mathbf{a}})$, which recovers Eq. (12) with

$$p_t^{\mathbf{0}} = \frac{1}{2}\left(1+\sum_{\substack{\mathbf{a}\\ \mathbf{a}\neq\mathbf{0}}}x_{\mathbf{a}}G_t^{\mathbf{a}}\right), \quad p_t^{\mathbf{a}} = \frac{x_{\mathbf{a}}}{2}(1-G_t^{\mathbf{a}}). \tag{B12}$$

Like in the previous model, it is not possible to find a general simple expression for the rates $\gamma_t^{\mathbf{a}}$ in terms of these probabilities. Manageable expressions arise in the following situations.

Particular cases. If the noise is the same for all "directions" $G_t^{\mathbf{a}} = G_t$, from Eqs. (15) and (B12), after some algebra [55], we get the rates

$$\gamma_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}}\frac{\dot{g}_t}{2}H_{\mathbf{ab}}\left[\tanh\left(\frac{g_t}{2}+\zeta_{\mathbf{b}}\right)-1\right], \tag{B13}$$

where the scalar functions read

$$g_t = \ln(1/G_t), \quad \dot{g}_t = -\frac{1}{G_t}\frac{dG_t}{dt} \tag{B14}$$

and $\zeta_{\mathbf{b}}$ is defined by Eq. (B9) with, instead of Eq. (B7), $p_\infty^{\mathbf{0}} = 1/2$ and $p_\infty^{\mathbf{a}} = x_{\mathbf{a}}/2$.

For a stationary *Gaussian white noise*, where $\langle\langle\xi_t\xi_{t'}\rangle\rangle = \Phi\delta(t-t')$, Eq. ((B11)) becomes $G(t) = \exp(-\Phi t)$. It is simple to check that in this situation Eq. (B13) recovers the solution (B8) of the previous model with $\varphi = \phi$. These results show that there are different underlying models that may lead to the same system density-matrix evolution. This degeneracy is not universal and clearly depends on the underlying parameters.

For a stationary *symmetric dichotomic noise* with amplitude A and switching rate η, the characteristic noise function [Eq. (B11)] is

$$G_t = e^{-\eta t}\left[\cosh(\chi t) + \frac{\eta}{\chi}\sinh(\chi t)\right], \quad \chi \equiv \sqrt{\eta^2 - A^2}. \tag{B15}$$

In contrast to the previous cases, here, the rates defined by Eq. (B13) may develop divergences. In fact, the functions (B14) become

$$g_t = \ln(1/G_t), \quad \dot{g}_t = \frac{A^2}{\eta + \chi[1/\tanh(\chi t)]}. \tag{B16}$$

Hence, divergent rates are found whenever $\eta < A$.

3. Statistical mixtures of Markovian evolutions

Departures with respect to a Markovian regime emerge whenever the system evolution is written as the statistical superposition of different Markovian propagators. Hence, we write

$$p_t^{\mathbf{a}} = \sum_{k=1}^{n} q_k p_k^{\mathbf{a}}(t), \tag{B17}$$

where $\{q_k\}$ are normalized positive weights ($\sum_{k=1}^n q_k = 1$) and each set of probabilities $\{p_k^{\mathbf{a}}(t)\}$ is associated with a Markovian solution of Eq. (2) with time-independent positive rates $\{\gamma_k^{\mathbf{a}}\}$.

From Eq. (15), the non-Markovian evolution is characterized by the rates

$$\gamma_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}} \frac{\sum_{k=1}^n q_k \mu_k^{\mathbf{b}} \exp\left(t\mu_k^{\mathbf{b}}\right)}{\sum_{k'=1}^n q_{k'} \exp\left(t\mu_{k'}^{\mathbf{b}}\right)}, \tag{B18}$$

where $\mu_k^{\mathbf{b}}$ are eigenvalues of the k-Markovian dynamics, $\mu_k^{\mathbf{b}} = \sum_{\mathbf{c}} H_{\mathbf{bc}}\gamma_k^{\mathbf{c}}$. The specific properties of these rates strongly depend on the considered Markovian evolutions and statistic weights.

Particular cases. In the *two-state case*, $n = 2$, the probabilities are $p_t^{\mathbf{a}} = q_1 p_1^{\mathbf{a}}(t) + q_2 p_2^{\mathbf{a}}(t)$, where each solution is associated with the rates $\gamma_1^{\mathbf{a}}$ and $\gamma_2^{\mathbf{a}}$ and $q_1 + q_2 = 1$. From Eq. (B18), after some algebra [55], we get

$$\gamma_t^{\mathbf{a}} = \frac{1}{2}\left(\gamma_1^{\mathbf{a}} + \gamma_2^{\mathbf{a}}\right) + \frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}}\Delta_{\mathbf{b}} \tanh(t\Delta_{\mathbf{b}} + \zeta), \tag{B19}$$

where the parameters are

$$\Delta_{\mathbf{b}} \equiv \frac{1}{2}\sum_{\mathbf{c}} H_{\mathbf{bc}}\left(\gamma_1^{\mathbf{c}} - \gamma_2^{\mathbf{c}}\right), \quad \zeta \equiv \frac{1}{2}\ln\left(\frac{q_1}{q_2}\right). \tag{B20}$$

In this case, many rates may also be negative at all times (see particular cases in Appendix C).

In the other extreme, a *continuous-state case* can be considered. Thus, Eq. (B17) is rewritten as

$$p_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}}\left\langle \prod_{\mathbf{c}} \exp(tH_{\mathbf{bc}}\gamma^{\mathbf{c}})\right\rangle, \tag{B21}$$

where we used the explicit expression (13) and the replacement $\sum_{k=1}^n q_k \to \langle\cdots\rangle$. The symbol $\langle\cdots\rangle$ denotes an average over the set of random rates $\{\gamma^{\mathbf{c}}\}$, with each "realization" defining a Markov solution. Assuming that all rates are independent random variables, it follows that $\langle\cdots\rangle \to \int_0^\infty d\gamma^{\mathbf{c}} \cdots P(\gamma^{\mathbf{c}})$, where $P(\gamma^{\mathbf{c}})$ is the corresponding probability density. By assuming an *exponential probability density* $P(\gamma^{\mathbf{c}}) = \tau_{\mathbf{c}} \exp(-\gamma^{\mathbf{c}}\tau_{\mathbf{c}})$, by using $\gamma^{\mathbf{0}} = -\sum_{\mathbf{c}(\mathbf{c}\neq\mathbf{0})}\gamma^{\mathbf{c}}$ [see Eq. (7)], we get

$$p_t^{\mathbf{a}} = \frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}} \prod_{\substack{\mathbf{c} \\ \mathbf{c}\neq\mathbf{0}}} \frac{\tau_{\mathbf{c}}}{\tau_{\mathbf{c}} + (1 - H_{\mathbf{bc}})t}, \tag{B22}$$

where we have used $H_{\mathbf{b0}} = 1$. From Eq. (15), the corresponding rates associated with the non-Markovian evolution are

$$\gamma_t^{\mathbf{a}} = -\frac{1}{4^N}\sum_{\mathbf{b}} H_{\mathbf{ab}} \sum_{\substack{\mathbf{c} \\ \mathbf{c}\neq\mathbf{0}}} \frac{(1 - H_{\mathbf{bc}})}{\tau_{\mathbf{c}} + (1 - H_{\mathbf{bc}})t}. \tag{B23}$$

We notice that both $\{p_t^{\mathbf{a}}\}$ and $\{\gamma_t^{\mathbf{a}}\}$ develop a power-law behavior. In spite of this feature the rates are positive at all times, $\gamma_t^{\mathbf{a}} > 0$ ($\mathbf{a} \neq \mathbf{0}$). While most of the memory witnesses [5,6] associate this property with a Markovian regime, from operational approaches it is possible to detect and infer the presence of memory effects [48,49].

APPENDIX C: BIPARTITE AND TRIPARTITE ETERNAL NON-MARKOVIAN EVOLUTIONS

In addition to the previous examples, the developed approach allows us to show that master equations characterized by eternal non-Markovian effects are quite common for multipartite systems. As an example, we consider the statistical superposition of two different Markovian dynamics characterized by the rates $\gamma_1^{\mathbf{a}}$ and $\gamma_2^{\mathbf{a}}$ and equal weights [$q_1 = q_2$ in Eq. (B19)]. Taking $\gamma_1^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},\underline{\mathbf{a}}} - \delta_{\mathbf{a},\mathbf{0}})$ and $\gamma_2^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},\underline{\mathbf{b}}} - \delta_{\mathbf{a},\mathbf{0}})$ and using $(H_{\alpha\underline{\mathbf{a}}} - H_{\alpha\underline{\mathbf{b}}})/2 = (\pm 1, 0)$ and $H_{\alpha\underline{\mathbf{a}}}H_{\alpha\underline{\mathbf{b}}} = H_{\alpha\underline{\mathbf{c}}}$, from Eq. (B19) we recover the rates defined in Eq. (22). When each (vectorial) rate involves different Pauli channels, more complex expressions are obtained.

As a first example, take a *bipartite* system ($N = 2$) with

$$\gamma_1^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},10} + \delta_{\mathbf{a},01} - 2\delta_{\mathbf{a},00}), \tag{C1a}$$

$$\gamma_2^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},20} + \delta_{\mathbf{a},02} - 2\delta_{\mathbf{a},00}). \tag{C1b}$$

Thus, each dynamics is defined by a local (single) dephasing mechanism acting alternatively in the x and y directions. From Eq. (B19) we obtain

$$\gamma_t^{\mathbf{a}_0} = \tfrac{1}{2}\gamma, \quad \gamma_t^{\mathbf{a}_\pm} = \pm\tfrac{1}{4}\gamma\tanh(2\gamma t), \tag{C2}$$

where $\mathbf{a}_0$ and $\mathbf{a}_\pm$ correspond to the following Pauli strings: $\mathbf{a}_0 = (10), (01), (20), (02)$, $\mathbf{a}_+ = (11), (22)$, and $\mathbf{a}_- = (30), (03), (12), (21)$. Furthermore,

$$\gamma_t^{33} = -\frac{\gamma}{4}[2\tanh(\gamma t) - \tanh(2\gamma t)] = -2\gamma\frac{\sinh^4(\gamma t)}{\sinh(4\gamma t)},$$

while $\gamma_t^{\mathbf{a}} = 0$ if $\mathbf{a} \neq (\mathbf{a}_0, \mathbf{a}_+, \mathbf{a}_-)$. There are 11 non-null rates out of the 15 possible ones, with 5 of them being negative at all times.

As a second example we consider a *tripartite* system ($N = 3$), where

$$\gamma_1^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},110} + \delta_{\mathbf{a},101} + \delta_{\mathbf{a},011} - 3\delta_{\mathbf{a},000}), \tag{C3a}$$

$$\gamma_2^{\mathbf{a}} = \gamma(\delta_{\mathbf{a},220} + \delta_{\mathbf{a},202} + \delta_{\mathbf{a},022} - 3\delta_{\mathbf{a},000}). \tag{C3b}$$

Hence, each Markovian evolution corresponds to dephasing in the x and y directions but now considers all pairs of bipartite dephasing operators. From Eq. (B19) we get

$$\gamma_t^{\mathbf{a}_+} = \frac{1}{4}\gamma[2 + \tanh(2\gamma t)], \tag{C4a}$$

$$\gamma_t^{\mathbf{a}_-} = -\frac{1}{4}\gamma\tanh(2\gamma t), \tag{C4b}$$

where $\mathbf{a}_\pm$ correspond to the following Pauli strings: $\mathbf{a}_+ = (110), (101), (011), (220), (202), (022)$ and $\mathbf{a}_- = (330), (303), (033), (123), (132), (213), (231), (312), (321)$, where $\gamma_t^{\mathbf{a}} = 0$ if $\mathbf{a} \neq \mathbf{a}_+, \mathbf{a}_-$. In this case, out of 63 possible rates, 15 are non-null, with 9 of them being negative at all times.

APPENDIX D: CPF CORRELATION CALCULUS

For a system coupled to incoherent degrees of freedom [Eq. (18)], the (bipartite) system-environment state $\rho_t^{se} =$

$\sum_{\mathbf{h}} \rho_t^{\mathbf{h}}|\mathbf{h})$, from Eqs. (19) and (33), reads

$$\rho_t^{se} = \sum_{\alpha} (S_\alpha \rho_0 S_\alpha)\, \mathbb{F}_\alpha(t)|q_0). \tag{D1}$$

This evolution defines the system-environment dynamics between measurements. The measurement of operator $S_{\underline{\mathbf{m}}}$ [Eq. (31)] leads to the transformation $\rho^{se} = \sum_{\mathbf{h}} \rho^{\mathbf{h}}|\mathbf{h}) \to |m\rangle\langle m||q_m)$, where $|q_m) = \sum_{\mathbf{h}} \langle m|\rho_t^{\mathbf{h}}|m\rangle/\mathrm{Tr}[\langle m|\rho_t^{\mathbf{h}}|m\rangle]|\mathbf{h})$. With these ingredients, the calculation of the joint probability can be performed in a standard way. We get

$$\frac{P(z,y,x)}{P(x)} = \sum_{\alpha,\beta} |\langle z|\sigma_\alpha|y\rangle|^2 |\langle y|S_\beta|x\rangle|^2 (1|\mathbb{F}_\alpha(\tau)\mathbb{F}_\beta(t)|q_0), \tag{D2}$$

where $P(x) = \langle x|\rho_0|x\rangle$ and $(1| \equiv \sum_{\mathbf{h}} (\mathbf{h}|$. This result is valid for arbitrary Hermitian system observables.

Using the Bayes rule, the conditional probabilities that define the CPF correlation [Eq. (32)] can be written as $P(z,x|y) = P(z,y,x)/P(y)$, where $P(y) = \sum_{z,x} P(z,y,x)$. Furthermore, $P(z|y) = \sum_x P(z,x|y)$, and $P(x|y) = \sum_z P(z,x|y)$. From Eq. (D2), using

$$\sum_z z|\langle z|S_\alpha|y\rangle|^2 = \langle y|S_\alpha S_{\underline{\mathbf{z}}} S_\alpha|y\rangle, \tag{D3a}$$

$$\sum_x x|\langle y|S_\beta|x\rangle|^2 P(x) = \langle y|S_\beta S_{\underline{\mathbf{x}}} \rho_{\underline{\mathbf{x}}} S_\beta|y\rangle, \tag{D3b}$$

$$\sum_x |\langle y|S_\beta|x\rangle|^2 P(x) = \langle y|S_\beta \rho_{\underline{\mathbf{x}}} S_\beta|y\rangle, \tag{D3c}$$

where the system state $\rho_{\underline{\mathbf{x}}}$ is

$$\rho_{\underline{\mathbf{x}}} \equiv \sum_x P(x)|x\rangle\langle x| = \sum_x \langle x|\rho_0|x\rangle|x\rangle\langle x|, \tag{D4}$$

the CPF correlation can be written as

$$C_{pf}(t,\tau)|_y = \frac{1}{P(y)^2} \sum_{\alpha,\beta,\gamma} \Theta^{\alpha\beta\gamma}|_y \Lambda_{\alpha\beta\gamma}(t,\tau). \tag{D5}$$

The coefficients $\Theta^{\alpha\beta\gamma}|_y$ are

$$\Theta^{\alpha\beta\gamma}|_y = \langle y|S_\alpha S_{\underline{\mathbf{z}}} S_\alpha|y\rangle\langle y|S_\beta S_{\underline{\mathbf{x}}}\rho_{\underline{\mathbf{x}}} S_\beta|y\rangle\langle y|\sigma_\gamma \rho_{\underline{\mathbf{x}}} S_\gamma|y\rangle,$$

while the time dependence follows from

$$\Lambda_{\alpha\beta\gamma}(t,\tau) = +(1|\mathbb{F}_\alpha(\tau)\mathbb{F}_\beta(t)|q_0)(1|\mathbb{F}_\gamma(t)|q_0) - (1|\mathbb{F}_\alpha(\tau)\mathbb{F}_\gamma(t)|q_0)(1|\mathbb{F}_\beta(t)|q_0),$$

where $|q_t) = \sum_\alpha \mathbb{F}_\alpha(t)|q_0)$ and the probability $P(y)$ is

$$P(y) = \sum_\alpha (1|\mathbb{F}_\alpha(t)|q_0)\langle y|S_\alpha \rho_{\underline{\mathbf{x}}} S_\alpha|y\rangle. \tag{D6}$$

Expression (D5) is valid for arbitrary observables $\sigma_{\underline{\mathbf{m}}}$ [Eq. (31)]. In general, they can be written as linear combinations of Pauli strings $S_{\mathbf{a}}$. Assuming, for simplicity, that each $S_{\underline{\mathbf{m}}}$ corresponds to a unique Pauli string operator, from Eq. (5) it follows that

$$\langle y|S_\alpha S_{\underline{\mathbf{z}}}\sigma_\alpha|y\rangle = H_{\alpha\underline{\mathbf{y}}}\delta_{\underline{\mathbf{z}},\underline{\mathbf{y}}} a_y, \tag{D7a}$$

$$\langle y|S_\beta S_{\underline{\mathbf{x}}}\rho_{\underline{\mathbf{x}}} S_\beta|y\rangle = \frac{1}{2^N}(H_{\beta\underline{\mathbf{y}}}\delta_{\underline{\mathbf{y}},\underline{\mathbf{x}}} a_y + \langle x\rangle), \tag{D7b}$$

$$\langle y|S_\gamma \rho_{\underline{\mathbf{x}}}\sigma_\gamma|y\rangle = \frac{1}{2^N}(1 + H_{\gamma\underline{\mathbf{y}}}\delta_{\underline{\mathbf{y}},\underline{\mathbf{x}}} a_y\langle x\rangle), \tag{D7c}$$

where $\langle x\rangle \equiv \mathrm{Tr}[S_{\underline{\mathbf{x}}}\rho_{\underline{\mathbf{x}}}]$. By introducing these equalities in Eq. (D5), after some algebra we get Eq. (34). Generalization to arbitrary observables can be worked out in a similar way from Eq. (D5).

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