Many-body dynamics with time-dependent interaction

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In this work we study the many-body dynamics of Bose-Einstein condensates subject to an arbitrary time-varying scattering length. By employing a variational ansatz which assumes the majority of the particles are condensed, we derive an effective Bogoliubov-like Hamiltonian that governs the dynamics of thermal particles. Crucially, we show that there exists a hidden symmetry in this Hamiltonian that can map the many-body dynamics to the precession of an SU(1,1) "spin" and also allows an exact dynamical solution for this precession in an arbitrary "magnetic field." As a demonstration, we calculate the situation where the scattering length is sinusoidally modulated. We show that the noncompactness of the SU(1,1) group naturally leads to solutions with exponentially growth of Bogoliubov modes and causes instabilities.

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I. INTRODUCTION

The ability to accurately control various parameters in cold atomic gases allows the investigation of quantum matter under extreme conditions that are beyond reach in other physical systems.

Among these parameters, the tunable interaction strength via magnetic Feshbach resonances [1–3] is a key ingredient for many fascinating quantum phenomena such as the physics of BEC-BCS crossover [4–7], superfluid to Mott insulator transition [8–11], and the few-body Efimov effect [12–16]. The tunable interaction not only allows the study of the equilibrium physics of quantum gases but also allows the investigation of many exotic out-of-equilibrium physics with time-varying interaction strength [17–22]. For example, the Bose fireworks experiment recently carried out by the Chicago group shows that a Bose condensate emits matter-wave jets and forms striking fireworks patterns while subject to periodic modulated interactions [19–21]. Moreover, the progress in experimental methods such as optical Feshbach resonance [23] allows rapid and spatial modulations of the scattering length between atoms. Such rapid control of interaction strength has provided the experimental investigation of various exotic dynamic behavior in quantum gases including the recently discovered relation between quantum chaos and out-of-time correlators [24–26].

In this work, we focus on the dynamic problem of a homogeneous Bose-Einstein condensate subject to an *arbitrary* time-varying scattering length. We assume that the system is initially prepared in the ground state of a weakly interacting Hamiltonian such that the condensate fraction $N_0/N \simeq 1$, and we focus on the short-time dynamics in which the condensate fraction remains close to unity. In this limit, one might naively anticipate that the dynamics of the system could be described by a mean-field level Gross-Pitaevskii (GP) equation with time-varying coupling constant $g(t)$. However, it is straightforward to show that the solution to the time-dependent GP equation is homogeneous as long as the system is initially in the ground state [27]. This is related to the fact that the ground-state solution (i.e., the saddle point) of a time-independent GP equation does not rely on the interaction strength g. Thus, it is necessary to go beyond the mean-field theory and consider the role of *quantum fluctuations*.

It is worth noting that even though the mean-field dynamics of the ground state (saddle point) is trivial, one may still analyze the stability properties of it via the time-dependent GP equation. For example, in the study of Faraday instabilities of periodic driven BEC [28–30], one can perform a stability analysis by adding a small classical perturbation $w(t)\cos(\mathbf{k}\cdot\mathbf{r})$ to the ground-state solution.

In the corresponding static problem, the next order correction is known as the Lee-Huang-Yang correction, which can be obtained with the Bogoliubov theory. Inspired by this correspondence, we propose a variational ansatz which accounts for the next order quantum correction to the dynamic problem. We show that the dynamics of the variational wave function is governed by a Bogoliubov-like Hamiltonian. Crucially, we find that the Hamiltonian possesses a hidden SU(1,1) symmetry which not only allows an exact solution to the time-dependent Schrödinger equation but also maps the dynamic problem to an SU(1,1) "spin" moving in a time-varying magnetic field. The SU(1,1) "spin" model closely resembles a normal SU(2) spin in an external field as its dynamics can be viewed as a point moving on a hyperboloid (see Fig. 1) in parameter space which resembles an SU(2) Bloch sphere. To further demonstrate our method, we also calculate the dynamics of a system with periodically modulated scattering length.

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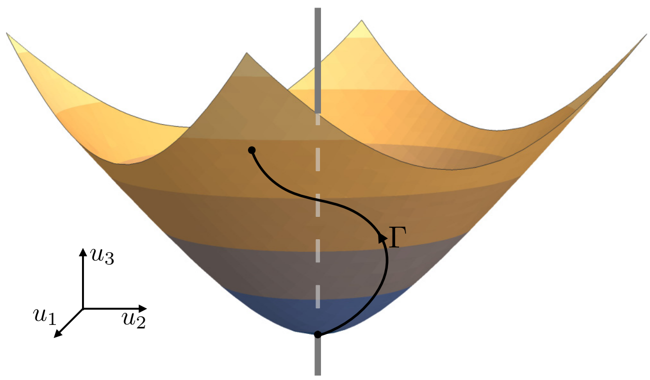

FIG. 1. A schematic diagram showing the dynamics of the SU(1,1) spin on the "Bloch" hyperboloid. Γ represents the trajectory of $\mathbf{u}(t)$. The Berry curvature on the hyperboloid is identical to the field of a line of magnetic monopole represented by the gray line.

II. MODEL

We consider a Hamiltonian which describes a system of bosons interacting via short-range interaction:

$$H(t) = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2} g(t) \sum_{\mathbf{k},\mathbf{q},\mathbf{q}'} a^\dagger_{\mathbf{q}+\mathbf{k}} a^\dagger_{\mathbf{q}'} a_{\mathbf{q}} a_{\mathbf{q}'+\mathbf{k}}. \tag{1}$$

Here $a^\dagger_{\mathbf{k}}(a_{\mathbf{k}})$ are the bosonic creation (annihilation) operator with momentum $\mathbf{k}$ and mass m; $g(t)$ is an *arbitrary* time-varying coupling constant which is related to the s-wave scattering length a_s by $g(t) = 4\pi a_s(t)/m$ (we set $\hbar$ and the volume of the system to 1). The dynamic theory we develop in this work does *not* rely on the specific form of the dispersion $\epsilon_{\mathbf{k}}$ as long as the system has an inversion symmetry i.e., $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$, and without loss of generality, we set $\epsilon_{\mathbf{0}} = 0$.

To proceed, we assume that during the dynamic process the majority of the bosons still condense in the zero-momentum state, i.e., $N_0(t) = \langle a^\dagger_{\mathbf{0}} a_{\mathbf{0}} \rangle \simeq N \gg 1$. Therefore, one may approximate the time-dependent wave function by

$$|\Psi(t)\rangle = |\psi(t)\rangle_{\mathbf{k}\neq\mathbf{0}} \otimes e^{\sqrt{N_0}(a^\dagger_0 + a_0)} |0\rangle, \tag{2}$$

where the wave function $|\Psi(t)\rangle$ is decomposed into a product state of $|\psi(t)\rangle_{\mathbf{k}\neq\mathbf{0}}$ which represents the state of noncondensed thermal bosons and a coherent state of N_0 condensed particles.

To determine the "best" variational wave function $|\Psi(t)\rangle$, we use the Frenknel least action principle [31,32] for dynamic systems and minimize the action $S = \int dt \langle \Psi(t)| i\partial_t - H(t)|\Psi(t)\rangle$ [33]. This leads to a time-dependent Schrödinger equation

$$i\partial_t |\psi(t)\rangle = H_B |\psi(t)\rangle, \tag{3}$$

with

$$H_B(t) = \sum_{\mathbf{k}\neq\mathbf{0}} [\epsilon_{\mathbf{k}} + g(t)n] a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{g(t)n}{2} (a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + \text{H.c.}) + \frac{g(t)nN}{2} + O(N^{-1/2}). \tag{4}$$

We see that the dynamics of thermal particles are governed by a Bogoliubov-like Hamiltonian $H_B(t)$.

It is worth noting that simply diagonalizing $H_B(t)$ via Bogoliubov transformation does not solve the dynamic problem as its instantaneous eigenstate is not the solution to Eq. (3). The solution to the dynamic problem actually relies on the hidden dynamic symmetry of Hamiltonian H_B.

Note that the $\mathbf{k}$ part in H_B couples only to $-\mathbf{k}$, and it can be rewritten as

$$H_B = \sum_{\mathbf{k}\neq 0} \{ g(t) n A^{\mathbf{k}}_1 + [\epsilon_{\mathbf{k}} + g(t)n] A^{\mathbf{k}}_3 \} + E_0. \tag{5}$$

Here $E_0 = -\sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} + gn) + g(t)nN/2$, $A^{\mathbf{k}}_1$ and $A^{\mathbf{k}}_3$ are defined as $A^{\mathbf{k}}_1 = \frac{1}{2}(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + \text{H.c.})$ and $A^{\mathbf{k}}_3 = \frac{1}{2}(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{-\mathbf{k}} a^\dagger_{-\mathbf{k}})$.

It was pointed out by Chen *et al.* [34] that $A^{\mathbf{k}}_1$ and $A^{\mathbf{k}}_3$ can fit into an SU(1,1) algebra by including an extra operator $A^{\mathbf{k}}_2 = \frac{1}{2i}(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} - \text{H.c.})$. Together with this operator, their commutators form a close algebra:

$$[A^{\mathbf{k}}_1, A^{\mathbf{k}}_2] = -iA^{\mathbf{k}}_3,\ [A^{\mathbf{k}}_2, A^{\mathbf{k}}_3] = iA^{\mathbf{k}}_1,\ [A^{\mathbf{k}}_3, A^{\mathbf{k}}_1] = iA^{\mathbf{k}}_2. \tag{6}$$

Note that Eq. (6) differ from the common SU(2) algebra of a spin system by a minus sign. As we will see in the following, there is a close resemblance between the dynamics of Bogoliubov systems and the dynamics of an SU(2) spin in a time-dependent magnetic field.

III. SU(1,1) SPIN MODEL

Note that in the Bogoliubov Hamiltonian $H_B(t)$ all the $(\mathbf{k}, -\mathbf{k})$ subspaces with different $\mathbf{k}$ are decoupled. The original model is separated into many SU(1,1) spins labeled by $(\mathbf{k}, -\mathbf{k})$. This allows us to deal with one SU(1,1) spin at one time. For generality, in the following we will consider a model that consists of all the A_i components:

$$H^{\mathbf{k}}_h = \mathbf{h} \cdot \mathbf{A}^{\mathbf{k}} = h_1 A^{\mathbf{k}}_1 + h_2 A^{\mathbf{k}}_2 + h_3 A^{\mathbf{k}}_3. \tag{7}$$

Here $\mathbf{h} = (h_1, h_2, h_3)^T$ is an arbitrary time-dependent vector, and $H^{\mathbf{k}}_h$ can be reduced to H_B by letting $h_1 = 2g(t)n$, $h_2 = 0$ and $h_3 = 2[\epsilon_{\mathbf{k}} + g(t)n]$.

The SU(1,1) symmetry leads to three time-dependent invariants for $H^{\mathbf{k}}_h$. To see this, we consider operator S in the form of $S = \sum_i u_i(t) A^{\mathbf{k}}_i$. In order to make S a time-dependent invariant under $H^{\mathbf{k}}_h$, we have

$$\frac{d}{dt} S = i[H^{\mathbf{k}}_h, S] + \frac{\partial S}{\partial t} = 0. \tag{8}$$

Due to the closed commutation relations, the above equation leads to a set of linear equations for u_i,

$$\dot{\mathbf{u}} = \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ h_2 & -h_1 & 0 \end{pmatrix} \mathbf{u}, \tag{9}$$

with $\mathbf{u} = (u_1, u_2, u_3)^T$.

Any $\mathbf{u}(t)$ satisfying Eq. (9) represents an invariant $\mathbf{u} \cdot \mathbf{A}^{\mathbf{k}}$ for $H^{\mathbf{k}}_h$. There are three linear-independent solutions of this differential equation, which correspond to three independent invariants.

IV. REMARK ABOUT $H^{\mathbf{k}}_h$

From Eq. (9), one can prove that $\|\mathbf{u}\|^2 \equiv -u_1^2 - u_2^2 + u_3^2$ is a constant by showing that $\frac{d}{dt}\|\mathbf{u}\| = 0$. This means that the three-dimensional vector $\mathbf{u}(t)$ is restricted on the surface of a hyperboloid defined by $-u_1^2 - u_2^2 + u_3^2 = \text{const}$. This may be viewed as the SU(1,1) analog of the Bloch sphere in the SU(2) spin case.

Without loss of generality, we consider the solution of $\mathbf{u}(t)$ on the upper unit sheet of the hyperboloid as shown in Fig. 1. The corresponding invariant can be parametrized as $S(t) = \mathbf{u}\cdot\mathbf{A}^{\mathbf{k}} = \sinh\theta\cos\phi A_1^{\mathbf{k}} + \sinh\theta\sin\phi A_2^{\mathbf{k}} + \cosh\theta A_3^{\mathbf{k}}$. Using the commutation relations in Eq. (6), we can diagonalize it via the SU(1,1) rotation:

$$e^{iA_3^{\mathbf{k}}\phi}e^{iA_2^{\mathbf{k}}\theta}S(t)e^{-iA_2^{\mathbf{k}}\theta}e^{-iA_3^{\mathbf{k}}\phi} = A_3^{\mathbf{k}}. \tag{10}$$

Since $A_3^{\mathbf{k}} = \frac{1}{2}(n_{\mathbf{k}} + n_{-\mathbf{k}} + 1)$, the eigenstates of $S(t)$ are thus parametrized by two integers $\mathbf{n} = (n_+, n_-)$ with $n_\pm$ the number of bosons in $\mathbf{k}$ and $-\mathbf{k}$ states. They are given by $|\mathbf{n}\rangle = |n_+, n_-\rangle = \frac{1}{\sqrt{n_+!n_-!}}e^{iA_3^{\mathbf{k}}\phi}e^{iA_2^{\mathbf{k}}\theta}a_{\mathbf{k}}^{\dagger n_+}a_{-\mathbf{k}}^{\dagger n_-}|0\rangle$.

The instantaneous eigenstates of invariant $S(t)$ are useful because they are proportional to the solution to the Schrödinger equation $|\Phi\rangle$. According to Lewis's theory for time-dependent invariants [35,36], we have

$$|\Phi(t)\rangle = e^{-i\varphi(t)}|\mathbf{n}\rangle. \tag{11}$$

Here $|\Phi\rangle$ satisfies $[i\partial_t - H_h^{\mathbf{k}}(t)]|\Phi(t)\rangle = 0$. The phase $\varphi(t)$ contains a dynamical phase and a geometric phase with $\varphi(t) = \varphi_{\rm dyn}(t) - \varphi_{\rm g}(t)$:

$$\varphi_{\rm dyn} = \int_{t_0}^{t} d\tau\langle\mathbf{n}|H_h^{\mathbf{k}}(\tau)|\mathbf{n}\rangle, \quad \varphi_{\rm g} = i\int_{t_0}^{t} d\tau\langle\mathbf{n}|\partial_\tau|\mathbf{n}\rangle. \tag{12}$$

Suppose the initial state of the system is the ground state of $\mathbf{h}_0\cdot\mathbf{A}^{\mathbf{k}}$, the initial condition for Eq. (9) is then set as $\mathbf{u}(0) = \mathbf{h}_0$. We can then obtain the solution of the time-dependent Schrödinger equation by solving $\mathbf{u}(t)$ and substitute it into Eq. (11) with $n_+ = n_- = 0$.

V. REMARKS ABOUT $\varphi_{\rm g}$

By changing variable t to $\mathbf{u}$, we can show that the geometric phase φ_g depends only on the trajectory of $\mathbf{u}$,

$$\varphi_{\rm g} = i\int_\Gamma d\mathbf{u}\cdot\langle\mathbf{n}|\nabla_{\mathbf{u}}|\mathbf{n}\rangle = \int_\Gamma \mathcal{A}_\theta d\theta + \mathcal{A}_\phi\, d\phi, \tag{13}$$

where Γ is the trajectory of $\mathbf{u}$ on the hyperboloid as shown in Fig. 1. The Berry connection $\mathcal{A}_i$ is

$$\mathcal{A}_\theta = i\langle\mathbf{n}|\partial_\theta|\mathbf{n}\rangle = 0, \tag{14}$$

$$\mathcal{A}_\phi = i\langle\mathbf{n}|\partial_\phi|\mathbf{n}\rangle = -C_{\mathbf{n}}\cosh\theta, \tag{15}$$

with charge $C_{\mathbf{n}} = (n_+ + n_- + 1)/2$.

As is well known, the Berry curvature of an SU(2) spin is identical to the field of a Dirac monopole positioned at the center of the Bloch sphere. In the SU(1,1) dynamic theory, the Berry curvature in $\mathbf{u}$-space is given by $\nabla\times\mathbf{A} = C_{\mathbf{n}}\frac{\hat{e}_\rho}{\rho}$ with $\rho = \sqrt{u_1^2 + u_2^2}$ the radial coordinate and $\hat{e}_\rho = (u_1\hat{e}_1 + u_2\hat{e}_2)/\rho$ the unit vector along radial direction. This Berry curvature is equal to the field of a *line of Dirac monopoles* positioned on the u_3-axis with uniform linear density $d = C_{\mathbf{n}}/2$ as shown in Fig. 1. The fact that the monopole line is infinitely long is a consequence of the noncompactness of the SU(1,1) group [37].

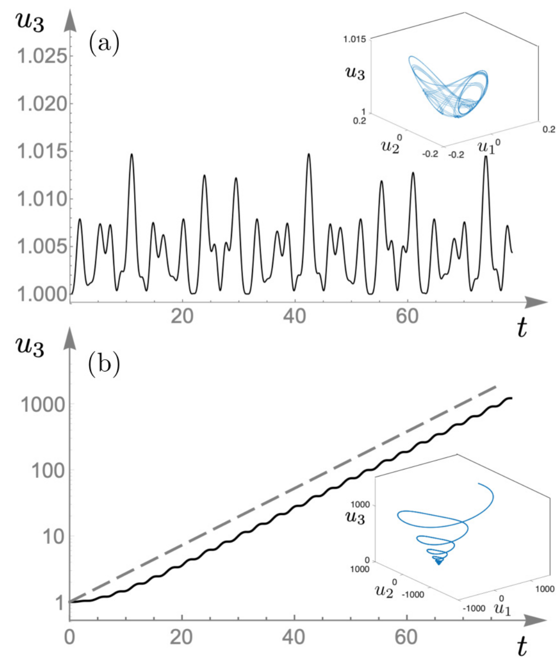

FIG. 2. Sold lines: Typical long-term behavior for $u_3(t)$. The time t is plotted in units of $1/\omega$. For both plots, we have $g_0 = 0$, $\delta g = 0.1\omega/n$. In (a), we set $\epsilon_{\mathbf{k}} = 1.2\omega$, which leads to a bounded oscillating behavior. In (b), we set $\epsilon_{\mathbf{k}} = 0.5\omega$ and find that u_3 grows exponentially in the long term (the y-axis is in log scale). Dashed line: $e^{\lambda t}$ with $\lambda \sim 0.103\omega$, the Lyapunov exponent calculated by the Floquet theory. One can see its long-term trend nicely agrees with $u_3(t)$. The insets show actual trajectories of $\mathbf{u}(t)$ in both cases.

VI. BOSE GAS WITH PERIODICALLY DRIVEN $g(t)$

In the following, we consider a specific form of time-varying interaction strength with $g(t) = g_0 + \delta g\sin\omega t$ and focus on the long-term behavior of the system. Such sinusoidal modulation is probably the most simple case and has already been implemented in several cold atom experiments [17,19,22,38].

In the case of the weakly interacting Bose gas $h_1 = 2g(t)n$, $h_2 = 0$, and $h_3 = 2[\epsilon_{\mathbf{k}} + g(t)n]$, the coupled linear equations (9) can be further simplified into a single differential equation for $u_{31} \equiv u_3 - u_1$ [39],

$$\dddot{u}_{31} + 4\epsilon_{\mathbf{k}}[\epsilon_{\mathbf{k}} + 2g(t)n]\dot{u}_{31} + 4\epsilon_{\mathbf{k}}\dot{g}(t)nu_{31} = 0. \tag{16}$$

One may check that the above equation is equivalent to the coupled equations (9).

For the periodic driven case, the Floquet theorem asserts that the solution to Eq. (9) must take the form of $\mathbf{u}(t) = e^{-iE_F t}\mathbf{p}(t)$ with E_F the quasi-energy and $\mathbf{p}(t)$ a periodic function in t. The quasi-energy $E_F = \alpha + i\lambda$ is in general complex, and its imaginary part controls the stability of the system. For a real quasi-energy, i.e., $\lambda = 0$, the vector $\mathbf{u}$ is always bounded, meaning the condensate emits only a finite number of thermal particles with momentum $\pm\mathbf{k}$. On the other hand, if the quasi-energy E_F is complex, the $\mathbf{u}$ grows exponentially in the long term, meaning the condensate will keep emitting thermal particles until the variational wave function (2) breaks down. As one can see, the imaginary part λ plays an important role in controlling the growth speed of the thermal modes,

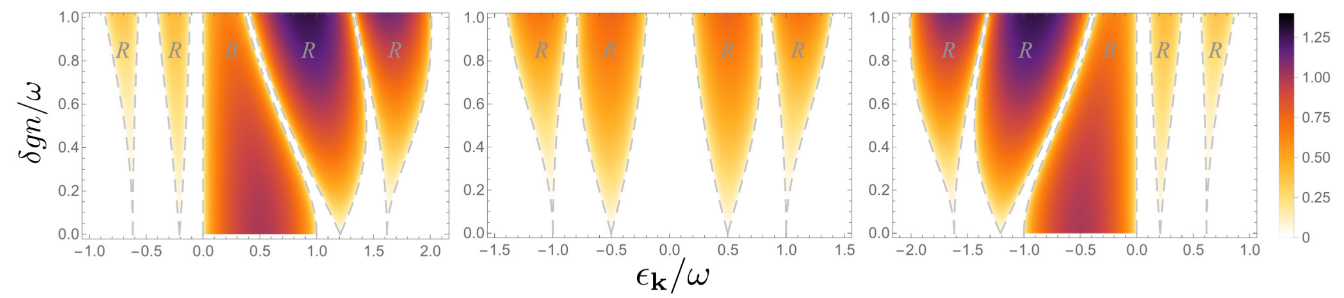

FIG. 3. Stability diagram for Bose gas with oscillating interaction strength $g(t) = g_0 + \delta g \sin \omega t$. From left to right: $g_0 = -0.5\omega/n$, $g_0 = 0$, $g_0 = 0.5\omega/n$. The white area marks the stable region with a vanishing Lyapunov exponent. The colored area marks the unstable regions. Dashed lines represent the transition curves that separate two regions. The Lyapunov exponent is shown via the color map. "B"s and "R"s in the instability lobes stand for Bogoliubov and resonance, which categorizes two different origins of the instabilities.

which can thus be interpreted as the Lyapunov exponent of the system. Recent developments in quantum chaotic systems have shown a close connection between the Lyapunov exponent of a quantum system and its out-of-time correlator [24–26]. In the following we will show that the Lyapunov exponent λ of our system can also be calculated by utilizing the SU(1,1) symmetry and mapping the system to a classical time-dependent harmonic oscillator.

We solve Eq. (9) numerically for different $\epsilon_{\mathbf{k}}$ (this is equivalent to focus on Bogoliubov modes with different momenta $\mathbf{k}$). The result is plotted in Fig. 2, which shows that u_3 indeed grows in the form of $e^{\lambda t}$. This is in contrast to the dynamics of an SU(2) spin as all the components of the SU(2) spin are bounded. As one can see from the insets, the exponentially growing solutions are related to the noncompactness of the SU(1,1) group, which is the main difference between the SU(1,1) and SU(2) groups.

To calculate λ, we further show that the third-order equation (16) is related to a second-order one:

$$\ddot{v} + \epsilon_{\mathbf{k}}[\epsilon_{\mathbf{k}} + 2g(t)n]v = 0. \tag{17}$$

Namely, if v_1, v_2 are the two solutions of Eq. (17), $u = v_1 v_2$ is then the solution of Eq. (16). Thus the three linear independent solutions for Eq. (16) are given by v_1^2, $v_1 v_2$, and v_2^2, with v_1, v_2 the linear independent solutions of Eq. (17) [40,41]. For $g(t) = g_0 + \delta g \sin \omega t$, Eq. (17) reduces to a Mathieu equation. The Mathieu equation can be used to describe the classical dynamics of a parametric oscillator, whose long-term Lyapunov exponent [42] may be calculated by the standard Whittaker-Hill formula [43].

It is worth noting that the Mathieu equation has already been used in the study of the dynamic Faraday instability of BEC within the mean-field approach [28–30]. For the instability analysis, both methods agree with each other and give the same result. We remark on two crucial differences between the mean-field approach and our beyond-mean-field approach. First, as previously mentioned, the mean-field approach always requires a small perturbation around the homogeneous saddle point solution of the GP equation; i.e., if the system is initially prepared *exactly* in its ground state, it never forms a Faraday pattern. Whereas in the beyond-mean-field approach, the ground state is *essentially* instable, as the quantum fluctuation (zero-point motion near the saddle point) plays the role of the perturbation. Second, from the path integral point of view, doing a perturbation analysis in the mean-field approach is equivalent to expanding the classical action $S[\psi]$ and approximating it by a quadratic one near the saddle point (i.e., a classical time-dependent harmonic oscillator). Whereas the Bogoliubov Hamiltonian H_B in our approach can be regarded as the corresponding quantum model. The fact that the dynamics of a time-dependent quantum harmonic oscillator can be solved by mapping it to a classical one is already known and well understood [35,36]. Nevertheless, there are certain quantum characters that can be captured only by the beyond-mean-field approach including the quantum zero-point energy (LHY correction) and the dynamical and geometric phase in the dynamic process.

We plot the Lyapunov exponent λ as a function of $\epsilon_{\mathbf{k}}$ and δg in Fig. 3. One can see that the system develops several instability lobes while turning on the modulation δg. These lobes are caused by two types of instability—the resonance instability and the Bogoliubov instability. The resonance instability lobes emerge from $\sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2g_0 n)} = n\omega/2$ for small modulation strength δg and keep growing while increasing δg. Note that $\sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2g_0 n)}$ is the energy for Bogoliubov mode in the unperturbed system. This indicates that this instability appears because the driven frequency is in resonance with two Bogoliubov excitations (one $\mathbf{k}$ and one $-\mathbf{k}$) of the unperturbed system. The Bogoliubov instability lobes exist even when there is no interaction strength modulation and shrink with increasing δg. They appear when $\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2g_0 n) < 0$, corresponding to the system having imaginary energy for the Bogoliubov mode. Such instability is an intrinsic instability of the unperturbed system and hence is named Bogoliubov instability. The fact that the Bogoliubov instability lobes shrink with increasing δg suggests that we may actually use the temporally modulated interaction to stabilize condensates that are originally unstable with static interactions (e.g., bosons with attractive interaction).

It can also be seen from Fig. 3 that in a regular uniform BEC, where $\epsilon_{\mathbf{k}} \in (0, +\infty)$ there is always instability associated with some momentum pair $(\mathbf{k}, -\mathbf{k})$. This instability causes an exponentially growing Bogoliubov phonon mode with momentum $(\mathbf{k}, -\mathbf{k})$ and eventually leads to the formation of Faraday patterns with spatial period π/k as observed in Ref. [17].

To conclude, we have developed a beyond-mean-field theory to describe the dynamics of Bose-Einstein condensates with time-varying interaction strength. Utilizing the Frenknel least action principle for a variational dynamical wave function, we proved that the noncondensate part of the system can be well described by a Bogoliubov-like Hamiltonian. Furthermore, by identifying a hidden SU(1,1) symmetry of the system, we show that the dynamic problem of bosons can be mapped to the problem of an SU(1,1) spin in a time-varying magnetic field. We explicitly constructed the time-dependent invariants of this SU(1,1) spin model in an *arbitrary* time-varying magnetic field. These invariants are crucial for the construction of the *exact* solution to the original time-dependent Schrödinger equation. Interestingly, the Berry curvature of the SU(1,1) spin is found to be identical to the field of a line of Dirac monopoles. Experiments that can generate such a gauge field in a BEC have been proposed for years but not yet realized [44]. Thus the model we describe in this work might provide an alternative and feasible method to create and simulate such a novel configuration of gauge fields. As an example of our theory, we calculated the dynamics of weakly interacting bosons with sinusoidally modulated interaction strength. Such a model has been extensively studied experimentally [17,22,28–30] because of its relation to the Faraday instability. We show that there exists a quantum-classical correspondence between our beyond-mean-field theory and the previously studied mean-field approach.

Note added. Recently, two preprints [45,46] which deal with a similar dynamic problem also appeared. Reference [45] studies the momentum distribution and structure factor of a Bose gas with periodically modulated scattering length, and Ref. [46] focuses on the geometrizing of the Bogoliuv Hamiltonian.

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