Quantum-fluctuation-induced superfluid density in two-dimensional spin-orbit-coupled Bose-Einstein condensates

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(Received 7 December 2020; revised 3 August 2021; accepted 5 August 2021; published 24 August 2021)

We study a two-dimensional spin-$\frac{1}{2}$ interacting Bose-Einstein condensate with a Rashba-Dresselhaus spin-orbit coupling. The analytical expression of the superfluid density is derived from the linear response theory and Bogoliubov approximation. We find that the superfluid density can be divided into three terms, which originate from the condensate, the interaction between the condensate and the Bogoliubov excitations, and only the excitations. We show that the condensate-excitation interaction changes the bare mass in the superfluid density into the effective mass. In the isotropic spin-orbit coupling limit, the effective mass diverges, and the superfluid density is predicted to be zero within previous studies; however, our work shows that the excitation-induced superfluid density remains nonzero, which means that quantum fluctuations rescue the superfluidity. Our results show the importance of the quantum fluctuations in understanding the superfluid properties of spin-orbit-coupled Bose-Einstein condensates.

DOI: 10.1103/PhysRevA.104.023320

I. INTRODUCTION

Spin-orbit coupling (SOC) plays an important role in a variety of physical systems and is responsible for plenty of fascinating phenomena, such as the quantum spin Hall effect [1,2], topological superfluids [3–5], and topological insulators [6]. Recently, different types of SOCs have been realized in ultracold atomic gases [7–10]. The exquisite controllability of atomic systems opens up the possibility for exploring novel quantum states of matter and has attracted a lot of attention. In the past decades, the spin-orbit-coupled Bose gases have been studied extensively [11]. The ground state and excitation properties have been investigated using various approaches [12–21]. The system exhibits three typical phases [14,19,20], i.e., plane-wave, stripe, and zero-momentum phases, in different parameter regions.

The SOC modifies the single-particle dispersion such that the effective mass and density of states are enhanced. As a result, the interaction effects become more important, and the beyond-mean-field effects might be noticeable. Bogoliubov excitations are also affected by the SOC; besides the gapless phonon excitation, there exists a branch of gapped quasiparticle excitation. The multiband structure was recently shown to play an important role in understanding fermionic superfluidity of flat- or quasiflat-band systems in which the effective mass is large [22,23]. For example, it has been suggested that the geometric superfluid density contributes much to the high superconducting transition temperature of twisted bilayer graphene [24–26]. The superfluid density of spin-orbit-coupled fermionic superfluids has also been investigated [27,28]. It is thus natural to investigate the multiband and beyond-mean-field effects in spin-orbit-coupled Bose-Einstein condensates (BECs).

Since the Galilean invariance is broken by SOC, the superfluid density can be different from the particle density even at zero temperature [29]. Previous studies [30–32] on spin-orbit-coupled BECs predict that the superfluid density becomes zero when the effective mass diverges, while the condensate fraction is still very large for weak interactions. However, in these works the effects of multibands and quantum fluctuations on the superfluid density of spin-orbit-coupled bosons are not fully taken into account.

In this work we go beyond the mean-field analysis and investigate the superfluid properties of a two-dimensional (2D) interacting Bose gas with an in-plane Rashba-Dresselhaus SOC. We calculate the superfluid density using the linear response theory and Bogoliubov approximation and find that in general, the superfluid density consists of three terms: The first term comes from the Bose condensate and is proportional to the condensate fraction. The second term is from the interaction between the condensate and the Bogoliubov excitations, which is always negative and cancels part of the first term. The third term originates from the Bogoliubov excitations, which takes the same form as in the fermionic case [28], and is related to both interband and intraband processes. In the isotropic SOC limit, the effective mass diverges, and the first two terms cancel out completely. However, the third term remains nonzero, which means that it is the quantum fluctuation that makes the superfluidity possible. Our result is thus qualitatively different from previous predictions that the superfluidity will be destroyed in the isotropic SOC limit [30]. Experimentally, our prediction can be tested by measuring the sound velocity [33].

II. HAMILTONIAN

We consider a 2D spin-$\frac{1}{2}$ Bose gas with a Rashba-Dresselhaus SOC, which is described by the Hamiltonian $H = H_0 + H_{\text{int}}$. The single-particle Hamiltonian H_0 and the

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interaction Hamiltonian $H_{\rm int}$ are given by [14,30]

$$H_0 = \int d^2\mathbf{r}\,\psi^\dagger\left[\frac{\mathbf{p}^2+\kappa_0^2}{2}+\kappa_0(p_x\sigma_x+\lambda p_y\sigma_y)\right]\psi \quad (1)$$

and

$$H_{\rm int} = \int d^2\mathbf{r}\,(g_{\uparrow\uparrow}n_\uparrow n_\uparrow + g_{\downarrow\downarrow}n_\downarrow n_\downarrow + 2g_{\uparrow\downarrow}n_\uparrow n_\downarrow), \quad (2)$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)^T$ is the spinor, and $\mathbf{p} = (p_x, p_y)$ is the momentum operator. κ_0 is the SOC strength and is taken to be positive. σ_x and σ_y are the Pauli matrices. λ $(0 \leqslant \lambda \leqslant 1)$ describes the anisotropy of the in-plane Rashba-Dresselhaus SOC; when $\lambda = 1$, the system reduces to the isotropic case [17]. $n_\sigma = \psi_\sigma^\dagger\psi_\sigma$ is the particle density for spin σ. $g_{\sigma\sigma}$ $(\sigma =\uparrow, \downarrow)$ and $g_{\uparrow\downarrow}$ are the intraspecies and interspecies s-wave couplings, respectively. Experimentally, a quasi-2D system can be realized by applying a confinement potential along the z direction and a weak trap in the x-y plane. In the weak confinement limit, the confinement length l_z is much larger than the three-dimensional scattering length $a_{\sigma\sigma'}$, and the interaction parameters $g_{\sigma\sigma'}$ can be written as $g_{\sigma\sigma'} = \hbar^2\tilde{g}_{\sigma\sigma'}/m$, where $\hbar$ is the reduced Plank constant, m is the atom mass, and $\tilde{g}_{\sigma\sigma'} = \sqrt{8\pi}a_{\sigma\sigma'}/l_z$ is dimensionless and much smaller than 1 [34]. In this paper we set $\hbar$ and m to unity and take $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = g$ and $g_{\uparrow\downarrow} = g'$.

Without interactions, the single-particle spectrum has two branches,

$$\epsilon_\pm(\mathbf{k}) = \tfrac{1}{2}\left[\left(\sqrt{k_x^2+\lambda^2k_y^2}\pm\kappa_0\right)^2+(1-\lambda^2)k_y^2\right]; \quad (3)$$

the ground state is doubly degenerate when $0 \leqslant \lambda < 1$ with the energy minima located at $\mathbf{k} = (\pm\kappa_0, 0)$. For the isotropic case with $\lambda = 1$, the lowest energy has degenerate states on the circle with $\sqrt{k_x^2+k_y^2} = \kappa_0$. The effective mass is anisotropic, with

$$\frac{1}{m_{{\rm eff},x}} = 1, \qquad \frac{1}{m_{{\rm eff},y}} = 1-\lambda^2. \quad (4)$$

In the isotropic SOC limit, the effective mass $m_{{\rm eff},y}$ diverges.

When considering the interactions, mean-field studies of the ground state have shown different phases. Depending on the relative magnitude of g and g', the bosons can condense into a plane-wave phase with a single momentum $(g > g')$ or a density-stripe state of two opposite momenta $(g < g')$ [14].

III. SUPERFLUID DENSITY

In this work, we focus on the region $g > g'$. Our starting point is the plane-wave state with momentum $\mathbf{k}_0 = (\kappa_0, 0)$, and the condensate wave function is [14,30]

$$\phi(\mathbf{r}) = \sqrt{\frac{n_0}{2}}e^{i\kappa_0x}\begin{pmatrix}1\\-1\end{pmatrix}, \quad (5)$$

where n_0 is the condensate density.

To study the quantum fluctuations, first, we expand the bosonic field operator as

$$\psi_\sigma(\mathbf{r}) = \phi_\sigma(\mathbf{r}) + \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{k}\sigma}, \quad (6)$$

where V is the volume of the system and we take it to be unity; substituting Eq. (6) into the Hamiltonian and keeping the quadratic terms, we obtain the Bogoliubov Hamiltonian

$$H = \frac{1}{2}\sum_{\mathbf{k}}\Psi_{\mathbf{k}}^\dagger\mathcal{H}(\mathbf{k})\Psi_{\mathbf{k}}, \quad (7)$$

with $\Psi_{\mathbf{k}}^\dagger = [\psi^\dagger_{\mathbf{k}+\mathbf{k}_0,\uparrow}, \psi^\dagger_{\mathbf{k}+\mathbf{k}_0,\downarrow}, \psi_{-\mathbf{k}+\mathbf{k}_0,\uparrow}, \psi_{-\mathbf{k}+\mathbf{k}_0,\downarrow}]$ and

$$\mathcal{H}(\mathbf{k}) = \begin{bmatrix}h(\mathbf{k}+\mathbf{k}_0) & \Delta\\ \Delta & h^*(-\mathbf{k}+\mathbf{k}_0)\end{bmatrix}, \quad (8)$$

with $h(\mathbf{k}+\mathbf{k}_0) = \frac{(\mathbf{k}+\mathbf{k}_0)^2+\kappa_0^2}{2} + gn_0 - g'n_0\sigma_x + \kappa_0[(k_x+k_0)\sigma_x + \lambda k_y\sigma_y]$ and $\Delta = gn_0 - g'n_0\sigma_x$.

The superfluid density can be obtained by the linear response theory, which was applied to study the geometric superfluid density in multiband attractive Fermi-Hubbard systems recently [23]. The geometric contribution, which is associated with the off-diagonal matrix elements of the current operator, is shown to play an important role in the superfluid density, especially in models with flat bands where the effective mass diverges. A similar approach can be generalized to bosonic systems.

According to the linear response theory, the superfluid density ρ_{ij} is obtained by taking the zero-momentum limit of the transverse component of the current-current function $K_{ij}(\mathbf{q}, \omega)$, i.e., $\rho_{ij} = K_{ij}(\mathbf{q}\to 0, \omega = 0)$ and

$$K_{ij}(\mathbf{q},\omega) = \langle T_{ij}\rangle - i\int_0^\infty dt\, e^{i(\omega+i0^+)t} \times\left\langle\left[j_i^p(\mathbf{q},t), j_j^p(-\mathbf{q},0)\right]\right\rangle, \quad (9)$$

where T_{ij} is the diamagnetic current operator and j_i^p is the paramagnetic current operator, given by

$$T_{ij} = \delta_{ij}\sum_{\mathbf{k},\sigma}\psi^\dagger_{\mathbf{k},\sigma}\psi_{\mathbf{k},\sigma} \quad (10)$$

and

$$j_i^p(\mathbf{q},t) = \sum_{\mathbf{k}}\Psi_{\mathbf{k}}^\dagger\partial_i\mathcal{H}(\mathbf{k}+\mathbf{q}/2)P_+\Psi_{\mathbf{k}+\mathbf{q}}. \quad (11)$$

Here $\partial_i \equiv \partial_{k_i}$, and $P_+ = (\tau^3 + I)/2$ is an upper projection operator, with I being the identity matrix and τ^3 being the third Pauli matrix acting on the particle-hole space. Note that when we calculate the response function, the condensate contribution in j_i^p vanishes in the thermodynamic limit, while in T_{ij} it contributes a finite value. We have

$$\langle T_{ij}\rangle = n\delta_{ij}, \quad (12)$$

where n is the total particle density, and for later convenience we express it as $n = n_0 + \delta n$, with δn being the condensate depletion,

$$\delta n = -\sum_k {\rm Tr}\left[G'(k)\partial_i^2\mathcal{H}(\mathbf{k})P_+\right] \quad (13)$$

$$= \sum_k {\rm Tr}[G'(k)\partial_i\mathcal{H}(\mathbf{k})G'(k)\partial_i\mathcal{H}(\mathbf{k})P_+], \quad (14)$$

where $k = (i\omega_n, \mathbf{k})$; $\omega_n = 2\pi nT$ is the Matsubara frequency, with T being the temperature; and $G'(k) = [i\omega_n\tau^3 - \mathcal{H}(\mathbf{k})]^{-1}$ is the imaginary-time Green's function for the Bogoliubov

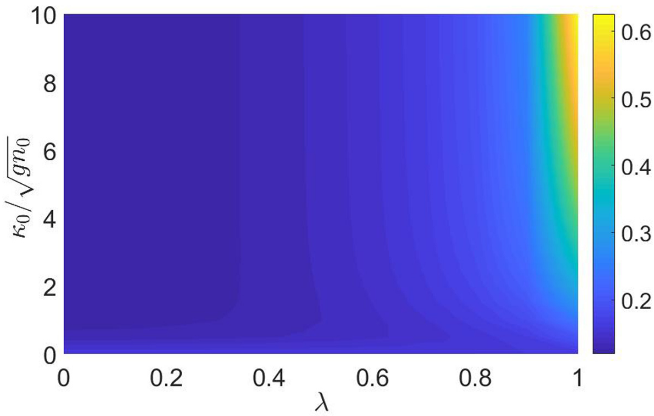

FIG. 1. Quantum depletion $\delta n/(gn_0)$ as a function of the SOC strength κ_0 and the anisotropy parameter λ. The interspecies interaction strength is taken to be $g'/g = 0.5$.

excitations. As $\partial_i^2\mathcal{H}(\mathbf{k})P_+ = P_+$, Eq. (13) is valid. To get Eq. (14) from Eq. (13), we use the fact that $\partial_i G'(k) = G'\partial_i\mathcal{H}G'$ and then integrate Eq. (13) by parts. The advantage of Eq. (14) is that it has the same structure as the paramagnetic current correlation function.

Let $|n, s\rangle$ be the eigenvector of the matrix $\tau^3\mathcal{H}$ with the eigenvalue E_n^s, i.e., $\tau^3\mathcal{H}|n, s\rangle = E_n^s|n, s\rangle$, and let $s = \pm$ denote the positive and negative eigenvalues. The wave vectors satisfy $\langle m, t|\tau^3|n, s\rangle = s\delta_{mn}\delta_{st}$. Then $G(i\omega, \mathbf{k})$ can be written as

$$G'(i\omega, \mathbf{k}) = \sum_{n,s} \frac{s|n, s\rangle\langle n, s|}{i\omega - E_n^s}. \qquad (15)$$

By summing over the Matsubara frequencies ω, from Eq. (14) we can obtain

$$\delta n = \sum_{n,s}\sum_{m,t}\sum_{\mathbf{k}} st\frac{\tilde{n}(E_n^s) - \tilde{n}(E_m^t)}{E_n^s - E_m^t} \times \langle n, s|\partial_i\mathcal{H}(\mathbf{k})P_+|m, t\rangle\langle m, t|\partial_i\mathcal{H}(\mathbf{k})|n, s\rangle, \qquad (16)$$

where $\tilde{n}(x) = 1/[\exp(x/T) - 1]$ is the Bose-Einstein distribution. In this work, all the quantities are calculated at zero temperature. As in the fermionic case, the prefactor should be understood as $-\partial_{E_n}\tilde{n}(E_n)$ when $n = m$, which vanishes in the zero-temperature limit since the Bogoliubov dispersion is gapless only at $\mathbf{k} = (0, 0)$. The effect of SOC on quantum depletion is shown in Fig. 1, where $\delta n/(gn_0)$ is plotted [35]. As one can see, the depletion increases with the increasing of the SOC strength and anisotropy. The sharp increase for the isotropic SOC case is because the single-particle ground state is circularly degenerate, and the noninteracting density of states diverges as in the one-dimensional case [36]; therefore, the interaction effects are enhanced [37]. In the weakly interacting limit, $g \ll 1$ [34], and as a result, $\delta n/n_0$ is largely suppressed even when $\lambda = 1$. So we expect that the Bogoliubov approximation, which has been widely used to investigate the spin-orbit-coupled BEC (see, e.g., [14,18,21,37–39]), is still valid in the vicinity of the $\lambda = 1$ limit when the interaction is weak.

The paramagnetic response function is given by

$$\langle j_i^p(\mathbf{q})j_i^p(-\mathbf{q})\rangle = \sum_k \mathrm{Tr}[G(i\omega, \mathbf{k})\partial_i\mathcal{H}(\mathbf{k} + \mathbf{q}/2)\tau^3 \times G(i\omega, \mathbf{k} + q)\partial_i\mathcal{H}(\mathbf{k} + \mathbf{q}/2)\tau^3P_+], \qquad (17)$$

where

$$G(i\omega, \mathbf{k}) = G_0\delta_{\omega,0}\delta_{\mathbf{k},\mathbf{0}} + G'(i\omega, \mathbf{k}) \qquad (18)$$

and $G_0 = -|\phi(\mathbf{r})\rangle\langle\phi(\mathbf{r})|$ comes from the condensate wave function. There are two types of contributions in the paramagnetic response function: One is the correlation between the condensate and the excitations, and the other comes solely from the excitations.

Performing the Matsubara frequency summation and taking the zero-momentum limit, we obtain the superfluid density

$$\rho_{ii}^s = \rho_{ii}^{s,C} + \rho_{ii}^{s,C-E} + \rho_{ii}^{s,E}. \qquad (19)$$

The first two terms, $\rho_{ii}^{s,C}$ and $\rho_{ii}^{s,C-E}$, come from the condensate and the interaction between the condensate and quasiparticle excitations, respectively. The third term $\rho_{ii}^{s,E}$ is totally determined by the excitations. Explicitly,

$$\rho_{ii}^{s,C} = n_0, \qquad (20)$$

$$\rho_{ii}^{s,C-E} = \frac{n_0}{2}\lim_{\mathbf{q}\to 0}(\langle\phi|\partial_i\mathcal{H}(\mathbf{q}/2)\tau^3G'(0, \mathbf{q}) \times \partial_i\mathcal{H}(\mathbf{q}/2)P_+|\phi\rangle + (\mathbf{q} \leftrightarrow -\mathbf{q})), \qquad (21)$$

$$\rho_{ii}^{s,E} = 2\sum_{n,s}\sum_{m,t}\sum_{\mathbf{k}} st\frac{\tilde{n}(E_n^s) - \tilde{n}(E_m^t)}{E_n^s - E_m^t} \times \langle n, s|\partial_i\mathcal{H}(\mathbf{k})P_-|m, t\rangle\langle m, t|\partial_i\mathcal{H}(\mathbf{k})P_+|n, s\rangle, \quad (22)$$

where $P_- = I - P_+$ is the projection operator to the hole space.

Now we compare our result with that of the fermionic counterpart. In the spin-orbit-coupled fermionic superfluid, $\rho_{ii}^{s,C}$ and $\rho_{ii}^{s,C-E}$ are absent because they are related to the Bose condensate. The $\rho_{ii}^{s,E}$ term takes the same form as the result for the fermionic SOC superfluid [28], and the only difference is that the Fermi-Dirac distribution is replaced by the Bose-Einstein distribution. As in the fermionic case, we can separate the contribution into the conventional one which depends only on the band dispersion and the geometric one which depends also on the wave function.

IV. RESULTS AND DISCUSSION

The contribution of condensate-excitation interaction to the superfluid density, Eq. (21), depends only on the zero-momentum limit of the Green's function and can be calculated analytically. We find that only the gapped modes contribute to the result. Combining it with the condensate contribution, Eq. (20), we obtain

$$\rho_{ii}^{s,C} + \rho_{ii}^{s,C-E} = \frac{n_0}{m_{\mathrm{eff},i}}, \qquad (23)$$

which reduces to the result obtained previously [30]. Equation (23) provides a different understanding of the effective mass

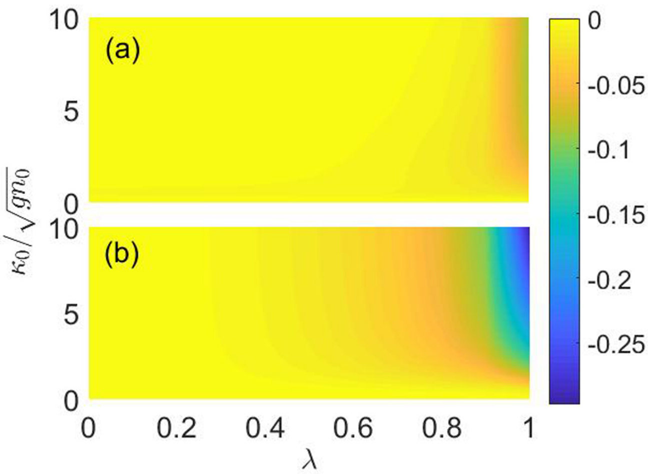

FIG. 2. The difference between the superfluid density $\rho_{ii}^{s,E}$ and the quantum depletion δn is shown as a function of the SOC strength κ_0 and the anisotropy λ (a) for $(\rho_{xx}^{s,E} - \delta n)/(gn_0)$ and (b) for $(\rho_{yy}^{s,E} - \delta n)/(gn_0)$. The interspecies interaction strength is fixed to $g'/g = 0.5$.

in the superfluid density, that the interaction between the condensate and Bogoliubov excitations modifies the bare mass into the effective mass.

The total superfluid density can thus be written as

$$\rho_{ii}^s = \frac{n_0}{m_{\text{eff},i}} + \rho_{ii}^{s,E}. \tag{24}$$

Before presenting the numerical results for the general cases, we first consider two limiting cases: (1) In the absence of SOC (i.e., $\kappa_0 = 0$), the system is Galilean invariant, and the zero-temperature superfluid density reduces to the well-known result with $\rho_{ii}^s = n$ [29]. This implies that the contribution of excitations $\rho_{ii}^{s,E}$ is equal to the quantum depletion in the presence of Galilean invariance. In other words, a nonzero $\rho_{ii}^{s,E} - \delta n$ indicates the breaking of Galilean invariance. (2) In the $\lambda = 0$ limit, the SOC is nonzero only in the x direction. In this case we can first apply a spin rotation along the y axis such that σ^x is changed to σ^z, and the SOC can be eliminated with a unitary transformation [11]. The quantum depletion can be calculated analytically (for details see the Appendix),

$$\delta n = \frac{1}{4\pi}\left((g+g')n_0 + \frac{(g-g')n_0}{1+x+\sqrt{x(2+x)}}\right), \tag{25}$$

with $x = 2\kappa_0^2/[(g-g')n_0]$. We also find that the superfluid density is the same as the total particle density n.

For general parameters, we calculate $\rho_{ii}^{s,E}$ numerically. Figure 2 shows the difference between the excitation term of the superfluid density and the depletion, i.e., $(\rho_{ii}^{s,E} - \delta n)/(gn_0)$, as a function of the SOC strength and anisotropy. We find that away from the two limiting cases, $\rho_{ii}^{s,E}$ is always less than δn, and although the effective mass in the x direction remains unchanged, the superfluid density ρ_{xx}^s is smaller than the total density. As mentioned above, this is a result of the breaking of Galilean invariance in the x direction.

As we can see from Fig. 2, the difference is not very sensitive to the parameters unless λ gets close to 1. To explain this, we note that the system has a Z_2 symmetry for λ: After a 2π rotation of the spin along the z axis, λ is changed to $-\lambda$. This symmetry implies that all the physical properties, including the quantum depletion and superfluid density, are even functions of λ. A similar argument also applies for κ_0. Expanding the difference in terms of λ and κ_0 to the lowest order, we obtain $\rho_{ii}^{s,E} - \delta n \propto \kappa_0^2\lambda^2$. Because of the quadratic dependence, when λ is away from the isotropic point, the difference is small and not very sensitive to both λ and κ_0.

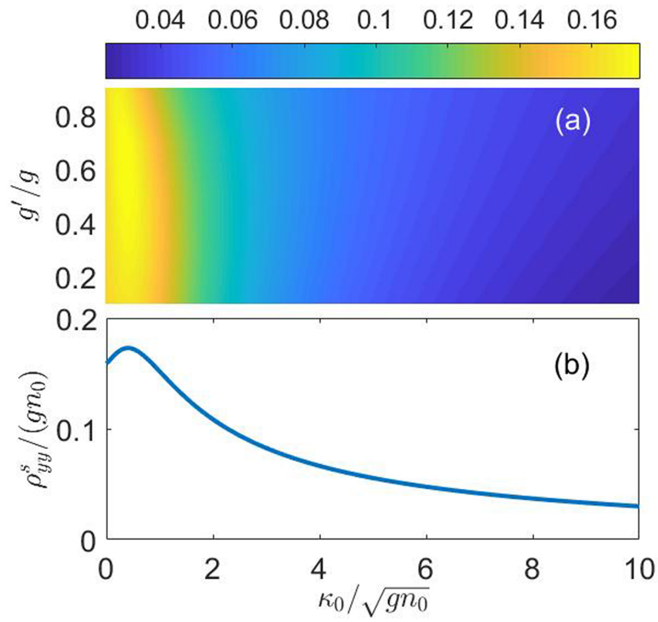

FIG. 3. (a) The superfluid density in the y direction $\rho_{yy}^s/(gn_0)$ as a function of the SOC strength $\kappa_0/\sqrt{gn_0}$ and the interspecies interaction strength g'/g with isotropic SOC. (b) The result for $g'/g = 0.5$.

Then we focus on the most interesting case, i.e., $\lambda = 1$. In this limit, although the condensate fraction is large for weak interactions, its contribution to the superfluid density in the y direction is completely canceled out, and the superfluidity comes solely from the excitations. In Fig. 3(a) we illustrate the excitation-induced superfluid density ρ_{yy}^s as a function of the SOC and the interspecies interaction strengths. In the $\kappa_0 = 0$ limit, ρ_{yy}^s is the same as the quantum depletion with $\delta n \approx 0.16gn_0$ [see Eq. (25)]. We find that ρ_{yy}^s reaches a maximum at finite SOC strength [see Fig. 3(b)]; similar behavior has been observed in the three-dimensional equal-strength Rashba-Dresselhaus spin-orbit-coupled BEC [40].

Our result for the isotropic SOC is qualitatively different from the result in [30], where the superfluidity was predicted to be destroyed. However, in our work we find that the excitations play a crucial role and rescue the superfluidity. In Ref. [30], the superfluid density was obtained by integrating out the massive quadratic fluctuations, and high-order fluctuations were not taken into account. We expect the same result would be obtained if a one-loop correction were calculated. Our result is also different from that of the BEC with an equal mixture of Rashba and Dresselhaus SOCs, where the mean-field superfluid density is found to vanish at the phase boundary between the plane-wave phase and zero-momentum phase [31]. Quantum corrections modify the phase boundary quantitatively, but the superfluid density remains zero at the shifted boundary [40]. In contrast, in our work, the isotropic SOC limit is not a phase-transition point, and the superfluid density remains nonzero for all the parameters we study.

A nonvanishing superfluid density results in a nonzero sound velocity [31,40], which can be measured in current cold-atom experiments [33]. This provides a way to test our theoretical prediction.

V. CONCLUSION

We studied the superfluid properties of a spin-orbit-coupled BEC in two dimensions with the linear response theory and Bogoliubov approximation. We found that the superfluid density can be divided into three parts, which originate from the condensate, the interaction between the condensate and quasiparticle excitations, and only the excitations. The first two terms can be expressed by the effective mass. The third term, which is caused by the quantum fluctuations, has the same form as the superfluid density in a spin-orbit-coupled fermionic superfluid and is closely related to multiband processes [28]. Away from the isotropic limit, this term is less important because it is much smaller than the term expressed by the effective mass. But it becomes more and more important when approaching the isotropic limit. At the isotropic SOC limit, although the condensate fraction is large, its contribution to the superfluid density is completely canceled out, and the total superfluid density comes solely from the Bogoliubov excitations. In conclusion, our work provides insights into the connection between superfluidity and BECs and demonstrates the importance of quantum fluctuations in understanding superfluidity. We became aware that a similar effect of quantum-fluctuation-induced superfluid density was obtained in flat-band systems [41] very recently.

In this work, we focused on the plane-wave phase; the method can be extended to the stripe phase. The quantum-fluctuation corrections to other physical quantities are also important and deserve future investigations.

ACKNOWLEDGMENTS

L.Y. thanks L. Liang and X. Cui for valuable discussions and thanks Y. Yao for helpful suggestions. This work is supported by the Science Challenge Project (Grant No. TZ2018003).

APPENDIX: QUANTUM DEPLETION AND SUPERFLUID DENSITY IN THE $\lambda = 0$ LIMIT

In this Appendix we calculate analytically the quantum depletion and superfluid density in the $\lambda = 0$ limit. As explained in the main text, we can perform a momentum-independent unitary transformation to change the Bogoliubov Hamiltonian [Eq. (8)] to a block-diagonalized form $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ = \begin{bmatrix} \frac{\mathbf{k}^2}{2} + g_+ n_0 & g_+ n_0 \\ g_+ n_0 & \frac{\mathbf{k}^2}{2} + g_+ n_0 \end{bmatrix} \tag{A1}$$

and

$$\mathcal{H}_- = \begin{bmatrix} \frac{(\mathbf{k}+2\mathbf{k_0})^2}{2} + g_- n_0 & g_- n_0 \\ g_- n_0 & \frac{(\mathbf{k}-2\mathbf{k_0})^2}{2} + g_- n_0 \end{bmatrix}, \tag{A2}$$

with $g_\pm = g \pm g'$.

The quantum depletion is

$$\delta n = \int \frac{d^2\mathbf{k}}{(2\pi)^2} (v_+^2 + v_-^2), \tag{A3}$$

where

$$v_\pm^2 = \frac{1}{2}\left(\frac{\epsilon_\pm}{E_\pm} - 1\right), \tag{A4}$$

with $\epsilon_+ = \mathbf{k}^2/2 + g_+ n_0$, $\epsilon_- = \mathbf{k}^2/2 + 2\kappa_0^2 + g_- n_0$, and $E_\pm = \sqrt{\epsilon_\pm^2 - (g_\pm n_0)^2}$. The integral can be calculated analytically, and the result is given as Eq. (25) in the main text.

For $\kappa_0 = 0$, the quantum depletion is $\delta n = g n_0/(2\pi) \approx 0.16 g n_0$. Note that the quantum depletion is proportional to $g n_0$ in 2D systems, and for this reason, in the main text we plot $\delta n/(g n_0)$ and $\rho^s/(g n_0)$.

The superfluid density can also be calculated analytically in the $\lambda = 0$ limit. We find that the contribution from the paramagnetic response function vanishes in the zero-temperature limit, and therefore, the superfluid density is the same as the total particle density.

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