Path-integral approach to the thermodynamics of bosons with memory: Partition function and specific heat

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For a system of bosons that interact through a class of general memory kernels, a recurrence relation for the partition function is derived within the path-integral formalism. This approach provides a generalization to previously known treatments in the literature of harmonically coupled systems of identical particles. As an example the result is applied to the specific heat of a simplified model of an open quantum system of bosons, harmonically coupled to a reservoir of distinguishable fictitious masses.

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I. INTRODUCTION

One of the peculiar features of the path-integral approach in statistical mechanics is the appearance of retarded interactions between particles, after integrating out the degrees of freedom of the environment that the particles couple to. This most famously appears in Feynman's variational treatment of the polaron problem [1], where the effective action functional of an electron interacting with a bath of phonons is obtained by integrating out the phonon degrees of freedom. The resulting action functional describes the electron interacting with itself at previous times through an interaction term that is moreover nonquadratic in the electron coordinate, prohibiting an analytical solution. A variational solution can be found by proposing a model action functional where this interaction term is replaced by a quadratic retarded interaction of the electron with itself, mediated by some memory kernel that depends on variational parameters. In Feynman's initial treatment of the problem, the model action is obtained by integrating out a harmonically coupled fictitious particle to the electron [2], which yields a memory kernel as a function of two variational parameters.

This approach has since then known various extensions towards either generalizations of the model action, applications to multiple particles, or entirely different physical systems [3,4]. In the context of a single solid-state polaron, instead of obtaining a model action by integrating out fictitious particles, which inevitably restricts the variational freedom of the memory kernel, an approach directly proposing the most general quadratic single-particle model action with memory has been studied in Ref. [5]. On the other hand, extensions towards multiple particles [6,7] or towards an arbitrary number of identical particles [8] have so far relied on using restricted model actions that are constructed by integrating out harmonically coupled fictitious particles. Even in the absence of a fictitious system the canonical treatment of identical particles in the path-integral formalism significantly complicates the expressions for the partition and correlation functions [9,10]. This naturally raises the question as to how these approaches could be extended to many identical particles using action functionals with general memory kernels, yielding an all-encompassing treatment. In this paper we answer the first part of this question by presenting a derivation of the partition function for such a general action functional. When necessary to emphasize that the memory kernels arise from the influence of an external system, we will also refer to this quantity as the reduced partition function.

It is important to emphasize that in the context of variational models, the environment often plays merely the role of an intermediary used to obtain a variationally suitable expression for the retarded interactions. However, the thermodynamics that follows from the reduced partition function, interpreted as a model for an open quantum system, has been shown to display particularly interesting behavior in itself [11,12]. As an example of an application we will show how our expression for the partition function with memory could be used to generalize the study of the specific heat for identical particles in this direction as well.

In this paper we consider the following action functional (we will work in units of $\hbar = 1$):

$$\begin{aligned} S^{(N)}[\overline{\mathbf{r}}, x, y, \overline{\boldsymbol{\kappa}}] &= \frac{m}{2}\sum_i^N \int_0^\beta \dot{\mathbf{r}}_i(\tau)^2 d\tau \\ &+ \frac{m}{2}\sum_i^N \int_0^\beta d\tau \int_0^\beta d\sigma x(\tau-\sigma)\mathbf{r}_i(\tau)\cdot\mathbf{r}_i(\sigma) \\ &+ \frac{m}{2N}\sum_{i,j}^N \int_0^\beta d\tau \int_0^\beta d\sigma [y(\tau-\sigma)-x(\tau-\sigma)]\mathbf{r}_i(\tau)\cdot\mathbf{r}_j(\sigma) \\ &- m\sum_i^N \int_0^\beta d\tau \mathbf{r}_i(\tau)\cdot\boldsymbol{\kappa}_i(\tau). \end{aligned} \tag{1}$$

This action contains the most general quadratic many-particle potential terms. For the functional arguments the notations $\overline{\mathbf{r}} = \mathbf{r}_1, \ldots, \mathbf{r}_N$ and $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_N$ are used. This (Euclidean) action functional describes N particles with mass m at temperature $(k_B\beta)^{-1}$ that interact through memory kernels

$x(\tau-\sigma)$ and $y(\tau-\sigma)$. The memory kernels generally represent the effect of some external system or medium that induces retarded interactions and would arise after integrating out the external system coupled to the particles. However, here they are taken to be completely general and can also be defined to include harmonic trapping potentials. In addition, we introduce a set of completely general vector source functions $\boldsymbol{\kappa}_i(\tau)$, which may represent time-dependent external forces on the particles but will mainly prove to be useful for calculating expectation values. Expression (1) can be rewritten to note that each particle interacts with itself through the memory kernel $\frac{1}{N}[(N-1)x(\tau-\sigma)+y(\tau-\sigma)]$ and with any other particle through the memory kernel $\frac{1}{N}[y(\tau-\sigma)-x(\tau-\sigma)]$, and hence the two can be tuned independently.

We restrict the memory kernels to be symmetric $(x(\tau),\ y(\tau)) = (x(-\tau),\ y(-\tau))$ and β-periodic $(x(\beta-\tau),\ y(\beta-\tau)) = (x(\tau),\ y(\tau))$. These are general properties of bosonic Green's functions [13] which are also assumed in the treatment for the single polaron in Ref. [5] and naturally arise in systems with a harmonic coupling to an external system [2,6–8,14–16]. In addition, we will assume that $\int_0^\beta x(\tau)d\tau \neq 0$ and $\int_0^\beta y(\tau)d\tau \neq 0$, so that we do not need to introduce a finite volume in our treatment—a technical step that occurs when taking the free-particle limit as the harmonic oscillator frequency tends to zero. We specifically consider three-dimensional systems and in further notation $d=3$, unless specified otherwise.

The goal of this work is to obtain a recurrence relation for the partition function of bosons described by the general action functional (1). To provide an example, we will apply our result to study the specific heat of the identical oscillator extension of the system in Ref. [11]. Our approach generalizes the previously known results for a system of harmonically coupled identical oscillators in Refs. [9,10], which corresponds to a specific choice of memory kernels in (1). First, in Sec. II we will extend the calculation performed in Ref. [17] to a many-particle system to obtain the distinguishable particle propagator corresponding to Eq. (1). Next, in Sec. III we will discuss which steps of Refs. [9,10] need to be generalized to take memory effects for identical particles into account. Therefore, in a way, this work can be seen as an application of the methods in Ref. [17] to generalize the approach in Ref. [9]. Finally, in Sec. IV we will apply the results to consider the specific heat of an open quantum system of bosons, where the effects of the environment are represented by a harmonic coupling to fictitious masses.

II. PROPAGATOR

Before taking the permutation symmetries of identical particles into account, first the many-particle propagator for N distinguishable particles has to be calculated:

$$K_N[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\beta|\overline{\mathbf{r}}_0,0)=\int_{\overline{\mathbf{r}}_0,0}^{\overline{\mathbf{r}}_T,\beta}\mathcal{D}\overline{\mathbf{r}}\,e^{-S^{(N)}[\overline{\mathbf{r}},x,y,\overline{\boldsymbol{\kappa}}]}. \tag{2}$$

The boundary points are indicated by $\overline{\mathbf{r}}_T=\overline{\mathbf{r}}(\beta)$ and $\overline{\mathbf{r}}_0=\overline{\mathbf{r}}(0)$. To emphasize that the expression for the propagator is still a functional of the memory kernels and source functions, this dependence on x, y, and $\overline{\boldsymbol{\kappa}}$ is indicated in the square brackets. The calculation of the propagator for $N=1$ has been performed in Ref. [17], and we largely base our derivation for the many-particle case in the rest of this section on the methods presented in Refs. [9,17].

For a quadratic action functional given by expression (1), the path integral can be expanded around the classical paths that minimize the action functional to write

$$\begin{aligned}&K_N[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\beta|\overline{\mathbf{r}}_0,0)\\&\quad=K_N[x,y,\mathbf{0}](0,\beta|0,0)e^{-S_{\rm cl}[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\overline{\mathbf{r}}_0)}.\end{aligned} \tag{3}$$

Here, $S_{\rm cl}[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\overline{\mathbf{r}}_0)$ is the action functional (1) evaluated along the classical paths that are found as solutions to the following set of integrodifferential equations:

$$\ddot{\mathbf{R}}(\tau)-\int_0^\beta y(\tau-\sigma)\mathbf{R}(\sigma)d\sigma+\mathbf{K}(\tau)=0, \tag{4}$$

$$\begin{aligned}&\ddot{\mathbf{r}}_i(\tau)-\int_0^\beta x(\tau-\sigma)\mathbf{r}_i(\sigma)d\sigma\\&\quad-\int_0^\beta[y(\tau-\sigma)-x(\tau-\sigma)]\mathbf{R}(\sigma)d\sigma+\boldsymbol{\kappa}_i(\tau)=0.\end{aligned} \tag{5}$$

The center-of-mass coordinate $\mathbf{R}=\frac{1}{N}\sum_{i=1}^N\mathbf{r}_i$ decouples together with the center-of-mass source term $\mathbf{K}=\frac{1}{N}\sum_{i=1}^N\boldsymbol{\kappa}_i$ yielding an equation that has already been solved in Ref. [17]. Having obtained a solution to Eq. (4), the last two terms in Eq. (5) can be seen as an effective source term, which allows us to solve Eq. (5) using the same approach. Substitution of the solutions into the action functional yields $S_{\rm cl}(\overline{\mathbf{r}}_T,\overline{\mathbf{r}}_0)$, which can then be used to derive the fluctuation factor $K_N[x,y,\mathbf{0}](0,\beta|0,0)$ in the same way as in Ref. [17]. This lengthy calculation can be somewhat shortened by writing the paths in terms of fluctuations around the center of mass, for which the derivation is presented in Appendix A.

As shown in Appendix A, the many-body propagator (2) factorizes in terms of single-particle propagators just as in the case of a harmonically coupled system [9]:

$$\begin{aligned}&K_N[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\beta|\overline{\mathbf{r}}_0,0)\\&\quad=\frac{K[y,\sqrt{N}\mathbf{K}](\sqrt{N}\mathbf{R}_T,\beta|\sqrt{N}\mathbf{R}_0,0)}{K[x,\sqrt{N}\mathbf{K}](\sqrt{N}\mathbf{R}_T,\beta|\sqrt{N}\mathbf{R}_0,0)}\\&\qquad\times\prod_{j=1}^N K[x,\boldsymbol{\kappa}_j](\mathbf{r}_{j,T},\beta|\mathbf{r}_{j,0},0).\end{aligned} \tag{6}$$

The propagators on the right-hand side of Eq. (6) are the single-particle propagators for which the action functional (1) depends on a single memory kernel, making the notation of Eq. (2) somewhat redundant. Hence let us separately define the single-particle propagator as a functional of only the memory kernel $x(\tau-\sigma)$:

$$K[x,\boldsymbol{\kappa}](\mathbf{r}_T,\beta|\mathbf{r}_0,0)=\int_{\mathbf{r}_0,0}^{\mathbf{r}_T,\beta}\mathcal{D}\mathbf{r}e^{-S^{(1)}[\mathbf{r},x,\boldsymbol{\kappa}]}, \tag{7}$$

where

$$\begin{aligned}S^{(1)}[\mathbf{r},x,\boldsymbol{\kappa}]=&\int_0^\beta\frac{m\dot{\mathbf{r}}^2}{2}d\tau\\&+\frac{m}{2}\int_0^\beta d\tau\int_0^\beta d\sigma x(\tau-\sigma)\mathbf{r}(\tau)\cdot\mathbf{r}(\sigma)\\&-m\int_0^\beta d\tau\mathbf{r}(\tau)\cdot\boldsymbol{\kappa}(\tau).\end{aligned} \tag{8}$$

In what follows, we will decompose the memory kernels and the source terms in their Fourier components x_n, y_n, and $\boldsymbol{\kappa}_n$, respectively, using the convention $f(\tau) = \sum_{n=-\infty}^{\infty} f_n e^{i\nu_n\tau}$, with $\nu_n = 2\pi n/\beta$ being the bosonic Matsubara frequencies. Following the method of Ref. [17] and assuming the same stability conditions, we derive the following expression for the single-particle propagator with memory:

$$K[x,\boldsymbol{\kappa}](\mathbf{r}_T,\beta|\mathbf{r}_0,0) = \left(\frac{m}{2\pi\beta}\right)^{d/2}\left(\frac{4}{\beta^3 x_0\Delta_x}\right)^{d/2}\frac{1}{\prod_{k=1}\left(1+\frac{\beta x_k}{\nu_k^2}\right)^d}\exp\left[-\frac{m}{2\beta}A_x(\mathbf{r}_T-\mathbf{r}_0)^2-\frac{m}{2\beta}\frac{1}{\Delta_x}(\mathbf{r}_T+\mathbf{r}_0)^2\right.$$
$$+\frac{2m}{\beta}\frac{1}{\Delta_x}\sum_n\frac{\boldsymbol{\kappa}_n}{\nu_n^2+\beta x_n}\cdot(\mathbf{r}_T+\mathbf{r}_0)-\frac{2m}{\beta}\left(\frac{\beta}{2}\sum_{n\neq 0}\frac{i\nu_n}{\nu_n^2+\beta x_n}\boldsymbol{\kappa}_n\right)\cdot(\mathbf{r}_T-\mathbf{r}_0)$$
$$\left.-\frac{2m}{\beta}\frac{1}{\Delta_x}\left(\sum_n\frac{\boldsymbol{\kappa}_n}{\nu_n^2+\beta x_n}\right)^2+\frac{2m}{\beta}\left(\frac{\beta^2}{4}\sum_n\frac{\boldsymbol{\kappa}_n\cdot\boldsymbol{\kappa}_{-n}}{\nu_n^2+\beta x_n}\right)\right]. \tag{9}$$

In Eq. (9) we have chosen a slightly different notation from that of Ref. [17] to define the following dimensionless functionals of the memory kernel x:

$$A_x = \sum_{n=-\infty}^{\infty}\frac{\beta x_n}{\nu_n^2+\beta x_n}, \tag{10}$$

$$\Delta_x = \frac{4}{\beta^2}\sum_{n=-\infty}^{\infty}\frac{1}{\nu_n^2+\beta x_n}. \tag{11}$$

In what follows we will generally assume $A_x > 0$ and $\Delta_x > 0$ to restrict ourselves to propagators (9) that are convergent for any combination of the boundary points. Note that due to the previous assumption of $x_0 \neq 0$ and $y_0 \neq 0$ the functionals are well defined when written in this form. Nevertheless, taking the limit $x_0, y_0 \to 0$ in the propagators still yields the appropriate expression, and this distinction will only become of importance in the partition function further on.

III. PARTITION FUNCTION FOR IDENTICAL PARTICLES WITH MEMORY

The path-integral approach is naturally extended to the treatment of identical particles by taking all possible permutations of the end points into account [2]. In this way, the canonical partition function for bosons is written as

$$\mathcal{Z}(N) = \frac{1}{N!}\sum_P\int d\overline{\mathbf{r}}\int_{\overline{\mathbf{r}},0}^{P[\overline{\mathbf{r}}],\beta}\mathcal{D}\overline{\mathbf{r}}'\, e^{-S^{(N)}[\overline{\mathbf{r}}',x,y,\overline{\kappa}]}. \tag{12}$$

The path integral counts all possible paths from an ordered set of initial points $\overline{\mathbf{r}} = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ to a final set of points $P[\overline{\mathbf{r}}] = \{P\mathbf{r}_1, P\mathbf{r}_2, \ldots, P\mathbf{r}_N\}$, where the coordinates are reordered by a permutation P on a set of N, using the commonly used notation $P\mathbf{r}_1 = \mathbf{r}_{P(1)}$. All possible values of the set $\overline{\mathbf{r}}$ are then integrated out, and the sum over all possible permutations P is finally taken. The treatment can be straightforwardly extended to fermions by adding a factor $(-1)^P$ that provides a minus sign to all odd permutations.

The propagator (6) exhibits the same factorization pattern as a harmonically coupled system of oscillators, and hence initially the approach of Ref. [9] can be followed. The integration over all possible boundary points $\overline{\mathbf{r}}$ can be extended to include the center-of-mass variable through the introduction of a delta function,

$$\int d\overline{\mathbf{r}} \to \int d\mathbf{R}\int d\overline{\mathbf{r}}\,\delta\left(\mathbf{R}-\frac{1}{N}\sum_i\mathbf{r}_i\right), \tag{13}$$

which is then written in its Fourier representation [9]. This allows us to separate the contribution of the center-of-mass propagators in Eq. (6) as follows:

$$\mathcal{Z}(N) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\,\mathcal{Z}_R(N,\mathbf{k})\mathcal{Z}_r(N,\mathbf{k}), \tag{14}$$

where

$$\mathcal{Z}_R(N,\mathbf{k}) = \int d\mathbf{R}\, e^{i\mathbf{k}\cdot\mathbf{R}}\frac{K[y,\mathbf{0}](\sqrt{N}\mathbf{R},\beta|\sqrt{N}\mathbf{R},0)}{K[x,\mathbf{0}](\sqrt{N}\mathbf{R},\beta|\sqrt{N}\mathbf{R},0)} \tag{15}$$

and

$$\mathcal{Z}_r(N,\mathbf{k}) = \frac{1}{N!}\sum_P\int d\overline{\mathbf{r}}\prod_{j=1}^N K[x,\mathbf{0}](P\mathbf{r}_j,\beta|\mathbf{r}_j,0)e^{-i\mathbf{k}\cdot\mathbf{r}_j/N}. \tag{16}$$

Note that we set the source functions $\boldsymbol{\kappa}_i = 0$, as their main purpose was in deriving the fluctuation factor, and from now on we consider the action functional (1) without source terms. The integral in expression (15) converges under the restriction $\Delta_x > \Delta_y$ and can be readily computed as a Gaussian integral, and calculating expression (16) will prove to be the main challenge. Following the standard approaches [2,9], any permutation P can be partitioned into M_ℓ disjoint permutation cycles of length ℓ, which allows us to write

$$\mathcal{Z}_r(N,\mathbf{k}) = \sum_{M_1,M_2,\ldots,M_N}^{*}\prod_{\ell=1}^N\frac{1}{\ell^{M_\ell}(M_\ell)!}h_\ell(\mathbf{k})^{M_\ell}, \tag{17}$$

where the $*$ symbol above the summation symbol indicates a constrained summation that has to obey $\sum_{\ell=1}^N \ell M_\ell = N$. In this representation the nested N-dimensional integral in expression (16) factorizes as a product of ℓ-fold integrals that correspond to each permutation cycle:

$$h_\ell(\mathbf{k}) = \int d\mathbf{r}_1\cdots\int d\mathbf{r}_\ell\, K[x,\mathbf{0}](\mathbf{r}_1,\beta|\mathbf{r}_\ell,0)\cdots$$
$$\times K[x,\mathbf{0}](\mathbf{r}_3,\beta|\mathbf{r}_2,0)K[x,\mathbf{0}](\mathbf{r}_2,\beta|\mathbf{r}_1 0)e^{-i\frac{1}{N}\mathbf{k}\cdot\sum_{j=1}^\ell\mathbf{r}_j}. \tag{18}$$

The next step is to obtain an expression for $h_\ell(\mathbf{k})$, which requires the computation of an ℓ-dimensional integral in expression (18). While high-dimensional Gaussian integrals can always in principle be calculated by converting them into a linear algebra problem of finding a determinant of an ℓ-dimensional matrix, finding an explicit expression for the latter is not always equally straightforward. In the approach of Ref. [9], which we have thus far followed very closely, the integral (18) is calculated by relying on the composition property of the propagators. If the composition property holds, then $h_\ell(\mathbf{k})$ becomes the single-particle partition function of exactly the same system as described by the single-particle propagator, but at an inverse temperature $\ell\beta$ and with additional delta kicks to account for the $\mathbf{k}$ exponent. This partition function can then be readily computed with standard path integration methods. This trick is not applicable here, as the propagator with memory (9) does not obey the composition property. This can be easily seen by noting that the action functional (1) cannot just be split into a sum of two parts on respective time intervals. In Appendix B we show how integral (18) can be directly computed and obtain the following result in d dimensions:

$$h_\ell(\mathbf{k}) = Q_x^{\ell d} \frac{1}{\left|2\sinh\left(\frac{\ell}{2}\operatorname{arccosh}\left[\frac{A_x\Delta_x+1}{A_x\Delta_x-1}\right]\right)\right|^d} \times \exp\left(-\frac{\ell k^2\beta}{8N^2m}\Delta_x\right), \tag{19}$$

where

$$Q_x = \frac{1}{\prod_{k=1}\left(1+\frac{\beta x_k}{\nu_k^2}\right)}\left(\frac{1}{\beta^3 x_0}\frac{4}{|A_x\Delta_x-1|}\right)^{1/2}. \tag{20}$$

The functional form of $h_\ell(\mathbf{k})$ is very similar to that found in Ref. [9]. The main differences are that the oscillator-frequency-dependent parts are now replaced by expressions containing Δ_x and A_x, functionals of the memory kernel, appearing in the argument of the hyperbolic sine and the exponential. An additional factor Q_x appears, which equals 1 when the memory kernel x corresponds to a harmonic oscillator without memory.

The choice of writing expression (19) in terms of the hyperbolic sine has the advantage of being maximally illustrative in regard to how changes due to memory arise on top of previously known expressions in Ref. [9]. However, due to this choice some particular care should be taken when $\Delta_x A_x < 1$. In this case each of the two factors in the determinant (B14) in Appendix B can become negative, and the complex modulus should be added after taking the square root if the factors are to be separated as in (19) and (20). For simple harmonic oscillator systems, and the model system considered in Sec. IV, $\Delta_x A_x > 1$ and this subtlety can be safely ignored.

The expression for the partition function (14) can now be computed. The center of mass $\mathcal{Z}_R(N,\mathbf{k})$ can be calculated from the propagators, and now that the $\mathbf{k}$ dependence of $h_\ell(\mathbf{k})$ is known, the $\mathbf{k}$ integral in (14) can be performed. After some algebraic work, one obtains

$$\mathcal{Z}(N) = \mathbb{Z}(N) Q_x^{Nd}\left(\frac{\beta x_0}{\beta y_0}\right)^{d/2}\prod_{k=1}^{\infty}\left(\frac{1+\frac{\beta x_k}{\nu_k^2}}{1+\frac{\beta y_k}{\nu_k^2}}\right)^d, \tag{21}$$

with $\mathbb{Z}$ given by

$$\mathbb{Z}(N) = \sum_{M_1,M_2,\ldots,M_N}^{*}\prod_{\ell=1}^{N}\frac{1}{\ell^{M_\ell}(M_\ell)!}\frac{1}{\left|2\sinh\left(\frac{\ell}{2}\operatorname{arccosh}\left[\frac{A_x\Delta_x+1}{A_x\Delta_x-1}\right]\right)\right|^{M_\ell d}}. \tag{22}$$

Note that due to the presence of the additional factor in expression (21) it is now the product $Q_x^{Nd}\mathbb{Z}(N)$ that represents the partition function in the absence of two-body interactions, extending the result of Ref. [9]. Following the approach in Ref. [9], the constrained summation (22) can be transformed into a recurrence relation:

$$\mathbb{Z}(N) = \frac{1}{N}\sum_{k=0}^{N-1}\mathbb{Z}(k)\left|2\sinh\left[\frac{(N-k)}{2}\operatorname{arccosh}\left(\frac{A_x\Delta_x+1}{A_x\Delta_x-1}\right)\right]\right|^{-d}. \tag{23}$$

The recurrence relation requires an initial value, and it can be seen that $\mathbb{Z}(0)=1$ yields the correct $\mathcal{Z}(1)$ result according to expression (21). Alternatively, the factor Q_x could be absorbed in the definition of $\mathbb{Z}(N)$, but then the recurrence would have to start from $\mathbb{Z}(0)=Q_x^{Nd}$.

As a consistency check, consider the specific choice $x(\tau-\sigma)=w^2\delta(\tau-\sigma)$ and $y(\tau-\sigma)=\Omega^2\delta(\tau-\sigma)$ for which the action functional (1) exactly corresponds with the system of coupled oscillators in Ref. [9]. The different Matsubara sums and products in Eq. (21) can now be readily computed to find

$$\left(\frac{\beta x_0}{\beta y_0}\right)^{d/2}\prod_{k=1}^{\infty}\left(\frac{1+\frac{\beta x_k}{\nu_k^2}}{1+\frac{\beta y_k}{\nu_k^2}}\right)^d = \frac{\sinh\left(\frac{\beta w}{2}\right)^d}{\sinh\left(\frac{\beta\Omega}{2}\right)^d} \tag{24}$$

and $Q_x=1$. In particular, the resulting hyperbolic cosine from

$$\frac{A_x\Delta_x+1}{A_x\Delta_x-1} = \cosh(\beta w) \tag{25}$$

allows us to cancel the inverse hyperbolic cosine in the weight factor of the recurrence relation (23). Substituting these results, the expression for the partition function in Ref. [9] is exactly retrieved in this limit.

IV. EXAMPLE APPLICATION: OPEN QUANTUM SYSTEM OF IDENTICAL OSCILLATORS

In this section we present a brief example application of the derived expressions to a stylized model of an open quantum system of identical particles. We consider the system depicted in Fig. 1 of noninteracting bosons in a harmonic trap with frequency Ω, coupled to an environment. The effect of the environment is modeled as a harmonic coupling with frequency W of each boson to a fictitious particle with mass M. This model corresponds to a particular equal-particle case of the more general models studied in Refs. [18,19] for distinguishable particles of the system, which we will here consider for bosons. Note that because of the Bose statistics that have to be imposed, this model is more than simply N unrelated copies of a two-particle system.

The fictitious particles are taken to be uncoupled and distinguishable, which could represent an environment with a far slower relaxation than the bosonic system. The (Euclidean) Lagrangian of the full system corresponding to the partition

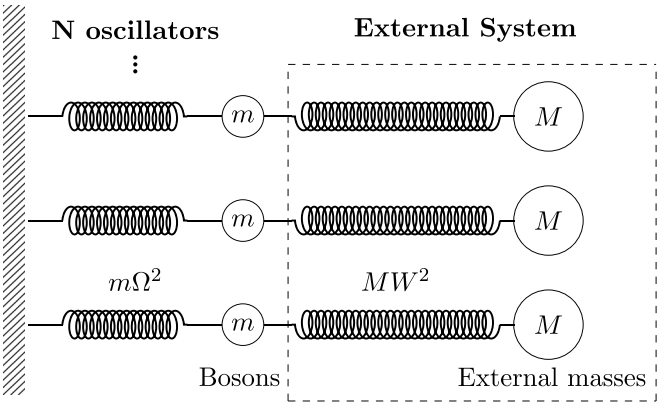

FIG. 1. A one-dimensional depiction of the system described by (26).

function $\mathcal{Z}_{\rm tot}$ is therefore given by

$$L_{\rm tot} = \sum_{i=1}^{N}\left(\frac{m}{2}\dot{\mathbf{r}}_i^2 + \frac{m\Omega^2}{2}\mathbf{r}_i^2 + \frac{M}{2}\dot{\mathbf{Q}}_i^2 + \frac{MW^2}{2}(\mathbf{r}_i - \mathbf{Q}_i)^2\right), \tag{26}$$

and the Lagrangian of the external system is defined as

$$L_{\rm f} = \sum_{i=1}^{N}\left(\frac{M}{2}\dot{\mathbf{Q}}_i^2 + \frac{MW^2}{2}\mathbf{Q}_i^2\right). \tag{27}$$

It is important to note that through (27) we adopt the view of the external system as defined in Ref. [11]. The external system is considered to be the whole of the fictitious particles and springs with constant MW^2 that are attached to the degrees of freedom of interest. The fictitious particles can be integrated out as in Ref. [14] to obtain the reduced partition function:

$$\mathcal{Z}(N) = \frac{\mathcal{Z}_{\rm tot}(N)}{\mathcal{Z}_{\rm f}(N)}, \tag{28}$$

where $\mathcal{Z}_{\rm f}$ is the partition function of the external system corresponding to (27). Expression (28) is exactly the identical-particle extension of one of the stylized models of an open quantum system considered in Ref. [11]. The resulting $\mathcal{Z}(N)$ can now be cast in the form of (12), where the memory kernels in the action functional (1) are given by

$$x(\tau-\sigma) = y(\tau-\sigma) = \frac{MW^2}{m}\left[\frac{W^2 + \frac{m}{M}\Omega^2}{W^2}\delta(\tau-\sigma) - \frac{W\cosh(W[|\tau-\sigma| - \beta/2])}{2\sinh(W\beta/2)}\right]. \tag{29}$$

This is the simplest translationally noninvariant model that provides a memory kernel $x(\tau-\sigma)$ with nontrivial memory effects for the recurrence relation (23). The functionals A_x and Δ_x are obtained after computing the Matsubara summations in expressions (10) and (11):

$$A_x = \frac{\beta\omega_+}{2}\coth\left(\frac{\beta\omega_+}{2}\right)\gamma_+ + \frac{\beta\omega_-}{2}\coth\left(\frac{\beta\omega_-}{2}\right)\gamma_-, \tag{30}$$

$$\Delta_x = \frac{2}{\beta\omega_+}\coth\left(\frac{\beta\omega_+}{2}\right)\gamma_+ + \frac{2}{\beta\omega_-}\coth\left(\frac{\beta\omega_-}{2}\right)\gamma_-, \tag{31}$$

where

$$\omega_\pm^2 = \frac{\frac{m+M}{m}W^2 + \Omega^2 \pm \sqrt{\left(\frac{m+M}{m}W^2 + \Omega^2\right)^2 - 4W^2\Omega^2}}{2}, \tag{32}$$

$$\gamma_\pm = \frac{1}{2}\left[1 \pm \frac{\Omega^2 + \left(\frac{M}{m} - 1\right)W^2}{\omega_+^2 - \omega_-^2}\right]. \tag{33}$$

The frequencies $\omega_\pm$ that diagonalize the full system [11] therefore naturally appear in the calculation.

Since for this system $y_n = x_n$, the interaction factor in front of (21) cancels out, and the partition function $\mathcal{Z}(N)$ is written as a product of only two factors, Q_x^{Nd} and the recurrence part $\mathbb{Z}(N)$. The Matsubara product in (20) can be computed for the specific memory kernel (29), which allows us to write

$$Q_x = \frac{2\sinh\left(\frac{\beta W}{2}\right)}{\beta W}\frac{\beta\omega_+}{2\sinh\left(\frac{\beta\omega_+}{2}\right)}\frac{\beta\omega_-}{2\sinh\left(\frac{\beta\omega_-}{2}\right)} \times\left(\frac{1}{\beta^3 x_0}\frac{4}{\Delta_x A_x - 1}\right)^{1/2}, \tag{34}$$

with Δ_x and A_x known from (31) and (30). In three dimensions the recurrence relation (23) for $\mathbb{Z}(N)$ has no known solution and has to be computed numerically. As shown in the approach of Ref. [9] a numerically stable implementation is obtained by defining

$$b = e^{-q}, \quad q = \operatorname{arccosh}\left[\frac{\Delta_x A_x + 1}{\Delta_x A_x - 1}\right], \tag{35}$$

and without loss of generality proposing the following way of writing the recurrence factor:

$$\mathbb{Z}(N) = \prod_{j=1}^{N}\rho_j\frac{b^{\frac{3}{2}}}{(1-b^j)^3}. \tag{36}$$

This fixes the first coefficient $\rho_1 = 1$, and after substitution of (36) into (23) a recurrence relation for ρ_N is found:

$$\rho_N = \frac{1}{N}\frac{(1-b^N)^3}{(1-b)^3}\left[1 + \sum_{k=0}^{N-2}\frac{(1-b)^3}{(1-b^{(N-k)})^3}\prod_{j=k+1}^{N-1}\frac{(1-b^j)^3}{\rho_j}\right]. \tag{37}$$

Due to the additional factor in the expression for the partition function $\mathcal{Z}(N) = Q_x^{Nd}\mathbb{Z}(N)$, the internal energy and specific heat of the system are written as a sum of two terms:

$$U(N) = U_Q(N) + \mathbb{U}(N) = -3N\partial_\beta\ln(Q_x) - \partial_\beta\ln(\mathbb{Z}), \tag{38}$$

$$C(N) = C_Q(N) + \mathbb{C}(N) = 3Nk_B\beta^2\partial_\beta^2\ln(Q_x) + k_B\beta^2\partial_\beta^2\ln(\mathbb{Z}). \tag{39}$$

Analytical expressions for U_Q and C_Q can straightforwardly be calculated from the factor Q_x in (34). The recurrence relations for $\mathbb{U}(N)$ and $\mathbb{C}(N)$ are obtained after computing the partial derivatives of $\ln(\mathbb{Z})$ by combining (36) with (37):

$$\frac{\mathbb{U}(N)}{\partial_\beta q} = \frac{1}{N}\frac{1}{\rho_N}\frac{(1-b^N)^3}{(1-b)^3}\left(\frac{\mathbb{U}(N-1)}{\partial_\beta q} + \frac{3}{2}\frac{1+b}{1-b} + \sum_{k=0}^{N-2}\frac{(1-b)^3}{(1-b^{(N-k)})^3}\left[\frac{\mathbb{U}(k)}{\partial_\beta q} + \frac{3(N-k)}{2}\frac{1+b^{(N-k)}}{1-b^{(N-k)}}\right]\prod_{j=k+1}^{N-1}\frac{(1-b^j)^3}{\rho_j}\right) \tag{40}$$

and

$$\mathbb{C}(N)k_B^{-1} = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{(1-b^{(N-k)})^3}\prod_{j=k+1}^{N}\frac{(1-b^j)^3}{\rho_j}\left(k_B^{-1}\mathbb{C}(k) + \beta^2\left[\frac{3(N-k)}{2}\frac{1+b^{(N-k)}}{1-b^{(N-k)}}\partial_\beta q + \mathbb{U}(k) - \mathbb{U}(N)\right]\right.$$
$$\left.\times\left[\frac{3(N-k)}{2}\frac{1+b^{(N-k)}}{1-b^{(N-k)}}\partial_\beta q + \mathbb{U}(k)\right] + \beta^2 3(N-k)^2\frac{b^{(N-k)}}{(1-b^{(N-k)})^2}(\partial_\beta q)^2 - \beta^2\frac{3(N-k)}{2}\frac{1+b^{(N-k)}}{1-b^{(N-k)}}\partial_\beta^2 q\right). \tag{41}$$

Here, the recurrence formulas are initiated from $\mathbb{U}(0) = 0$ and $\mathbb{C}(0) = 0$, and the partial derivatives $\partial_\beta q$ and $\partial_\beta^2 q$ can be analytically computed from (35) since Δ and A are known.

The specific heat (39) is shown in Fig. 2 as a function of the temperature, measured with respect to the critical temperature in the absence of the external system $k_B T_c = \hbar\Omega[N/\zeta(3)]^{1/3}$, with $\zeta(x)$ being the Riemann zeta function. We can clearly observe the main bosonic condensation peak slightly below the critical temperature, which at weak coupling corresponds exactly to the result in Ref. [9]. The sharpness of the peak fades towards stronger coupling with the external system but nevertheless remains visibly present. In addition to the main condensation peak, at an intermediate coupling strength an anomalous dip and peak are observed at low temperatures. These anomalous features in the specific heat of open quantum systems have been studied for distinguishable particles in Refs. [11,19,20], where it is shown that the specific heat can even become negative for certain systems. This is explained in Refs. [11,19] by the fact that the specific heat (39) is the difference of the specific heats of the system and the trapped fictitious particles as defined in the partition function (28), and a more extensive interpretation can be found in Ref. [20].

We can also note that the high- and low-temperature limits of the specific heat are in agreement with Ref. [19]. From expression (34) we can see that at high temperatures for $\beta \to 0$, Q_x approaches a finite value and hence the first part of the specific heat $C_Q(N)$ in (39) goes to zero. In the same limit the recurrence part of the partition function can be shown to diverge as $\mathbb{Z} \sim \beta^{-Nd}$, from which follows $C(N) = 3Nk_B$. In the low-temperature limit $\beta \to \infty$, one can show that in the presence of the environment, $\mathbb{Z}$ remains finite and Q_x becomes an exponential function of β, from which follows $C(N) = 0$.

An overview of the structure of the main condensation peak and the anomalous dip is presented in Fig. 3. For both a light and a heavy mass M of the fictitious particles, remnants of the bosonic condensation peak remain visible up to strong coupling with the external system. At low temperatures and

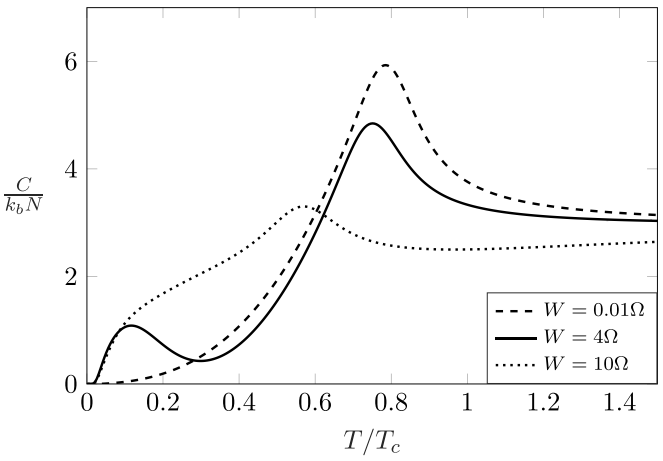

FIG. 2. Specific heat per particle as a function of the temperature of $N = 100$ noninteracting bosons in a harmonic potential, harmonically coupled to external masses for $M = m$. The results are shown for three coupling strengths $W = [0.01\Omega, 4\Omega, 10\Omega]$ plotted by the dashed, solid, and dotted lines, respectively

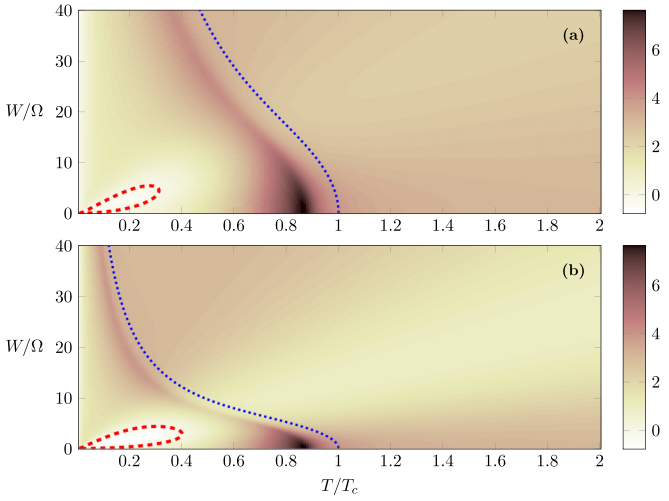

FIG. 3. Color map of the specific heat per particle $C/(k_B N)$ for $N = 500$ bosons, for (a) $M = m$ and (b) $M = 10m$. The dashed loop in the bottom left corner of each color map indicates the region where the specific heat becomes negative. The dotted line indicates the effective temperature obtained from (43).

weaker coupling the anomalous dip can be seen as a region of lighter shading. In contrast to the single-particle case for this system [11], we find that the anomalous dip can drop below zero for bosons in Fig. 3, where the dashed loop indicates a region of negative specific heat.

As can also be seen from Fig. 3, coupling with the external system significantly lowers the effective critical temperature of the bosons. This can be understood by noting that the generalized bosonic recurrence relation (23) is nothing else than the recurrence relation for harmonically trapped bosons where the trap frequency is replaced by a temperature-dependent quantity:

$$\tilde{\Omega}(T) = \frac{1}{\beta}\operatorname{arccosh}\left[\frac{\Delta_x A_x + 1}{\Delta_x A_x - 1}\right], \tag{42}$$

which allows us to define an effective critical temperature as the solution of

$$\frac{\tilde{T}_c}{T_c} = \frac{\tilde{\Omega}(\tilde{T}_c)}{\Omega}. \tag{43}$$

The results are plotted as the dotted lines in Fig. 3 and agree well with the behavior of the condensation peak. It is important to note that only the recurrence part is correctly reproduced by substituting $\Omega \to \tilde{\Omega}(T)$ in the harmonic oscillator result. The factor Q_x in front of the partition function (21) is not retrieved this way because it is entirely absent in the harmonic case. As the latter, however, can be taken out of the recurrence relation, it is no surprise that it should play no significant role in the inherently bosonic features of the system, and the behavior of the condensation peak is accurately reproduced by (43).

V. CONCLUSION

In this paper we presented an approach that incorporates the effects of retarded interactions in the path-integral formalism for identical particles. First, the many-body propagator for distinguishable particles was derived and shown to exhibit the same factorization pattern in terms of single-particle propagators as seen in harmonically coupled systems without retardation [9]. However, the main difference is that the single-particle propagators no longer obey the composition property when the system has memory. This complicates the computation of a class of integrals appearing in the derivation of the partition function, for which we obtain explicit expressions by utilizing the properties of circulant matrices. The resulting expression for the partition function is a functional applicable to a general class of memory kernels and is shown to reduce to the known result for harmonically coupled systems without memory in the appropriate limit.

The results were then applied to study the specific heat of noninteracting bosons in a harmonic trap coupled to an external system of fictitious masses. This provides the simplest model system that yields nontrivial memory effects in the condensation recurrence relation. We show that the presence of the environment shifts the bosonic condensation to lower temperatures and significantly smooths out the Bose condensation peak in the specific heat, which nevertheless remains visible even at strong coupling. To better understand these types of open systems and, in particular, to calculate the density and the pair correlation function, expressions for the identical-particle one- and two-point generating functionals are required. The results presented here pave the way to compute these quantities. These will in turn allow one to study the autocorrelation functions and occupation numbers and formulate the most general harmonic variational approach for identical particles.

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APPENDIX A: DERIVATION OF THE DISTINGUISHABLE PARTICLE PROPAGATOR

For the single-particle limit of (1), the classical action is calculated in Ref. [17]. For completeness and due to slightly different notations, we briefly summarize the calculation below. Consider the single-particle action functional:

$$\begin{aligned} S^{(1)}[\mathbf{r}, x, \boldsymbol{\kappa}] = &\int_0^\beta \frac{m\dot{\mathbf{r}}^2}{2} d\tau \\ &+ \frac{m}{2}\int_0^\beta d\tau \int_0^\beta d\sigma x(\tau-\sigma)\mathbf{r}(\tau)\cdot\mathbf{r}(\sigma) \\ &- m\int_0^\beta d\tau \mathbf{r}(\tau)\cdot\boldsymbol{\kappa}(\tau). \end{aligned} \tag{A1}$$

The classical path is found as the solution to the following integrodifferential equation with boundary conditions $\mathbf{r}_T = \mathbf{r}(\beta)$ and $\mathbf{r}_0 = \mathbf{r}(0)$:

$$\ddot{\mathbf{r}}(\tau) - \int_0^\beta x(\tau-\sigma)\mathbf{r}(\sigma)d\sigma + \boldsymbol{\kappa}(\tau) = 0. \tag{A2}$$

In Ref. [17], the following Fourier decomposition is proposed:

$$\begin{aligned} \mathbf{r}_{\mathrm{cl}}(\tau) = &\mathbf{r}_0 + (\mathbf{r}_T - \mathbf{r}_0)\frac{\tau}{\beta} - \frac{\mathbf{A}_0}{2}\tau(\tau-\beta) \\ &+ \sum_{n\neq 0}\frac{\mathbf{A}_n}{\nu_n^2}(e^{i\nu_n\tau} - 1), \end{aligned} \tag{A3}$$

where after substitution into (A2), the following solutions are found (assuming $x_0 \neq 0$; otherwise the appropriate limit should be taken):

$$\mathbf{A}_0 = \frac{4}{\beta^2\Delta_x}\left(\sum_n \frac{\boldsymbol{\kappa}_n}{\nu_n^2 + \beta x_n} - \frac{1}{2}(\mathbf{r}_T + \mathbf{r}_0)\right), \tag{A4}$$

$$\mathbf{A}_n = \frac{\beta x_n}{\nu_n^2 + \beta x_n}\mathbf{A}_0 + \frac{1}{\left(1 + \frac{\beta x_n}{\nu_n^2}\right)}\left(\boldsymbol{\kappa}_n + x_n\frac{\mathbf{r}_T - \mathbf{r}_0}{i\nu_n}\right). \tag{A5}$$

The coefficients can be substituted into (A3) to obtain an explicit expression for the classical solution $\mathbf{r}_{\mathrm{cl}}(\tau)$ and its Fourier components $\mathbf{r}_n$. After integrating the kinetic energy by parts and writing the remaining source term integral in Fourier

space, the classical action can be written as

$$S_{\rm cl}^{(1)}[x,\boldsymbol{\kappa}](\mathbf{r}_T,\mathbf{r}_0) = \tfrac{m}{2}[\dot{\mathbf{r}}_{\rm cl}(\beta)\cdot\mathbf{r}_T - \dot{\mathbf{r}}_{\rm cl}(0)\cdot\mathbf{r}_0] - \tfrac{m\beta}{2}\textstyle\sum_n \mathbf{r}_n\cdot\boldsymbol{\kappa}_{-n}. \tag{A6}$$

By taking the derivative of (A3) and substituting its boundary points to find the first part, and performing the Fourier sum using $\mathbf{r}_n = \frac{\boldsymbol{\kappa}_n - \mathbf{A}_n}{\beta x_n}$ to find the second part, the single-particle classical action becomes

$$S_{\rm cl}^{(1)}[x,\boldsymbol{\kappa}](\mathbf{r}_T,\mathbf{r}_0) = \frac{m}{2\beta}A_x(\mathbf{r}_T-\mathbf{r}_0)^2 + \frac{m}{2\beta}\frac{1}{\Delta_x}(\mathbf{r}_T+\mathbf{r}_0)^2 - \frac{2m}{\beta}\frac{1}{\Delta_x}(\mathbf{r}_T+\mathbf{r}_0)\cdot\sum_n\frac{\boldsymbol{\kappa}_n}{\nu_n^2+\beta x_n} + \frac{2m}{\beta}(\mathbf{r}_T-\mathbf{r}_0)\cdot\left(\frac{\beta}{2}\sum_n\frac{i\nu_n\boldsymbol{\kappa}_n}{\nu_n^2+\beta x_n}\right) + \frac{2m}{\beta}\frac{1}{\Delta_x}\left(\sum_n\frac{\boldsymbol{\kappa}_n}{\nu_n^2+\beta x_n}\right)^2 - \frac{2m}{\beta}\left(\frac{\beta^2}{4}\sum_n\frac{\boldsymbol{\kappa}_n\cdot\boldsymbol{\kappa}_{-n}}{\nu_n^2+\beta x_n}\right). \tag{A7}$$

For the source terms, some care should be taken regarding pointwise convergence when performing calculations in Fourier space, as pointed out in Ref. [17]. For example, when considering a source function $\boldsymbol{\kappa}(\tau) = f\delta(\tau-\sigma)$ for $\sigma = 0$ or $\sigma = \beta$, the correct result should be derived by considering $\sigma \in]0,\beta[$ and respectively taking the limit of $\sigma \to 0^+$ or $\sigma \to \beta^-$ rather than direct substitution due to discontinuities at the edge. Taking care of the appropriate limits, the known results for, e.g., the harmonic oscillator or the kicked particle are readily obtained from (A7).

To obtain the many-particle extension of this result for the action functional (1), a similar but lengthier calculation was performed starting from Eqs. (4) and (5) by first finding $\mathbf{R}_{\rm cl}(\tau)$ with the previous method and then using this result to solve the equation for $\mathbf{r}_{\rm cl}^{(i)}(\tau)$. However, in line with Ref. [9], a somewhat shorter argument yielding the same result can be formulated by switching to the variable $\mathbf{u}_i = \mathbf{r}_i - \mathbf{R}$ at the level of the classical equations:

$$\ddot{\mathbf{R}}(\tau) - \int_0^\beta y(\tau-\sigma)\mathbf{R}(\sigma)d\sigma + \mathbf{K}(\tau) = 0, \tag{A8}$$

$$\ddot{\mathbf{u}}_i(\tau) - \int_0^\beta x(\tau-\sigma)\mathbf{u}_i(\sigma)d\sigma + \boldsymbol{\kappa}_i(\tau) - \mathbf{K}(\tau) = 0, \tag{A9}$$

with boundary conditions $\mathbf{u}_{i,(T,0)} = \mathbf{r}_{i,(T,0)} - \mathbf{R}_{(T,0)}$. In addition, the solution is subject to the constraint $\sum_i \mathbf{u}_i(\tau) = \mathbf{0}$. The many-body classical action corresponding to (1), written in terms of the coordinates $\mathbf{u}_i$ and $\mathbf{R}$, yields

$$S_{\rm cl}[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\overline{\mathbf{r}}_0) = \sum_{i=1}^N S_{\rm cl}^{(1)}[x,\boldsymbol{\kappa}_i-\mathbf{K}](\mathbf{u}_{i,T},\mathbf{u}_{i,0}) + S_{\rm cl}^{(1)}[y,\sqrt{N}\mathbf{K}](\sqrt{N}\mathbf{R}_T,\sqrt{N}\mathbf{R}_0). \tag{A10}$$

Here, we have used the property $\sum_i \mathbf{u}_i(\tau) = \mathbf{0}$ to drop a number of terms and add an additional source term in $\mathbf{K}(\tau)$ to obtain the difference of source terms $\boldsymbol{\kappa} - \mathbf{K}$ in the first term of (A10). Through direct substitution of the boundary conditions $\mathbf{u}_{i,(T,0)} = \mathbf{r}_{i,(T,0)} - \mathbf{R}_{(T,0)}$ and source term $\boldsymbol{\kappa}_i - \mathbf{K}$ into (A7), one can easily confirm that

$$S_{\rm cl}[x,y,\overline{\boldsymbol{\kappa}}](\overline{\mathbf{r}}_T,\overline{\mathbf{r}}_0) = \sum_{i=1}^N S_{\rm cl}^{(1)}[x,\boldsymbol{\kappa}_i](\mathbf{r}_{i,T},\mathbf{r}_{i,0}) + S_{\rm cl}^{(1)}[y,\sqrt{N}\mathbf{K}](\sqrt{N}\mathbf{R}_T,\sqrt{N}\mathbf{R}_0) - S_{\rm cl}^{(1)}[x,\sqrt{N}\mathbf{K}](\sqrt{N}\mathbf{R}_T,\sqrt{N}\mathbf{R}_0). \tag{A11}$$

Next, we have to find the fluctuation factor of the propagator $K_N[x,y,\mathbf{0}](0,\beta|0,0)$ as defined in Sec. II. While the decomposition of the classical action (A11) strongly suggests a similar factorization for the fluctuation factor, let us present a complete overview of the calculation. Following the approach in Ref. [17], we consider the many-particle fluctuation factor $K_N[\lambda x,\lambda y,\overline{\mathbf{0}}](0,\beta|0,0)$, where the memory kernels are scaled by a variable λ, and define

$$J(\lambda) = \ln\{K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)\}. \tag{A12}$$

The logarithm of the fluctuation factor $J(1)$ can then be written as

$$J(1) = J(0) + \int_0^1 d\lambda\frac{\partial J(\lambda)}{\partial\lambda} = J(0) + \int_0^1 d\lambda\frac{\frac{\partial}{\partial\lambda}K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)}{K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)}, \tag{A13}$$

where of course $J(0) = \frac{Nd}{2}\ln(\frac{m}{2\pi\beta})$ is the known free-particle result in d dimensions. In path-integral notation (2), one can write

$$\frac{\partial}{\partial\lambda}K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0) = -\int_{0,0}^{0,\beta}\mathcal{D}\overline{\mathbf{r}}\left(\frac{m}{2}\sum_i^N\int_0^\beta d\tau\int_0^\beta d\sigma x(\tau-\sigma)\mathbf{r_i}(\tau)\cdot\mathbf{r_i}(\sigma) + \frac{m}{2N}\sum_{i,j}\int_0^\beta d\tau\int_0^\beta d\sigma[y(\tau-\sigma)-x(\tau-\sigma)]\mathbf{r_i}(\tau)\cdot\mathbf{r_j}(\sigma)\right)e^{-S^{(N)}[\overline{\mathbf{r}},\lambda x,\lambda y,\mathbf{0}]}. \tag{A14}$$

By making use of functional derivatives with respect to the source terms $\boldsymbol{\kappa}_i$ and taking them out of the path integral, the propagator fraction in the λ integral of (A13) can be written as

$$\frac{\frac{\partial}{\partial\lambda}K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)}{K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)} = -\left(\frac{1}{2m}\sum_i^N\int_0^\beta d\tau\int_0^\beta d\sigma x(\tau-\sigma)\frac{\delta}{\delta\boldsymbol{\kappa}_i(\tau)}\cdot\frac{\delta}{\delta\boldsymbol{\kappa}_i(\sigma)}\right.$$
$$\left.+\frac{1}{2Nm}\int_0^\beta d\tau\int_0^\beta d\sigma[y(\tau-\sigma)-x(\tau-\sigma)]\sum_i\frac{\delta}{\delta\boldsymbol{\kappa}_i(\tau)}\cdot\sum_j\frac{\delta}{\delta\boldsymbol{\kappa}_j(\sigma)}\right)e^{-S_{\rm cl}[\lambda x,\lambda y,\bar{\boldsymbol{\kappa}}](0,0)}\Big|_{\bar{\boldsymbol{\kappa}}=0}. \tag{A15}$$

Since $S_{\rm cl}[\lambda x,\lambda y,\bar{\boldsymbol{\kappa}}](0,0)$ is known, the functional derivatives can be straightforwardly performed to obtain

$$\frac{\frac{\partial}{\partial\lambda}K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)}{K_N[\lambda x,\lambda y,\mathbf{0}](0,\beta|0,0)} = \frac{d}{2}(N-1)\left[\left(\sum_n\frac{1}{\nu_n^2+\lambda\beta x_n}\right)^{-1}\sum_n\frac{\beta x_n}{\left(\nu_n^2+\lambda\beta x_n\right)^2}-\sum_n\frac{\beta x_n}{\nu_n^2+\lambda\beta x_n}\right]$$
$$+\frac{d}{2}\left[\left(\sum_n\frac{1}{\left(\nu_n^2+\lambda\beta y_n\right)^2}\right)^{-1}\sum_n\frac{\beta y_n}{\left(\nu_n^2+\lambda\beta y_n\right)^2}-\sum_n\frac{\beta y_n}{\nu_n^2+\lambda\beta y_n}\right]. \tag{A16}$$

The λ integral in (A13) can now be analytically computed to finally obtain the many-body fluctuation factor:

$$K_N[x,y,\mathbf{0}](0,\beta|0,0) = K[x,\mathbf{0}](0,\beta|0,0)^{(N-1)}K[y,\mathbf{0}](0,\beta|0,0), \tag{A17}$$

where the single-particle fluctuation factor in d dimensions is given by

$$K[x,\mathbf{0}](0,\beta|0,0) = \left(\frac{m}{2\pi\beta}\right)^{\frac{d}{2}}\left(\frac{4}{\beta^3x_0\Delta_x}\right)^{\frac{d}{2}}\left(\frac{1}{\prod_{k=1}^\infty\left(1+\frac{\beta x_k}{\nu_k^2}\right)}\right)^d. \tag{A18}$$

This result together with (3) and (A11) proves the factorization of the propagator in (6).

APPENDIX B: EXPLICIT EVALUATION OF CLOSED-LOOP GAUSSIAN INTEGRALS

Let us start by defining a shorthand notation for the single-particle propagator (9) with $\boldsymbol{\kappa}_i = 0$:

$$K[x,\mathbf{0}](\mathbf{r}_T,\beta|\mathbf{r}_0,0) = \mathcal{A}^d\exp[-a(\mathbf{r}_T-\mathbf{r}_0)^2-b(\mathbf{r}_T+\mathbf{r}_0)^2], \tag{B1}$$

where $a=\frac{m}{2\beta}A_x$, $b=\frac{m}{2\beta}\frac{1}{\Delta_x}$, and

$$\mathcal{A} = \left(\frac{m}{2\pi\beta}\right)^{1/2}\left(\frac{4}{\beta^3x_0\Delta_x}\right)^{1/2}\frac{1}{\prod_{k=1}\left(1+\frac{\beta x_k}{\nu_k^2}\right)}. \tag{B2}$$

It follows from expression (18) and (B1) that the cyclic integral $h_\ell(\mathbf{k})$ factorizes as a product of each dimensional component $h_\ell(\mathbf{k}) = h_\ell(k_x)h_\ell(k_y)h_\ell(k_z)$, where each factor is of the form

$$h_\ell(k_z) = \mathcal{A}^\ell\int_{-\infty}^\infty dz_1\cdots\int_{-\infty}^\infty dz_\ell K[x,0](z_1,\beta|z_\ell,0)\cdots$$
$$\times K[x,0](z_3,\beta|z_2,0)K[x,0](z_2,\beta|z_1 0)e^{-i\frac{1}{N}k_z\sum_{j=1}^\ell z_j}. \tag{B3}$$

Here, the notation for $K[x,0](z_1,\beta|z_\ell,0)$ as a function of scalar points z_T and z_0 rather than vector variables refers to the propagator (B1) in one dimension, $d=1$. After substitution of the propagators, expression (B3) can also be calculated using the well-known Gaussian integral formula:

$$h_\ell(k_z) = \mathcal{A}^\ell\int_{-\infty}^\infty dz_1\cdots\int_{-\infty}^\infty dz_\ell\exp(-\mathbf{z}^T\mathcal{C}\mathbf{z}-\mathbf{B}^T\mathbf{z})$$
$$= \mathcal{A}^\ell\sqrt{\frac{\pi^\ell}{\det(\mathcal{C})}}\exp\left(\frac{1}{4}\mathbf{B}^T\mathcal{C}^{-1}\mathbf{B}\right), \tag{B4}$$

where we invoke a vector notation for $\mathbf{z}^T=(z_1,\ldots,z_\ell)$, $\mathbf{B}^T=\frac{ik_z}{N}(1,\ldots,1)$ and define the $\ell\times\ell$ dimensional matrix as

$$\mathcal{C} = \begin{pmatrix} 2(a+b) & (b-a) & 0 & \ldots & (b-a)\\ (b-a) & 2(a+b) & (b-a) & \ldots & \ldots\\ 0 & (b-a) & 2(a+b) & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & (b-a)\\ (b-a) & \ldots & 0 & (b-a) & 2(a+b)\end{pmatrix}. \tag{B5}$$

The matrix $\mathcal{C}$ is a circulant matrix, characterized by the property that any row or column is obtained by shifting the previous one by a single space (using periodic boundary conditions at the edges). Every circulant matrix has the same set of $j=\{0,1,\ldots,\ell-1\}$ eigenvectors [21]:

$$\mathbf{y}_j^T = \frac{1}{\sqrt{\ell}}\left(\rho_j^0,\rho_j^1,\ldots,\rho_j^{\ell-1}\right),\quad \text{where } \rho_j = e^{\frac{2\pi i}{\ell}j}, \tag{B6}$$

with corresponding eigenvalues for this particular matrix [21],

$$\lambda_j = 2(a+b)+2(b-a)\cos\left(\frac{2\pi j}{\ell}\right). \tag{B7}$$

The goal now is to calculate both the determinant of $\mathcal{C}$ and the quadratic form $\mathbf{B}^T\mathcal{C}^{-1}\mathbf{B}$ of its inverse to obtain an explicit

expression of (B4). An expression for the determinant is readily written down as the product over all eigenvalues:

$$\det(\mathcal{C}) = \prod_{j=0}^{\ell-1}\left[2(a+b)+2(b-a)\cos\left(\frac{2\pi j}{\ell}\right)\right]$$
$$= [2(a-b)]^{\ell}\prod_{j=0}^{\ell-1}\left[\frac{a+b}{a-b}-\cos\left(\frac{2\pi j}{\ell}\right)\right]. \quad \text{(B8)}$$

Consider the strictly positive real numbers a and b and assume $a \neq b$. We can now define $\tilde{z} = \operatorname{arccosh}(\frac{a+b}{a-b})$. For $\frac{a+b}{a-b} > 1$, $\tilde{z}$ is real and uniquely defined. However, any $\frac{a+b}{a-b} < 1$ lies exactly on the branch cut of the arccosh function, and $\tilde{z}$ is complex and uniquely defined only up to the choice of whether the branch cut is approached from above or below the real axis. Either of the two choices work, and as we will show, both yield the same result. Having converted $\frac{a+b}{a-b}$ in this form, the cosines in (B8) can now be added:

$$\det(\mathcal{C}) = [2(a-b)]^{\ell}\prod_{j=0}^{\ell-1}\left[\cos(i\tilde{z})-\cos\left(\frac{2\pi j}{\ell}\right)\right]$$
$$= [4(a-b)]^{\ell}\prod_{j=0}^{\ell-1}\sin\left(\frac{\pi j}{\ell}+\frac{i\tilde{z}}{2}\right)\prod_{j=0}^{\ell-1}\sin\left(\frac{\pi j}{\ell}-\frac{i\tilde{z}}{2}\right). \quad \text{(B9)}$$

We encountered a very concise proof of the resulting sine product series in Ref. [22]. First note that the following polynomial in c can be decomposed in terms of its roots:

$$c^{\ell}-1 = \prod_{j=0}^{\ell-1}\left(c-e^{\frac{2\pi i}{\ell}j}\right). \quad \text{(B10)}$$

Setting $c = e^{2iz}$, this can be applied to factorize the sine function as follows:

$$\sin(\ell z) = \frac{e^{-i\ell z}}{2i}(e^{2i\ell z}-1) = \frac{e^{-i\ell z}}{2i}\prod_{j=0}^{\ell-1}\left(e^{2iz}-e^{\frac{2\pi i}{\ell}j}\right). \quad \text{(B11)}$$

After some algebraic manipulations on (B11), one readily obtains for any complex z

$$\prod_{j=0}^{\ell-1}\sin\left(\frac{\pi j}{\ell}+z\right) = \frac{1}{2^{\ell-1}}\sin(\ell z), \quad \text{(B12)}$$

which is the known result found in tables of product series [23]. Using this result in (B9) yields

$$\det(\mathcal{C}) = 4(a-b)^{\ell}\sinh\left(\frac{\ell}{2}\tilde{z}\right)^2. \quad \text{(B13)}$$

Let us now go back to the ambiguity of defining $\tilde{z}$ along the branch cut. If $-1 < \frac{a+b}{a-b} < 1$, then $\tilde{z}$ is purely imaginary and only changes sign across the branch cut, which clearly does not affect (B13). If $\frac{a+b}{a-b} < -1$, then the real part of $\tilde{z}$ remains constant along the branch cut and the imaginary part jumps from π to $-\pi$, which does not change (B13) for an integer ℓ. Therefore any choice gives the same result, and we can unambiguously write

$$\det(\mathcal{C}) = 4(a-b)^{\ell}\sinh\left[\frac{\ell}{2}\operatorname{arccosh}\left(\frac{a+b}{a-b}\right)\right]^2. \quad \text{(B14)}$$

We want to emphasize that when $a-b<0$, each of the two factors in (B14) become negative for odd cycles ℓ, but the determinant always remains strictly positive and hence the square root in (B4) is well defined and real.

Next we have to find the quadratic form of the inverse matrix $\mathbf{B}^T\mathcal{C}^{-1}\mathbf{B}$. For this, we note that the matrix $\mathcal{C}$ is diagonalized as $D = Q^*\mathcal{C}Q$ [21], where Q is the matrix with the normalized eigenvectors (B6) and D is the matrix with eigenvalues (B7) on the diagonal. It readily follows that

$$\mathbf{B}^T\mathcal{C}^{-1}\mathbf{B} = \mathbf{B}^T Q D^{-1} Q^*\mathbf{B} = -\frac{k_z^2\ell}{N^2}\frac{1}{4b}. \quad \text{(B15)}$$

The determinant (B13) and quadratic form of the inverse (B15) now yield

$$h_\ell(k_z) = \mathcal{A}^{\ell}\left(\frac{\pi^{\ell}}{4(a-b)^{\ell}\sinh\left[\frac{\ell}{2}\operatorname{arccosh}\left(\frac{a+b}{a-b}\right)\right]^2}\right)^{1/2}$$
$$\times \exp\left(-\frac{k_z^2\ell}{N^2}\frac{1}{16b}\right). \quad \text{(B16)}$$

After substitution of a, b, and $\mathcal{A}$, and taking the dimensionality into account, we exactly obtain expression (19) in Sec. III.

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