State space structure of tripartite quantum systems

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State space structure of tripartite quantum systems is analyzed. In particular, it has been shown that the set of states separable across all the three bipartitions [say, $\mathcal{B}^{int}(ABC)$] is a strict subset of the set of states having positive partial transposition across the three bipartite cuts [say, $\mathcal{P}^{int}(ABC)$] for all the tripartite Hilbert spaces $\mathbb{C}_A^{d_1}\otimes\mathbb{C}_B^{d_2}\otimes\mathbb{C}_C^{d_3}$ with $\min\{d_1,d_2,d_3\}\geqslant 2$. The claim is proved by constructing a state belonging to the set $\mathcal{P}^{int}(ABC)$ but not to $\mathcal{B}^{int}(ABC)$. For $(\mathbb{C}^d)^{\otimes 3}$ with $d\geqslant 3$, the construction follows from a specific type of multipartite unextendible product bases. However, such a construction is not possible for $(\mathbb{C}^2)^{\otimes 3}$ since bipartite systems $\mathbb{C}^2\otimes\mathbb{C}^n$ do not allow any unextendible product bases for arbitrary n [Phys. Rev. Lett. **82**, 5385 (1999)]. For the three-qubit system, we therefore come up with a different construction.

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I. INTRODUCTION

Hilbert space quantum mechanics provides an extremely precise mathematical description for microscopic phenomena. It associates tensor product Hilbert spaces with composite quantum systems and results in entangled quantum states with no analog in classical physics [1–3]. The advent of quantum information theory [4] identifies several important applications of entanglement (see [5,6] and references therein). Characterization, identification, and quantification of quantum entanglement are thus questions of great practical interest. One of the most widely used tests for bipartite states' entanglement verification is the positive partial transposition (PPT) criterion. States having negative partial transposition (NPT) are always entangled [7], whereas PPT implies separability only for the systems with composite dimensions not greater than six [8]. In other words, being PPT is a necessary and sufficient condition for separability in the composite systems with dimensions $\leqslant 6$. Consequently, for higher dimensional systems, one has several hierarchical convex-compact subsets of states within the set of allowed quantum states. Identifying these subsets as well as their boundaries is essential to understand the intricacy of quantum state space structure and entanglement properties of the quantum states. Having the convex-compact structures, these sets allow the classic Minkowski-Hahn-Banach separation theorem to characterize their several essential features [9].

The complexity of the situation increases rapidly with an increase in the number of component subsystems comprising the composite system [10]. For instance, separability/PPT-ness can be considered across different bipartite cuts, and accordingly one ends up with different convex-compact subsets of states. Several intriguing structures consequently emerge. For instance, a three-qubit state may not be fully separable even if it is separable across every possible bipartite cut and hence contains multipartite entanglement [11]. On the other hand, for such a system the set of biseparable states is strictly contained within the PPT mixture set [12]. In this work, we show that the set of states that are separable across all possible bipartitions are contained strictly within the set of states that are PPT across all possible bipartite cuts. We prove this claim for an arbitrary tripartite Hilbert space $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ with $\min\{d_1,d_2,d_3\}\geqslant 2$. The proof is constructive. We construct states that are PPT across all three bipartitions but inseparable across some bipartite cuts. Construction for $(\mathbb{C}^d)^{\otimes 3}$ with $d\geqslant 3$ follows from a specific kind of unextendible product bases. However, such a construction is not possible for three-qubit Hilbert space as there cannot be any set of mutually orthogonal product states in $\mathbb{C}^2\otimes\mathbb{C}^n$ that are not completable [13,14]. To prove the claim for three-qubit, we therefore come up with a different construction.

The paper is organized as follows: in Sec. II we note the notations and recall some preliminary results; all our findings are listed in Sec. III; finally in Sec. IV we give our conclusions along with some open questions for further research.

II. NOTATIONS AND PRELIMINARIES

A quantum system S is associated with a complex separable Hilbert space $\mathcal{H}_S$ over complex field [4,15]. We will consider only finite-dimensional systems throughout this paper, hence $\mathcal{H}_S$ will be isomorphic to some complex space $\mathbb{C}_S^d$, where d is the dimension of the complex vector space. The system's state is described by a density operator ρ_S (positive operator with unit trace) acting on $\mathbb{C}_S^d$. A collection of density operators forms a convex compact set $\mathcal{D}(S)$ embedded in $\mathbb{R}^{d^2-1}$. We will also sometimes specify this set as $\mathcal{D}(d)$ to distinguish between systems with different dimensions. The extreme points of $\mathcal{D}(S)$ are called pure states, and they satisfy the condition $\rho_S^2=\rho_S$. Let $\mathcal{E}_{\mathcal{D}}(S)$ denotes the set of all extremal points of $\mathcal{D}(S)$. Such an extremal state can also be considered as a rank-1 projector, i.e., $\rho_S=|\psi\rangle_S\langle\psi|$ for some

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$|\psi\rangle_S \in \mathbb{C}_S^d$ whenever $\rho_S \in \mathcal{E}_{\mathcal{D}}(S)$. States that are not pure are called mixed states, and they allow convex decomposition in terms of pure states, i.e., $\forall\, \rho_S \notin \mathcal{E}_{\mathcal{D}}(S)$, $\exists\, \sigma_S^i \in \mathcal{E}_{\mathcal{D}}(S)$ s.t. $\rho_S = \sum p_i \sigma_S^i$, where $p_i > 0$ and $\sum p_i = 1$.

Hilbert space $\mathbb{C}_{AB\cdots}$ of a composite system consisting of component subsystems $A, B, \ldots$ is given by the tensor product of the Hilbert spaces associated with the component subsystems, i.e., $\mathbb{C}_{AB\cdots}^d = \mathbb{C}_A^{d_A} \otimes \mathbb{C}_B^{d_B} \otimes \cdots$. Here $d_A, d_B, \ldots$ denotes the dimension of the component subsystems, while the dimension of the composite system is $d = d_A \times d_B \times \cdots$. While the axiomatic formulations of quantum mechanics contain this tensor product postulate [16–18], recent developments indicate that this assumption can be logically derived from the state postulate and the measurement postulate [19].

A bipartite state $\rho_{AB} \in \mathcal{D}(AB)$ is called a pure product state if and only if $\rho_{AB} = |\chi\rangle_{AB}\langle\chi|$, where $|\chi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$ for some $|\psi\rangle_A \in \mathbb{C}_A$ and $|\phi\rangle_B \in \mathbb{C}_B$. The convex hull of these product states will be denoted as $\mathcal{S}(AB)$, and the states in $\mathcal{S}(AB)$ are generally called separable states. A state $\rho_{AB}^{ent} \in \mathcal{D}(AB)$ but $\rho_{AB}^{ent} \notin \mathcal{S}(AB)$ is called an entangled state. For a bipartite system one can define another convex compact set, the Peres set $\mathcal{P}(AB)$: the set of states with positive-partial-transpose (PPT) [7]. A bipartite state ρ_{AB} belongs to the set $\mathcal{P}(AB)$ whenever $\rho_{AB}^{T_k} \geqslant 0$, where T_k denotes (partial) transposition with respect to the kth subsystem with $k \in \{A, B\}$.

For a bipartite system, the following set inclusion relation is immediate:

$$\mathcal{S}(AB) \subseteq \mathcal{P}(AB) \subsetneq \mathcal{D}(AB). \tag{1}$$

Equality between the first two sets holds for composite systems of dimensions not greater than six [8]. For the higher dimensions $\mathcal{S}(AB)$ is known to be a proper subset of $\mathcal{P}(AB)$ [11,14,20–23].

Moving to the tripartite system, a pure state $|\chi\rangle_{ABC}$ is called fully product if it is of the form $|\chi\rangle_{ABC} = |\psi\rangle_A \otimes |\phi\rangle_B \otimes |\eta\rangle_C$ for some $|\psi\rangle_A \in \mathbb{C}_A^{d_A}$, $|\phi\rangle_B \in \mathbb{C}_B^{d_B}$, and $|\eta\rangle_C \in \mathbb{C}_C^{d_C}$. The convex hull of pure product states will be denoted as $\mathcal{F}(ABC)$, and a state belonging to this set is generally called a fully separable state. A tripartite state is called biseparable across the A|BC cut if it is of the form $|\chi\rangle_{ABC} = |\psi\rangle_A \otimes |\phi\rangle_{BC}$, where the state $|\phi\rangle_{BC}$ is allowed to be entangled across the B|C cut. The convex hull of states biseparable across the A|BC cut will be denoted as $\mathcal{B}(A|BC)$. Similarly, one can define the sets $\mathcal{B}(B|CA)$ and $\mathcal{B}(C|AB)$ that are biseparable across the B|CA and C|AB cuts, respectively. A state belonging in the convex hull of the sets $\mathcal{B}(A|BC)$, $\mathcal{B}(B|CA)$, and $\mathcal{B}(C|AB)$ is generally called biseparable, and we denote the convex hull of the sets as $\mathcal{B}^{ch}(ABC)$:

$$\mathcal{B}^{ch}(ABC) := \text{Convex Hull}\{\mathcal{B}(A|BC), \mathcal{B}(B|CA), \mathcal{B}(C|AB)\}. \tag{2}$$

We can also consider the intersection of these three sets, which we will denote as $\mathcal{B}^{int}(ABC)$:

$$\mathcal{B}^{int}(ABC) := \mathcal{B}(A|BC) \cap \mathcal{B}(B|CA) \cap \mathcal{B}(C|AB). \tag{3}$$

Common intuition might lead to the conclusion that the set $\mathcal{B}^{int}(ABC)$ is identical to the set of fully separable states and hence contains no entanglement. However, the construction of SHIFTS UPB in [11] establishes that this is not the case. Thus a tripartite state can contain entanglement even when it is separable across every possible bipartition. The convex sets of the PPT states over a particular cut can also be defined, and the convex hull as well as the intersection of distinct such sets can then be considered. We will respectively denote them as $\mathcal{P}(A|BC)$, $\mathcal{P}(B|CA)$, $\mathcal{P}(C|AB)$, $\mathcal{P}^{ch}(ABC)$, and $\mathcal{P}^{int}(ABC)$, where

$$\mathcal{P}^{ch}(ABC) := \text{Convex Hull}\{\mathcal{P}(A|BC), \mathcal{P}(B|CA), \mathcal{P}(C|AB)\}; \tag{4a}$$

$$\mathcal{P}^{int}(ABC) := \mathcal{P}(A|BC) \cap \mathcal{P}(B|CA) \cap \mathcal{P}(C|AB). \tag{4b}$$

In the present work, we aim to explore the set inclusion relations among these different convex sets.

The following inclusion relation is apparent in tripartite systems:

$$\mathcal{F}(ABC) \subsetneq \mathcal{B}^{ch}(ABC) \subsetneq \mathcal{P}^{ch}(ABC). \tag{5}$$

Here we emphasize the fact that $\mathcal{B}^{ch}(ABC)$ is a proper subset of $\mathcal{P}^{ch}(ABC)$ for all tripartite systems, unlike in the bipartite case. Thus we can have tripartite states which are PPT entangled. The PPT entangled states exhibit an intriguing irreversible feature: their preparation under local quantum operations assisted with classical communications (LOCC) requires a nonzero amount of a maximally entangled state to be shared between the subsystems [24], but no maximally entangled state can be distilled from them under LOCC [25,26]. Despite being undistillable, PPT entangled states find several applications, such as activating entanglement distillation for other entangled states [27,28], multipartite entanglement manipulation [29], information processing [30], private key distillation [31–33], and quantum metrology [34].

III. RESULTS

Let us first recall some already known structures. It seems tempting to assume that the set $\mathcal{B}^{int}(ABC)$ should be identical to the set of fully separable states $\mathcal{F}(ABC)$. Quite surprisingly, even for the simplest case of the three-qubit system, this intuition is not correct. It turns out that $\mathcal{F}(2 \otimes 2 \otimes 2)$ is a strict subset of $\mathcal{B}^{int}(2 \otimes 2 \otimes 2)$. An example of a state belonging in $\mathcal{B}^{int}(2 \otimes 2 \otimes 2)$ but not in $\mathcal{F}(2 \otimes 2 \otimes 2)$ follows from the construction of the unextendible product basis (UPB) in $(\mathbb{C}^2)^{\otimes 3}$ [11]:

$$\mathcal{U}_{PB}^{\textbf{Shifts}} \equiv \left\{ \begin{array}{ll} |S_1\rangle := |0, 1, +\rangle, & |S_2\rangle := |1, +, 0\rangle \\ |S_3\rangle := |+, 0, 1\rangle, & |S_4\rangle := |-, -, -\rangle \end{array} \right\}, \tag{6}$$

where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $|x, y, z\rangle \in (\mathbb{C}^2)^{\otimes 3}$ stands as a short-hand notation for $|x\rangle_A \otimes |y\rangle_B \otimes |z\rangle_C$. Let us consider the three-qubit state

$$\rho_{SU} := \frac{1}{4}\left(\mathbf{I}_8 - \sum_{i=1}^{4} |S_i\rangle\langle S_i|\right). \tag{7}$$

Since the construction follows from the SHIFTS UPB [11], we use the initials "SU" as a subindex to denote the resulting state. From the property of SHIFTS UPB, it follows that $\rho_{SU} \notin \mathcal{F}(2 \otimes 2 \otimes 2)$. However, as shown in [11], the state is biseparable across all the three bipartite cuts and hence $\rho_{SU} \in \mathcal{B}^{int}(2 \otimes 2 \otimes 2)$. In other words, being separable across all possible bipartitions, the state ρ_{SU} contains multipartite entanglement.

The authors in [35,36] find simple criteria to certify genuine entanglement in a multipartite state. To this aim, the authors in [36] have made the conjecture that for the three-qubit system, the set of the PPT-mixture will be identical to the set of biseparable states, i.e., $\mathcal{P}^{ch}(2 \otimes 2 \otimes 2) = \mathcal{B}^{ch}(2 \otimes 2 \otimes 2)$. However, Ha and Kye disproved this conjecture by constructing three-qubit genuinely entangled states which are PPT [12]. Motivated by the results of Refs. [12,36], here we address a different question. We ask whether a state being PPT across all the three bipartitions implies that it is biseparable across all the three bipartite cuts. The recent results of [37,38] are worth mentioning at this point. In particular, the authors in [37] have provided a family of N-partite almost diagonal symmetric entangled states that are PPT with respect to each bipartition but nevertheless are entangled. However, their construction is for odd N greater than 3. So the question remains unanswered for the general tripartite systems. Our question can be reformulated as to whether $\mathcal{B}^{int}(ABC)$ and $\mathcal{P}^{int}(ABC)$ are the same set. In the following sections, we answer this question in the negative and prove that $\mathcal{B}^{int}(ABC)$ is a proper subset of $\mathcal{P}^{int}(ABC)$.

Theorem 1. $\mathcal{B}^{int}(ABC) \subsetneq \mathcal{P}^{int}(ABC)$ for $\mathbb{C}_A^{d_1} \otimes \mathbb{C}_B^{d_2} \otimes \mathbb{C}_C^{d_3}$ with $\min\{d_1, d_2, d_3\} \geqslant 2$.

A state being separable in some bipartition must be PPT across that bipartition. It therefore follows that a state belonging to the set $\mathcal{B}^{int}(ABC)$ must also belong to the set $\mathcal{P}^{int}(ABC)$. To prove the strict inclusion relation, we provide an explicit construction of states, ρ_{ABC}, that belong to $\mathcal{P}^{int}(ABC)$ but not to $\mathcal{B}^{int}(ABC)$. We first discuss the construction in $d \otimes d \otimes d$ with $d \geqslant 3$ and then in $2 \otimes 2 \otimes 2$ dimensions.

A. Construction in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geqslant 3$

For $d = 3$, the construction follows from a recently proposed UPB in $(\mathbf{C}^3)^{\otimes 3}$ [39]. We will work with the computational basis $\{|p, q, r\rangle \mid p, q, r = 0, 1, 2\}$ for the Hilbert space $(\mathbf{C}^3)^{\otimes 3}$. Consider the following *twisted* orthogonal product basis (t-OPB):

$$\mathbb{B}_0 := \{|\psi\rangle_{kkk} \equiv |k, k, k\rangle \mid k \in \{0, 1, 2\}\}, \tag{8a}$$
$$\mathbb{B}_1 := \{|\psi(i, j)\rangle_1 \equiv |0, \eta_i, \xi_j\rangle\}, \tag{8b}$$
$$\mathbb{B}_2 := \{|\psi(i, j)\rangle_2 \equiv |\eta_i, 2, \xi_j\rangle\}, \tag{8c}$$
$$\mathbb{B}_3 := \{|\psi(i, j)\rangle_3 \equiv |2, \xi_j, \eta_i\rangle\}, \tag{8d}$$
$$\mathbb{B}_4 := \{|\psi(i, j)\rangle_4 \equiv |\eta_i, \xi_j, 0\rangle\}, \tag{8e}$$
$$\mathbb{B}_5 := \{|\psi(i, j)\rangle_5 \equiv |\xi_j, 0, \eta_i\rangle\}, \tag{8f}$$
$$\mathbb{B}_6 := \{|\psi(i, j)\rangle_6 \equiv |\xi_j, \eta_i, 2\rangle\}, \tag{8g}$$

where $i, j \in \{0, 1\}$, and $|\eta_i\rangle := |0\rangle + (-1)^i |1\rangle$, $|\xi_j\rangle := |1\rangle + (-1)^j |2\rangle$. Consider the state $|S\rangle := (|0\rangle + |1\rangle + |2\rangle)^{\otimes 3}$. Note that $|S\rangle$ is orthogonal neither to the states in $\mathbb{B}_0$ nor to the states $\{|\psi(0, 0)\rangle_l\}_{l=1}^6$. It is orthogonal to the remaining states, and accordingly the set of states

$$\mathcal{U}_{PB}^{[3]} := \left\{ \bigcup_{l=1}^{6} \{\mathcal{B}_l \setminus |\psi(0, 0)\rangle_l\} \bigcup |S\rangle \right\} \tag{9}$$

forms a UPB in $(\mathbf{C}^3)^{\otimes 3}$ [39]. The cardinality of $\mathcal{U}_{PB}^{[3]}$ is 19 and the eight-dimensional subspace orthogonal to this UPB is fully entangled. The normalized projector on this fully entangled subspace is a rank-8 density operator $\rho^{[3]}(8) \in \mathcal{D}((\mathbf{C}^3)^{\otimes 3})$, given by

$$\rho^{[3]}(8) := \frac{1}{8}\left(\mathbb{I}_{27} - \sum_{|\psi\rangle \in \mathcal{U}_{PB}^{[3]}} |\tilde{\psi}\rangle \langle \tilde{\psi}| \right). \tag{10}$$

Here $|\tilde{x}\rangle$ denotes the normalized state proportional to the unnormalized ray vector $|x\rangle$.

Proposition 1. The state $\rho^{[3]}(8) \in \mathcal{P}^{int}(3 \otimes 3 \otimes 3)$ but $\rho^{[3]}(8) \notin \mathcal{B}^{int}(3 \otimes 3 \otimes 3)$.

Proof. Partial transposition (PT) acts linearly, and on a product state $\rho_{AB} = |\phi\rangle_A \langle\phi| \otimes |\chi\rangle_B \langle\chi|$ it acts as $\rho_{AB}^{T_B} := (\mathbf{I} \otimes T)[|\phi\rangle_A \langle\phi| \otimes |\chi\rangle_B \langle\chi|] := |\phi\rangle_A \langle\phi| \otimes |\chi^\star\rangle_B \langle\chi^\star|$ [14], where $|\chi^\star\rangle = \sum \alpha_i^\star |i\rangle$ for $|\chi\rangle = \sum \alpha_i |i\rangle$ with $\{|i\rangle\}$ being an orthonormal basis. Therefore we have $\rho_3^{T_x}(8) \geqslant 0$ for $x \in \{A, B, C\}$. In fact, since all the coefficients of the states in t-OPB $\{\mathbb{B}_l\}_{l=0}^6$ are real, therefore $\rho_3^{T_x}(8) = \rho^{[3]}(8)$ for all x. This implies that the state $\rho^{[3]}(8)$ is PPT across all bipartitions, and consequently $\rho^{[3]}(8) \in \mathcal{P}^{int}(3 \otimes 3 \otimes 3)$. A state to belong in the set $\mathcal{B}(A|BC)$ must allow separable decomposition across this cut. However, as pointed out in [39], one can have only four mutually orthogonal states that are separable across the A|BC cut and orthogonal to the states in $\mathcal{U}_{PB}^{[3]}$.[1] These four states are given by $\{|\psi^-\rangle_{24}, |\psi^-\rangle_{56}, |\psi^-\rangle_{(0)1}, |\psi^-\rangle_{(2)3}\}$, where $|\psi^-\rangle_{lm} := |\psi(0, 0)\rangle_l - |\psi(0, 0)\rangle_m$ and $|\psi^-\rangle_{(k)l} := 4|k, k, k\rangle - |\psi(0, 0)\rangle_l$. Since the state $\rho^{[3]}(8)$ is of rank 8, there is a deficit of separable states across $A|BC$ cut, and thus $\rho^{[3]}(8)$ does not allow a separable decomposition across this cut implying $\rho^{[3]}(8) \notin \mathcal{B}(A|BC)$. From the symmetry of the construction, it follows that $\rho^{[3]}(8)$ belongs neither to $\mathcal{B}(B|CA)$ nor to $\mathcal{B}(C|AB)$, and consequently, it follows that $\rho^{[3]}(8) \notin \mathcal{B}^{int}(ABC)$. This completes the proof. ■

Proposition 1 can be generalized for arbitrary higher-dimensional Hilbert spaces $(\mathbb{C}^d)^{\otimes 3}$ using the UPBs constructed in [39]. For the general construction, we refer readers to [39]. Here we recall only the UPB of $(\mathbb{C}^4)^{\otimes 3}$ since the construction for even dimensions is different from the odd dimensional case. The t-OPB in $(\mathbb{C}^4)^{\otimes 3}$ is given by

$$\mathbb{B}_0 := \{|\psi\rangle_{kkk} \equiv |k, k, k\rangle \mid k \in \{0, 3\}\}, \tag{11a}$$

[1] Using this construction, the authors in [39] have introduced the concept of unextendible biseparable basis (UBB). Construction of UBB is crucial as the subspace orthogonal to it turns out to be a genuinely entangled subspace. This construction is also relevant to the study of genuine quantum nonlocality without the entanglement (GQNWE) phenomenon [40–42]. GQNWE is a true multiparty generalization of the seminal quantum nonlocality without the entanglement [13] phenomenon, which has been studied beyond the quantum scenario very recently [43].

$$\mathbb{B}_0' := \{|\psi(l,m,p)\rangle \equiv |\phi_l,\phi_m,\phi_p\rangle\}, \tag{11b}$$

$$\mathbb{B}_1 := \{|\psi(i,j)\rangle_1 \equiv |0,\eta_i,\xi_j\rangle\}, \tag{11c}$$

$$\mathbb{B}_2 := \{|\psi(i,j)\rangle_2 \equiv |\eta_i,3,\xi_j\rangle\}, \tag{11d}$$

$$\mathbb{B}_3 := \{|\psi(i,j)\rangle_3 \equiv |\xi_j,0,\eta_i\rangle\}, \tag{11e}$$

$$\mathbb{B}_4 := \{|\psi(i,j)\rangle_4 \equiv |\xi_j,\eta_i,3\rangle\}, \tag{11f}$$

$$\mathbb{B}_5 := \{|\psi(i,j)\rangle_5 \equiv |3,\xi_j,\eta_i\rangle\}, \tag{11g}$$

$$\mathbb{B}_6 := \{|\psi(i,j)\rangle_6 \equiv |\eta_i,\xi_j,0\rangle\}, \tag{11h}$$

where $l,m,p \in \{0,1\}$, $|\phi_0\rangle := |1\rangle + |2\rangle$, $|\phi_1\rangle := |1\rangle - |2\rangle$; $i,j \in \{0,1,2\}$, $|\eta_0\rangle = |0\rangle + |1\rangle + |2\rangle$, and $|\eta_1\rangle$ and $|\eta_2\rangle$ are a linear combination of $\{|0\rangle, |1\rangle, |2\rangle\}$ such that $\{|\eta_i\rangle\}_{i=0}^2$ are mutually orthogonal; $|\xi_0\rangle = |1\rangle + |2\rangle + |3\rangle$ and $|\xi_1\rangle$ and $|\xi_2\rangle$ are a linear combination of $\{|1\rangle, |2\rangle, |3\rangle\}$ such that $\{|\xi_i\rangle\}_{i=0}^2$ are mutually orthogonal. Considering $|S\rangle := (|0\rangle + |1\rangle + |2\rangle + |3\rangle)^{\otimes 3}$ the UPB is given by

$$\mathcal{U}_{PB}^{[4]} := \left\{ \bigcup_{l=1}^{6} \{\mathcal{B}_l \setminus |\psi(0,0)\rangle_l\} \bigcup \{\mathcal{B}_0' \setminus |\psi(0,0,0)\rangle\} \bigcup |S\rangle \right\}. \tag{12}$$

The rank-8 state $\rho^{[4]}(8)$ belonging to $\mathcal{P}^{int}(4\otimes 4\otimes 4)$ but not to $\mathcal{B}^{int}(4\otimes 4\otimes 4)$ is given by

$$\rho^{[4]}(8) := \frac{1}{8}\left(\mathbb{I}_{64} - \sum_{|\psi\rangle \in \mathcal{U}_{PB}^{[4]}} |\tilde{\psi}\rangle\langle\tilde{\psi}| \right). \tag{13}$$

This finalizes the construction in $(\mathbb{C}^d)^{\otimes 3}$ for $d \geqslant 3$. However, a similar construction is not possible for the three-qubit Hilbert space. In the next subsection, we will discuss a different construction for this case.

B. Construction in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

We would first like to point out a fundamental difference between the three-qubit unextendible product basis $\mathcal{U}_{PB}^{\textbf{Shifts}}$ and the UPBs $\mathcal{U}_{PB}^{[3]}$ and $\mathcal{U}_{PB}^{[4]}$ (and their generalization). If biseparable states are allowed along with the states in $\mathcal{U}_{PB}^{\textbf{Shifts}}$, one can construct a complete orthogonal basis for $(\mathbb{C}^2)^{\otimes 3}$. For instance, consider the two-qubit states $|a\rangle = |1,+\rangle$, $|b\rangle = |+,0\rangle$, $|c\rangle = |0,1\rangle$, $|d\rangle = |-,-\rangle$. Then the states

$$\$(A|BC) \equiv \begin{Bmatrix} |\kappa_1\rangle := |0\rangle\,|a^\perp\rangle\,,\ |\kappa_2\rangle := |1\rangle\,|b^\perp\rangle \\ |\kappa_3\rangle := |+\rangle\,|c^\perp\rangle\,,\ |\kappa_4\rangle := |-\rangle\,|d^\perp\rangle \end{Bmatrix}$$

are separable across the A|BC cut, where $|a^\perp\rangle, |b^\perp\rangle \in \text{Span}\{|a\rangle, |b\rangle\}$ and $|c^\perp\rangle, |d^\perp\rangle \in \text{Span}\{|c\rangle, |d\rangle\}$ with $|x^\perp\rangle$ denoting the state orthogonal to $|x\rangle$. The states in $\$(A|BC)$ along with the states in $\mathcal{U}_{PB}^{\textbf{Shifts}}$ form a complete basis for $(\mathbb{C}^2)^{\otimes 3}$. On the other hand, as discussed in the proof of Proposition 1, the set $\mathcal{U}_{PB}^{[3]}$ (or its generalization $\mathcal{U}_{PB}^{[d]}$) is not completable by appending only biseparable states. This fact plays a crucial role in proving the nonbiseparability of the state $\rho^{[3]}(8)$ and its generalization. This fact, however, does not hold for a three-qubit product basis as it is not possible to have a set of orthogonal product states in $\mathbb{C}^2 \otimes \mathbb{C}^n$ which is uncompletable [11,14]. We therefore look for a different method to construct our required state in the three-qubit Hilbert space.

At this stage, we check the state of [12], used to disprove the conjecture in [36]. This particular state is PPT across the A|BC cut and hence belongs to $\mathcal{P}(A|BC)$. However, being NPT across the other cuts, it does not belong to $\mathcal{P}^{int}(2\otimes 2\otimes 2)$ and hence fails to fulfill our purpose.

We then consider the PPT-bound entangled state of $\mathbb{C}^2 \otimes \mathbb{C}^4$ as constructed by Horodecki [20]. This particular state can also be thought of as a state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Let us recall Horodecki's construction of a three-qubit state. For that, first consider the states

$$|\psi^1\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |00\rangle_{BC} + |1\rangle_A \otimes |01\rangle_{BC}), \tag{14a}$$

$$|\psi^2\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |01\rangle_{BC} + |1\rangle_A \otimes |10\rangle_{BC}), \tag{14b}$$

$$|\psi^3\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |10\rangle_{BC} + |1\rangle_A \otimes |11\rangle_{BC}), \tag{14c}$$

$$|\phi^{(b)}\rangle_{ABC} = |1\rangle_A \otimes \left(\sqrt{\frac{1+b}{2}}\,|00\rangle + \sqrt{\frac{1-b}{2}}\,|11\rangle \right)_{BC}, \tag{14d}$$

where $0 \leqslant b \leqslant 1$. Consider the density operator defined by

$$\chi_{ABC} := \frac{2}{7}\sum_{i=1}^{3} \mathbb{P}[\psi^i_{ABC}] + \frac{1}{7}\mathbb{P}[011], \tag{15}$$

where $\mathbb{P}[x] := |x\rangle\langle x|$. A straightforward calculation yields that the state χ_{ABC} is NPT across the A|BC cut. Consider a new density operator,

$$\sigma_{ABC}^{(b)} := \frac{7b}{7b+1}\chi_{ABC} + \frac{1}{7b+1}\mathbb{P}[\phi_{ABC}^{(b)}]. \tag{16}$$

The state $\sigma_{ABC}^{(b)}$ turns out to be PPT across A|BC cut for the whole range of the parameter b. Here the state $\mathbb{P}[\phi_{ABC}^{(b)}]$ can be thought of as the noise part that absorbs the NPT-ness of χ_{ABC}. Furthermore, applying the range criterion of entanglement (Theorem 2 of [20]), it turns out that for $0 < b < 1$ the state is entangled, whereas it is separable for $b = 0, 1$. Therefore, for $b \in (0,1)$ the state $\sigma_{ABC}^{(b)} \in \mathcal{P}(A|BC)$ but $\sigma_{ABC}^{(b)} \notin \mathcal{B}(A|BC)$. Matrix representation of $\sigma_{ABC}^{(b)}$ in the computational basis reads as

$$\sigma_{ABC}^{(b)} \equiv \frac{1}{7b+1}\begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{pmatrix}. \tag{17}$$

Here we follow the lexicographic order from left to right and from up to down. Explicit calculation further yields that $\sigma_{ABC}^{(b)}$ is NPT across the other two cuts for $b \in [0,1]$, and hence it does not belong to $\mathcal{P}^{int}(2\otimes 2\otimes 2)$. It is important to note that the construction of $\sigma^{(b)}_{ABC}$ is not party symmetric. So we consider a party symmetric state $\eta^{(b)}_{ABC}$ given by

$$\eta^{(b)}_{ABC} := \frac{1}{3}\left(\sigma\frac{(b)}{ABC} + \sigma\frac{(b)}{BCA} + \sigma\frac{(b)}{CAB}\right). \tag{18}$$

We use a symbol like $\sigma\frac{(b)}{ABC}$ to denote that the ordering of party index does matter. For instance, the state $\sigma\frac{(b)}{ABC}$ is the same as $\sigma^{(b)}_{ABC}$ of Eq. (16), whereas $\sigma\frac{(b)}{BCA}$ is same as the state in Eq. (16) but with the role of the party indices changed as $A \to B,\ B \to C,\ \mathbb{C} \to A$; the state $\sigma\frac{(b)}{CAB}$ is defined similarly.

For a certain range of the parameter b, the state $\eta^{(b)}_{ABC}$ turns out to be PPT across all three bipartitions. However, we find that the rank of this state as well as its partial transposition across different cuts turn out to be 8. Therefore, the range criterion does not directly apply to establish the inseparability of this state. To obtain a lower rank density operator, we thus consider the following operator:

$$\begin{aligned} h^{(b)}_{ABC} := \eta^{(b)}_{ABC} - \mu\left(v_1 v_1^{\mathrm{T}} + v_2 v_2^{\mathrm{T}}\right) + \nu\left(v_3 v_4^{\mathrm{T}} + v_4 v_3^{\mathrm{T}}\right) \\ + \epsilon\left(v_5 v_6^{\mathrm{T}} + v_6 v_5^{\mathrm{T}}\right), \end{aligned} \tag{19}$$

where $\mu := \frac{b}{3(1+7b)}$, $\nu = \frac{1+3b}{6+42b}$, $\epsilon := \frac{2b}{3(1+7b)}$; T denotes matrix transposition; and

$$v_1 := (01000010)^{\mathrm{T}}, \quad v_2 := (00100100)^{\mathrm{T}},$$
$$v_3 := (01000000)^{\mathrm{T}}, \quad v_4 := (00100000)^{\mathrm{T}},$$
$$v_5 := (00000010)^{\mathrm{T}}, \quad v_6 := (00000100)^{\mathrm{T}}.$$

Although the matrix $h^{(b)}_{ABC}$ is positive semidefinite, it does not have a unit trace. A proper normalization yields us the density operator

$$\rho^{[2]}_{ABC}(b) := \frac{3+21b}{3+17b} h^{(b)}_{ABC}. \tag{20}$$

Straightforward calculations lead us to the following observations regarding the state $\rho^{[2]}_{ABC}(b)$:

O-1 : $\rho^{[2]}_{ABC}(b)$ is a valid density operator for $b \in [0, 1]$, i.e., for all values of the parameter b, $\rho^{[2]}_{ABC}(b) \in \mathcal{D}(2 \otimes 2 \otimes 2)$.

O-2 : Partial transposition of $\rho^{[2]}_{ABC}(b)$ with respect to A is positive semidefinite for parameter values $b \in [0, 1]$, i.e., $[\rho^{[2]}_{ABC}(b)]^{\mathrm{T}_A} \geqslant 0$ and consequently $\rho^{[2]}_{ABC}(b) \in \mathcal{P}(A|BC)$ for all $b \in [0, 1]$.

O-3 : Partial transpositions of $\rho^{[2]}_{ABC}(b)$ with respect to B and C are positive semidefinite for parameter values $b \in (\sim 0.8184, 1] := \mathfrak{R} \subset [0, 1]$, i.e., $[\rho^{[2]}_{ABC}(b)]^{\mathrm{T}_x} \geqslant 0$ for $x \in \{B, C\}$ and consequently $\rho^{[2]}_{ABC}(b) \in \mathcal{P}(B|CA)$ and $\rho^{[2]}_{ABC}(b) \in \mathcal{P}(C|AB)$ for all $b \in \mathfrak{R}$.

We thus arrive at the following proposition:

Proposition 2. For the parameter values $b \in \mathfrak{R}$, the state $\rho^{[2]}_{ABC}(b) \in \mathcal{P}^{int}(2 \otimes 2 \otimes 2)$ but $\rho^{[2]}_{ABC}(b) \notin \mathcal{B}^{int}(2 \otimes 2 \otimes 2)$, and hence $\mathcal{B}^{int}(2 \otimes 2 \otimes 2) \subsetneq \mathcal{P}^{int}(2 \otimes 2 \otimes 2)$.

Proof. Proof of the first part follows immediately from the observations O-2 and O-3. Since $\rho^{[2]}_{ABC}(b) \in \mathcal{P}(A|BC)$ for $b \in [0, 1]$ and $\rho^{[2]}_{ABC}(b) \in \mathcal{P}(B|CA), \mathcal{P}(C|AB)$ for $b \in \mathfrak{R}$, therefore $\rho^{[2]}_{ABC}(b) \in \mathcal{P}^{int}(2 \otimes 2 \otimes 2)$ for $b \in \mathfrak{R}$.

We now prove that the state $\rho^{[2]}_{ABC}(b)$ is not separable across the AB|C cut. For that, we first write the matrices of the density operator $\rho^{[2]}_{ABC}(b)$ and its partial transposition with respect to C,

$$\rho^{[2]}_{ABC}(b) \equiv \Theta \begin{pmatrix} \Gamma & 0 & 0 & \frac{\Gamma}{3} & 0 & \frac{\Gamma}{3} & \frac{\Gamma}{3} & 0 \\ 0 & \Lambda & \Lambda & 0 & 0 & 0 & 0 & \Omega \\ 0 & \Lambda & \Lambda & 0 & 0 & 0 & 0 & \Omega \\ \frac{\Gamma}{3} & 0 & 0 & \Gamma & \frac{\Gamma}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Gamma}{3} & \Delta & 0 & 0 & \Omega \\ \frac{\Gamma}{3} & 0 & 0 & 0 & 0 & \frac{2\Gamma}{3} & \frac{2\Gamma}{3} & 0 \\ \frac{\Gamma}{3} & 0 & 0 & 0 & 0 & \frac{2\Gamma}{3} & \frac{2\Gamma}{3} & 0 \\ 0 & \Omega & \Omega & 0 & \Omega & 0 & 0 & \zeta \end{pmatrix}, \tag{21}$$

$$\left[\rho^{[2]}_{ABC}(b)\right]^{\mathrm{T}_c} \equiv \Theta \begin{pmatrix} \Gamma & 0 & 0 & \Lambda & 0 & 0 & \frac{\Gamma}{3} & 0 \\ 0 & \Lambda & \frac{\Gamma}{3} & 0 & \frac{\Gamma}{3} & 0 & 0 & \Omega \\ 0 & \frac{\Gamma}{3} & \Lambda & 0 & 0 & \frac{\Gamma}{3} & 0 & 0 \\ \Lambda & 0 & 0 & \Gamma & 0 & 0 & \Omega & 0 \\ 0 & \frac{\Gamma}{3} & 0 & 0 & \Delta & 0 & 0 & \frac{2\Gamma}{3} \\ 0 & 0 & \frac{\Gamma}{3} & 0 & 0 & \frac{2\Gamma}{3} & \Omega & 0 \\ \frac{\Gamma}{3} & 0 & 0 & \Omega & 0 & \Omega & \frac{2\Gamma}{3} & 0 \\ 0 & \Omega & 0 & 0 & \frac{2\Gamma}{3} & 0 & 0 & \zeta \end{pmatrix}, \tag{22}$$

where $\Gamma := \frac{b}{1+7b}$, $\Lambda := \frac{1+3b}{6(1+7b)}$, $\Delta := \frac{1+5b}{6(1+7b)}$, $\zeta := \frac{1+b}{2(1+7b)}$, $\Omega := \frac{2b+\sqrt{1-b^2}}{6(1+7b)}$, and $\Theta := \frac{3+21b}{3+17b}$. A vector ω lying in the range of $\rho^{[2]}_{ABC}(b)$ can be expressed as

$$\begin{aligned} \omega = (A, B, B, C; D, E, E, F)^{\mathrm{T}}, \\ \text{where}\, A, B, C, D, E, F \in \mathbb{C}. \end{aligned} \tag{23}$$

However, if $\rho^{[2]}_{ABC}(b)$ has to be separable across the AB|C cut, then according to the *range criterion* [20], there exists a set of product vectors $\{\psi_i \otimes \phi_k\}$ spanning the range space of $\rho^{[2]}_{ABC}(b)$ such that $\{\psi_i \otimes \phi_k^*\}$ span the range space of $[\rho^{[2]}_{ABC}(b)]^{\mathrm{T}_C}$; or any of the vectors $\{\psi_i \otimes \phi_k\}$ ($\{\psi_i \otimes \phi_k^*\}$) belongs to the range of $\rho^{[2]}_{ABC}(b)$ ($[\rho^{[2]}_{ABC}(b)]^{\mathrm{T}_C}$), with $\psi_i \in \mathbb{C}^4$ and $\phi_k \in \mathbb{C}^2$. Without any loss of generality, the elements of the set $\{\psi_i \otimes \phi_k\}$ can be written as

$$v_1 = (\alpha, \beta, \gamma, \delta)^{\mathrm{T}} \otimes (1, 0)^{\mathrm{T}}, \tag{24a}$$
$$v_2 = (\alpha, \beta, \gamma, \delta)^{\mathrm{T}} \otimes (0, 1)^{\mathrm{T}}, \tag{24b}$$
$$v_3 = (\alpha, \beta, \gamma, \delta)^{\mathrm{T}} \otimes (1, t)^{\mathrm{T}}, \tag{24c}$$

where $\alpha, \beta, \gamma, \delta, t \in \mathbb{C}$; & $t \neq 0$. Comparing Eqs. (24) with Eq. (23) we obtain

$$v_1 = (A, 0, D, 0)^{\mathrm{T}} \otimes (1, 0)^{\mathrm{T}}, \tag{25a}$$
$$v_2 = (0, C, 0, F)^{\mathrm{T}} \otimes (0, 1)^{\mathrm{T}}, \tag{25b}$$
$$v_3 = (A, tA, D, tD)^{\mathrm{T}} \otimes (1, t)^{\mathrm{T}}. \tag{25c}$$

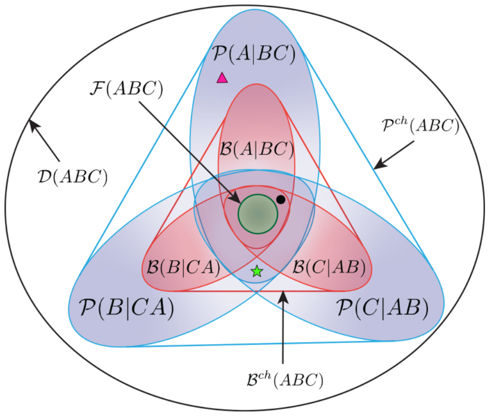

FIG. 1. A set inclusion diagram among the different convex sets of states for tripartite quantum systems. For instance, $\mathcal{B}(A|BC)$ denotes the set of separable states across the A|BC cut, whereas $\mathcal{P}(A|BC)$ stands for the set of PPT states across the same cut. The set of fully separable states $\mathcal{F}(ABC)$ (green region) is a strict subset of $\mathcal{B}(A|BC) \cap \mathcal{B}(B|CA) \cap \mathcal{B}(C|AB)$ even for the three-qubit Hilbert space. The • representing the state ρ_{SU} of Eq. (7) (first identified in [11]) establishes this strict inclusion relation. On the other hand, even for the simplest tripartite system $\mathcal{B}^{ch}(2 \otimes 2 \otimes 2) \subsetneq \mathcal{P}^{ch}(2 \otimes 2 \otimes 2)$. The ▲ representing the state identified in [12] establishes this particular strict inclusion relation. For any tripartite Hilbert space $\mathbb{C}_A^{d_1} \otimes \mathbb{C}_B^{d_2} \otimes \mathbb{C}_C^{d_3}$ with $\min\{d_1, d_2, d_3\} \geqslant 2$, $\mathcal{B}(A|BC) \cap \mathcal{B}(B|CA) \cap \mathcal{B}(C|AB) \subsetneq \mathcal{P}(A|BC) \cap \mathcal{P}(B|CA) \cap \mathcal{P}(C|AB)$. The ★ representing the states described in Propositions 1 and 2 and in Remark 1 establishes this fact.

Therefore, the partial complex conjugations of vectors in Eq. (25) are obtained as

$$v_1^* = (A, 0, D, 0)^{\mathrm{T}} \otimes (1, 0)^{\mathrm{T}}, \tag{26a}$$
$$v_2^* = (0, C, 0, F)^{\mathrm{T}} \otimes (0, 1)^{\mathrm{T}}, \tag{26b}$$
$$v_3^* = (A, tA, D, tD)^{\mathrm{T}} \otimes (1, t^*)^{\mathrm{T}}. \tag{26c}$$

These vectors should span the range of $[\rho_{ABC}^{[2]}(b)]^{\mathrm{T}_C}$. Now consider the vector $u = (0, 0, \frac{b}{3+17b}, 0, 0, \frac{2b}{3+17b}, \frac{2b+\sqrt{1-b^2}}{6+34b}, 0)^{\mathrm{T}}$ in the range of $[\rho_{ABC}^{[2]}(b)]^{\mathrm{T}_C}$. This particular vector cannot be spanned by $\{v_1^*, v_2^*, v_3^*\}$ and hence leads to the fact that $\rho_{ABC}^{[2]}(b)$ is inseparable across the AB|C cut. Thus, $\rho_{ABC}^{[2]}(b) \notin \mathcal{B}(C|AB)$ and hence $\rho_{ABC}^{[2]}(b) \notin \mathcal{B}^{int}(2 \otimes 2 \otimes 2)$ for $b \in (\sim 0.8184, 1]$. This completes the proof. ■

Remark 1. Proposition 1 and Proposition 2 yield the proof for Theorem 1 for $(\mathbb{C}^d)^{\otimes 3}$ with $d \geqslant 3$ and $(\mathbb{C}^2)^{\otimes 3}$, respectively. Given these two Propositions, it is also not hard to see that Theorem 1 also holds for any tripartite Hilbert space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ with $\min\{d_1, d_2, d_3\} \geqslant 2$. For arbitrary d_1, d_2, d_3, construct the state $\rho_{ABC}^{[d_m]}$ either as of Proposition 1 if $d_m := \min\{d_1, d_2, d_3\} \geqslant 3$ or as of Proposition 2 if $d_m = 2$. Clearly, $\rho_{ABC}^{[d_m]} \in \mathcal{P}^{int}(d_1 \otimes d_2 \otimes d_3)$ but $\rho_{ABC}^{[d_m]} \notin \mathcal{B}^{int}(d_1 \otimes d_2 \otimes d_3)$.

Theorem 1 reveals a nontrivial geometric implication regarding the state space structure of tripartite Hilbert spaces by establishing a proper set inclusion relation among different convex sets of states (see Fig. 1).

IV. CONCLUDING REMARKS AND FUTURE OUTLOOK

We have studied the intricate state space structure of multipartite quantum systems. In particular, we have shown that the intersection of three sets of biseparable states (across three different bipartite cuts) is a strict subset of the intersection of three sets of PPT states for tripartite Hilbert spaces $\mathbb{C}_A^{d_1} \otimes \mathbb{C}_B^{d_2} \otimes \mathbb{C}_C^{d_3}$ with $\min\{d_1, d_2, d_3\} \geqslant 2$. We establish this strict set inclusion relation by explicit construction of the states that belong to the set $\mathcal{P}^{int}(d_1 \otimes d_2 \otimes d_3)$ but not to $\mathcal{B}^{int}(d_1 \otimes d_2 \otimes d_3)$. At this point, the work by Eggeling and Werner [10] is worth mentioning. There the authors studied the state space structure for tripartite quantum systems by considering a particular class of states that commute with unitaries of the form $U \otimes U \otimes U$. Whereas for a three-qubit system, it turns out that $\mathcal{P}^{int}(2 \otimes 2 \otimes 2) = \mathcal{B}^{int}(2 \otimes 2 \otimes 2)$ if we limit within the $U \otimes U \otimes U$ invariant class; our results show that this is not the case for generic state space. The present study thus provides understanding of the multipartite state space structure as well as the multipartite entanglement behavior and adds to the previous results established in [10–12].

Our work welcomes further questions for future study. First, we have shown only that the convex sets of states $\mathcal{P}^{int}(ABC)$ and $\mathcal{B}^{int}(ABC)$ are not identical. Now, according to the classic Minkowski-Krein-Milman theorem, we know that every convex (and compact) set in Euclidean space (or more generally in a locally convex topological vector space) is the convex hull of its extreme points [44] (see also [45]). Since $\mathcal{P}^{int}(ABC) \neq \mathcal{B}^{int}(ABC)$, the sets of extreme points of these sets are also different, i.e., $\mathcal{E}_{\mathcal{P}^{int}}(ABC) \neq \mathcal{E}_{\mathcal{B}^{int}}(ABC)$. Characterizing these extreme points will provide us a more detailed picture regarding the tripartite state space structure. In this respect, the work of [46] is worth mentioning, where the authors have shown that a $(d-3)/2$-simplex is sitting on the boundary between the set $\mathcal{P}(AB)$ and the set of non-PPT states for the Hilbert space $\mathbb{C}_A^d \otimes \mathbb{C}_B^d$ for odd d with $d \geqslant 3$. There are in-fact some efficient methods in literature to check extremality of $\mathcal{P}(AB)$ [47–50]. Our study also motivates questions on quantum dynamics. For the bipartite case, researchers have identified entanglement breaking completely positive trace-preserving maps (channels) $\mathcal{N}$, such that $(\mathcal{I} \otimes \mathcal{N})[\rho_{AB}] \in \mathcal{S}(AB)\, \forall\, \rho_{AB} \in \mathcal{D}(AB)$ [51,52]. Similarly, here one might be interested in the classes of channels that map any tripartite state to the sets $\mathcal{F}(ABC)/\mathcal{B}^{int}(ABC)/\mathcal{P}^{int}(ABC)$, i.e., $(\mathcal{I} \otimes \mathcal{I} \otimes \mathcal{N})[\rho_{ABC}] \in \mathcal{F}(ABC)/\mathcal{B}^{int}(ABC)/\mathcal{P}^{int}(ABC)\, \forall\, \rho_{ABC} \in \mathcal{D}(ABC)$. Finally, comprehending the state space structure for an arbitrary number of systems is far from complete.

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