Emergence of the Born rule in strongly driven dissipative systems

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To understand the dynamical origin of the measurement in quantum mechanics, several models have been put forward which have a quantum system coupled to an apparatus. The system and the apparatus evolve in time, and the Born rule for the system to be in various eigenstates of the observable is naturally obtained. In this paper, we show that the effect of the drive-induced dissipation in an open quantum system can lead to the Born rule, even if there is no separate apparatus. The applied drive needs to be much stronger than the system-environment coupling. In this condition, we show that the dynamics of the driven-dissipative system could be reduced to a Milburn-like form, using a recently proposed fluctuation-regulated quantum master equation [A. Chakrabarti and R. Bhattacharyya, Phys. Rev. A **97**, 063837 (2018).]. The irreversible part of the dynamics is caused by the drive-induced dissipation. The resulting mixed state is identical to that obtained by using the Born rule.

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I. INTRODUCTION

The Born rule provides the probability of the outcome of a measurement of an observable on a quantum system [1]. This rule is introduced as a postulate on the process of measurement in the axiomatic formulation of quantum mechanics [2]. The rule states that the act of measuring an observable on a given normalized state $|\psi\rangle$ results in the system collapsing to one of the eigenstates of the observable (say, $|\phi_i\rangle$) with the probability of such an outcome given by $|\langle\phi_i|\psi\rangle|^2$. The nonanalytic nature of the measurement, in the form of a collapse, prompted the development of several dynamical models which aimed to show that the Born rule was but a natural outcome of the time evolution of a coupled system and apparatus, with much ingenuity involved in constructing the apparatus and the coupling of the system to the apparatus [3–9].

Among the earliest to attempt a dynamical model, von Neumann formulated the measurement process through a coupling between two entities. One is the observed system and the other is the measuring apparatus; the observer does not directly measure the system but infers the state of the system by observing that of the apparatus (referred to as the pointer variable) [3]. The system and the apparatus evolve together and reach a steady state where eigenstates of the system and the apparatus are entangled. Each state of the apparatus uniquely identifies an eigenstate of the system. The creation of the entangled state between the system and the apparatus is known as the premeasurement step [11]. Subsequent to this step, a projective measurement is required, in which an observable of the system is measured in an orthonormal basis composed of the eigenstates of the observable, and the measurement projects the system onto one of these eigenstates with probability given by the square of the length of the projection and returns the corresponding eigenvalue. The system and the apparatus evolve under a unitary propagator. As such, the collapse of the state function is not within the scope of von Neumann's premeasurement model [10].

Coleman and Hepp proposed an exactly solvable model relying on unitary evolution [4]. In this model, a fast particle passes through a long row (assumed infinite) of noninteracting spins, which are flipped one after another, to induce an observable macroscopic signature. Suitable choice of this spin-flipping local potential results in the emulation of the Born rule. Among the other unitary approaches, the model by Cini is also exactly solvable and is constructed using a spin-$1/2$ particle interacting with a spin-L particle as the apparatus [6]. However, being completely unitary, these models do not describe the collapse.

In general, the measurement in quantum mechanics is an irreversible process. Therefore, the postmeasurement state of a quantum system could be described by a mixed state density matrix. We note that the irreversibility also arises naturally in open quantum systems or driven-dissipative systems. Thus, the need for an environment in modeling the measurement process was felt. Several dynamical models of measurement were proposed that use quantum master equations or, in more general terms, use the notion of the environment.

Zurek, in the early 1980s, showed that von Neumann's scheme may be extended using an apparatus coupled to the environment [11,12]. The environment is assumed to have many degrees of freedom. After a combined evolution of the apparatus and the environment under a suitably chosen coupling, one takes a trace over the environment. The resulting state is a mixed state with different apparatus states (pointer variables) having different probabilities as per the Born rule. This decoherence-assisted process of selection of the pointer states is named as einselection [10–12].

Motivated by the fact that open quantum systems show irreversible dynamics, Green proposed modeling of an apparatus with coupling to the thermal baths [7]. In this model, a

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two-level system is considered, which is brought into interaction with separate detectors. Each detector has two sets of oscillators at different temperatures. The oscillators are coupled by an interaction with the particle. As such, the particle's states could be detected by a temperature change.

Gaveau and Schulman proposed a static variant of the Coleman-Hepp model where the system was a spin-$1/2$ particle and the apparatus was a one-dimensional Ising spin model [8]. Later Allahverdyan and others proposed the Curie-Weiss model [9]. A spin-$1/2$ particle (the system) is coupled to an Ising ferromagnet (the apparatus) by a quartic infinite-range Ising interaction in this model. The apparatus is allowed to interact with a phonon bath through a spin-boson coupling. The reduced density matrix of the system dynamically evolves to the form predicted by the Born rule, with the magnet reflecting the state of the particle.

In all the approaches described above, the system and the apparatus are made to evolve together. The apparatus is often modeled in a rather elaborate way, such as in the Curie-Weiss model. Now, with the increasing importance of quantum information processing, incorporating such a measuring apparatus in the simulations of quantum circuits becomes cumbersome since each qubit would require its own apparatus. This paper shows that it is possible to have a dynamical model without an explicit invocation of an apparatus, provided one uses the recently observed drive-induced dissipation (DID) within the framework, as described below.

Driven-dissipative dynamics with non-Bloch behavior, which is a manifestation of the DID, has also been observed experimentally in a variety of systems [13–17]. Motivated by such observations, a variant of the Markovian quantum master equation has recently been proposed by Chakrabarti and Bhattacharyya [18] which shows that the dissipator has a contribution from the drive as well. Its formulation requires an explicit introduction of the fluctuations in the environment, which provides a regulator in the dissipator. The presence of the regulator ensures that the DID could be calculated as a simple closed-form expression. The master equation is named as the fluctuation-regulated quantum master equation (FRQME). We note that in recent years, FRQME has been used to predict the optimal clock speed of qubit gates and the nonlinearity of the light shifts [19,20].

If one considers an apparatus connected to a system, then typically, the premeasurement occurs, followed by a projective measurement in the form of a collapse. Subsequent trace over the apparatus subspace results in the mixed density matrix described by the Born rule. In this paper, we show that applying a time-dependent drive on the system serves a similar purpose provided one includes the higher-order effect of the drive in the form of DID. We use FRQME to include the DID in the dynamics of the system and show that this results in the emergence of the Born rule.

If we consider the drive to be much stronger than the system-environment interaction, we would expect that the system would reach a quasisteady state, much before the system-environment coupling begins to influence the system density matrix. As a result, starting from a pure state, the system ends up with a mixed state, and the final density matrix reduces to a statistical mixture of the eigenstates of the drive Hamiltonian with probabilities being the same as that predicted by the Born rule. The operator of the drive Hamiltonian serves as the observable being measured.

II. FLUCTUATION-REGULATED QUANTUM MASTER EQUATION

In this formulation, one considers the standard settings of a driven open quantum system along with an explicit introduction of the thermal fluctuation acting on the environment. The form of the thermal fluctuations is chosen to be diagonal in the eigenbasis $\{|\xi_j\rangle\}$ of the static Hamiltonian of the environment, represented by $\mathscr{H}_{\rm E}(t) = \sum_j f_j(t)|\xi_j\rangle\langle\xi_j|$, where $f_j(t)$'s are assumed to be independent, Gaussian, δ-correlated stochastic variables with zero mean and standard deviation κ [18], i.e., $\overline{f_j(t)} = 0$, $\overline{f_j(t_1)f_j(t_2)} = \kappa^2\delta(t_1 - t_2)$. This ensures that the fluctuations would destroy the coherences in the environment but do not change the equilibrium population distribution of the environment. Next, we move to the interaction representation with respect to the static Hamiltonians of the system and the environment, and denote the Hamiltonians with upright H symbols. To arrive at the regulator from the thermal fluctuations a finite propagator is constructed, which is infinitesimal in terms of the drive and system-environment coupling Hamiltonians (together denoted by $H_{\rm eff}$), but remains finite in the instances of the fluctuations of the environment. To fulfill this condition, only the first-order contribution of $H_{\rm eff}$ is taken in the construction of the propagator $U(t_1, t)$ within the time interval t to t_1, but we consider many instances of the fluctuation taking place in that interval and retain all possible higher-order terms of $H_{\rm E}$. In other words, the timescale of the fluctuations of the environment is assumed to be much faster compared to the timescale over which the system evolves. Finally, we get a finite propagator of the following form:

$$U(t_1) \approx U_{\rm E}(t_1) - i\int_t^{t_1} H_{\rm eff}(t_2)U_{\rm E}(t_2)dt_2 \tag{1}$$

where $U_{\rm E}(t_1) = \mathbb{I} - i\int_t^{t_1} H_{\rm E}(t_2)U_{\rm E}(t_2)dt_2$.

Next the Born approximation [21] is used; i.e., at the beginning of the coarse-graining interval, the total density matrix of the system-environment pair can be factorized into that of the system and the environment as $\rho(t) = \rho_{\rm S}(t)\otimes\rho_{\rm E}^{\rm eq}$. This approximation and the assumptions regarding the nature of the fluctuation provide the desired regulator in the second order under an ensemble average as

$$\overline{U_{\rm E}(t_1)\tilde{\rho}(t)U_{\rm E}^\dagger(t_2)} = \rho_{\rm S}(t)\otimes\rho_{\rm E}^{\rm eq}e^{-\frac{1}{2}\kappa^2|t_1-t_2|}. \tag{2}$$

A regular coarse-graining procedure [22] is subsequently carried out to obtain the FRQME in the following form:

$$\begin{aligned}\frac{d}{dt}\rho_{\rm S}(t) = &-i\,{\rm Tr}_{\rm E}[H_{\rm eff}(t), \rho_{\rm S}(t)\otimes\rho_{\rm E}^{\rm eq}]^{\rm sec}\\ &-\int_0^\infty d\tau\,{\rm Tr}_{\rm E}[H_{\rm eff}(t), [H_{\rm eff}(t-\tau), \rho_{\rm S}(t)\otimes\rho_{\rm E}^{\rm eq}]]^{\rm sec}\,e^{-\frac{|\tau|}{\tau_c}}\end{aligned} \tag{3}$$

where $\tau_c = 2/\kappa^2$ is the characteristic timescale of the decay of the autocorrelation of the fluctuations and the superscript "sec" stands for secular approximation that involves ignoring

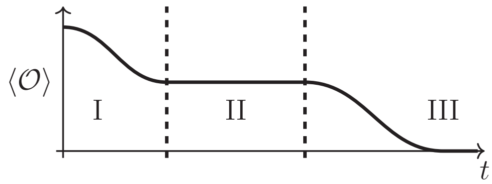

FIG. 1. A schematic diagram of the nature of the evolution of an observable of the system. In regions I and II, the first-order commutator and the dissipators $\mathcal{D}_{\rm S}$ dominate the evolution of the system, provided $\mathcal{D}_{\rm S} \gg \mathcal{D}_{\rm SE}$. In region III, the dissipators $\mathcal{D}_{\rm S}$ and $\mathcal{D}_{\rm SE}$ together dictate the evolution of the system. Region II is a quasisteady state where the Born rule emerges. To analyze the behavior of the system in regions I and II, we neglect the dissipator from the system-environment coupling.

the fast oscillating terms in the quantum master equation. We note that since $H_{\rm eff}$ contains the drive term, hence the DID originates from the double commutator under the integral in the above equation.

The FRQME is in Gorini-Kossakowski-Lindblad-Sudarshan form and yields a trace-preserving, completely positive dynamical map. This FRQME predicts simpler forms of DID, which have been shown to be the absorptive Kramers-Kronig pairs of the well-known Bloch-Siegert and light shift terms. The predicted nature of DID from the FRQME has also been verified experimentally [20,23].

III. THE MODEL

We consider a strongly driven system which is weakly coupled to its environment. We use FRQME to follow the Markovian dynamics of this system. We note that for our system $H_{\rm eff}$ is given by $H_{\rm SE} + H_{\rm S}$. For a simple Jaynes-Cummings-type system-environment coupling, we have $\mathrm{Tr}_{\rm E}\{H_{\rm SE}\,\rho\} = 0$ and the FRQME reduces to

$$\frac{d\rho_{\rm S}}{dt} = -i[H_{\rm S}, \rho_{\rm S}]^{\rm sec} - \mathcal{D}_{\rm S}\rho_{\rm S} - \mathcal{D}_{\rm SE}\rho_{\rm S} \tag{4}$$

where $\mathcal{D}_{\rm S}$ and $\mathcal{D}_{\rm SE}$ are the Lindbladians from $H_{\rm S}$ and $H_{\rm SE}$, respectively, where the dissipator $\mathcal{D}$ includes the double commutator terms. We note that the cross terms between the two Hamiltonians vanish since $\mathrm{Tr}_{\rm E}\{H_{\rm SE}\,\rho\} = 0$. For a strong drive which results in $\mathcal{D}_{\rm S} \gg \mathcal{D}_{\rm SE}$, it is expected that the system would reach a quasisteady state with respect to the commutator and the dissipator $\mathcal{D}_{\rm S}$, and would be influenced by $\mathcal{D}_{\rm SE}$ at a much later stage (region III), as depicted in Fig. 1.

The drive Hamiltonian $H_{\rm S}$ is chosen to be time independent, and its operator part contains only the observable of interest. This choice is made on the ground that if we perform rotating wave approximation on a resonant linearly polarized oscillating field, or use a resonant circularly polarized oscillating field, both would result in the same Hamiltonian. Under this condition, the drive Hamiltonian $H_{\rm S}$ in the interaction representation would be in secular form and hence the superscript "sec" is dropped in the remaining part of the paper.

As such, for the strong drive, Eq. (4) reduces to the following effective form for regions I and II:

$$\frac{d\rho_{\rm S}}{dt} = -i\,[H_{\rm S}, \rho_{\rm S}]\, - \tau_c[H_{\rm S}, [H_{\rm S}, \rho_{\rm S}]]. \tag{5}$$

This is the form of FRQME that we shall use in the remaining part of the paper.

Let the eigenvalues and the eigenvectors of $H_{\rm S}$ be $\{\lambda_i\}$ and $\{|\phi_i\rangle\}$, respectively. Therefore, $H_{\rm S}$ can be written as $H_{\rm S} = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$.

Let $|\psi_i\rangle = \sum_j c^i_j|\phi_j\rangle$ and

$$\begin{aligned}\rho_{\rm S} &= \sum_i p_i|\psi_i\rangle\langle\psi_i| \\ &= \sum_{j,k}\sum_i p_i c^i_j c^{i\,*}_k\,|\phi_j\rangle\langle\phi_k| \\ &= \sum_{j,k} a_{jk}|\phi_j\rangle\langle\phi_k|\end{aligned}$$

where $a_{jk} = \sum_i p_i c^i_j c^{i\,*}_k$. We rewrite Eq. (5) in terms of its (i, j)th elements as

$$\frac{d}{dt}\rho_{\rm S}|_{ij} = -i[H_{\rm S}, \rho_{\rm S}]_{ij} - \tau_c[H_{\rm S}, [H_{\rm S}, \rho_{\rm S}]]_{ij}. \tag{6}$$

Let the (i, j)th element of the density matrix be given by $(\rho_{\rm S})_{ij} = \langle\phi_i|\rho_{\rm S}|\phi_j\rangle = a_{ij}$. We can express Eq. (6) in terms of a_{ij} as

$$\dot{a}_{ij} = [-i\Delta\lambda_{ij} - \tau_c\Delta\lambda^2_{ij}]a_{ij} \tag{7}$$

where $\Delta\lambda_{ij} = (\lambda_i - \lambda_j)$. It is clear that $\dot{a}_{ii} = 0$, that means the diagonal elements do not evolve with time.

The solution of Eq. (7) is

$$a_{ij}(t) = a_{ij}(0)e^{-i\Delta\lambda_{ij}t}e^{-\tau_c\Delta\lambda^2_{ij}t}. \tag{8}$$

A. Nondegenerate case

If $\lambda_i \neq \lambda_j$, then $a_{ij}(t \to \infty) = 0$ and $a_{ii}(t \to \infty) = a_{ii}(0) = $ const. Therefore, all off-diagonal elements will vanish and only diagonal elements will survive in the limit $t \to \infty$ and the density matrix can be expressed as

$$\rho_{\rm S}(t \to \infty) = \sum_i\sum_m p_m|c^m_i|^2|\phi_i\rangle\langle\phi_i|. \tag{9}$$

B. Degenerate case

If $\lambda_i = \lambda_j$, then $\dot{a}_{ij} = 0$ and $a_{ij}(t \to \infty) = a_{ij}(0) = $ const. Therefore, both diagonal and off-diagonal elements remain constant in the limit $t \to \infty$ and the density matrix can be expressed as

$$\rho_{\rm S}(t \to \infty) = \sum_{i,j}\sum_m p_m c^m_i c^{m\,*}_j|\phi_i\rangle\langle\phi_j|. \tag{10}$$

The above form of the density matrix is identical to the form predicted by the Born rule.

IV. EXAMPLES

We exemplify the emergence of the Born rule for a single-qubit system and also for a multiqubit system. For the latter, we employ a drive that has a degenerate eigensystem.

A. Single qubit

For the single-qubit system, we begin with the system in a pure state given by

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \tag{11}$$

and the initial density matrix of the system is given by $\rho_0 = |\psi\rangle\langle\psi|$. To emulate the evolution of the system under a strong drive, we apply a pulse with flip angle κ on the system about the y axis. The Hamiltonian corresponding to this pulse is expressed as

$$H_{\mathrm{s}} = \omega_1\frac{\sigma_y}{2} \tag{12}$$

and the time required to apply this pulse is $\frac{\kappa}{\omega_1}$.

The eigenvectors of the operator part of the drive Hamiltonian H_{s} are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle),\ |\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle), \tag{13}$$

and the corresponding nondegenerate eigenvalues are $\pm 1/2$, respectively. We note that the initial state $|\psi\rangle$ can also be expressed as $|\psi\rangle = \sum_i c_i|\phi_i\rangle$, where, $c_1 = \langle\phi_1|\psi\rangle = (\cos\frac{\theta}{2} - ie^{i\varphi}\sin\frac{\theta}{2})/\sqrt{2}$ and $c_2 = \langle\phi_2|\psi\rangle = (\cos\frac{\theta}{2} + ie^{i\varphi}\sin\frac{\theta}{2})/\sqrt{2}$.

Equation (5) can be expressed in the Liouville space as follows:

$$\frac{d\hat{\rho}_{\mathrm{s}}}{dt} = [-i\hat{\hat{\mathcal{L}}}^{(1)}_{\mathrm{drive}} - \tau_c\hat{\hat{\mathcal{L}}}^{(2)}_{\mathrm{drive}}]\hat{\rho}_{\mathrm{s}} = \hat{\hat{\Gamma}}\hat{\rho}_{\mathrm{s}} \tag{14}$$

where $\hat{\hat{\mathcal{L}}}^{(1)}_{\mathrm{drive}}$ is the Liouville superoperator or Liouvillian for the corresponding $[H_{\mathrm{s}}, \rho_{\mathrm{s}}]$ term and $\hat{\hat{\mathcal{L}}}^{(2)}_{\mathrm{drive}}$ is the Liouvillian for the corresponding $[H_{\mathrm{s}}, [H_{\mathrm{s}}, \rho_{\mathrm{s}}]]$ term which is responsible for the second-order DID.

Solving this differential equation (14), we can write the system density matrix at a later time t as

$$\hat{\rho}_{\mathrm{s}}(t) = e^{\hat{\hat{\Gamma}}t}\hat{\rho}_{\mathrm{s}}(0) \tag{15}$$

where $\hat{\rho}_{\mathrm{s}}(0)$ is the initial density matrix and $e^{\hat{\hat{\Gamma}}t}$ is the propagator in Liouville space.

We construct the superoperator $\hat{\hat{\Gamma}}$ using Eq. (14) and construct the Liouville space propagator as $U = e^{\hat{\hat{\Gamma}}\frac{\kappa}{\omega_1}}$ which acts on the initial state ρ_0 to produce the final density matrix.

When we apply this propagator U on the initial density matrix ρ_0, the final density matrix becomes

$$\rho_{\mathrm{s}} = U\rho_0 = \begin{pmatrix} 1 - e^{-\omega_1\tau_c\kappa}a & -i\sin\varphi\sin\theta + e^{-\omega_1\tau_c\kappa}b \\ i\sin\varphi\sin\theta + e^{-\omega_1\tau_c\kappa}b & 1 + e^{-\omega_1\tau_c\kappa}a \end{pmatrix}, \tag{16}$$

where $a = (\sin\theta\sin\kappa\cos\varphi - \cos\kappa\cos\theta)$, $b = (\cos\kappa\cos\varphi\sin\theta + \cos\theta\sin\kappa)$. The dissipator $\mathcal{D}_{\mathrm{s}}$ as described earlier provides the decaying terms in the above. In the limit $\omega_1\tau_c\kappa \to \infty$, the final density matrix becomes

$$\rho_{\mathrm{s}} = \frac{1}{2}\begin{pmatrix} 1 & -i\sin\theta\sin\varphi \\ i\sin\theta\sin\varphi & 1 \end{pmatrix}. \tag{17}$$

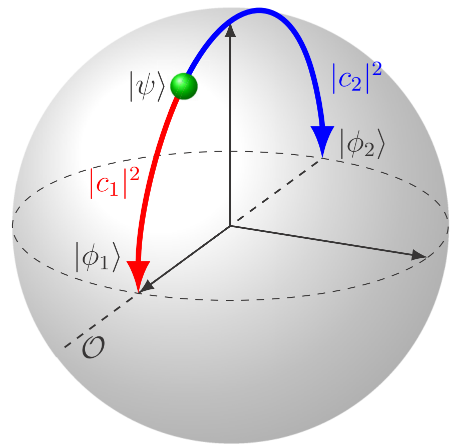

FIG. 2. Schematic depiction of the journey to the final states starting from $|\psi\rangle = \sum_i c_i|\phi_i\rangle$ (the green point) where $\{\phi_i\}$ are the eigenstates of the observable $\mathcal{O}$. $|c_i|^2$ is the probabilities of the path $|\psi\rangle \to |\phi_i\rangle$. The final state is a mixed state density matrix in accordance with the Born rule.

We can express the above ρ_{s} as

$$\rho_{\mathrm{s}} = \sum_{i=1,2}|c_i|^2|\phi_i\rangle\langle\phi_i| \tag{18}$$

where $|c_1|^2 = |\langle\phi_1|\psi\rangle|^2 = \frac{1}{2}(1+\sin\theta\sin\varphi)$, and $|c_2|^2 = |\langle\phi_2|\psi\rangle|^2 = \frac{1}{2}(1-\sin\theta\sin\varphi)$. This defines a mixed state density matrix of the eigenstates of the drive. As such, the drive projects the initial state of the system onto one of its eigenstates with probabilities given by $|c_1|^2$ and $|c_2|^2$, respectively. Therefore, our result agrees with the Born rule. Figure 2 shows a schematic diagram of the evolution of the system on a Bloch sphere, with the classical paths described by colored arrows. The system moves from a pure state to a mixed state.

B. Degenerate observable and multiqubit

We extend our analysis to an observable with a degenerate eigensystem. We choose a two-qubit system in an entangled state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{19}$$

We intend to measure $\sigma_y \otimes \mathbb{I}$ on this system, such that the measurement takes place only on the first qubit. The eigenvectors of this observable are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - i|11\rangle),\ |\phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - i|10\rangle),$$
$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + i|11\rangle),\ |\phi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|10\rangle), \tag{20}$$

and the corresponding eigenvalues are $-1, -1, 1, 1$, respectively.

The initial density matrix of the system is given by $\rho_0 = |\psi\rangle\langle\psi|$. Like the single-qubit case, the operation of the observable is emulated by a pulse with flip angle κ on the first qubit about the y axis. The Hamiltonian corresponding to this pulse is expressed as

$$H_s = \omega_1 \frac{\sigma_y}{2} \otimes \mathbb{I} \tag{21}$$

and the time required to apply this pulse is κ/ω_1.

The system evolves under the Liouvillian obtained using Eq. (14) and the final density matrix assumes the form

$$\rho_s = U\rho_0 = \frac{1}{4}\begin{pmatrix} (1+e^{-2\omega_1\tau_c\kappa}C) & -e^{-2\omega_1\tau_c\kappa}S & e^{-2\omega_1\tau_c\kappa}S & (1+e^{-2\omega_1\tau_c\kappa}C) \\ -e^{-2\omega_1\tau_c\kappa}S & (1-e^{-2\omega_1\tau_c\kappa}C) & (-1+e^{-2\omega_1\tau_c\kappa}C) & -e^{-2\omega_1\tau_c\kappa}S \\ e^{-2\omega_1\tau_c\kappa}S & (-1+e^{-2\omega_1\tau_c\kappa}C) & (1-e^{-2\omega_1\tau_c\kappa}C) & e^{-2\omega_1\tau_c\kappa}S \\ (1+e^{-2\omega_1\tau_c\kappa}C) & -e^{-2\omega_1\tau_c\kappa}S & e^{-2\omega_1\tau_c\kappa}S & (1+e^{-2\omega_1\tau_c\kappa}C) \end{pmatrix}, \tag{22}$$

where $S = \sin\kappa$, and $C = \cos\kappa$.

In the limit $\omega_1\tau_c\kappa \rightarrow \infty$, the final density matrix becomes

$$\rho_s = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

We can express the above ρ_s as

$$\rho_s = \sum_{i,j\in\{1,2\}} c_i c_j^* |\phi_i\rangle\langle\phi_j| + \sum_{i,j\in\{3,4\}} c_i c_j^* |\phi_i\rangle\langle\phi_j| \tag{24}$$

where $c_1 = \langle\phi_1|\psi\rangle = \frac{i}{2}$, $c_2 = \langle\phi_2|\psi\rangle = \frac{1}{2}$, $c_3 = \langle\phi_3|\psi\rangle = -\frac{i}{2}$, $c_4 = \langle\phi_4|\psi\rangle = \frac{1}{2}$.

The off-diagonal terms will be present in the final form of ρ_s because of the degeneracy between $|\phi_1\rangle, |\phi_2\rangle$ and $|\phi_3\rangle, |\phi_4\rangle$. We can rewrite this as a mixture of the linear superpositions of the states in the following form:

$$\rho_s = (|c_1|^2 + |c_2|^2)|\Phi_1\rangle\langle\Phi_1| + (|c_3|^2 + |c_4|^2)|\Phi_2\rangle\langle\Phi_2|$$

where $|\Phi_1\rangle = (c_1|\phi_1\rangle + c_2|\phi_2\rangle)/\sqrt{|c_1|^2 + |c_2|^2}$ and $|\Phi_2\rangle = (c_3|\phi_3\rangle + c_4|\phi_4\rangle)/\sqrt{|c_3|^2 + |c_4|^2}$ are the normalized superposition states. The application of the drive causes the system to collapse in the degenerate eigensubspaces formed by $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$, $|\phi_4\rangle$ with probabilities being $(|c_1|^2 + |c_2|^2)$ and $(|c_3|^2 + |c_4|^2)$, respectively. Therefore, the results of the present example are also consistent with the Born rule.

V. DISCUSSIONS

According to von Neumann, at the premeasurement stage, an entangled superposition of the system and apparatus is created that can be described by unitary time evolution. He also provided a detailed mathematical derivation to show that it is possible to arrive at such an entangled state by a correct choice of Hamiltonian. After the premeasurement, the superposition decoheres and reduces to a single product state of the two. Zurek, in his seminal paper, summarized the process, as given below [11]. For the initial-state vectors $|\psi\rangle$ and $|A_\circ\rangle$ of the system and the apparatus, respectively, an evolution under an appropriately chosen Hamiltonian leads to the following entangled form:

$$\begin{aligned} |A_\circ\rangle \otimes |\psi\rangle &= \left\{\sum_i a_i |A_i\rangle\right\} \otimes \left\{\sum_i c_i |\phi_i\rangle\right\} \\ &\longrightarrow \sum_i c_i |A_i\rangle \otimes |\phi_i\rangle. \end{aligned} \tag{25}$$

In the above, $|A_i\rangle$ and $|\phi_i\rangle$ are the eigenstates of the pointer variable of the apparatus and the observable of the system, respectively. The pointer variable of an apparatus has a one-to-one correspondence with a single observable of the system, such that a particular pointer variable helps measure a particular observable of the system.

The last step is the collapse of the wave function. If one traces out the apparatus, it would be a mixed state of the eigenstates of the observable of the system with respective probabilities given by the Born rule. Von Neumann named it as projective measurement, but it lacks explicit mathematics explaining why it occurs [3,10]. Our model provides a detailed mechanism of the collapse in the eigenstates of the observable, leading to the Born rule. As such, in a way, our approach supplants the premeasurement and the projective measurement by a single step. A strong drive, with the drive operator as the system observable, helps realize this step. The drive must act for a time much shorter than the relaxation time of the system. We discuss more on the latter part in the paragraphs below.

The FRQME is unique among Markovian master equations since it evaluates the dissipative effect of the drive; other master equations deal with the drive only in the first order. The drive, as shown in our earlier works, can have an absorptive Kramers-Kronig pair to the familiar dispersive shift terms, such as the light shifts and the Bloch-Siegert shifts [18,20]. As a result, FRQME involves a dissipator from the drive irrespective of the strength of the drive. For a weak drive, this dissipator is negligible, and hence the FRQME provides solutions identical to that of the regular Markovian QMEs. To arrive at the Born rule, the coherences between the eigenstates of the observable (here, the drive) must dissipate. While DID always guarantees this, the dissipator from the system-environment coupling does not. As such, the regular QMEs cannot show the collapselike short-term behavior of the system under a strong drive, an expected behavior.

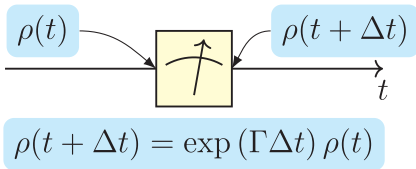

FIG. 3. A schematic diagram of the model of the measurement. After the measurement, $\rho(t + \Delta t)$ is of the form predicted by the Born rule. $\Gamma\Delta t$ should be sufficiently large for the system to reach a steady state.

We note that *the act of the measurement* must be much faster than the characteristic timescale of the relaxation. The system's evolution may have more than one characteristic rate due to the presence of two dissipators in the FRQME. We have shown one possible scenario when one dissipator is much stronger than the other using a schematic diagram in Fig. 1. When the drive is much stronger than the system-environment coupling, the system goes to a quasisteady state in region II in accordance with the Born rule. But if we wait for a longer time, system-environment coupling comes into play and one obtains a different mixed state density matrix with probability factors which do not satisfy the Born rule.

Such a deviation from the Born rule is also observed if the drive and the system-environment coupling are of comparable strength. In that case, one expects a competition between the decoherence due to the drive and system-environment interaction. We have shown earlier that such a scenario gives rise to the existence of the optimal clock speed for qubit gates in open quantum systems [19]. Here, the timescale separation vanishes, and region II is vanishingly small with no sustained quasisteady state.

So, a clear separation of the timescale is a requirement to realize a projective measurement within a finite timespan. Unlike an instantaneous collapse, this realization of the Born rule takes a finite timespan governed solely by the drive strength. For a stronger drive, the timespan of the region I is shorter. Hence region II captures the Born rule as a quasisteady state for a longer period of time. We note that the DID is scaled by the term τ_c that carries a signature from the environmental fluctuations and determines how fast one can reach a steady state. So, higher τ_c means faster collapse. We note that this τ_c is expected to be inversely proportional to temperature. Therefore, the emulation of the Born rule is favored at a lower temperature.

The principal feature of our model is that the first-order effect of the drive-in tandem with the DID leads the system to a mixed state. We note that the focus is on the creation of the mixed state through irreversible dynamics. The lack of an explicit apparatus means that we may not be able to *register* the outcome of a specific measurement, but that is not what this model intends to achieve. Even if one does not register the outcome of a measurement, a probabilistic mixed state description is reached and one can apply this repeatedly to model the measurement many times without having to *reset* the apparatus, as shown in the schematics in Fig. 3. This is one of the major advantages of this model.

In the year 1991, Milburn proposed a model for intrinsic decoherence based on a simple modification of a unitary Schrödinger evolution and derived an equation of motion for the density matrix of closed quantum systems as a substitution for the Schrödinger equation [24]. This is known as the Milburn equation. For sufficiently small fundamental time steps with terms up to second order being considered, the Milburn equation reduces to the FRQME of the form given by Eq. (5). A few years later, Bužek and Konôpka applied the Milburn equation to an open quantum system consisting of a two-level atom interacting with single- and multimode electromagnetic fields [25]. They reported that for very strong system-environment coupling, Rabi oscillation is completely suppressed, and the system collapses to a statistical mixture of the ground and excited states. We note that their study focuses on the overdamped nature of the system but is not a model of the measurement process. On the other hand, our model emulates the collapse part of the measurement by presenting a dynamical treatment of how an open quantum system behaves when a drive is applied to it. As per our model, measuring an observable is equivalent to evolving the system under that observable (which happens to be the drive) with a large amplitude, i.e., evolving the system strongly under the observable. This naturally leaves the system in a mixed state formed by the eigenstates of the observable with probabilities given by the Born rule.

VI. CONCLUSION

In this paper, we demonstrate that the DID from a strong drive can result in the emergence of the Born rule in a system weakly coupled to the environment. We assume that the dissipator from the drive is much larger than the dissipator from the system-environment coupling. The resulting dynamics is best analyzed in the eigenbasis of the drive, where the evolution destroys the coherences. Thus the final density matrix is in a mixed state and is diagonal in this representation for a nondegenerate observable. For an observable with degenerate eigenvalues, the coherences in the degenerate subspace survive in conformity with the Born rule. This dynamic model emulates the Born rule and could be used repeatedly on a system.

In quantum information processing, measurements are often included in quantum circuits. Such measurements could be on multiple qubits and occur more than once in the circuit. Our model could be very useful in simulating the dynamical behavior of a realistic open quantum system that has multiple occurrences of measurements. One would obtain the mixed state representation of the system at the end of the circuit operation, with the added advantage of not having to *reset* the apparatus. As examples, we have demonstrated the measurement operation for single- and multiqubit arrangements.

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