Bell nonlocality and the reality of the quantum wave function

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The status of the quantum wave function is one of the most debated issues in quantum foundations—whether it corresponds directly to the reality or just represents knowledge or information about some aspect of reality. We propose a ψ*-ontology* theorem that excludes a class of ontological explanations where the quantum wave function is treated as mere information. Our result, unlike the acclaimed Pusey-Barrett-Rudolph's theorem, does not presume the absence of *holistic* ontological properties for product quantum preparations. At the core of our derivation, we utilize the seminal no-go result by Bell that rules out any *local realistic* world view for quantum theory. We show that the observed phenomenon of quantum nonlocality cannot be incorporated in a class of ψ-epistemic models. Using the well-known Clauser-Horne-Shimony-Holt inequality, we obtain a threshold bound on the degree of epistemicity above which the ontological models are not compatible with quantum statistics.

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I. INTRODUCTION

What does a quantum wave function stand for? Does it represent the state of reality of the physical system (ψ-ontic doctrine) or does it merely provide information about the system (ψ-epistemic doctrine)? This question has been at the core of the quantum foundational debate ever since the advent of the theory [1]. A mathematically precise formulation of this question, within a broad class of realist approaches to quantum theory, can be made in the ontological model framework of Harrigan and Spekkens [2] (see, also [3,4]). Epistemicity, in this framework, is defined (as well as quantified) through the amount of overlap between probability distributions over the ontic states resulting from different quantum preparations.

In a fascinating development, Pusey, Barrett, and Rudolph (PBR) have shown that a ψ-epistemic interpretation contradicts the prediction of quantum theory in any model where independently prepared systems have independent physical states [named the preparation independence (PI) assumption] [5]. This result drew the attention of the quantum foundations community and, within a few days, several researchers reported similar theorems [6–9], commonly called ψ-ontology theorems [10], derived under different assumptions. Subsequently, several criticisms were raised regarding the assumptions used in those ψ-ontology theorems [11–14] (see, also, [10]). In particular, the authors in [14] have shown that the physical rationale for composition principles such as PI overreaches and thus places the no-go theorem put forward by PBR into jeopardy.

Interestingly, Maroney came up with a new kind of ψ-ontology theorem that uses no compositional assumption and rules out a class of ontological models with a certain degree of epistemicity [15,16]. Subsequently, several other results were obtained excluding ψ-epistemic models with an increasingly lower degree of epistemicity and consequently imposing a higher degree of onticity on the quantum wave function [17–21]. However, all these theorems apply to Hilbert spaces of dimension strictly greater than two.

In this work, we derive a ψ-ontology theorem that excludes the maximally ψ-epistemic model as well as nonmaximal ones with a certain degree of epistemicity. Importantly, our theorem does not assume the ontic composition principles of PI. In fact, it does not presuppose that the ontic state space for quantum product preparations should be a Cartesian product of their individual ontic state spaces only. In other words, we consider that the two or more quantum systems prepared even in a product state can possess *holistic* ontic features accessible only through global measurements, which broadens the scope of our result over PBR's theorem. Furthermore, unlike the ψ-ontology theorems reported in Refs. [15–21], the present theorem applies to qubit Hilbert space too. Quite importantly, our result demonstrates an interesting connection between the degree of epistemicity and Bell nonlocality [22–25]. While it was already recognized that ψ-complete and ψ-ontic models for quantum theory are inconsistent with the concept of *locality* [1,2], our result establishes the fact in the reverse direction. It shows that the phenomenon of Bell nonlocality prefers a ψ-ontic interpretation for the quantum wave function as ψ-epistemic models having epistemicity above a threshold degree cannot incorporate the observed quantum nonlocality.

II. FRAMEWORK

We first recall the ontological model framework as developed in [2] (see, also, [3,4,10,13]). Such a model consists of

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a space Λ of ontic states that completely identify the possible physical properties of a system. A quantum system prepared in the state $|\psi\rangle$ is associated with a probability distribution $\mu(\lambda|\psi)$ over Λ, where each realization of the preparation $|\psi\rangle$ results in an ontic state $\lambda \in \Lambda$ sampled with probability measure $\mu(\lambda|\psi)$. The probability distribution of the resulting ontic states is called the epistemic state associated with $|\psi\rangle$, and $\mathrm{Supp}[\mu(\lambda|\,|\psi\rangle)] \equiv \Lambda_\psi := \{\lambda \in \Lambda \,|\, \mu(\lambda|\psi) > 0\}$ is called the ontic support of $|\psi\rangle$. For all $|\psi\rangle$, we must have $\int_{\Lambda_\psi} d\lambda \mu(\lambda|\psi) = 1$. Whenever an observable M is measured on a system, the possible outcomes are the eigenvalues ϕ_k with the associated eigenvectors $|\phi_k\rangle$, i.e., $M = \sum_k \phi_k |\phi_k\rangle \langle\phi_k|$. Here we restrict our attention to rank-1 projective measurements only, although quantum theory allows a more general measurement process described by the positive operator valued measure (POVM). Given a system in the ontic state λ, the probability of obtaining the kth outcome is given by a response function $\xi(\phi_k|\lambda, M) \in [0, 1]$. A generic ontological model keeps open the possibility for this outcome response to be contextual [3]; whenever the measurement context is not important, we will denote the response function simply as $\xi(\phi_k|\lambda)$. An *outcome deterministic* (OD) ontological model demands $\xi(\phi_k|\lambda, M) \in \{0, 1\}\, \forall\, k, \lambda, M$. Denoting $\mathrm{Supp}[\xi(\psi|\lambda)] := \{\lambda \in \Lambda \,|\, \xi(\psi|\lambda) > 0\}$ and $\mathrm{Core}[\xi(\psi|\lambda)] := \{\lambda \in \Lambda \,|\, \xi(\psi|\lambda) = 1\}$, the following set of inclusion relations is immediate:

$$\Lambda_\psi \subseteq \mathrm{Core}[\xi(\psi|\lambda)] \subseteq \mathrm{Supp}[\xi(\psi|\lambda)].$$

An OD model satisfies $\mathrm{Core}[\xi(\psi|\lambda)] = \mathrm{Supp}[\xi(\psi|\lambda)]$, whereas a model with $\Lambda_\psi = \mathrm{Core}[\xi(\psi|\lambda)]$ is termed *reciprocal* [13]. Interestingly, the authors in [13] have also established that

$$\text{Maximally } \psi - \text{epistemic} \Leftrightarrow \text{OD} \wedge \text{Reciprocal}. \quad (1)$$

An operational transformation procedure T at the ontological level corresponds to a transition matrix $\Gamma_T(\lambda, \tilde{\lambda})$ denoting the probability density for a transition from the ontic state λ to the ontic state $\tilde{\lambda}$. In the prepare and measure scenario, the reproducibility of the Born rule at the operational level demands $\int_\Lambda d\lambda \xi(\phi_k|\lambda, M)\mu(\lambda|\psi) = |\langle\phi_k|\psi\rangle|^2 := \Pr(\phi_k|\psi)$.

III. DEGREE OF EPISTEMICITY

In a maximally ψ-epistemic model, the quantum overlap $|\langle\psi|\phi\rangle|^2$ for any two state vectors $|\psi\rangle$ and $|\phi\rangle$ is completely accounted for by the overlap between the corresponding epistemic distributions $\mu(\lambda|\psi)$ and $\mu(\lambda|\phi)$, i.e., $\int_{\Lambda_\phi} \mu(\lambda|\psi) d\lambda = |\langle\psi|\phi\rangle|^2$ [15]. Maroney's theorem [15] and the subsequent results [16–21] exclude the maximally ψ-epistemic model and a class of ontological models with increasingly lower degree of epistemicity. To quantify the degree of epistemicity of an ontological model, please note that $\int_{\Lambda_\phi} d\lambda \mu(\lambda|\psi) = \int_{\Lambda_\phi} d\lambda \xi(\phi|\lambda)\mu(\lambda|\psi) \leqslant \int_\Lambda d\lambda \xi(\phi|\lambda)\mu(\lambda|\psi) = |\langle\phi|\psi\rangle|^2$. The first equality is due to the fact that $\Lambda_\phi \subseteq \mathrm{Core}[\xi(\phi|\lambda)]$. One can express the above inequality as an equality of the following form:

$$\int_{\Lambda_\phi} d\lambda \mu(\lambda|\psi) = \Omega(\phi, \psi)\, |\langle\phi|\psi\rangle|^2, \quad (2)$$

where $\Omega(\phi, \psi)$ captures degree of epistemicity of a model and for any pair of nonorthogonal quantum states, $\Omega(\phi, \psi) \in [0, 1]$. For a maximally ψ-ontic model, $\Omega(\phi, \psi) = 0$ for all nonorthogonal pairs of state, while $\Omega(\phi, \psi) = 1$ for all such pairs in a maximally ψ-epistemic model (see [15] for more elaboration). The larger the departure of $\Omega(\phi, \psi)$ from its maximum possible value, the more the model is ψ-ontic and, consequently, the less ψ-epistemic it is.

IV. ONTIC COMPOSITION

The discussions so far consider ontological models for a single system only. More involved situations arise for composite systems. Consider System-A and System-B with ontic state space Λ^A and Λ^B, respectively. For a composite quantum system prepared in a state ρ_{AB}, a naive classical thinking suggests the joint ontic state λ_{joint} to be in $\Lambda_A \times \Lambda_B$, i.e., $\Lambda_{\rho_{AB}} \subseteq \Lambda^A \times \Lambda^B$. However, the lesson from the seminal Bell's theorem indicates a more intricate description of the ontic state for a composite system. Violation of any local realistic inequality by some joint quantum preparation ρ_{AB} necessitates some "nonlocal" ontic state space Λ^{NL}. More precisely, this nonlocal variable captures the essence of a nonlocal correlation in the sense of Bell [22,23]. At this point, the reader should be reminded that these nonlocal correlations are perfectly compatible with the no-signaling principle that prohibits the instantaneous transfer of information. Therefore, we can say that $\Lambda_{\rho_{AB}} \subseteq (\Lambda^A \times \Lambda^B) \cup \Lambda^{NL}$, where Λ^{NL} accounts for a Bell-type local realistic inequality violation. Importantly, all product states are Bell local (as is the case for a separable state and *local* entangled states [26–28]). But, does it assert that such a product quantum preparation should not contain any *holistic* ontic feature, i.e., their ontic support ought to be a subset of $\Lambda^A \times \Lambda^B$? PBR in their derivation have considered such an assumption. In fact, their PI assumption is even restrictive as it explicitly spells $\mu(\lambda_A, \lambda_B|\psi_A \otimes \psi_B) = \mu(\lambda_A|\psi_A)\mu(\lambda_B|\psi_B)$ [5]. However, such an assumption is conservative as it considers the (local) reality of the product measurements only. It is possible that a composite system even prepared in a product state contains properties that are accessible through global measurements only, e.g., a Bell basis measurement. More dramatically, the phenomena of "nonlocality without entanglement" indeed indicates such a situation even without involving any entanglement in the measurement basis [29–31]. Thus, for a bipartite product (also for separable) quantum state ρ_{AB}, the ontic support should be considered as $\Lambda_{\rho_{AB}} \subseteq \Lambda^A \times \Lambda^B \times \Lambda^G$, where Λ^G carries strictly relational holistic information about the two systems and remains hidden under local measurements, i.e., not accessible through local measurements. Here we make a clear distinction between Λ^G and Λ^{NL}. Both are nonclassical features of ontological state space for a bipartite quantum system. But they correspond to two completely distinct operational nonclassical features of quantum theory. Λ^{NL} contains variables that are nonlocal strictly in the sense of Bell inequality violation, which is revealed through local measurements performed on spatially separated subsystems of the composite system—no global measurement is involved or required in this case. This is clearly not the case for ontic elements belonging to Λ^G, which accounts for the global properties corresponding to nonclassical joint measurements allowed in quantum theory due to its richer bipartite effect space structure containing entangled effects in addition to product effects. These global measurements with entangled effects correspond to holistic properties or observables of the composite system whose outcomes can never be realized by locally measuring the individual subsystems. At this point, one may consider the possibility that the product quantum states have ontic support within Λ^{NL} in such a way the nonlocal effect gets averaged out in the operational statistics. However, our following proposition (see Appendix A for the proof) puts a *no-go* on such assertion.

Proposition 1. In a maximally ψ-epistemic model (more generally, in any outcome deterministic ontological model), quantum product preparations do not possess any nonlocal ontic state.

At this point, we leave this question of whether the proposition can be extended for any ψ-epistemic model open for further investigation. However, for other ψ-epistemic models (apart from those considered in Proposition 1), the nonexistence of a nonlocal ontic state in the support of quantum product preparation can be justified using the principle of *Occam's razor*, which researchers in quantum foundations have applied in a different context [32]. For an entangled state ρ_{AB} exhibiting Bell nonlocality, the ontic description will be $\Lambda_{\rho_{AB}} \subseteq (\Lambda^A \times \Lambda^B \times \Lambda^G) \cup \Lambda^{NL}$ with $\Lambda_{\rho_{AB}} \cap \Lambda^{NL} \neq \emptyset$. The Cartesian product structure assumed above is not necessary for our argument, but it provides a simple way to present the idea.

V. A ψ-ONTOLOGY THEOREM

We are now in a position to prove our ψ-ontology theorem. To this aim, we consider a quantum copying machine $\mathbb{M}$ that perfectly copies the states $|0\rangle$ and $|1\rangle$. The pioneering "no-cloning" theorem does not prohibit the existence of such a machine as the copying states are mutually orthogonal [33]. The action of $\mathbb{M}$ can be described by a unitary evolution $U_{\mathbb{M}}$ satisfying $U_{\mathbb{M}} |i\rangle |r\rangle = |i\rangle |i\rangle$, where $|r\rangle$ is some fixed reference state and $i \in \{0, 1\}$. The machine has two input ports, i.e., one fed with particles R prepared in some reference state $|r\rangle$ and the other fed with system S prepared in state $|0\rangle$ or $|1\rangle$.

The action of this copying machine is worth analyzing in the ontological picture. Whenever S is prepared in the state $|i\rangle$, a composite ontic state $\lambda_{\text{joint}} = (\lambda^S, \lambda^R, \lambda^G)$ is fed into the machine, where $\lambda^S \in \Lambda_i \subseteq \Lambda^S$, $\lambda^R \in \Lambda_r \subseteq \Lambda^R$, and $\lambda^G \in \Lambda^G$, i.e., $\Lambda_{ir} := \{\mu(\lambda_{\text{joint}}|\,|ir\rangle) > 0\} \subseteq \Lambda^S \times \Lambda^R \times \Lambda^G$. After the successful completion of cloning, the machine yields the outcome $|ii\rangle$, i.e., it yields an ontic state that belongs in $\Lambda_{ii} \subseteq \Lambda^S \times \Lambda^R \times \Lambda^G$. Let us now feed the above copying machine with the system state prepared in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. At the ontological level, the machine receives some ontic state $\lambda_{\text{joint}} \in \Lambda^S \times \Lambda^R \times \Lambda^G$ sampled with probability distribution $\mu(\lambda_{\text{joint}}|\,|+r\rangle) = \mu(\lambda^S, \lambda^R, \lambda^G|\,|+r\rangle)$. Let us consider a maximally ψ-epistemic model underlying quantum theory. This will imply $\Lambda_+ = (\Lambda_0 \cap \Lambda_+) \cup (\Lambda_1 \cap \Lambda_+)$ and $\Lambda_{+r} = (\Lambda_{0r} \cap \Lambda_{+r}) \cup (\Lambda_{1r} \cap \Lambda_{+r}) \subseteq \Lambda_{0r} \cup \Lambda_{1r}$. Since the ontic state obtained after the machine's action only depends on the input ontic state fed into the machine, $\text{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}}[|+r\rangle])] \subseteq \text{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}}[|0r\rangle])] \cup \text{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}}[|1r\rangle])] \equiv \Lambda_{00} \cup \Lambda_{11}$. Since a product preparation cannot have any nonlocal ontic reality, i.e., $\Lambda_{00} \cap \Lambda^{NL} = \emptyset = \Lambda_{11} \cap \Lambda^{NL}$ (see Proposition 1), it, in turn, implies $\text{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}}[|+r\rangle])] \cap \Lambda^{NL} = \emptyset$.

But the above conclusion is in direct contradiction with predictions of quantum theory. Due to the linearity of the machine's action, one will obtain the output $|\phi^+\rangle = (|0\rangle |0\rangle + |1\rangle |1\rangle)/\sqrt{2}$ whenever the machine $\mathbb{M}$ is supposed to copy the state $|+\rangle$. Being the maximally entangled state, it exhibits maximum violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [34] for a suitable measurement choice on its local parts. Hence the entire ontic support of the output pairs cannot be contained within $\Lambda^S \times \Lambda^R \times \Lambda^G$; in other words, $\Lambda_{\phi^+} \cap \Lambda^{NL} \neq \emptyset$. At this point, it is important to note that like PBR, we have neither considered ontic state space for a product preparation to be a Cartesian product of their individual state space nor presumed the *strong* assumption of "preparation independence." We have also not invoked the assumption of "local independence," i.e., $\int d\lambda^G \mu(\lambda^S, \lambda^R, \lambda^G|\,|ir\rangle) = \mu(\lambda^S|i)\mu(\lambda^R|r)$ as considered in [12], and the assumption of "ontic indifference" used in [6].

A similar proof runs if we feed any state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ into the machine $\mathbb{M}$ instead of the state $|+\rangle$, where $\mathbb{C} \ni \alpha, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$. In a maximally ψ-epistemic theory, the ontic support Λ_ψ entirely gets shared between Λ_0 and Λ_1. Whereas for $|+\rangle$ these two shares are equal, for $|\psi\rangle$ the shares are proportional to the quantum overlaps. A similar argument in this case implies $\text{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}}[|\psi r\rangle])] \cap \Lambda^{NL} = \emptyset$. But this is again in contradiction with the quantum prediction as the machine's action on $|\psi r\rangle$ yields the state $\alpha\,|00\rangle + \beta\,|11\rangle$, which is known to be Bell nonlocal [35].

Our result thus establishes that a maximally ψ-epistemic model cannot account for the Bell nonlocal correlations. In the "orthodox" interpretation of quantum theory, the wave function ψ alone provides the complete description of reality, which itself can be considered as a ψ-complete ontological model. Researchers have already acknowledged that Einstein, at the Solvay conference, had shown the incompatibility of a ψ-complete model with locality through a simple argument [1]. In Einstein's own words [36], "One arrives at very implausible theoretical conceptions, if one attempts to maintain the thesis that the statistical quantum theory is in principle capable of producing a complete description of an individual physical system...." Extending this result, the authors in [2] have shown that "Any ψ-ontic ontological model that reproduces the quantum statistics (QSTAT) violates locality."

In brief, these two results together assert "ψ-complete and/or ψ-ontic $\Rightarrow \neg$ locality." Our theorem can be viewed as a converse of this claim as it shows that "Nonlocality (in the sense of Bell) $\Rightarrow \neg$ maximal ψ-epistemicity." Here we recall the result of Ref. [17], where it has been shown that "maximally ψ-epistemic $\Rightarrow$ Kochen-Specker noncontextual." Therefore, Kochen-Specker contextuality [37] excludes the maximally ψ-epistemic model for quantum systems with Hilbert space dimension strictly greater that two. Similarly, our theorem shows that the Bell nonlocality rules out the maximally ψ-epistemic ontological model. Manifestly, the scope of our theorem is broader than that in [17] as it excludes the maximally ψ-epistemic model even for a qubit system.

So far, we have shown that Bell nonlocality prohibits the ontological model that is maximally ψ-epistemic. Naturally, the question arises, What about the nonmaximally ψ-epistemic models [see Eq. (2)]? Interestingly, we will now show that a broader class of ontological models having a certain degree of epistemicity can also be excluded from a similar reasoning.

Whenever the state $|+\rangle$ is fed into the machine copying the state $|0\rangle$ and $|1\rangle$ perfectly, the resulting state being maximally entangled exhibits Bell nonlocality. Thus, part of the ontic support Λ_+ must lie outside $\Lambda_0 \cup \Lambda_1$, which accounts for the observed CHSH violation and, consequently, this will impose a bound on the degree of epistemicity. For a suitable choices of measurements, the maximally entangled state can exhibit CHSH violation up to $2\sqrt{2}$, i.e., $\langle\phi^+|\,\mathbb{CHSH}\,|\phi^+\rangle = 2\sqrt{2}$; $\mathbb{CHSH}$ denotes the CHSH expression or operator. The quantum reproducibility condition, therefore, demands

$$\int_{\Lambda_+} d\lambda^S \int_{\Lambda^R}\int_{\Lambda^G} d\lambda^R d\lambda^G \mu(\lambda_{\text{joint}}|\,|+r\rangle)\mathbb{CHSH}_{\mathbb{M}[\lambda_{\text{joint}}]} = \langle\phi^+|\,\mathbb{CHSH}\,|\phi^+\rangle = 2\sqrt{2}.$$

Here, $\lambda_{\text{joint}} \equiv (\lambda^S, \lambda^R, \lambda^G)$ is the input ontic state and $\mathbb{M}[\lambda_{\text{joint}}]$ is the output ontic state after the machine's action. The domain of integration for the variable λ^S can be divided into three disjoint parts, i.e., $\int_{\Lambda_+} d\lambda^S = \int_{\Lambda_0\cap\Lambda_+} d\lambda^S + \int_{\Lambda_1\cap\Lambda_+} d\lambda^S + \int_{\Lambda_+\setminus(\Lambda_0\cup\Lambda_1)} d\lambda^S$.

Note that whenever $\lambda^S \in \Lambda_i \cap \Lambda_+$, the joint ontic state $\lambda_{\text{joint}} = (\lambda^S, \lambda^R, \lambda^G)$ belonging in Λ_{+r} as well as in Λ_{ir} cannot lead to the observed nonlocality since $\mathbb{M}[\lambda_{\text{joint}}] \in \Lambda_{ii}$. However, there may exist $(\lambda^S, \lambda'^R, \lambda'^G) \in \Lambda_{+r}$ such that $(\lambda^S, \lambda'^R, \lambda'^G) \notin \Lambda_{ir}$, where $\lambda^S \in \Lambda_i \cap \Lambda_+$ for some $i \in \{0, 1\}$. In such a case, the machine can distinguish the input preparation $|+\rangle$ or $|i\rangle$ by accessing λ'^R and/or λ'^G and the observed nonlocal behavior can be well explained. At this point, we assume that

$$\forall\,\lambda^S \in \Lambda_i \cap \Lambda_+,$$
$$(\lambda^S, \lambda'^R, \lambda'^G) \in \Lambda_{+r} \Rightarrow (\lambda^S, \lambda'^R, \lambda'^G) \in \Lambda_{ir}. \quad (3)$$

Importantly, this is a strictly weaker assumption than the ontic composition principle of PI used by PBR. To argue this, first note that for $\lambda^S \in \Lambda_+ \cap \Lambda_i$, the Bayes' rule allows us to write $\mu(\lambda^S, \lambda^R, \lambda^G|\,|+r\rangle) = \mu(\lambda^S|\,|+r\rangle)\mu(\lambda^R|\lambda^S, |+r\rangle)\mu(\lambda^G|\lambda^S, \lambda^R, |+r\rangle)$. We can also write $\mu(\lambda^S, \lambda^R, \lambda^G|\,|ir\rangle) = \mu(\lambda^S|\,|ir\rangle)\mu(\lambda^R|\lambda^S, |ir\rangle)\mu(\lambda^G|\lambda^S, \lambda^R, |ir\rangle)$. Clearly, the assumptions

$$\mu(\lambda^R|\lambda^S, |+r\rangle) = \mu(\lambda^R|\lambda^S, |ir\rangle), \quad (4a)$$
$$\mu(\lambda^G|\lambda^S, \lambda^R, |+r\rangle) = \mu(\lambda^G|\lambda^S, \lambda^R, |ir\rangle) \quad (4b)$$

will imply Eq. (3). Unlike the PI assumption, Eqs. (4a) and (4b) do not prohibit correlation at the ontic level for two operationally independent preparations and hence they are weaker assumptions [38]. Furthermore, the conditions in Eqs. (4a), (4b) suffice for our purpose, but they are not at all the necessary requirements. The following weaker conditions:

$$\mu(\lambda^R|\lambda^S, |ir\rangle) = 0 \Rightarrow \mu(\lambda^R|\lambda^S, |+r\rangle) = 0, \quad (5a)$$
$$\mu(\lambda^G|\lambda^S, \lambda^R\,|ir\rangle) = 0 \Rightarrow \mu(\lambda^G|\lambda^S, \lambda^R\,|+r\rangle) = 0, \quad (5b)$$

serve well for our purpose. Therefore, $\Lambda_+ \setminus (\Lambda_0 \cup \Lambda_1)$ should have a nonzero measure in order to explain the observed nonlocality.

The area of the domain, $\Lambda_+ \setminus (\Lambda_0 \cup \Lambda_1)$, required to explain the observed quantum nonlocality imposes a bound on the degree of epistemicity of the underlying ontological model. Assuming $\Omega(+,0) = \Omega(+,1) := \Omega$, it turns out that $\Omega \leqslant 2-\sqrt{2}$ (see Appendix B). Therefore, a threshold amount of onticity is required in the ontological model to incorporate the observed CHSH violation in quantum theory. Taking a more general consideration of a cloning machine $\mathbb{M}_\phi$ that perfectly copies the states $|\phi\rangle$ and $|\phi^\perp\rangle$, one can obtain the following general bound: $|\alpha|^2\Omega(\phi,\psi) + |\beta|^2\Omega(\phi^\perp,\psi) \leqslant 2-\sqrt{1+4|\alpha|^2|\beta|^2}$, where $|\psi\rangle = \alpha\,|\phi\rangle + \beta\,|\phi^\perp\rangle$ and $\alpha \neq 0, 1$ (see Appendix B).

VI. DISCUSSIONS

The PBR theorem has initiated a surge in research interest regarding the reality of the quantum wave function [6–10,39–44]. Between the two competitive views—ψ-ontic vs ψ-epistemic—it favors the former. To this claim, it uses an assumption, called PI, regarding the ontology of a composite system. Establishing such a powerful doctrine about the reality of the quantum state, the theorem has gone through detailed scrutiny and the PI assumption has gained several criticisms [11–14]. In particular, the criticism in Ref. [14] is quite severe. The authors there consider the ontic composition assumption to be weaker than the assumption of "preparation independence" and reject the vast class of deterministic hidden-variables theories, including those consistent on their targeted domain. This result challenges the compositional aspect of the real or ontic states one might wish to assume through preparation independence while modeling a tensor-product quantum state. It therefore motivates renewed aspirations to establish the ψ-ontic nature of the quantum wave function from a more rational assumption or using no such assumption at all. At this point, our theorem starts contributing. Our result established that if we take the maximally ψ-epistemic doctrine, then quantum nonlocality cannot be explained in such a model. Importantly, unlike PBR, we do not consider any compositional assumption. In fact, we consider that ontic state space for product preparation can be more general than the Cartesian product of their individual ontic states as they can possess composite ontic properties. In this regard, the toy model of Spekkens [45] is worth mentioning. The model is maximally ψ-epistemic by construction. Though it reproduces a number of phenomena as observed in quantum theory, it is a Bell local model. Our theorem establishes a general result in this direction as it shows that no maximally ψ-epistemic model can incorporate the nonlocal behavior of quantum theory. Furthermore, we show that the observed phenomenon of quantum nonlocality excludes not only the maximally ψ-epistemic models, but it also imposes a bound on the degree of epistemicity of the underlying models. The extent of our theorem is also broader than the ψ-ontology theorems in [15–21] as these results apply to quantum systems with dimensions greater than two but remain silent for a qubit system. Here it is worth mentioning the theorem proved by Aaronson *et al.* [46]. There the authors have considered only those ψ-epistemic models in which every pair of nonorthogonal states has ontic overlap of nonzero measure. However, our theorem does not limit itself to such pairwise ψ-epistemic models; i.e., given any two nonorthogonal states, the theorem dictates an upper bound on their ontic overlap without assuming anything about the degree of ontic overlap between all other pairs of nonorthogonal states. Our result opens up some research possibilities. In our work, we have considered the CHSH inequality. It would be interesting to study whether more stringent restriction(s) on the underlying ontological models can be obtained from other classes of local realistic inequalities. It may also be interesting to study what new kind of restriction genuine quantum nonlocality would impose on the nature of the quantum wave function.

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APPENDIX A: PROOF OF PROPOSITION 1

Before going to the main part of the proof, let us first discuss what conclusion one can make regarding the evolution of the underlying ontic states when some measurement is performed on some operational quantum preparation.

Single system. Consider that a qubit is prepared in the state $|0\rangle$, the $+1$ eigenstate of σ_z. At the ontological level, the system is prepared in some ontic state $\lambda \in \Lambda_0 := \{\lambda|\mu(\lambda|\,|0\rangle > 0\} \subset \Lambda$. Suppose σ_x measurement is performed on system.

In such a scenario, what can we say about the postmeasurement ontic state (say) λ'? Can we give a deterministic evolution from the premeasurement ontic state λ to the postmeasurement ontic state λ'? The answer to this question is no. However, we can make some conclusion regarding λ'. Given that on the premeasurement ontic state λ, the measurement σ_x is performed, we can say that the postmeasurement ontic state λ' must belong to $(\Lambda_+ \cup \Lambda_-)$, i.e., $\Lambda_0 \ni \lambda \xrightarrow{\sigma_x} \lambda' \in (\Lambda_+ \cup \Lambda_-)$.

Suppose that on the quantum preparation $|0\rangle$, the σ_z measurement is performed instead of the σ_x measurement. Clearly, the operational state remains identical as a measurement does not disturb its eigenkets. In this case, can we make the conclusion that the postmeasurement ontic state λ' is the same as the premeasurement ontic state λ? Again the answer is no. However, we can make the conclusion that λ' must belong to the support Λ_0. In other words, a measurement that does not disturb the operational preparation can disturb the ontic preparation. Such a disturbance is indeterministic in general, but the final ontic state remains within the support of the operational preparation, i.e., $\Lambda_0 \ni \lambda \xrightarrow{\sigma_z} \lambda' \in \Lambda_0$.

Therefore, in general, we can say a measurement, disturbing or nondisturbing on an operational preparation, should not assume any deterministic state-update rule for the underlying ontic states. The only thing that can be specified is the set of possible postmeasurement ontic states for a given premeasurement ontic state.

Bipartite system. Consider a two-qubit bipartite system $\mathcal{S}_{AB}$. As discussed in the main text, the ontic state space of such a system has three parts: $\Lambda_A \times \Lambda_B$ corresponding to local reality, Λ^G carrying strictly relational information about the two systems that remains hidden under local measurements, and Λ^{NL} corresponds to reality resulting in the Bell nonlocal feature. Since in the following we will consider local measurements on the bipartite system, without loss of any generality we only consider the joint ontic states $\lambda_{\text{joint}} \in (\Lambda_A \times \Lambda_B) \cup \Lambda^{NL}$. Any $\lambda_{\text{joint}} \in (\Lambda_A \times \Lambda_B)$ can be thought of as $\lambda_{\text{joint}} \equiv (\lambda_A, \lambda_B)$, where $\lambda_A \in \Lambda^A$ and $\lambda_B \in \Lambda^B$. On the other hand, a $\lambda_{\text{joint}} \in \Lambda^{NL}$ must violate at least one of the conditions of outcome independence (OI) and parameter independence.

Outcome independence:

$$p(a|b, x, y, \lambda_{\text{joint}}) = p(a|x, y, \lambda_{\text{joint}}),\ \forall\, a, b, x, y; \quad \text{(A1a)}$$

$$p(b|a, x, y, \lambda_{\text{joint}}) = p(b|x, y, \lambda_{\text{joint}}),\ \forall\, a, b, x, y. \quad \text{(A1b)}$$

Here, a, b respectively, denote Alice's and Bob's outcome for their respective local measurements x and y.

Parameter independence:

$$p(a|x, y, \lambda_{\text{joint}}) = p(a|x, \lambda_{\text{joint}}),\ \forall\, a, x, y; \quad \text{(A2a)}$$

$$p(b|x, y, \lambda_{\text{joint}}) = p(b|y, \lambda_{\text{joint}}),\ \forall\, b, x, y. \quad \text{(A2b)}$$

In an OI model, a nonlocal ontic state $\lambda_{\text{joint}} \in \Lambda^{NL}$ must violate the condition of parameter independence, which can happen in three ways:

(i) Violation of Eq. (A2a) for some choice of a, x, y. Such a λ_{joint} we will represent as $\lambda_{\text{joint}} \equiv (\lambda_{A\leftarrow B}, \lambda_B)$, as Bob's measurement choice affects Alice's marginal outcome probabilities.

(ii) Violation of Eq. (A2b); $\lambda_{\text{joint}} \equiv (\lambda_A, \lambda_{A\rightarrow B})$.

(iii) Violation of Eq. (A2a) and Eq. (A2b); $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$.

For a better clarification, specific examples of these four types of λ_{joint} are discussed below.

Case (1). $\lambda_{\text{joint}} \equiv (\lambda_A, \lambda_B) \in \Lambda_A \times \Lambda_B$ Table I.

Such a λ_{joint} can be described by the Cartesian product of ontic states $\lambda_A \in \Lambda_A$ and $\lambda_B \in \Lambda_B$ corresponding to the following tables respectively.

TABLE I. Note that λ_{joint} satisfies the condition of parameter independence. None of Alice's outcomes depend on the measurement context(s) chosen by Bob, and vice versa. Therefore, observables of both A and B possess local reality (independent of any remote context chosen by another observer).

No.	$xy(\downarrow)/ab(\rightarrow)$	$+1,+1$	$+1,-1$	$-1,+1$	$-1,-1$
1	$\sigma_x \otimes \sigma_x$	1	0	0	0
2	$\sigma_x \otimes \sigma_y$	0	1	0	0
3	$\sigma_y \otimes \sigma_x$	0	0	1	0
4	$\sigma_y \otimes \sigma_y$	0	0	0	1

No.	$x(\downarrow)/a(\rightarrow)$	+1	−1
1	σ_x	1	0
2	σ_y	0	1
No.	$y(\downarrow)/b(\rightarrow)$	+1	−1
1	σ_x	1	0
2	σ_y	0	1

Case (2-i). $\lambda_{\text{joint}} \equiv (\lambda_{A\leftarrow B}, \lambda_B) \in \Lambda^{NL}$ Table II. In this case, observables of the A system do not possess any local reality, i.e., it cannot be described by an ontic state belonging to its local ontic space Λ^A. However, observables of the B system take values independent of the measurement contexts chosen by Alice, and hence they possess local ontic reality, i.e., B has its local ontic state $\lambda_B \in \Lambda^B$ corresponding to the following table:

No.	$y(\downarrow)/b(\rightarrow)$	+1	−1
1	σ_x	1	0
2	σ_y	1	0

Case (2-ii). $\lambda_{\text{joint}} \equiv (\lambda_A, \lambda_{A\rightarrow B}) \in \Lambda^{NL}$ Table III. In this case, the outcome of the observables on the A system do not depend on Bob's measurement choice. Therefore, although B does not possess local reality (i.e., context independent reality), the system A is in a local ontic state $\lambda^A \in \Lambda^A$ corresponding to the following table:

No.	$x(\downarrow)/a(\rightarrow)$	+1	−1
1	σ_x	1	0
2	σ_y	1	0

Case (2-iii). $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$. In such joint ontic state, parameter independence is violated both from A to B and from B to A. Specification of such a λ_{joint} requires two different tables: one of [*Type (2-ii)* nonlocal] for the case when Alice measures first, Bob measures second. And the other table will be of [*Type (2-i)* nonlocal] for the case when Bob measures his system before Alice measures her system. We are now in a position to prove Proposition 1 stated in the main text.

Proof. Consider the product quantum preparation $|0\rangle_A\,|0\rangle_B$. Contrary to Proposition 1, assume that there exists some nonlocal joint ontic state $\lambda_{\text{joint}} \in \Lambda^{NL}$ lying in the support Λ_{00}. Since we are considering the ontological model to be maximally ψ-epistemic, the model must be outcome deterministic and reciprocal [13]. Thus the nonlocal λ_{AB} must violate at least one of the assumptions of parameter independence and outcome independence (OI) [47,48]. Since outcome deterministic models satisfy OI [49], λ_{AB} must violate parameter independence. Note that nonlocal correlations have a classical-like explanation if we assume that the agents sacrifice their *free choice* or measurement independence [41,50–53]. However, here we are considering that the agents enjoy their free choice.

Consider a $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$ [*Type (2-iii)* nonlocal]. Such a λ_{joint} requires two different tables of assignments: one for when Alice measures first (which would be [*Type (2-ii)* nonlocal]) and a different one for when Bob measures first (which would be [*Type (2-i)* local]). But, for simplicity, we consider a fixed order of their measurements: Alice measures her part of the system before Bob performs any measurement on his part, so that a single table of assignments is sufficient. A typical example of such $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B} \in \Lambda_{00}$ is given in Table IV.

Note that in accordance with our chosen order of measurements, the table of assignment for $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$ is of [*Type (2-ii)* nonlocal]. Here, we do not provide any table of [*Type (2-i)* nonlocal] associated with $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$ that considers the case when Alice and Bob measure in the reverse temporal order (i.e., Bob first and Alice second) since it is straightforward to construct such a table.

On this premeasurement joint ontic state, let Alice first measure σ_z on her system. This will cause all observables at Bob's end to take values in accordance with the chosen context of σ_z at Alice's side. Therefore, postmeasurement

TABLE II. Note that Alice's outcome for the σ_y measurement depends on Bob's measurement choice(s) (compare the third and fourth rows). However, Bob's outcomes are independent of the measurement contexts chosen by Alice.

No.	$xy(\downarrow)/ab(\rightarrow)$	+1, +1	+1, −1	−1, +1	−1, −1
1	$\sigma_x\otimes\sigma_x$	1	0	0	0
2	$\sigma_x\otimes\sigma_y$	1	0	0	0
3	$\sigma_y\otimes\sigma_x$	1	0	0	0
4	$\sigma_y\otimes\sigma_y$	0	0	1	0

TABLE III. Note that the σ_y's value for Bob depends on the measurement context chosen by Alice (compare the second and fourth rows). Importantly, Alice's outcomes do not depend on Bob's measurement choice(s).

No.	$xy(\downarrow)/ab(\rightarrow)$	+1, +1	+1, −1	−1, +1	−1, −1
1	$\sigma_x\otimes\sigma_x$	1	0	0	0
2	$\sigma_x\otimes\sigma_y$	1	0	0	0
3	$\sigma_y\otimes\sigma_x$	1	0	0	0
4	$\sigma_y\otimes\sigma_y$	0	1	0	0

TABLE IV. $\lambda_{\text{joint}} \equiv \lambda_{A\leftrightarrow B}$ the table shows only the value assignments for the case when Alice measures first, Bob second.

No.	$xy(\downarrow)/ab(\rightarrow)$	+1, +1	+1, −1	−1, +1	−1, −1
1	$\sigma_z\otimes\sigma_z$	1	0	0	0
2	$\sigma_z\otimes\sigma_x$	1	0	0	0
3	$\sigma_x\otimes\sigma_z$	1	0	0	0
4	$\sigma_x\otimes\sigma_x$	1	0	0	0
5	$\sigma_x\otimes\sigma_y$	1	0	0	0
6	$\sigma_y\otimes\sigma_x$	0	1	0	0
7	$\sigma_y\otimes\sigma_y$	1	0	0	0
8	$\sigma_z\otimes\sigma_y$	1	0	0	0
9	$\sigma_y\otimes\sigma_z$	1	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

TABLE V. $\lambda'_B \in \Lambda_B$ in the postmeasurement joint ontic state $\lambda'_{\text{joint}} = (\lambda'_{A\leftarrow B}, \lambda'_B)$ after Alice's measurement. Notice that A to B parameter dependence is destroyed due to Alice's measurement and as a result, system B attains local reality.

No.	$y(\downarrow)/b(\rightarrow)$	$+1$	-1
1	σ_z	1	0
2	σ_x	1	0
3	σ_y	1	0
⋮	⋮	⋮	⋮

λ'_{joint} will be of *Type (2-i)* nonlocal, i.e., $(\lambda'_{A\leftarrow B}, \lambda'_B)$, where λ'_B is described in Table V.

The measurement σ_z on the A system thus leads to the following ontic transformation:

$$\lambda^{\text{joint}} \equiv \lambda_{A\leftrightarrow B} \xrightarrow{\sigma_z\otimes\mathbb{I}} \lambda'_{\text{joint}} \equiv (\lambda'_{A\leftarrow B}, \lambda'_B) \in \mathit{Type}(2-i). \tag{A3}$$

Following reasoning clarifies such a postmeasurement evolution of the ontic state. Before Alice's measurement, observables of system B possess no local reality, which is reflected in the premeasurement ontic state $\lambda_{A\leftrightarrow B}$. For instance, Bob's σ_x observable does not have context independent reality in the state $\lambda_{A\leftrightarrow B}$ (see Table IV). In other words, one cannot assign value to σ_x of B independent of Alice's choice of measurement (which is evident from the second and sixth rows of Table IV). But, as soon as Alice chooses one particular measurement context, Bob's observables take the corresponding values in accordance with Alice's chosen measurement context. To see this, consider the second row of the premeasurement state (Table IV) which asserts the following: "*If Alice measures σ_z on A, Bob will obtain $+1$ if he measures σ_x on B.*" When Alice has already performed the measurement σ_z, Bob's σ_x observable must take the value $+1$ in the postmeasurement state λ'_{joint}; otherwise, the proposition of the second row of the premeasurement state gets violated. If Alice had measured σ_y instead of σ_z, Bob's σ_x observable would have taken the value -1; otherwise, the proposition of the sixth row of the premeasurement state would have been violated. Since Alice measures σ_z, not σ_y, the possibility of Bob's σ_x value being -1 is no longer there in the postmeasurement state λ'_{joint}. Therefore, Alice's choice of a particular context assigns Bob's observable context independent local reality. Similarly, after Alice's σ_z measurement, the eighth row demands that B should have $\sigma_y = 1$ in the postmeasurement joint ontic state λ'_{joint}.

In general, in the postmeasurement joint ontic state λ'_{joint}, σ_n of B will take the value obeying the condition imposed by the row corresponding to $\sigma_z \otimes \sigma_n$ of the premeasurement joint ontic state $\lambda_{A\leftrightarrow B}$. Thus, all the observables of B take some fixed particular value, and therefore attain local reality, as soon as Alice chooses her measurement context. Therefore, in the postmeasurement state, although parameter independence can be violated from B to A, there is no such violation from A to B and hence, in λ'_{joint}, B is in the ontic state $\lambda'_B \in \Lambda_B$ (given in Table V).

Please note that we can only claim that the postmeasurement ontic state $\lambda'_{\text{joint}} \equiv (\lambda'_{A\leftarrow B}, \lambda'_B) \in \mathit{Type}(2-i)$, but which particular λ'_{joint} is not specified and, for our argument, is not required. Suppose Bob now performs σ_z measurement on the state λ'_{joint}. A similar reasoning as above will imply that the final ontic state $\lambda''_{\text{joint}} \equiv (\lambda''_A, \lambda''_B) \in \mathit{Type}(1)$. Thus, we have

$$\mathit{Type}\ (2-iii) \ni \lambda_{A\leftrightarrow B} \xrightarrow{\sigma_z\otimes\mathbb{I}} (\lambda'_{A\leftarrow B}, \lambda'_B) \in \mathit{Type}(2-i) \xrightarrow{\mathbb{I}\otimes\sigma_z} (\lambda''_A, \lambda''_B) \in \mathit{Type}(1). \tag{A4}$$

Here, in the second step of ontic evolution in Eq. (A4), we have made the following *assumption regarding ontic evolution* of a certain class of joint ontic states:

In any joint ontic state $\lambda^{AB}_{\text{joint}}$, if there exists no parameter dependence from $A \rightarrow B$ (or from $B \rightarrow A$), the local measurement on B (or A) cannot generate parameter dependence from $A \rightarrow B$ (or $B \rightarrow A$) in the postmeasurement state $\lambda'^{AB}_{\text{joint}}$. Due to this, in the second step of ontic evolution in (4), the following has not been the case: $(\lambda'_{A\leftarrow B}, \lambda'_B) \in \mathit{Type}(2-i) \xrightarrow{\mathbb{I}\otimes\sigma_z} (\lambda''_A, \lambda''_{A\rightarrow B}) \in \mathit{Type}(2-ii)$. More precisely, as there was no parameter dependence from $A \rightarrow B$ in $(\lambda'_{A\leftarrow B}, \lambda'_B)$, measuring σ_z on B cannot generate parameter dependence from $A \rightarrow B$ in the postmeasurement state.

As the local joint ontic states [i.e., $\lambda_{\text{joint}} \equiv (\lambda_A, \lambda_B) \in \mathit{Type}(1)$] do not possess parameter dependence in either direction (i.e., neither $A \rightarrow B$ nor $A \leftarrow B$), the assumption immediately leads to the following:

$$(\lambda_A, \lambda_B) \in \mathit{Type}(1) \xrightarrow{\sigma_z\otimes\mathbb{I}} (\lambda'_A, \lambda'_B) \in \mathit{Type}(1) \xrightarrow{\mathbb{I}\otimes\sigma_z} (\lambda''_A, \lambda''_B) \in \mathit{Type}(1).$$

Joint ontic states of system AB which are local remain local under local measurements done on subsystems A and/or B. [Note that as we have fixed a specific temporal order in which Alice measures first and Bob measures second, $\lambda_{A\leftrightarrow B} \in \mathit{Type}(2-iii)$ is essentially of $\mathit{Type}(2-ii)$ in which no parameter dependence exists from $B \rightarrow A$. Therefore, we could have as well used the assumption in the very first step of ontic evolution in (A4) to obtain $\mathit{Type}(2-iii) \ni \lambda_{A\leftrightarrow B} \xrightarrow{\sigma_z\otimes\mathbb{I}} (\lambda'_A, \lambda'_B) \in \mathit{Type}(1)$].

Therefore, from Eq. (A4), for any premeasurement $\lambda_{\text{joint}} \in \Lambda^{NL}$ that was assumed to lie within the ontic support of $|00\rangle$, the postmeasurement joint ontic state after Alice and Bob perform their local measurements will be $\lambda_{\text{joint}} \equiv (\lambda''_A, \lambda''_B) \in \Lambda_A \times \Lambda_B$. Therefore, the postmeasurement quantum state cannot contain any $\lambda_{\text{joint}} \in \Lambda^{NL}$ in its ontic support. But since the postmeasurement quantum state remains unchanged, i.e., it is again $|0\rangle_A |0\rangle_B$, this implies that ontic support of $|0\rangle_A |0\rangle_B$ cannot contain any $\lambda_{\text{joint}} \in \Lambda^{NL}$, which is in contradiction with our initial assumption.

It is not hard to see that one arrives at a similar contradiction for every product quantum preparation. Furthermore, the arguments also hold in any outcome deterministic ontological model instead of the maximally ψ-epistemic one only. This completes our proof. ■

APPENDIX B: GENERAL BOUND ON DEGREE OF EPISTEMICITY

Consider a quantum copying machine $\mathbb{M}_\phi$ that perfectly copies the state $|\phi\rangle$ and $|\phi^\perp\rangle$. While the machine is fed with

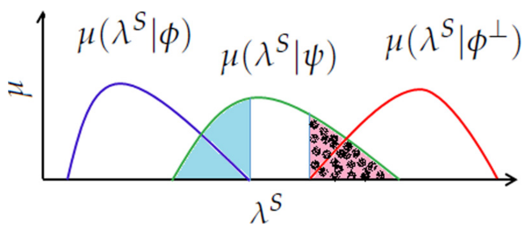

FIG. 1. Ontic support $\Lambda_\chi \equiv \mathrm{Supp}[\mu(\lambda|\chi)]$ for $|\chi\rangle \in \{|\phi\rangle, |\phi^\perp\rangle, |\psi\rangle = \alpha|\phi\rangle + \beta|\phi^\perp\rangle\}$. Along the x axis, ontic states for the system S are shown, and along the y axis, the probability distribution on these ontic states for different quantum preparations is plotted. In the case of a nonmaximally ψ-epistemic model, $\Lambda_\psi \cap (\Lambda_\phi \cup \Lambda_{\phi^\perp})^C \neq \emptyset$. The area of the sky-blue shaded region is given by $\int_{\Lambda_\phi} d\lambda^S \mu(\lambda^S|\psi)$ and the area of the pink shaded (black dotted) region is $\int_{\Lambda_{\phi^\perp}} d\lambda^S \mu(\lambda^S|\psi)$.

system S prepared in state $|\phi\rangle$ and $|\phi^\perp\rangle$ and the reference system R is prepared in some fixed state $|r\rangle$, its action at the ontological level is similar as discussed in the main text. If we fed $\mathbb{M}_\phi$ with $|\psi\rangle_S|r\rangle_R$, with $|\psi\rangle = \alpha|\phi\rangle + \beta|\phi^\perp\rangle$ ($\alpha \neq 0, 1$), then the linearity of quantum theory results in the output state $|\Theta\rangle_{SR} = \alpha|\phi\rangle_S|\phi\rangle_R + \beta|\phi^\perp\rangle_S|\phi^\perp\rangle_R$. For suitable choices of measurements, this state can exhibit CHSH inequality violation up to $2\sqrt{1+4|\alpha|^2|\beta|^2}$ [35]. Assuming maximally ψ-epistemicity, an argument similar to the one presented in the main text implies $\mathrm{Supp}[\mu(\lambda_{\text{joint}}|U_{\mathbb{M}_\phi}[|\psi r\rangle])] \cap \Lambda^{NL} = \emptyset$ and hence the observed nonlocality cannot be explained. To incorporate this nonlocality, we need to depart from ψ-maximal epistemicity. Now the quantum reproducibility condition for the observed nonlocality demands

$$\int_{\Lambda^S} d\lambda^S \int_{\Lambda^R}\int_{\Lambda^G} d\lambda^R d\lambda^G \mu(\lambda_{\text{joint}}|\,|\psi r\rangle)\mathbb{CHSH}_{\mathbb{M}_\phi[\lambda_{\text{joint}}]} = \langle\Theta_{SR}|\,\mathbb{CHSH}\,|\Theta_{SR}\rangle = 2\sqrt{1+4|\alpha|^2|\beta|^2}, \tag{B1}$$

where $\lambda_{\text{joint}} = (\lambda^S, \lambda^R, \lambda^G)$ and $\mathbb{M}_\phi[\lambda_{\text{joint}}]$ is the evolved ontic state after the machine action, and $\mathbb{CHSH}$ denotes the CHSH expression. As argued in the main text, $\mathbb{M}_\phi[\lambda_{\text{joint}}] \notin \Lambda^{NL}$ whenever $\lambda_{\text{joint}} \in (\Lambda_\psi \cap \Lambda_\phi) \cup (\Lambda_\psi \cap \Lambda_{\phi^\perp})$. Let us, therefore, consider that the ontic region $\Lambda_\psi \setminus (\Lambda_\phi \cup \Lambda_{\phi^\perp})$ has nonzero measure, which will be the case for nonmaximally ψ-epistemic models. Thus, the domain of integration for λ^S, in the left-hand side of Eq. (B1), can be divided into three disjoint regions: $\Lambda_\psi \cap \Lambda_\phi$, $\Lambda_\psi \cap \Lambda_{\phi^\perp}$, and $\Lambda_\psi \setminus (\Lambda_\phi \cup \Lambda_{\phi^\perp})$ (see Fig. 1). Thus, we have

$$\begin{aligned}&\int_{\Lambda_\psi\cap\Lambda_\phi} d\lambda^S \int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle)\mathbb{CHSH}_{\mathbb{M}_\phi[\lambda_{\text{joint}}]}\\ &= \int_{\Lambda_\psi\cap\Lambda_\phi} d\lambda^S \int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle) \times 2\\ &= 2\int_{\Lambda_\phi} d\lambda^S \mu(\lambda^S|\psi) = 2\Omega(\phi,\psi)|\langle\phi|\psi\rangle|^2\\ &= 2|\alpha|^2\Omega(\phi,\psi).\end{aligned} \tag{B2}$$

We have considered that a local ontic state yields the maximum possible value 2 for the CHSH expression, and used the fact that $\int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle) = \mu(\lambda^S|\psi r) = \mu(\lambda^S|\psi)$. Similar reasoning yields

$$\int_{\Lambda_\psi\cap\Lambda_{\phi^\perp}} d\lambda^S \int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle)\mathbb{CHSH}_{\mathbb{M}_\phi[\lambda_{\text{joint}}]} = 2|\beta|^2\Omega(\phi^\perp,\psi). \tag{B3}$$

Whenever $\lambda_{\text{joint}} \in \Lambda_\psi \setminus (\Lambda_\phi \cup \Lambda_{\phi^\perp})$, the evolved ontic state $\mathbb{M}_\phi[\lambda_{\text{joint}}]$ may lie in Λ^{NL} and, consequently, contributes to the observed quantum nonlocality. Assuming that all such λ_{joint} yield the maximum possible CHSH value (i.e., 4), we obtain

$$\begin{aligned}&\int_{\Lambda_\psi\setminus(\Lambda_\phi\cup\Lambda_{\phi^\perp})} d\lambda^S \int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle)\mathbb{CHSH}_{\mathbb{M}_\phi[\lambda_{\text{joint}}]}\\ &= \int_{\Lambda_\psi\setminus(\Lambda_\phi\cup\Lambda_{\phi^\perp})} d\lambda^S \int d\lambda^R d\lambda^G \mu(\lambda^S,\lambda^R,\lambda^G|\,|\psi r\rangle) \times 4\\ &= 4\left[\int_{\Lambda_\psi} d\lambda^S\mu(\lambda^S|\psi) - \int_{\Lambda_\phi} d\lambda^S\mu(\lambda^S|\psi) - \int_{\Lambda_{\phi^\perp}} d\lambda^S\mu(\lambda^S|\psi)\right]\\ &= 4[1 - |\alpha|^2\Omega(\phi,\psi) - |\beta|^2\Omega(\phi^\perp,\psi)].\end{aligned} \tag{B4}$$

Equations (B1)–(B4) together, therefore, imply

$$2\sqrt{1+4|\alpha|^2|\beta|^2} \leqslant 2\,|\alpha|^2\Omega(\phi,\psi) + 2\,|\beta|^2\Omega(\phi^\perp,\psi) + 4\,[1 - |\alpha|^2\Omega(\phi,\psi) - |\beta|^2\Omega(\phi^\perp,\psi)]. \tag{B5}$$

Here, instead of equality, we put inequality as some local $\mathbb{M}_\phi[\lambda_{\text{joint}}]$ can yield a CHSH value less than 2 and some nonlocal $\mathbb{M}_\phi[\lambda_{\text{joint}}]$ can yield a CHSH value less than 4. Simplifying the above expression, we obtain

$$|\alpha|^2\Omega(\phi,\psi) + |\beta|^2\Omega(\phi^\perp,\psi) \leqslant 2 - \sqrt{1+4|\alpha|^2|\beta|^2}. \tag{B6}$$

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