

Unobservable causal loops as a way to explain both the quantum computational speedup and quantum nonlocality

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We consider the reversible processes between two one-to-one correlated measurement outcomes which underlie both problem solving and quantum nonlocality. In the former case, the two outcomes are the setting and the solution of the problem; in the latter, they are those of measuring a pair of maximally entangled observables whose subsystems are space separate. We argue that the quantum description of these processes mathematically describes the correlation but leaves the causal structure that physically ensures it free of violating the time symmetry required of the description of a reversible process. It would therefore be incomplete and could be completed by time symmetrizing it. This is done by assuming that the two measurements evenly contribute to selecting the pair of correlated measurement outcomes. Time symmetrization leaves the ordinary quantum description unaltered but shows that it is the quantum superposition of unobservable time-symmetrization instances whose causal structure is completely defined. Each instance is a causal loop: Causation goes from the initial to the final measurement outcome and then back from the final to the initial outcome. In the speedup, all is as if the problem solver knew in advance half of the information about the solution she will produce in the future and could use this knowledge to produce the solution with fewer computation steps. In nonlocality, the measurement on either subsystem retrocausally and locally changes the state of both subsystems when the two were not yet spatially separate. This locally causes the correlation between the two future measurement outcomes.

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I. INTRODUCTION

The present work is a further step of the evolutionary approach [1–4]. In it, with others, we developed a retrocausal explanation of the quantum computational speedup still *ad hoc* for quantum computation. In the present work, we detach it from quantum computation and show that it derives from a more universal property of quantum correlation that also explains quantum nonlocality.

The potentially paradoxical character of the approach is clear from the abstract. The fact that the problem solver could benefit from the advanced knowledge of half of the information about the solution¹ she will produce and read in the future to produce the solution more efficiently is reminiscent of the inventor of the time machine sending back in time to herself, before she had the idea, the design of the machine. Although the latter is a favored theme of novels and movies, from a physical standpoint it is obviously nonsense. The present explanation of the quantum speedup is, so to speak, half of that nonsense. So, is it still nonsense or is it physically permissible? The present work deals with this question.

By the way, as quantum computation is still a specialized field and this work aims to be interdisciplinary, let us provide

an example of quantum computational speedup, that of the simplest instance of Grover quantum search algorithm [5].

Bob, the problem setter, hides a ball in a chest of four drawers. Alice, the problem solver, is to identify the location of the ball by opening drawers. In the classical and worst case, to solve the problem Alice has to open three drawers: If the last drawer opened contains the ball, she has solved the problem; if not, it must be in the only drawer not yet opened and she has solved the problem as well. In the quantum case, i.e., by Grover algorithm, she always identifies the location of the ball by opening just one drawer. There is a quantum computational speedup. Also this might appear paradoxical, of course.

At the level of feeling, saying that, in the quantum case, Alice opens just one drawer but in a quantum superposition of the four drawer numbers might mitigate that impression. In the following, we will try to put things together in an exact way.

Let us start from an only apparently remote lead. It is, of course, well known that Einstein, with Podolsky and Rosen [6], stated that the quantum description is incomplete because (in particular) it does not physically explain quantum nonlocality, namely the *spooky action at a distance*. By the way, this particular case was the real concern of Einstein and is the one of interest here.

In this work, we revamp that incompleteness issue starting from another lead. We consider the quantum description of quantum correlation (forgive the pun). We mean the reversible process between two one-to-one correlated measurement

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outcomes. Let us show in the first place that this process is common to the quantum speedup and quantum nonlocality.

In the speedup, the initial and final measurement outcomes are respectively the outcome of the initial measurement, which selects the problem setting out of a uniform quantum superposition of all the possible problem settings, and the corresponding solution of the problem selected by the final measurement (in the four-drawer example, they are both the number of the drawer with the ball). The reversible process between the two outcomes is, of course, the quantum algorithm.

In nonlocality, the two outcomes are those of measuring two maximally entangled observables whose subsystems are space separate.

By the way, note that in both cases that the process between the two measurement outcomes is reversible in spite of comprising the final measurement (which does not change the quantum state). In fact, there is a unitary transformation between the two one-to-one correlated outcomes.

Now, our thesis is that the *ordinary quantum description*² of quantum correlation is incomplete because it mathematically describes the correlation between the two measurement outcomes but not the causal structure that physically ensures it. Because of the reversibility of the quantum process, the process causal structure would be left free by the ordinary quantum description and also free of violating the time symmetry required of the description of a reversible process. This would say, first, that this description is incomplete and, second, that it is completed by time symmetrizing it. A basic assumption is that this requires evenly and nonredundantly sharing between the initial and the final measurements the selection of the information that specifies the sorted out pair of correlated measurement outcomes among all the possible pairs. This is, of course, an unorthodox assumption, but we will justify it in a detailed way in the next section. We should proceed as follows.

The initial measurement selects one of the possible halves of the information in question. The respective measurement outcome unitarily propagates toward the final measurement until becoming the state immediately before it. The latter measurement selects the remaining half of the information. The respective measurement outcome unitarily propagates toward the initial measurement, by the inverse of the previous unitary transformation, until becoming its definitive outcome. The latter propagation, which inherits both selections, is an instance of the time-symmetrized quantum process. Note that the causal structure of each time-symmetrization instance is completely defined and is a causal loop. Causality goes from the initial to the final measurement outcome and then back from the final to the initial measurement outcome. Since there is a plurality of possible instances, each corresponding to a way of taking half of the information in question, we should take their quantum superposition.

We will see that this time symmetrization leaves the ordinary quantum description unaltered, as it should be since

the description of a reversible process should be time symmetric to begin with. But, at the same time, it shows that this description is a quantum superposition of unobservable time-symmetrization instances whose causal structure is completely defined. These instances are unobservables because, of course, they vanish in their quantum superposition, namely in the ordinary quantum description (which leaves the causal structure that ensures the correlation free).

For the fact of describing the causal structure that physically ensures the correlation between the two measurement outcomes, with respect to the ordinary quantum description which leaves this causal structure free, they describe the quantum process in a more complete way: a way that immediately explains the quantum computational speedup and quantum nonlocality.

We will see that, in the speedup case, the outcome of the initial measurement encodes the problem to be solved by Alice and Alice's knowledge of the solution. The projection of the quantum state associated with the final measurement that selects the remaining half of the information, retrocausally, by the inverse of the time-forward unitary transformation, changes it. This change reduces the computational complexity of the problem to be solved and correspondingly increases Alice's knowledge of the solution. The latter goes from complete ignorance of the solution to knowledge of half of the information that specifies it (among all the possible solutions). Therefore, everything is as if Alice knew in advance half of the information about the solution she will produce and read in the future and could use this knowledge to produce the solution more efficiently than in the classical case. Of course, one can see the causal loop.

The explanation in question is quantitative in character, given an oracle problem, it allows us to compute the number of oracle queries needed to solve it in an optimal quantum way.³ It is the minimum number logically (classically) needed to find the solution given the advanced knowledge of half of the information that specifies it.

By the way, let us note that this is a completely technical consequence of a fundamental interpretation of quantum mechanics. It is a *synthetic* solution, i.e., axiomatically derived from fundamental principles, of the so-called *quantum query complexity problem*. It is the problem of finding whether the number of oracle queries needed to solve an oracle problem (forgive the pun) in an optimal quantum way is quadratic or exponential (or the like) in problem size. It is central to quantum computer science and still unsolved in the general case [7,8]. The fundamental principle presently at play is that the causal structure of a time-reversible process does not introduce a preferred time direction of causality.

In the nonlocality case, the time symmetrization of the quantum process between the two measurement outcomes revamps, with an improvement, the retrocausal explanation of quantum nonlocality given by Costa de Beauregard in 1953 [9].

²We call the usual quantum description *ordinary* to distinguish it from the complete quantum description that we are going to propose.

³We will explain in the technical part of the work what oracle problems and oracle queries are. For the time being, we can see them just as problems and computation steps.

Let us lay the situation down. Say that at time t_0 two photons are generated by parametric down-conversion in the state of maximally entangled polarizations and diverging momenta. At time t_1 , after their spatial separation, we measure the polarization of one photon. Of course, this instantly changes the polarization state also of the space-separate photon. There is the spooky action at a distance. At time t_2 , we measure, in the same basis, the polarization of the other photon. There is one-to-one correlation between the two measurement outcomes.

In the first place, Costa de Beauregard noted that the unitary transformation that connects the two measurement outcomes, in a mathematically equivalent way, can go from t_1 to t_2 either directly or via t_0 , when the two photons are not yet space separate. In the latter case, the unitary transformation goes first backward in time from t_1 to t_0 and then forward in time from t_0 to t_1 and eventually t_2 . In the former case, there is action at a distance, which is certainly unphysical. In the latter, everything is local and therefore physical provided that we consider retrocausality physical too.

His explanation assumes that, in causal order, the initial measurement only projects the reduced density operator of the polarization of the respective photon, leaving that of the space-separate photon unaltered. Then this projection, on its way toward the final measurement via t_0 , at time t_0 locally projects the polarization state of both photons. The resulting state, still on its way toward the final measurement, at the time of the initial measurement emulates action at a distance. In other words, the action from the first measurement outcome to the polarization state of the space-separate photon would be local, first through the unitary transformation from t_1 to t_0 and then through the unitary transformation from t_0 to t_1 . But the sum of the two times, one negative and the other positive, is zero; this is what emulates action at a distance.

According to the present approach, this explanation would have two drawbacks:

(i) Since the initial measurement alone selects the sorted out pair of measurement outcomes, it violates the time symmetry required of the reversible process between the two outcomes.

(ii) Correspondingly, it involves the (retrocausal) change of a past state of the ordinary quantum description, namely the state of maximal polarization entanglement of the two not yet space-separate photons.

We go now to the explanation provided by the present work. We should complete the ordinary quantum description of the reversible process that connects the two measurement outcomes via t_0 by time symmetrizing it. In each time-symmetrization instance, in causal order, the projection of the quantum state due to either measurement (which selects one of the two halves of the information about the outcomes pair) only projects the reduced density operator of the polarization of the photon on which the measurement is performed, leaving that of the other photon unaltered. Then this projection, going toward the other measurement via t_0 , at time t_0 locally projects the polarization state of both photons. As we will see, together, these two projections (one for each measurement) cause the one-to-one correlation between the two measurement outcomes.

The differences with respect to Costa de Beauregard explanation are as follows:

(i) There are no violations of the time symmetry required of the description of the reversible process between the two measurement outcomes.

(ii) Correspondingly, no past quantum state of the ordinary quantum description is changed.

The Costa de Beauregard explanation would be unphysical and the present explanation would be physical, although the basic idea is the same.

Another difference, of course, is that the present explanation is provided by the completion of the quantum description of quantum correlation, which also explains the quantum speedup. Costa de Beauregard's explanation is *ad hoc* for quantum nonlocality.

Note that the causal loop implicit in the time-symmetrization procedure this time generates two causal loops, one for each measurement. Each loop is the measurement that retrocausally changes the state of the two not yet space-separate subsystems in a way that in turn contributes to causing the correlation between the two measurement outcomes.

By the way, all the above would seem to answer Einstein *et al.*'s call in [6]. It shows (i) why the ordinary quantum description is incomplete (it would be so because it allows causal structures that violate the time symmetry required of a reversible process) and (ii) how to complete it (by time symmetrizing it). Our hidden (unobservable) variables, which complete the ordinary quantum description by telling the two photons, when they are not yet spatially separate, how to behave in the future measurements on them (as Einstein and the others required), are the time directions of the causations between the two measurement outcomes in the unobservable time-symmetrization instances.

While all the above would seem to exactly answer Einstein's expectations, it does so in a way that likely was not imagined at the time. It is, of course, an unorthodox way also at the present time. However, it benefits, so to speak, two paradigm shifts well successive to Einstein *et al.*'s time: time-symmetric quantum mechanics [10–16] and quantum computation [17], with the respective speedup [18,19]. By the way, the present unorthodox approach, in particular, has been inspired by Ref. [12] about the nonsequential behavior of the wave function, which we call here the ordinary quantum description.

We should eventually note that the quantum computational speedup and quantum nonlocality are the two known tasks that cannot be performed classically. By definition, the speedup is with respect to the classical algorithms; in the case of quantum nonlocality, we are dealing with quantum cryptography based on Bell's theorem [20], whose security has no classical parallel. This work shows that they both rely on quantum causal loops. This suggests the conjecture that the present unobservable form of quantum retrocausality has to do with the very difference between quantum and classical mechanics.

Given the unconventional character of the work, before getting lost in the formalizations, we provide in the next section a very detailed line of reasoning. It goes from the diagnosis of the incompleteness of the ordinary description of quantum correlation to the prescription of the way of completing it. In particular, it comprises our best justification of all the unorthodox assumptions taken in the present work.

II. LINE OF REASONING

We provide our line of reasoning, segmented as a numbered sequence of consequential steps. Since it is an unorthodox line of reasoning that goes against conventions, we have done our best to be precise.

(i) Our starting argument is that the ordinary quantum description of the reversible processes between two one-to-one correlated measurement outcomes mathematically describes the correlation but leaves the causal structure that physically ensures it free.

Naturally, this goes against the convention that causality only goes forward in time. Our justification for doing without this convention in the case of a reversible quantum process is twofold. On the one side, the convention in question is not implied by the dynamic equations of the reversible quantum process but is an add-on to them. On the other, it is a convention that clashes with the very definition of reversible process. Let us recall it: A deterministic process is time reversible if the time-reversed process satisfies the same dynamic equations as the original process. Of course, in the time-reversed process, causality goes backward in time, hence the clash.

(ii) The first critical step of the present line of reasoning is doing without that convention. Without it, the ordinary quantum description (the dynamic equations) does not provide any information about the causal structure that ensures the correlation between the two measurement outcome. It leaves this causal structure free.

In the first place, the initial measurement outcome can be considered the cause of the final measurement outcome or, time symmetrically, the final measurement outcome the cause of the initial one. We should keep in mind that between the two one-to-one correlated measurement outcomes there is a unitary transformation.

We note that, until now, choosing either time direction of causality does not add any information to the ordinary quantum description. It is like reading it from left to right or right to left. The information about the quantum process is the same in either case.

What is important is that, once accepted that causality can also go backward in time, in the quantum world the ambiguity about the time direction of causality can go down to a deeper level. Causality can go forward or backward in time between any two corresponding (correlated) parts of the two measurement outcomes and, independently, forward or backward in time between the two complementary parts.

Let us exemplify this in the four-drawer instance of Grover algorithm. Say that the number of the drawer with the ball selected by the initial measurement (out of the quantum superposition of the four drawer numbers) in binary notation is 01. The corresponding solution, selected by the final measurement, is naturally 01 too. The sorted-out pair of correlated measurement outcomes is thus (01, 01).

Under the convention that causality only goes forward in time, the initial measurement selects the number 01 and this causes the number 01 “selected” (just read) by the final measurement. As we will soon see, without that convention, the ordinary quantum description allows to assume a different picture. The left digit of the initial measurement outcome can be the cause of the left digit of the final measurement outcome

and the right digit of the final measurement outcome can be the cause of the right digit of the initial measurement outcome. Correspondingly, we should share the selection of the sorted-out pair of correlated measurement outcomes between the initial and final measurements. The initial measurement selects the left digit of the pair and the final measurement the right one.

Note that the ordinary quantum description leaves us free to choose which amount of information is selected by the initial measurement and which by the final measurement. The information that specifies this choice, among all the possible choices, is information about the quantum process that the ordinary quantum description, which leaves this choice free, does not provide. Once accepted the fact that causality in the quantum world can also go backward in time, this fact alone shows that the ordinary quantum description is incomplete. There is information about the quantum process that it does not provide.

(iii) The second critical step of the present line of reasoning, allowed by the previous one, is asking that the causal structure that ensures the correlation between the two measurement outcomes does not introduce a preferred time direction of causality. In other words, we ask that the initial and final measurements evenly (and of course nonredundantly) contribute to select the pair of correlated measurement outcomes among all the possible pairs. In equivalent terms, we require that each measurement selects one of the possible halves of the information that specifies the pair of outcomes among all the possible pairs and the other measurement selects the other half. Since there is a plurality of ways of taking half of the information, we should take their quantum superposition. At point (vii), we will see how to put all things together.

This would seem to be the natural thing to do since we are dealing with a time-reversible process. However, we can justify it in a stronger way, by comparing the case that the two measurements are one after the other with the case that they are simultaneous.

First, we note that it does not matter whether we perform the two measurements simultaneously or not. In the nonlocality case, we can postpone the initial measurement to the time of the final measurement, or anticipate the latter to the time of the initial measurement. In the case of the speedup, the initial measurement that selects the problem setting (e.g., the number of the drawer with the ball) can be postponed to the time of the final measurement (also here the respective observables commute). In this case, Alice performs the algorithm in a quantum superposition of all the possible problem settings. Then, simultaneously, Bob’s measurement selects the problem setting and Alice’s measurement the corresponding solution. In either the nonlocality or the speedup case, everything evidently remains the same: action at a distance and respectively the maximum speedup achievable by the quantum algorithm.

Now, in the case that the two measurements are simultaneous, the assumption that they evenly contribute to the selection of the sorted-out pair of outcomes becomes mandatory. As we will see in the formalizations, in both the speedup and the nonlocality case we are dealing with two measurements that simultaneously select two maximally entangled observables. There is a perfect symmetry between them and no reason for which they should differently contribute to the selection.

In order that everything remains the same when the two measurements are not simultaneous, one has to keep the causal structure of the simultaneous case. In other words, when made subsequent to each other, the two measurements should continue to evenly contribute to the selection of the outcome pair. In equivalent terms, we should require that the causal structure that physically ensures the correlation between the two measurement outcomes does not introduce a preferred time direction of causality.

(iv) The adoption of this requirement implies that the causal structures allowed by the ordinary quantum description can violate it. Of course, with the causal structure free, the amount of information about the outcomes pair selected by one measurement can be different from that selected by the other measurement. This has in turn the two following consequences.

(v) The ordinary quantum description of the processes in question is incomplete because it allows causal structures that violate the no preferred time direction of causality requirement.

(vi) To complete the ordinary quantum description, it is sufficient to time symmetrize it.

(vii) All the above already define how to time symmetrize the ordinary quantum description of the processes in question. We have already anticipated this in the introduction. We repeat it here in the example of the four-drawer problem.

The initial measurement, performed in a quantum superposition the four possible numbers of the drawer with the ball (*numbers* for short), selects one of the possible halves of the information that specifies the number (among the four possible numbers). For example, it selects the value 0 of the left digit of the number. The corresponding measurement outcome is therefore the superposition of the numbers 00 and 01. This superposition unitarily propagates toward the final measurement until becoming the state immediately before it (as a superposition of two tensor products, each a number beginning with 0 and the corresponding solution—that same number). The latter measurement selects the remaining half of the information (say the value 1 of the right digit, so that the number selected by the two measurements is 01). The respective measurement outcome unitarily propagates toward the initial measurement, by the inverse of the previous unitary transformation, until becoming its definitive outcome. The latter propagation, which inherits both selections, is an instance of the time-symmetrized quantum process.

Note that the causal structure of each time-symmetrization instance is completely defined and is a causal loop: Causation goes from the initial to the final measurement outcome and then back from the final to the initial outcome. Since there is a plurality of possible instances, each corresponding to a way of taking half of the information in question, we should take their quantum superposition.

By the way, we should note that the causal structure of these instances, whose formation essentially relies on quantum superpositions, would not be possible in a classical version of the algorithm, even under the idealization that there are reversible classical processes. We could not divide the evolution of the two-digit number into two evolutions, one for the left and the other for the right digit, perfectly isolated from one another as required if the time directions of causality are

to be opposite in them. In fact, oracle queries must work in an inseparable way on the two digits together. Going deeper into this topic could be an interesting research prospect.

(viii) To complete our line of reasoning, we must anticipate another result that will come out from the formalizations. We will see that the superposition of all the time-symmetrization instances yields the ordinary quantum description of the process between the two measurement outcomes back again.

Of course this is as it should be, because the quantum description of a reversible process should be time symmetric to begin with, so it is right that it remains unchanged under its time symmetrization.

Also note that such instances, with a completely defined causal structure, are unobservable because they vanish in the superposition of all of them, which is indeed the ordinary quantum description which leaves the causal structure that physically ensures the correlation between the two measurement outcomes free.

(ix) However, the fact that the time symmetrization of the ordinary quantum description gives us the same description back again does not mean that we are left with nothing in hand. On the contrary, we have found that the ordinary quantum description of the reversible process between two one-to-one correlated measurement outcomes, which leaves the causal structure that physically ensures the correlation free, is a quantum superposition of unobservable time-symmetrization instances whose causal structure is completely defined.

(x) The key feature of these instances is that they are causal loops.

(xi) Because of their completely defined causal structure, such causal loops describe the quantum process between the two measurement outcomes in a more complete way than their quantum superposition—the ordinary quantum description which leaves the causal structure free.

(xii) As we will see in the formalizations, they immediately and exactly explain the quantum computational speedup and quantum nonlocality.

(xiii) Of course, what completes the ordinary quantum description is the description of the quantum superposition it is made of, in fact, that of the unobservable quantum causal loops.

With this, we have finished exposing our line of reasoning. What emerges seems to be a very interesting form of quantum retrocausality.

Probably, the reason one is skeptical about the existence of retrocausality is its ability to change the past. This, in the macroscopic world, leads to well-known paradoxical consequences. However, the unobservable form of quantum retrocausality we are dealing with never changes the past as described by the ordinary quantum description, given that it vanishes in it. It should therefore be immune to the paradoxes that plague classical retrocausality.

III. TIME SYMMETRIZING THE ORDINARY QUANTUM DESCRIPTION OF THE COMPUTATIONAL SPEEDUP

This section is a review of the explanation of the quantum computational speedup developed in the evolutionary

approach [1–4] with further clarifications and in a form that immediately fits the notion of quantum causal loop.

Before proceeding, let us explain what an *oracle*, an *oracle problem*, and an *oracle query* are in the present context. An oracle is a black box that computes a function, for example, the Kronecker delta. The oracle problem is finding, by means of the oracle, a characteristic of that function, for example, the argument for which the Kronecker delta is 1. Alice, the problem solver, gives the oracle a value of the argument of the function; in jargon, she performs an *oracle query*. The oracle gives her back the corresponding value of the function. The whole procedure counts for a single computation step, or oracle query. The number of oracle queries needed to solve an oracle problem in an optimal quantum way is its *quantum query complexity*.

It is natural to think that, in order to understand what happens within a quantum process, one must have a description of the same as complete as possible. With respect to the usual description of quantum algorithms, which is limited to the process of solving the problem, we physically describe also (1) the process of setting the problem and (2) the fact that consequently the problem setting should be hidden from the problem solver (otherwise it would tell her the solution of the problem before she begins her problem-solving action). By the way, this is particularly evident in the four-drawer problem, where the problem setting is the number of the drawer with the ball, which is also the solution of the problem. Only after these two completions of the object of the quantum description, we will complete the quantum description itself by time symmetrizing it—this is step 3.

A. Step 1: extending the usual quantum description to the process of setting the problem

Let us consider the simplest instance of the oracle problem solved by Grover quantum search algorithm [5], known as

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B} & & t_1 \rightarrow t_2 \quad \text{time } t_2, \text{ meas. of } \hat{A} \\
 (|00\rangle_B + |01\rangle_B + |10\rangle_B + |11\rangle_B)|00\rangle_A & & \cdot \\
 \downarrow & & \\
 |01\rangle_B|00\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |01\rangle_B|01\rangle_A
 \end{array} \tag{1}$$

We assume that, in the initial state, the number of the drawer with the ball is completely indeterminate. This is represented by a uniform quantum superposition of all the basis vectors of register *B*, top left corner of Eq. (1) (by the way, the basis of this register will never be rotated). Register *A* is initially in an arbitrary sharp state standing for a blank blackboard.

At time t_1 , Bob measures \hat{B} in that initial state. This projects the initial superposition on an eigenstate of \hat{B} selected at random, say $|01\rangle_B$ (follow the vertical arrow on the left of the table). The corresponding eigenvalue, $b = 01$, is the problem setting selected by Bob, here the number of the drawer with the ball. By the way, Bob could unitarily change it into a desired number. For simplicity, we disregard this operation.

Grover’s problem from now on. Bob hides a ball in one of four drawers. Alice is to locate it by opening drawers. Opening drawer number a amounts to querying the oracle with the question: Is the ball in drawer a ? The oracle computes the Kronecker function $\delta(b, a)$, where b is the number of the drawer with the ball, and of course gives back 1 if $a = b$ and 0 otherwise. As is well known, the four-drawer instance of the Grover algorithm always yields the solution of the problem—locates the ball—with just one oracle query, performed in a quantum superposition of the four possible drawer numbers. Since in the classical case it may be necessary to perform up to three queries, there is a *quantum computational speedup*.

The usual description of quantum algorithms is limited to Alice’s problem-solving action. We extend it to Bob’s action of setting the problem.

Let us introduce the notation first:

We number the four drawers in binary notation: 00,01,10,11.

A quantum register *B*, under the control of the problem setter Bob, is meant to contain the *problem-setting* b (the number of the drawer with the ball). Its basis vectors are thus $|00\rangle_B, |01\rangle_B, |10\rangle_B, |11\rangle_B$.

A quantum register *A*, under the control of the problem solver Alice, is meant to contain a , the number of the drawer that Alice wants to open; it will eventually contain the solution of the problem (the number of the drawer with the ball b). Its basis vectors are thus $|00\rangle_A, |01\rangle_A, |10\rangle_A, |11\rangle_A$.

Let the observable \hat{B} , of eigenstates/eigenvalues respectively $|00\rangle_B, |01\rangle_B, |10\rangle_B, |11\rangle_B$ / 00,01,10,11, be the number contained in register *B*; the observable \hat{A} , of eigenstates/eigenvalues respectively $|00\rangle_A, |01\rangle_A, |10\rangle_A, |11\rangle_A$ / 00,01,10,11, be the number contained in register *A*.

The process of setting and solving the problem is represented by the following table (from now on we disregard normalization):

We denote by $\hat{U}_{1,2}$ the unitary part of Alice’s problem-solving action. It should send the input state $|01\rangle_B|00\rangle_A$ into the output state $|01\rangle_B|01\rangle_A = \hat{U}_{1,2}|01\rangle_B|00\rangle_A$, which contains in register *A* the solution of the problem—again the number of the drawer with the ball selected by Bob—right-pointing horizontal arrows.

Of course, $\hat{U}_{1,2}$ can be the unitary part of Grover algorithm, or even a less efficient quantum algorithm that solves Grover’s problem. However, to us, it can remain an unknown unitary transformation. We only need to know that there can be a unitary transformation between the input and output states, which is always the case with this kind of representation where the output has a full memory of the input. Note that the input and output states of $\hat{U}_{1,2}$ are known once the oracle

problem—the set of the problem-settings and the corresponding solutions—is known. In other words, that of Eq. (1) is an external description of the quantum algorithm—of what the quantum algorithm should do to solve the oracle problem—which only depends on the oracle problem. $\hat{U}_{1,2}$, the unitary part of the quantum algorithm, can remain an unknown unitary transformation.

Eventually, at time t_2 , Alice acquires the solution by measuring \hat{A} . The output state of $\hat{U}_{1,2}$, namely $|01\rangle_B|01\rangle_A$, with register A already in an eigenstate of \hat{A} , remains unaltered; there is then a unitary transformation between the initial and final measurement outcomes; the process between them is physically reversible as no information is lost in it.

B. Step 2: relativizing the extended quantum description to the problem solver

Register B contains the problem setting b . Its state represents the knowledge of the problem setting on the part of Bob and any external observer. Initially it represents complete ignorance of it; see the top left corner of Eq. (1). After the initial measurement of \hat{B} on the part of Bob, it represents full knowledge of it; see the bottom left corner of Eq. (1).

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B} & & \text{time } t_2, \text{ meas. of } \hat{A} \\
 \hline
 (|00\rangle_B + |01\rangle_B + |10\rangle_B + |11\rangle_B)|00\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A \quad (2) \\
 & & \downarrow \\
 & & |01\rangle_B|01\rangle_A
 \end{array}$$

To Alice, the initial quantum superposition of the four possible numbers of the drawer with the ball remains unaltered after the initial Bob's measurement of \hat{B} —the top left corner of the table. This superposition represents her complete ignorance of the number of the drawer with the ball selected by Bob. By $\hat{U}_{1,2}$ (horizontal arrows), this superposition unitarily evolves into the superposition of four tensor products, each the product of a number of the drawer with the ball and the corresponding solution (that same number but in register A). Eventually, the final measurement of \hat{A} projects the latter superposition on $|01\rangle_B|01\rangle_A$, the tensor product of the number of the drawer with the ball already selected by Bob and the corresponding solution (vertical arrow). Note that this projection is indifferently that due to the initial Bob's measurement, postponed.

⁴This is particularly evident in the present case, where the problem setting and the solution are both the number of the drawer with the ball. But it holds in general, since Alice always knows the function computed by the black box.

⁵Alternatively, we could postpone at the end of Alice's action the very measurement of \hat{B} . This can be done because the reduced density operator of register B remains unaltered throughout $\hat{U}_{1,2}$. Note that, with reference to point (iii) of Sec. II, this would make the two measurements simultaneous.

Therefore, we can see that the description of Eq. (1) works for Bob and any external observer, not for Alice. Immediately after the initial measurement, she would know the solution of the problem⁴ before she begins her problem-solving action. To Alice, the problem-setting selected by Bob must be hidden inside a black box.

To physically represent the concealment of the outcome of the initial measurement from Alice (the problem solver and observer of the final measurement), in the first place we must be able to have two different descriptions of the quantum algorithm, one with respect to Bob and any external observer and the other with respect to Alice. For this, we resort to relational quantum mechanics [21,22]. In it, quantum states are not absolute but relative to the observer. We *relativize* the description of the quantum algorithm with respect to Alice as follows.

To her, we postpone after the end of her problem-solving action the projection of the quantum state associated with the initial measurement (note that \hat{B} and \hat{A} commute). This is, of course, a mathematically legitimate operation provided that the two ends of the projection undergo the unitary transformation $\hat{U}_{1,2}$.⁵

While the description of the quantum algorithm with respect to Bob and any external observer is that of Eq. (1), that with respect to Alice becomes the following:

C. Step 3: completing the ordinary quantum description by time symmetrizing it

With probability one of reading the solution, there is the unitary transformation $\hat{U}_{1,2}$ between the initial and final measurement outcomes. The process between them is reversible in spite of comprising the final measurement, which does not change the quantum state. Correspondingly, its ordinary quantum description leaves the direction of causality unspecified. We have seen that this allows causal structures that violate the time symmetry required of the description of a reversible process (violate the requirement of the absence of a preferred time direction of causality), thus showing that the ordinary quantum description of the process in question is incomplete. This also says that the description is completed by time symmetrizing it.

We time symmetrize the description by imposing that the initial and final measurements evenly contribute to the selection of the information that specifies the sorted out pair of correlated measurement outcomes (among all the possible pairs). Since there is a plurality of ways of sharing the information in question into two halves, we should take their quantum superposition.

We should assume that the initial measurement of \hat{B} and the final measurement of \hat{A} in all the possible ways reduce to complementary partial measurements that evenly and nonredundantly contribute to the selection of the pair of outcomes.

For example, the initial measurement of \hat{B} could reduce to that of \hat{B}_l (the left binary digit—bit—of the number in register B) and the final measurement of \hat{A} to that of \hat{A}_r (the right bit of the number in register A), or vice versa. Together, the measurement of \hat{B}_l at time t_1 and that of \hat{A}_r at time t_2 select a well defined problem setting; say it is 01. How to arrange together the selections performed by the two successive measurements in order that, together, they determine the

process between the two outcomes is explained in the next subsection.

1. Time symmetrizing the description of the quantum algorithm to Bob and any external observer

The description of the quantum algorithm of Eq. (1) is that with respect to Bob and any external observer. We time symmetrize it by the following *time-symmetrization (TS) formalism*:

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B}_l & t_1 \rightleftharpoons t_2 & \text{time } t_2, \text{ meas. of } \hat{A}_r \\
 (|00\rangle_B + |01\rangle_B + |10\rangle_B + |11\rangle_B)|00\rangle_A & & \\
 \downarrow & \Rightarrow \hat{U}_{1,2} \Rightarrow & |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A \\
 (|00\rangle_B + |01\rangle_B)|00\rangle_A & & \downarrow \\
 |01\rangle_B|00\rangle_A & \Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow & |01\rangle_B|01\rangle_A
 \end{array} \tag{3}$$

The initial measurement of \hat{B}_l , selecting the 0 of 01, projects the initial quantum superposition on the superposition of the terms beginning with 0, vertical arrow on the left of the table. Under $\hat{U}_{1,2}$, the latter superposition evolves into the superposition of the two products *number of the drawer with the ball* ⊗ *corresponding solution* (right-pointing horizontal arrows). Then the final measurement of \hat{A}_r , selecting the 1 of 01, projects the superposition in question on the term ending in 1 (vertical arrow on the right of the table). The backward in time propagation, by $\hat{U}_{1,2}^\dagger$, of this output state, which inherits both selections, is an instance of the time-symmetrized description of the quantum algorithm with respect to Bob and any external observer (bottom line of the table). Moreover, it is also the superposition of all instances since their bottom lines are all identical to each other. One can see that the bottom line of the table remains the same regardless of how we evenly share the selection of the sorted out pair of measurement outcomes between the initial and final measurements. It is also identical to the bottom line of Eq. (1), namely to the ordinary description of the quantum algorithm, since the symbol “ $\Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow$ ” is equivalent to “ $\Rightarrow \hat{U}_{1,2} \Rightarrow$ ”. As anticipated in the introduction, the superposition of all the time-symmetrization instances gives the ordinary quantum description back again—in a trivial way in the present case.

The above zigzag diagram yields the time-symmetric causal structure that ensures the correlation between the two measurement outcomes in the case of the quantum algorithm with respect to the problem setter Bob and any external observer. We have given this for completeness. It will be the causal structure of the quantum algorithm with respect to Alice the one that immediately explains the quantum speedup.

By the way, with reference to point (ii) of Sec. II, one can see the essential role played by quantum superposition in order that the left digit 0 of the number of the drawer with the ball 01 selected by the initial measurement causes the corresponding digit of the final measurement outcome and the left digit 1 selected by the final measurement causes the corresponding digit of the initial measurement outcome.

2. Time symmetrizing the description of the quantum algorithm to Alice

We time symmetrize the description of the quantum algorithm with respect to Alice of Eq. (2). The following zigzag diagram yields the time-symmetrization instance for problem setting $b = 01$, the measurement of \hat{B} reduced to that of \hat{B}_l , and the measurement of \hat{A} to that of \hat{A}_r :

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B}_l & t_1 \rightleftharpoons t_2 & \text{time } t_2, \text{ meas. of } \hat{A}_r \\
 (|00\rangle_B + |01\rangle_B + |10\rangle_B + |11\rangle_B)|00\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A. \\
 & & \downarrow \\
 (|01\rangle_B + |11\rangle_B)|00\rangle_A & \Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow & |01\rangle_B|01\rangle_A + |11\rangle_B|11\rangle_A
 \end{array} \tag{4}$$

The projection of the quantum state associated with the initial measurement of \hat{B}_l must be postponed after the end of Alice’s problem-solving action, outside Eq. (4), which is limited to this action. In fact, any information about the problem setting must be hidden from her. The top line of the diagram is thus the same of Eq. (2). Then the measurement of \hat{A}_r , in the output state of $\hat{U}_{1,2}$, selecting the 1 of 01, projects it on the

superposition of the terms ending with 1 (vertical arrow). The propagation of the latter superposition—which inherits both selections—backward in time by $\hat{U}_{1,2}^\dagger$ (left-pointing horizontal arrows) is an instance of the time-symmetrized quantum algorithm to Alice. See the bottom line of Eq. (4).

In view of what will follow, note that we can speak as well of projection of the top line of the diagram on the bottom line.

Let us repeat this bottom line—i.e., the time-symmetrized quantum algorithm—below for convenience:

$$\begin{array}{ccc} \text{time } t_1 & t_1 \leftarrow t_2 & \text{time } t_2 \\ (|01\rangle_B + |11\rangle_B)|00\rangle_A & \Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow & |01\rangle_B|01\rangle_A + |11\rangle_B|11\rangle_A \end{array} \quad (5)$$

The above equation can also be read the other way around as

$$\begin{array}{ccc} \text{time } t_1 & t_1 \rightarrow t_2 & \text{time } t_2 \\ (|01\rangle_B + |11\rangle_B)|00\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |01\rangle_B|01\rangle_A + |11\rangle_B|11\rangle_A \end{array} \quad (6)$$

By the way, let us say for completeness that the output state $|01\rangle_B|01\rangle_A + |11\rangle_B|11\rangle_A$ will eventually be projected on $|01\rangle_B|01\rangle_A$ by the projection due to the initial measurement of \hat{B}_l postponed after the end of Alice's action, outside Eq. (4), which is limited to this action.

For $b = 01$, there are in total three time-symmetrization instances. In each of them, the problem setting selected by Bob, $b = 01$, pairs with another problem setting (it pairs with $b = 11$ in the above instance). The superposition of all instances of the time-symmetrized quantum algorithm [one is that of Eq. (6)], also for all the possible problem settings,⁶ gives back the unitary part of the original quantum algorithm to Alice, the top line of Eq. (2). Of course, the superposition of all the pairs of basis vectors of register B yields the superposition of all the basis vectors, namely the top left state of Eq. (2).

3. Interpretation

Equation (5), or identically (6), yields an instance of the time-symmetrized quantum algorithm with respect to Alice. From now, on we will refer to the latter table, which is a more familiar description of a quantum algorithm.

Note that, besides the element of a quantum superposition, each instance can be seen itself as a full-fledged quantum algorithm. In the instance we are dealing with, it suffices to think that actually Bob measures \hat{B}_l in the initial state and Alice \hat{A}_r in the output state of $\hat{U}_{1,2}$; of course, this selects the entire problem setting 01 too. The time symmetrization of the quantum algorithm in this case only consists in propagating forward in time the selection due to the initial measurement and backward in time that due to the final measurement. The time-symmetrized quantum algorithm in this case is just one time-symmetrization instance and it obviously involves the same number of oracle queries of the original quantum algorithm.

As one can see in Eq. (6), the computational complexity of Grover's problem reduces in the time-symmetrized quantum algorithm with respect to the problem solver Alice. Alice's problem is now locating the ball hidden in the pair of drawers {01, 11}; check the input and output states of $\hat{U}_{1,2}$ in that table. Let us call the new problem *the reduced problem*.

In equivalent terms, Eq. (6) tells us that Alice, immediately after the initial measurement, knows that the ball is in the pair

of drawers {01, 11}. In fact, in the description of the quantum algorithm with respect to her, the state of register B represents her state of knowledge of the number of the drawer with the ball. In the input state of her problem-solving action, this changes from knowledge that the ball is in one of four drawers (i.e., from complete ignorance of the number of the drawer with the ball), into knowledge that the ball is in one of two drawers—compare the states in the top left and bottom left corners of Eq. (4). This is because Alice is shielded from the half information about the number of the drawer with the ball coming to her from the initial measurement, not from the half coming to her back in time from the final measurement.

Summing up, all is as if Alice knew in advance half of the information about the solution of the problem she will produce and read in the future and could use this knowledge to reduce the computational complexity of the problem to be solved. Below, we show that this allows us to compute the number of oracle queries needed to solve Grover's problem in an optimal quantum way (of course, without knowing the quantum algorithm).

The reduced problem, as any other problem, can always be solved quantumly with the number of oracle queries required by a (reversible) classical algorithm, i.e., logically required.⁷ The question then becomes whether it could be solved with even fewer queries, namely itself with a quantum speedup. Now, under the reasonable assumption that the quantum speedup is essentially related to the reduction, under time symmetrization, of the computational complexity of the problem to be solved, the answer must be negative. In fact, the quantum algorithm that solves the reduced problem has been time symmetrized already. Further time symmetrizing it would leave it unaltered without further reducing the computational complexity of the problem.⁸

Summing up, it turns out that the number of queries needed to solve Grover's problem in an optimal quantum way is that needed to solve the corresponding reduced problem in an optimal logical (classical) way. In the present four-drawer instance of Grover's problem, this is only one query. In fact, the reduced problem, i.e., locating a ball hidden in a pair of drawers, can be classically solved by opening either drawer. Note that we have found the number of oracle queries needed to solve Grover's problem in an optimal quantum way without knowing Grover algorithm, namely $\hat{U}_{1,2}$.

More generally, we should consider the generic number of drawers $N = 2^n$, where n is the number of binary digits (bits) in the drawer number. Also note that in the variant provided by Long [23], the Grover algorithm gives the solution with certainty for any value of N . Everything we said for the four drawers still holds.

⁷What face each other here are not quantum and classical physics but quantum physics and classical logic.

⁸We have seen that Bob and Alice can effectively perform the partial measurements of the instance we are dealing with. The time symmetrization in this case consists in propagating forward in time the selection performed by Bob's measurement and backward in time that performed by Alice's measurement. Doing this a second time changes nothing.

⁶In the quantum algorithm to Alice, the problem setting is selected only after the unitary part of Alice's problem-solving action.

In order that the initial and final measurements evenly contribute to the selection of the pair of measurement outcomes, it suffices to evenly share between the two measurements the selection of the n bits that specify the number of the drawer with the ball. This means ascribing to the final measurement the selection of $n/2$ bits. Since Alice knows these bits in advance, the reduced problem is locating the ball in $2^{n/2}$ drawers. It can be solved logically (classically) with $O(2^{n/2})$ queries.

In summary, the time-symmetrization formalism foresees that the number of queries needed to solve Grover's problem in an optimal quantum way is that needed to solve the corresponding reduced problem in an optimal logical (classical) way, namely $O(2^{n/2})$ queries. Also, in the quantum case, we must make reference to an optimal quantum algorithm since a nonoptimal one could take any higher number of queries. That Grover's problem can be solved quantumly with any number of queries above that required by the optimal Grover/Long algorithm has been shown in Refs. [23,24].

All the above is in agreement with the fact that Grover algorithm (more precisely Grover/Long algorithm) is demonstrably optimal [23–25] and employs $O(2^{n/2})$ queries [5].

Let us eventually note that the present explanation of the quantum speedup ascribes it to a quantum causal loop: The final measurement of the solution changes back in time the initial state of knowledge of the solution on the part of Alice and this in turn allows Alice to produce the solution with fewer oracle queries, which closes the loop.

4. From Grover's problem to any oracle problem

The method to compute the number of oracle queries needed to solve Grover's problem in an optimal quantum way can immediately be extended to any oracle problem, as follows.

In any quantum algorithm relativized to Alice, we have the following:

- (i) an initial state which is the superposition of all the possible problem settings tensor product the blank blackboard that will eventually contain the solution of the problem and
- (ii) a state immediately before the final Alice's measurement of the solution, which is a superposition of tensor product, each a problem setting multiplying the corresponding solution.

In other words, provided that we change the size of the quantum registers, the diagrams developed for Grover algorithm hold for the process of solving any oracle problem.

The only possible difference with respect to Grover's case is that the solution is a noninvertible function of the problem setting.

This difference is eliminated by thinking that Alice, at the end, measures both the content of register A , which contains the solution, and that of register B , which contains the problem setting. This is legitimate since this Alice's measurement of the content of B occurs after the unitary part of her problem-solving action, namely when the reason for hiding from her the problem setting has fallen.

The time symmetrization of this process requires that, in each time-symmetrization instance, the two measurements of the content of register B (the initial measurement made by

Bob and the final one made by Alice⁹) reduce to two partial measurements that evenly and nonredundantly contribute to the selection of the problem setting and the corresponding solution. Eventually, we should take the quantum superposition of all the time-symmetrization instances.

Let us further introduce a simplification. Since the reduced density operator of register B remains unaltered throughout the unitary part of Alice's action, any final measurement of a content of B on the part of Alice can be anticipated to the time of the initial measurement. The requirement of evenly sharing between the initial Bob's measurement and the final Alice's measurement the selection of the pair of correlated measurement outcomes can be replaced by that of splitting the initial measurement of the content of register B into two partial measurements that evenly and nonredundantly contribute to selecting the problem setting and the corresponding solution. In each time-symmetrization instance, Alice knows in advance the information selected by either partial measurement. Taking this into account yields the following *advanced knowledge rule* that holds for any oracle problem:

We should split the initial measurement of the problem setting in a state of maximal indetermination of it into two partial measurements that evenly and nonredundantly contribute to the selection of the problem setting and the corresponding solution. Alice knows in advance the information acquired by either partial measurement. The number of queries needed to solve the oracle problem in an optimal quantum way is that needed to solve it in an optimal logical (classical) way given the advanced knowledge in question.

Let us consider for example the problem solved by Deutsch-Jozsa algorithm [26]. The set of functions computed by the black box is all the constant and *balanced* functions $f_{\mathbf{b}} : \{0, 1\}^n \rightarrow \{0, 1\}$ (balanced functions have an even number of zeros and ones). The following array of function tables gives four of the eight functions for $n = 2$:

\mathbf{a}	$f_{0000}(\mathbf{a})$	$f_{1111}(\mathbf{a})$	$f_{0011}(\mathbf{a})$	$f_{1100}(\mathbf{a})$...
00	0	1	0	1	...
01	0	1	0	1
10	0	1	1	0	...
11	0	1	1	0	...

The first column from the left contains the argument of the function, the second and third columns show the corresponding values of the two constant functions, the fourth column shows those of a balanced function, etc. Note that we have chosen as problem setting (and function suffix) \mathbf{b} the table of the function (the sequence of function values for increasing values of the argument).

Bob selects one of these functions—a valuation of \mathbf{b} —at random, Alice is to find whether $f_{\mathbf{b}}$ is constant or balanced by performing oracle queries. In the classical and worst case, an $\exp(n)$ number of queries is needed, in the quantum case just one.

⁹Alice's measurement of the solution is redundant with her measurement of the problem setting and can be ignored.

We apply the advanced knowledge rule to this algorithm. The outcome of the initial Bob's measurement is the problem setting \mathbf{b} . We should split this measurement into two partial measurements that evenly and nonredundantly contribute to the selection of \mathbf{b} and the corresponding solution; Alice knows in advance the information acquired by either partial measurement.

As one can see, this implies that these two partial measurements measure two halves of the bit-string \mathbf{b} that contain either all zeros or all ones; we call such halves *good half-tables*. For example, let $\mathbf{b} = 0011$; the table of the corresponding function is the fourth column of the above array of function tables. The two good half-tables are $f_{0011}(00) = 0$, $f_{0011}(01) = 0$ and $f_{0011}(10) = 1$, $f_{0011}(11) = 1$. If either half-table contained both zeros and ones, also the other would. This would determine the solution (that the function is balanced) in a redundant way (and, of course, Alice would know the solution of the problem without doing anything). The advanced knowledge rule would be violated.

Then, Alice knows in advance a good half-table, e.g., $f_{\mathbf{b}}(00) = 0$, $f_{\mathbf{b}}(01) = 0$. We have omitted the value of the function suffix because of course she only knows the result of measuring the content of the left two digits of the problem setting 0011 contained in register B , she does not know the suffix of the function. Once she knows in advance a good half-table, classically, performing a single oracle query for a value of the argument outside that half-table (here, e.g., for $\mathbf{a} = 10$) allows her to ascertain whether the function is constant or balanced, in agreement with what the Deutsch-Jozsa algorithm does.

Let us now go to Simon [27] and the quantum part of Shor [28] factorization algorithm. Both these algorithms concern finding the period of a particular set of periodic functions. Classically, this requires a number of oracle queries exponential in problem size, and quantumly the number becomes polynomial; there is thus an exponential speedup. Applying the advanced knowledge rule to these algorithms yields what follows.

One can choose as problem setting \mathbf{b} the table of two consecutive periods of the function. Naturally, the period of the function is identified by two consecutive values of the argument of the function such that the corresponding values of the function are the same. One can see that the notion of good half-table still applies and that a pair of good half-tables are the tables of the two periods.

Since Alice knows in advance a good half-table (i.e., the table of a period of the function) performing an oracle query for a value of the argument immediately outside that half-table necessarily yields a repeated value of the function, which allows her to find the period of the function. It should be noted that, according to the advanced knowledge rule, the algorithms of Simon and Shor would be suboptimal, as they require a number of oracle queries polynomial in problem size instead of a single query; see Ref. [4] for further detail.

Summing up, the advanced knowledge rule exactly accounts for the speedups of Grover and Deutsch-Jozsa algorithms, which are demonstrably optimal, and in good approximation for those of Simon and Shor algorithms, more general for those of the Abelian hidden subgroup [29]. As is well known, these are the major quantum algorithms discov-

ered until now and cover both the quadratic and exponential speedups.

We note that in all cases the speedup is ensured by a quantum causal loop. The final measurement of the problem setting on the part of the problem solver changes backward in time her initial state of complete ignorance of it (and of the corresponding solution) into knowledge of half of the information that specifies it. This knowledge can be used by the problem solver to reach the solution with fewer oracle queries, which closes the loop.

IV. TIME SYMMETRIZING THE ORDINARY QUANTUM DESCRIPTION OF NONLOCALITY

We go now quantum nonlocality. We assume that two photons B and A at time t_0 have been generated by parametric down-conversion in a state of spatial contiguity and in the polarization state:

$$\frac{1}{\sqrt{2}}(|0\rangle_B|1\rangle_A - |1\rangle_B|0\rangle_A); \quad (7)$$

0/1 are horizontal/vertical polarizations. Then, in the time interval (t_0, t_1) , the two photons get spatially separate in a unitary way. The polarization of photon B is measured at time t_1 , and that of photon A is measured in the same basis at time t_2 . There is one-to-one correlation between the two measurement outcomes and, correspondingly, a unitary transformation between them. This transformation can go from t_1 to t_2 either directly or via t_0 , first backward in time from t_1 to t_0 and then forward in time from t_0 to t_2 . We should keep in mind that the unitary transformation that connects the two measurement outcomes via t_0 is local and thus physical (of course, at the condition that we consider the present form of quantum retrocausality physical too). The one that goes directly from t_1 to t_2 , implying the spooky action at a distance, is certainly unphysical.

To explain quantum nonlocality by means of the time-symmetrization (TS) formalism, we should time symmetrize the reversible quantum process that goes from t_1 to t_2 via t_0 . We should evenly share between the two measurements the selection of the single bit that specifies the pair of correlated outcomes among the two possible pairs 01 and 10. To this end, we represent the polarization state (7) in the redundant way:

$$\frac{1}{2}(|00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A). \quad (8)$$

One can see that, if we take the XOR (exclusive OR) between the two bits of each register, we obtain the state $\frac{1}{\sqrt{2}}(|0\rangle_B|0\rangle_A + |1\rangle_B|1\rangle_A)$. If, in (8), we changed the two central plus signs into minus signs, we would have obtained $\frac{1}{\sqrt{2}}(|0\rangle_B|0\rangle_A - |1\rangle_B|1\rangle_A)$, which can trivially be brought to state (7). However, we will do without these changes that are irrelevant here since we never rotate any measurement basis. Now the problem is the same as that of the four-drawer instance of Grover algorithm: evenly (and nonredundantly) sharing between the measurements of \hat{B} and \hat{A} the selection of the two bits that specify the pair of correlated outcomes.

A. Costa de Beauregard's explanation

First, we provide the ordinary (usual) quantum description of nonlocality (we always ignore normalization):

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B} & & t_1 \rightarrow t_2 \quad \text{time } t_2, \text{ meas. of } \hat{A} \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A & & \cdot \\
 \downarrow & & \\
 |01\rangle_B|01\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |01\rangle_B|01\rangle_A
 \end{array} \tag{9}$$

At time t_1 , the measurement of \hat{B} (the content of register B) in the maximally entangled state in the top left of the table selects at random, say, the eigenvalue $b = 01$, thus projecting the state of the two registers on $|01\rangle_B|01\rangle_A$ (vertical arrow). The fact that this measurement instantly changes also the state of the space separate register A is of course the “spooky action at a distance.” The successive unitary transformation $\hat{U}_{1,2}$, between t_1 and t_2 , is the identity for the polarization state of the two photons. By $\hat{U}_{1,2}$, the measurement outcome $|01\rangle_B|01\rangle_A$ directly (not via t_0) propagates to immediately before the measurement of \hat{A} (the content of register A) at time t_2 – horizontal arrows. Then this latter measurement deterministically selects the correlated eigenvalue $a = 01$.

The following table provides the quantum process corresponding to Costa de Beauregard’s explanation of quantum nonlocality [9,30]. Formally, it suffices to replace, in Eq. (9), $\hat{U}_{1,2}$ by $\hat{U}_{1,0,2}$, where $\hat{U}_{1,0,2}$ is the unitary transformation that goes from the first to the second measurement outcome, one-to-one correlated with it, via t_0 . Of course, we have the mathematical identity $\hat{U}_{1,2} = \hat{U}_{1,0,2}$. Equation (9) becomes the following:

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B} & & t_1 \rightarrow t_0 \rightarrow t_2 \quad \text{time } t_2, \text{ mes. of } \hat{A} \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A & & \cdot \\
 \downarrow & & \\
 |01\rangle_B|01\rangle_A & \Rightarrow \hat{U}_{1,0,2} \Rightarrow & |01\rangle_B|01\rangle_A
 \end{array} \tag{10}$$

This time the outcome of measuring \hat{B} at time t_1 unitarily propagates first backward in time to t_0 and then forward in time to t_2 , becoming the state immediately before and after the measurement of \hat{A} .

The point is that Eq. (10) hosts Costa de Beauregard’s explanation of quantum nonlocality. One should assume that, in causal order, the measurement of \hat{B} at time t_1 only projects the reduced density operator of register B on $|01\rangle_B\langle 01|_B$, leaving that of register A unaltered. Then this projection propagates backward in time from t_1 to t_0 . At time t_0 , it locally projects the original maximally entangled state on the sharp state $|01\rangle_B|01\rangle_A$. Eventually the latter state propagates forward in time from t_0 to t_2 where it becomes the state immediately before and after the final measurement.

Note that, under this interpretation, the fact that the measurement of \hat{B} at time t_1 instantly changes also the state of the space separate register A is, so to speak, a coincidence. It does instantly change the state of register A , but by two causally successive local propagations, one backward in time from t_1 to t_0 and the other forward in time from t_0 to t_2 . Of course, these two propagations take the same amount of time, but one with a negative and the other with a positive sign, so that the overall time taken is zero. This is what emulates action at a distance.

Costa de Beauregard’s explanation received little attention by the majority of physicists [30]. It is difficult to say why, because one should explain silence. Likely, the notion that causality can go backward in time, even along the objectively ideal reversible processes of quantum mechanics, is hardly accepted. In the next subsection, we will rebuild a similar explanation by means of the time-symmetrization formalism, thus replacing usual retrocausality by its present unobservable form.

B. Application of the time-symmetrization formalism

First, we apply the time-symmetrization (TS) formalism to the usual description of quantum nonlocality of Eq. (9), where the outcome of the first measurement directly propagates, by $\hat{U}_{1,2}$, to the second measurement. Note that we are dealing with a description with respect to any observer. We do not have to hide anything from anyone here. We consider the time-symmetrization instance where the sorted out value of the two correlated measurement outcomes is 01, the measurement of \hat{B} at time t_1 reduces to that of \hat{B}_l (the left bit of the number contained in register B) and the measurement of \hat{A} at time t_2 to that of \hat{A}_r (the right bit of the number in register A). The corresponding zigzag diagram is as follows:

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B}_l & & t_1 \rightleftarrows t_2 \quad \text{time } t_2, \text{ meas. of } \hat{A}_r \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A & & \cdot \\
 \downarrow & & \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A & \Rightarrow \hat{U}_{1,2} \Rightarrow & |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A \\
 & & \downarrow \\
 |01\rangle_B|01\rangle_A & \Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow & |01\rangle_B|01\rangle_A
 \end{array} \tag{11}$$

The measurement of \hat{B}_l at time t_1 projects the initial-state superposition (top left of the table) on the superposition of the terms beginning with 0 (vertical arrow on the left of the table). The latter superposition, by $\hat{U}_{1,2}$, propagates to t_2 (right-pointing

horizontal arrows), where the measurement of \hat{A}_r projects it on the sharp state $|01\rangle_B|01\rangle_A$ ending with 1 (vertical arrow on the right of the table). This measurement outcome, by $\hat{U}_{1,2}^\dagger$, propagates to t_1 (left-pointing horizontal arrows). The propagation in question, which inherits both selections, is both a time-symmetrization instance and the superposition of all instances for measurement outcome 01, with the bottom lines of all these instances being identical to each other. They are also identical to the ordinary quantum description—bottom line of Eq. (9)—since “ $\Leftarrow \hat{U}_{1,2}^\dagger \Leftarrow$ ” is equivalent to “ $\Rightarrow \hat{U}_{1,2} \Rightarrow$ ”.

Now we apply the TS formalism to the description of the quantum process that goes from t_1 to t_2 via t_0 . It suffices to replace, in Eq. (11), $\hat{U}_{1,2}$ by the mathematically equivalent $\hat{U}_{1,0,2}$. This yields the following:

$$\begin{array}{ccc}
 \text{time } t_1, \text{ meas. of } \hat{B}_l & t_1 \rightleftarrows t_0 \rightleftarrows t_2 & \text{time } t_2, \text{ meas. of } \hat{A}_r \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A + |10\rangle_B|10\rangle_A + |11\rangle_B|11\rangle_A & & \\
 \downarrow & & \\
 |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A & \Rightarrow \hat{U}_{1,0,2} \Rightarrow & |00\rangle_B|00\rangle_A + |01\rangle_B|01\rangle_A \\
 |01\rangle_B|01\rangle_A & \Leftarrow \hat{U}_{1,0,2}^\dagger \Leftarrow & \downarrow \\
 & & |01\rangle_B|01\rangle_A
 \end{array} \quad (12)$$

Of course, also in this case the bottom lines of the time-symmetrization instances are all identical to each other and to that of the ordinary quantum description of Eq. (10); we can replace “ $\Leftarrow \hat{U}_{1,0,2}^\dagger \Leftarrow$ ” by “ $\Rightarrow \hat{U}_{1,0,2} \Rightarrow$ ”. The important point is that the projections of both reduced density operators, that of register B after the measurement of \hat{B}_l and that of register A after the measurement of \hat{A}_r , in their propagation toward the other measurement go via t_0 . At time t_0 they locally project the state of the entire quantum system. Also the present zigzag diagram hosts the local explanation of spatial nonlocality.

By the way, one might wonder how the projection of the reduced density operator of either register “knows” that it must propagate to the other measurement through t_0 . Our answer would be that it propagates in all the possible ways along the unitary evolutions that connect the two measurement outcomes.

Eventually, let us note that what explains spatial nonlocality are a couple of quantum causal loops, one for each measurement. The two measurements, respectively at times t_1 and t_2 , change back in time the state of the two registers at time t_0 . This in turn causes the correlation between the two future measurement outcomes.

Let us compare now the explanation of nonlocality provided by the TS formalism and that of Costa de Beauregard.

The latter explanation assumes that the entire outcome of measuring \hat{B} at time t_1 (which only changes the reduced density operator of register B) goes backward in time to the time t_0 the two subsystems were not spatially separate. This (i) violates the present requirement that the description of quantum correlation does not present a preferred time direction of causality, (ii) changes a past state as described by the ordinary quantum description, and (iii) can be considered an arbitrary assumption in the sense that it is only justified by the explanation it provides.

The TS formalism replaces the ordinary retrocausality of Costa de Beauregard’s explanation of Eq. (10) by the unobservable retrocausality of Eq. (12). The latter comes from time symmetrizing the ordinary quantum description of the reversible process that physically connects, via t_0 , the two measurement outcomes. In present assumptions, more than being legitimate, this time symmetrization is mandatory to complete the quantum description. Furthermore, the two not yet space-separate subsystems are told about how to behave in

the future measurements by hidden variables: the unobservable time directions of the causations that physically ensure the correlation between the two outcomes. This unobservable way of sending information backward in time that does not change the observable past should be acceptable.

V. DISCUSSION

In this section, we discuss the unconventional and unorthodox points of the present work. They can be summarized as follows:

(i) Opening to retrocausality.

(ii) Resuming the apparently forgotten problem of the incompleteness of the quantum description raised by Einstein, Podolsky, and Rosen in their famous EPR paper.

(iii) Tackling the same problem in the quantum speedup. Its ordinary quantum description is a reversible process between two one-to-one correlated measurement outcomes exactly as in the case of quantum nonlocality and thus must suffer from the same incompleteness.

(iv) Deriving a technical result—the rule for computing the number of oracle queries needed to solve an oracle problem in an optimal quantum way—in an axiomatic way from physical principles.

(v) Showing that the quantum capability of performing tasks that cannot be performed classically essentially relies on unobservable causal loops.

Any of the above points can be puzzling. We defend them as follows:

Point (i). The fact that retrocausality can change the past in the macroscopic world gives rise to well-known paradoxes. However, the present form of quantum retrocausality has a justification and a remedy. The justification is that retrocausality is generated by a mandatory operation: completing the ordinary quantum description by time symmetrizing it. The remedy is that it resides in each unobservable time-symmetrization instance and vanishes in their quantum superposition, namely in the ordinary quantum description. In other words, the quantum form of retrocausality generated by the time-symmetrization formalism is unobservable and does not change the past as described by the ordinary quantum description. It should therefore be immune to the paradoxes that plague the usual form.

Points (ii) and (iii). In hindsight, it is clear that the ordinary description of quantum correlation—hence of nonlocality and the speedup—is incomplete. Indeed, it cannot describe the causal loops on which both are based. By completing it, it does.

Point (iv). The number of oracle queries needed to solve an oracle problem in an optimal quantum way has been axiomatically derived from the principle that the quantum description of a process which is reversible from the initial to the final measurement outcome must be symmetric in time, i.e., must not entail a preferred time direction of causality. This derivation solves in a synthetic way the so-called quantum *query complexity problem*, which is central to quantum computer science and still unsolved in the general case [7,8]. The synthetic character of this derivation contrasts with the analytical (entirely mathematical) character of the derivations typical of the latter discipline. Our defense is that the two kinds of derivation should be equivalent if quantum mechanics is consistent.

In favor of the synthetic approach, we would invoke Grover's authority. In 2001 [31], he called for a synthetic demonstration of the optimality of his quantum search algorithm. He wrote, "It has been proved that the quantum search algorithm cannot be improved at all, i.e. any quantum mechanical algorithm will need at least $\mathcal{O}(\sqrt{N})$ steps to carry out an exhaustive search of N items. Why is it not possible to search in fewer than $\mathcal{O}(\sqrt{N})$ steps? The argument used to prove this are very subtle and mathematical. What is lacking is a simple and convincing two line argument that shows why one would expect this to be the case."

Our "two-line" argument would be that time symmetrizing the ordinary quantum description of the reversible process of setting and solving Grover's problem reduces its quantum computational complexity from $\mathcal{O}(N)$ to $\mathcal{O}(\sqrt{N})$. Further time symmetrizing the already time-symmetrized description would leave it unaltered and would not further reduce the problem complexity.

Point (v). We have shown that, in the two known cases of the quantum speedup and quantum nonlocality, the quantum ability to perform tasks that cannot be performed classically relies on causal loops. Accepting the idea that nonrelativistic quantum mechanics can host causal loops might be difficult. However, the fact that such loops reside in the unobservable time-symmetrization instances and vanish in their quantum superposition, namely in the ordinary quantum description, should make it easier.

VI. SUMMARY AND CONCLUSION

As the present work is rather unconventional, it may be useful to summarize it before concluding.

A. Summary

We have been dealing with the quantum description of the reversible processes between two one-to-one correlated measurement outcomes that characterize both the quantum speedup and quantum nonlocality. We have argued that the ordinary quantum description of these processes mathemati-

cally describes the correlation between the two outcomes but leaves the causal structure that physically ensures it free.

We have seen that, consequently, the selection of the information that specifies the sorted-out pair of outcomes among all the possible pairs can share in any way between the initial and final measurements.

In the present assumptions, the ways where this sharing is uneven violate the time symmetry required of the description of a time-reversible process: the fact it should not entail a preferred time direction of causality. This says that the ordinary quantum description, which allows such violations, is incomplete and is completed by time symmetrizing it. To this end, the selection of the information that specifies the pair of correlated measurement outcomes should be evenly (and nonredundantly) shared between the initial and final measurements. Since this can be done in a plurality of ways, we should take their quantum superposition.

This time symmetrization of the ordinary quantum description leaves it unaltered, as it should be, but at the same time shows that it is a quantum superposition of unobservable time-symmetrization instances whose causal structure is completely defined. Each instance is a causal loop. Causality goes from the initial to the final measurement outcome and then back from the latter to the initial measurement outcome.

These instances are unobservable since they vanish in the superposition of all instances. In fact, the latter is the ordinary quantum description, which leaves the causal structure that physically ensures the correlation between the two measurement outcomes free. Being completed descriptions of the quantum process, they describe it in a more complete way than the ordinary quantum description. In particular, they immediately explain the quantum speedup and quantum nonlocality.

In the case of the quantum speedup, the final measurement of the problem setting and the corresponding solution on the part of the problem solver changes backward in time the initial measurement outcome. The latter represents both the problem to be solved and the knowledge of the problem setting and the corresponding solution on the part of the problem solver. The computational complexity of the problem to be solved reduces. Correspondingly, the problem solver's complete ignorance of the problem setting and the corresponding solution changes into knowledge of half of the information that specifies them. The causal loop becomes the fact that all is as if the problem solver knew in advance half of the information about the problem setting and the corresponding solution she will read in the future and could use this knowledge to produce the solution with fewer oracle queries.

In the case of quantum nonlocality, the measurement on either subsystem, when the two are space separate, retro-causally and locally changes the state of both subsystems when they were not space separate. The causal loop implicit in the time-symmetrization formalism originates two causal loops, one for each measurement. Together, they locally cause the correlation between the two measurement outcomes.

B. Conclusion

In conclusion, the key finding of the present approach is that the ordinary quantum description of the processes between two one-to-one correlated measurement outcomes,

which leaves the causal structure that physically ensures the correlation free, is the quantum superposition of unobservable causal loops. Such causal loops, which of course vanish in the ordinary quantum description, describe the quantum process in a more complete way. In particular, they immediately explain the quantum speedup and quantum nonlocality.

Future work, at the fundamental level, could be to look for causal loops in any task that is quantumly possible and classically impossible and for the possibility of extending the present time-symmetrization formalism beyond one-to-one quantum correlation. Of course, if the probability of finding the solution is not exactly one, there can still be a quantum speedup and a causal loop that ensures it. At the technical level, the advanced knowledge rule could be used in the search for new quantum algorithms and to arrange oracle problems into quantum complexity classes. One could also see whether the present axiomatic derivation of the rule for finding the number of queries needed to solve an oracle problem in an optimal quantum way might suggest a corresponding analytic

derivation. About the possibility of cross fertilization between the foundations of physics and quantum computer science, see Ref. [32].

We hope that the present work will help to clear up the notion of retrocausality in physics, in its present form of unobservable quantum causal loops that vanish in the ordinary quantum description and thus cannot change the past as described by it. This notion could open up to physics a territory that was previously off limits.

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