

## Characterizing mixed-state entanglement through single-photon interference

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Entanglement verification and measurement is essential for experimental tests of quantum mechanics and also for quantum communication and information science. Standard methods of verifying entanglement in a bipartite mixed state require detection of both particles and involve coincidence measurement. We present a method that enables us to verify and measure entanglement in a two-photon mixed state without detecting one of the photons, i.e., without performing any coincidence measurement or postselection. We consider two identical sources, each of which can generate the same two-photon mixed state but they never emit simultaneously. We show that one can produce a set of single-photon interference patterns, which contain information about entanglement in the two-photon mixed state. We prove that it is possible to retrieve the information about entanglement from the visibility of the interference patterns. Our method reveals a distinct avenue for verifying and measuring entanglement in mixed states.

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### I. INTRODUCTION

Entanglement is a fascinating trait of quantum mechanics: in addition to its implications for the foundations of quantum mechanics, today entanglement is a key resource in quantum information science. Verification and measurement of entanglement in a quantum state is an ever-growing field of research [1,2]. Entanglement in two-particle (more generally bipartite) quantum states can be verified, for example, by the violation of Bell's inequalities [3–7], quantum state tomography [8], entanglement witnesses [9–14], and measurements employing multiple copies of the quantum state [15–20]. For a general bipartite quantum state, all these methods require detection of both particles (subsystems). Known methods of verifying entanglement by performing measurement on one subsystem require the bipartite state to be pure [21–24]. Whether the entanglement of a bipartite mixed state can be verified by detecting only one subsystem is a question of fundamental importance.

We show by an example that it is indeed possible to verify the entanglement in a two-particle mixed state without detecting one of the particles. No coincidence measurement or postselection is required in our method. In order to demonstrate our method, we choose a polarization-entangled mixed state, which can be obtained by generalizing two Bell states. We use two identical sources of the quantum state, but only

one pair of photons is produced at a time; i.e., multiple copies of the state are *not* produced. We employ an interferometric technique to show that an entanglement criterion, namely, the positive partial transpose criterion [25,26], can be tested and the entanglement can also be measured by means of the concurrence, a popular measure of two-qubit entanglement [27,28].

Here, we present a detailed theoretical analysis of the entanglement verification technique. Furthermore, the nonlinear interferometer [29–31] used in our scheme has recently found important applications to various branches of quantum science and technology, including quantum imaging [32,33], quantum spectroscopy [34], polarization control [35], and fundamental tests of quantum mechanics [36,37]. All the corresponding theoretical analyses apply to pure states only. Our theoretical analysis also shows how to treat a mixed state in such an interferometric arrangement. Our results are fully supported by experimental observations [38].

The article is organized as follows. In Sec. II we outline our entanglement analysis scheme. In Sec. III, we present the class of states we address and discuss the relevant entanglement criterion and measure. In Sec. IV, we present the analysis of our method and the main results, which also include illustrations by numerical examples. Finally, we summarize and conclude in Sec. V.

### II. OUTLINE OF THE SCHEME

We consider two identical sources,  $Q_1$  and  $Q_2$ , each of which can produce the same two-photon mixed state,  $\hat{\rho}$ ; however, they never emit simultaneously (Fig. 1). We denote the

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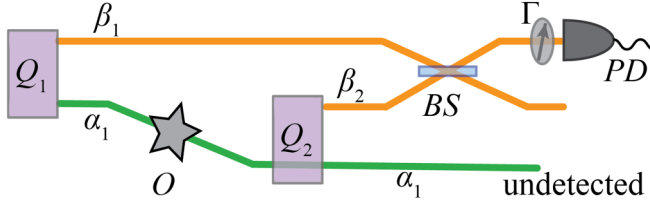


FIG. 1. Entanglement verification scheme. Two identical sources,  $Q_1$  and  $Q_2$ , individually generate the same two-photon state ( $\hat{\rho}$ ). Source  $Q_1$  can emit a photon pair ( $\alpha, \beta$ ) into propagation modes  $\alpha_1$  and  $\beta_1$ . Source  $Q_2$  is restricted to emit photon  $\alpha$  also in the mode  $\alpha_1$ . Photon  $\alpha$ , which is never detected, interacts with a device,  $O$ , between  $Q_1$  and  $Q_2$ . Source  $Q_2$  can emit photon  $\beta$  in propagation mode  $\beta_2$ . Modes  $\beta_1$  and  $\beta_2$  are combined by a beam splitter (BS) and an output of the BS is collected by a photon detector (PD). Another device ( $\Gamma$ ), placed before the PD, allows us to choose the measurement basis. Sources  $Q_1$  and  $Q_2$  emit probabilistically and never emit simultaneously. When it is impossible to know the source of a detected photon, single-photon interference is observed at the PD. For certain choices of basis, the entanglement of the two-photon state determines the visibility of the interference pattern. Information about the entanglement is retrieved from the single-photon interference patterns.

two photons by  $\alpha$  and  $\beta$ . Suppose that  $Q_1$  can emit photon  $\alpha$  into propagation mode  $\alpha_1$ . We ensure that  $Q_2$  can emit photon  $\alpha$  only in the same propagation mode ( $\alpha_1$ ). This is done by sending the beam of photon  $\alpha$  generated by  $Q_1$  through source  $Q_2$  and perfectly aligning the beam with the beam of photon  $\alpha$  that emerges from  $Q_2$ . Therefore, if one only observes photon  $\alpha$  that emerges from  $Q_2$ , one cannot identify the origin of the photon. We stress that this alignment does not result in stimulated emission (see, for example, [39,40]).

Sources  $Q_1$  and  $Q_2$  can emit photon  $\beta$  into distinct propagation modes  $\beta_1$  and  $\beta_2$ , respectively. These two modes are superposed by a beam splitter, BS, and one of the outputs of the BS is directed to a photon detector, PD. A device,  $\Gamma$ , is placed in front of the PD to choose appropriate measurement bases. It is important to note that only the single-photon counting rate (intensity) is measured in the experiment and no coincidence measurement is ever performed.

A device,  $O$ , is placed in the path of photon  $\alpha$  between  $Q_1$  and  $Q_2$ . (This device does not affect the emission rates at  $Q_1$  and  $Q_2$ .) Although  $\alpha$  is never detected, the interaction with  $O$  affects the interference pattern recorded by detecting photon  $\beta$  at the PD [29,30]. This striking phenomenon forms the basis of our entanglement verification scheme. We show that with the knowledge of this interaction, the information about the entanglement in the two-photon quantum state can be retrieved from single-photon interference patterns recorded in certain measurement bases. It is evident that the choice of devices  $O$  and  $\Gamma$  depends on the entangled degree of freedom.

In order to illustrate the scheme we work with a two-photon polarization-entangled state which is discussed in the next section.

### III. THE QUANTUM STATE

We consider a two-photon polarization-entangled mixed state that can be characterized by three free parameters. Such

a state can be expressed in the general form [41]

$$\hat{\rho} = I_H |H_\alpha, H_\beta\rangle \langle H_\alpha, H_\beta| + I_V |V_\alpha, V_\beta\rangle \langle V_\alpha, V_\beta| + (e^{-i\phi} \mathcal{S} \sqrt{I_H I_V} |H_\alpha, H_\beta\rangle \langle V_\alpha, V_\beta| + \text{H.c.}), \quad (1)$$

where  $0 \leq I_H \leq 1$ ,  $I_V = 1 - I_H$ ,  $\phi$  is a phase,  $0 \leq \mathcal{S} \leq 1$ , and  $H$  and  $V$  represent horizontal and vertical directions of polarization, respectively. It is evident that  $I_H$ ,  $I_V$ ,  $\mathcal{S}$ , and  $\phi$  are all real quantities. When  $\mathcal{S} = 1$ , the density operator ( $\hat{\rho}$ ) represents a pure state. When  $\mathcal{S} = 0$  and  $I_H = 1/2$  the state is diagonal in all bases, i.e., maximally mixed.

It is to be noted that  $\hat{\rho}$  can be obtained by generalizing the two following Bell States:  $|\Phi^+\rangle = (|H_\alpha, H_\beta\rangle + |V_\alpha, V_\beta\rangle)/\sqrt{2}$  and  $|\Phi^-\rangle = (|H_\alpha, H_\beta\rangle - |V_\alpha, V_\beta\rangle)/\sqrt{2}$ .

*Positive partial transpose (PPT) criterion.* Since we have a bipartite two-dimensional entangled state, the PPT criterion [25] can be applied to ensure separability or entanglement [26]. A partial transposition of the two-particle density matrix ( $\hat{\rho}$ ) is a transposition taken with respect to only one of the particles. The density operator  $\hat{\rho}$  has a positive partial transpose if and only if its partial transposition does not have any negative eigenvalues. According to the PPT criterion, a bipartite two-dimensional state is separable if and only if  $\hat{\rho}$  has positive partial transpose ([26]; see also [1]).

We take the partial transposition of  $\hat{\rho}$  [Eq. (1)] with respect to photon  $\alpha$  and find that the resulting matrix has the following eigenvalues:

$$I_H, \quad I_V, \quad \mathcal{S} \sqrt{I_H I_V}, \quad -\mathcal{S} \sqrt{I_H I_V},$$

where  $I_V = 1 - I_H$ . Since  $\mathcal{S}$ ,  $I_H$ , and  $I_V$  cannot take negative values, it follows from the PPT criterion that the state is entangled if and only if

$$\mathcal{S} \sqrt{I_H I_V} \neq 0. \quad (2)$$

*Concurrence.* The amount of entanglement in the two-qubit state ( $\hat{\rho}$ ) can be quantified by the concurrence [28]. In order to determine the concurrence one first needs to find the so-called spin-flipped density operator

$$\hat{\rho}^{\sim} = (\hat{\sigma}_y \otimes \hat{\sigma}_y) \hat{\rho}^* (\hat{\sigma}_y \otimes \hat{\sigma}_y), \quad (3)$$

where  $\hat{\sigma}_y$  is the second Pauli operator,  $\otimes$  implies Kronecker product, and the asterisk (\*) refers to the complex conjugation. The product  $\hat{\rho} \hat{\rho}^{\sim}$  has only real and non-negative eigenvalues. If the square roots of these eigenvalues, in decreasing order, are  $\lambda_1, \lambda_2, \lambda_3$ , and  $\lambda_4$ , the concurrence of  $\hat{\rho}$  is given by

$$\mathcal{C}(\hat{\rho}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}. \quad (4)$$

It follows from Eqs. (1), (3), and (4) that, in our case, the concurrence of the quantum state is

$$\mathcal{C}(\hat{\rho}) = 2\mathcal{S} \sqrt{I_H I_V} = 2\mathcal{S} \sqrt{I_H(1 - I_H)}. \quad (5)$$

Below we show that for the quantum state given by Eq. (1), the scheme allows us to test the PPT criterion as well as to measure the concurrence.

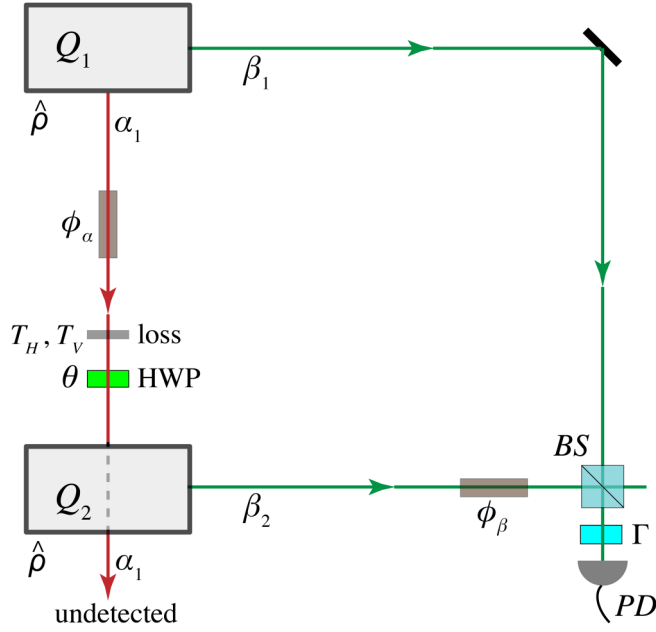


FIG. 2. Entanglement verification of a polarization-entangled state. Each source ( $Q_1$ ,  $Q_2$ ) individually produces the state  $\hat{\rho}$  [Eq. (1)]. Device  $O$  of Fig. 1 is now a half-wave plate (HWP) and device  $\Gamma$  projects photon  $\beta$  onto a horizontal ( $H$ ), vertical ( $V$ ), diagonal ( $D$ ), antidiagonal ( $A$ ), right-circular ( $R$ ), or left-circular ( $L$ ) polarization state. Photon  $\beta$  is detected at the PD and photon  $\alpha$  is not detected. For certain choices of HWP angle and certain choices of the measurement basis, the single-photon interference pattern recorded at the PD contains information of entanglement in state  $\hat{\rho}$ .

#### IV. ENTANGLEMENT VERIFICATION AND MEASUREMENT

##### A. Physical realization

The quantum state under consideration is entangled in polarization. In this case, we choose the device  $O$  of Fig. 1 to be a half-wave plate (HWP). As for device  $\Gamma$ , we use a combination of wave plates and a polarizer such that photon  $\beta$  can be projected onto the horizontal ( $H$ ), vertical ( $V$ ), diagonal ( $D$ ), antidiagonal ( $A$ ), right-circular ( $R$ ), or left-circular ( $L$ ) polarization state. The experimental setup is illustrated in Fig. 2.

Below we provide a detailed theoretical analysis explaining how the information about entanglement can be obtained from the single-photon interference patterns recorded at the PD.

##### B. Deriving the density operator

For the convenience of analysis, we rewrite Eq. (1) in the following form:

$$\hat{\rho}_j = \sum_{\mu, \nu} \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp(i\phi_{\mu\nu}) |\mu_\alpha^j, \mu_\beta^j\rangle \langle \nu_\alpha^j, \nu_\beta^j|, \quad (6)$$

where  $j = 1, 2$  refers to the sources  $Q_1$  and  $Q_2$ ;  $\mu = H, V$ ;  $\nu = H, V$ ;  $\phi_{\mu\nu} = 0$  for  $\mu = \nu$ , and  $\phi_{HV} = -\phi_{VH} = -\phi$ ; and

$$\mathcal{J}_{\mu\nu} = \mathcal{J}_{\nu\mu} = \begin{cases} 1 & \text{for } \mu = \nu \\ \mathcal{J} & \text{for } \mu \neq \nu. \end{cases} \quad (7)$$

We recall that sources produce only one photon pair in one detection run. The most general state that such an arrangement can produce is given by (Appendix A)

$$\hat{\rho}'_{\alpha\beta} = \sum_{j,k} \sum_{\mu,\nu} b_j b_k^* \sqrt{I_\mu I_\nu} P_{\mu\nu}^{jk} \exp(i\xi_{\mu\nu}^{jk}) |\mu_\alpha^j, \mu_\beta^j\rangle \langle \nu_\alpha^k, \nu_\beta^k|, \quad (8)$$

where both  $j = 1, 2$  and  $k = 1, 2$  refer to the sources,  $|b_j|^2$  is the probability of the photon pair being emitted by source  $Q_j$ , and  $|b_1|^2 + |b_2|^2 = 1$ ; the asterisk (\*) denotes complex conjugation, the phase  $\xi_{\mu\nu}^{jk}$  and the real positive quantity  $P_{\mu\nu}^{jk}$  must obey the relations (Appendix A)

$$\xi_{\mu\nu}^{jk} = -\xi_{\nu\mu}^{kj} \quad \forall j, k, \mu, \nu, \quad \text{and} \quad \xi_{\mu\nu}^{jj} = \phi_{\mu\nu} \quad \forall j, \quad (9a)$$

$$P_{\mu\nu}^{jk} = P_{\nu\mu}^{kj} \quad \forall j, k, \mu, \nu, \quad \text{and} \quad P_{\mu\nu}^{jj} = \mathcal{J}_{\mu\nu} \quad \forall j, \quad (9b)$$

and  $0 \leq P_{\mu\nu}^{jk} = P_{\nu\mu}^{kj} \leq 1$ .

We now impose the condition that photon pairs emitted by separate sources are fully coherent when they have the same polarization, i.e.,

$$P_{\mu^1}^{\mu^2} = P_{\mu^2}^{\mu^1} = 1 \quad \text{for } \mu = H, V. \quad (10)$$

This condition is easily attained in the laboratory by employing a pump laser with sufficiently long coherence length [38]. If we apply this condition, it follows from the positive semidefiniteness of  $\hat{\rho}'_{\alpha\beta}$  that (Appendix B)

$$P_{\mu^j}^{\nu^k} = \mathcal{J}_{\mu\nu}, \quad \forall j, k, \mu, \nu, \quad (11)$$

where  $\mathcal{J}_{\mu\nu}$  is given by Eq. (7).

From Eqs. (8) and (11), we find that the density operator representing the state of a photon pair in our system is given by

$$\hat{\rho}_{\alpha\beta} = \sum_{j,k} \sum_{\mu,\nu} b_j b_k^* \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp(i\xi_{\mu\nu}^{jk}) |\mu_\alpha^j, \mu_\beta^j\rangle \langle \nu_\alpha^k, \nu_\beta^k|. \quad (12)$$

We now mathematically represent the interaction of the HWP with photon  $\alpha$ . Ideally, there should not be any loss of photon in the propagation mode  $\alpha_1$  between  $Q_1$  and  $Q_2$ . However, due to experimental imperfections it is almost impossible to avoid slight misalignment of paths and probabilistic absorption of photon  $\alpha$  between  $Q_1$  and  $Q_2$ . In order to make our measurement scheme robust against such losses, we take them into account quantitatively. Let  $T_\mu$  be the probability amplitude of photon  $\alpha$  (polarized along direction  $\mu$ ) to arrive at  $Q_2$  from  $Q_1$ . Without any loss of generality, we can assume that  $T_\mu$  is real and therefore  $0 < T_\mu < 1$ . The combined effect of the HWP and the losses can be represented by the following relations involving field operators:

$$\hat{a}_{\alpha^2} = e^{i\phi_\alpha} [T_H (\hat{a}_{H^1} \cos 2\theta + \hat{a}_{V^1} \sin 2\theta) + R_H \hat{a}_{H_0}], \quad (13a)$$

$$\hat{a}_{\alpha^2} = e^{i\phi_\alpha} [T_V (\hat{a}_{H^1} \sin 2\theta - \hat{a}_{V^1} \cos 2\theta) + R_V \hat{a}_{V_0}], \quad (13b)$$

where  $\hat{a}$  represents a photon annihilation operator,  $\hat{a}_{\mu^j}^\dagger |\text{vacuum}\rangle = |\mu_\alpha^j\rangle$ , and  $\hat{a}_{\mu_0}$  can be interpreted as the field of a lost photon;  $R_\mu = \sqrt{1 - T_\mu^2}$ ; the HWP is set at angle  $\theta$ ; and  $\phi_\alpha$  is the phase gained due to the propagation through

air from  $Q_1$  to  $Q_2$  (assumed to be the same for all polarization directions).

We now proceed to derive the final form of the two-photon quantum state in the setup. Let us rewrite Eq. (13) in the compact form

$$\hat{a}_{\mu\alpha} = e^{i\phi_\alpha} \left[ \sum_{\lambda}^{H,V} \Lambda_{\mu\lambda}(\theta) \hat{a}_{\lambda\alpha} + R_{\mu} \hat{a}_{\mu_0} \right], \quad \mu = H, V, \quad (14)$$

where  $\Lambda_{HH}(\theta) = T_H \cos 2\theta$ ,  $\Lambda_{HV}(\theta) = T_H \sin 2\theta$ ,  $\Lambda_{VH}(\theta) = T_V \sin 2\theta$ , and  $\Lambda_{VV}(\theta) = -T_V \cos 2\theta$  are all real quantities. Using the facts that  $\hat{a}_{\mu\alpha}^\dagger |\text{vacuum}\rangle = |\mu_\alpha^j\rangle$  and  $\hat{a}_{\mu_0}^\dagger |\text{vacuum}\rangle = |\mu\rangle_0$ , we obtain from Eq. (14) the following transformation law for a ket:

$$|\mu_\alpha^j\rangle = e^{-i\phi_\alpha} \left[ \sum_{\lambda}^{H,V} \Lambda_{\mu\lambda}(\theta) |\lambda_\alpha^j\rangle + R_{\mu} |\mu\rangle_0 \right], \quad (15)$$

where  $\Lambda_{\mu\lambda}(\theta)$  is defined below Eq. (14). The final form of the density operator representing the two-photon quantum state is obtained by substituting Eq. (15) into Eq. (12). We denote this density operator by  $\hat{\rho}_{\alpha\beta}^{(f)}$  and provide its explicit form in Appendix C.

The density operator  $\hat{\rho}_{\alpha\beta}^{(f)}$  (Appendix C) can be used to determine the photon counting rate at the detector. Alternatively, one can also use the reduced density operator ( $\hat{\rho}_\beta$ ), which represents the state of photon  $\beta$  only. We use the latter approach in our analysis. We obtain the reduced density operator ( $\hat{\rho}_\beta$ ) by taking partial trace of  $\hat{\rho}_{\alpha\beta}^{(f)}$  over the subspace of photon  $\alpha$  and the loss modes. We find it to have the form

$$\begin{aligned} \hat{\rho}_\beta = & |b_1|^2 I_H |H_\beta^1\rangle\langle H_\beta^1| + |b_2|^2 I_H |H_\beta^2\rangle\langle H_\beta^2| \\ & + |b_1|^2 I_V |V_\beta^1\rangle\langle V_\beta^1| + |b_2|^2 I_V |V_\beta^2\rangle\langle V_\beta^2| \\ & + [b_1 b_2^* \cos 2\theta (I_H T_H \exp\{i(\phi_\alpha + \xi_{H^1}^2)\}) |H_\beta^1\rangle\langle H_\beta^2| \\ & - I_V T_V \exp\{i(\phi_\alpha + \xi_{V^1}^2)\}) |V_\beta^1\rangle\langle V_\beta^2| + \text{H.c.}] \\ & + [b_1 b_2^* \mathcal{S} \sqrt{I_H I_V} \sin 2\theta (T_V \exp\{i(\phi_\alpha + \xi_{H^1}^2)\}) |H_\beta^1\rangle\langle V_\beta^2| \\ & + T_H \exp\{i(\phi_\alpha + \xi_{V^1}^2)\}) |V_\beta^1\rangle\langle H_\beta^2| + \text{H.c.}] \end{aligned} \quad (16)$$

### C. Determining photon counting rates and visibility

We now show how to determine the single-photon counting rate. We recall that the propagation modes  $\beta_1$  and  $\beta_2$  are superposed by BS and one of the outputs of BS is sent through a device,  $\Gamma$ , which projects photon  $\beta$  onto a particular polarization state ( $H$ ,  $V$ ,  $D$ ,  $A$ ,  $R$ , or  $L$ ). Therefore, the positive-frequency part of the quantized electric field at the detector can be represented by

$$\hat{E}_{\mu\beta}^{(+)} = \hat{a}_{\mu\beta} + i e^{i\phi_\beta} \hat{a}_{\mu\beta}^2, \quad \mu = H, V, D, A, R, L, \quad (17)$$

where  $\hat{a}_{\mu\beta}^j$  is the annihilation operator corresponding to photon  $\beta$  with polarization  $\mu$  in beam  $\beta_j$ .

The single-photon counting rate (for a given polarization) at the detector can now be obtained by the standard formula

$$\mathcal{R}_\mu = \text{tr}\{\hat{\rho}_\beta \hat{E}_{\mu\beta}^{(-)} \hat{E}_{\mu\beta}^{(+)}\}, \quad (18)$$

where  $\hat{E}_{\mu\beta}^{(-)} = \{\hat{E}_{\mu\beta}^{(+)}\}^\dagger$ , and  $\hat{\rho}_\beta$  and  $\hat{E}_{\mu\beta}^{(+)}$  are given by Eqs. (16) and (17), respectively. We show below in Sec. IV D that the photon counting rates measured for various polarizations represent various interference patterns. The visibility of any such pattern is determined by the standard formula

$$\mathcal{V}_\mu = \frac{\mathcal{R}_\mu^{\max} - \mathcal{R}_\mu^{\min}}{\mathcal{R}_\mu^{\max} + \mathcal{R}_\mu^{\min}}, \quad (19)$$

where  $\mu$  represents the polarization of the detected photon ( $\beta$ ), and  $\mathcal{R}_\mu^{\max}$  and  $\mathcal{R}_\mu^{\min}$  are, respectively, the maximum and the minimum values of the single-photon counting rate.

In the next section (Sec. IV D), we show that the quantity  $\mathcal{S} \sqrt{I_H I_V}$ , which appears in the PPT criterion [Eq. (2)] and also in the formula of concurrence [Eq. (5)], can be determined from the visibility of the above-mentioned single-photon interference patterns.

### D. Signature of entanglement in single-photon interference patterns

We first consider the cases in which photon  $\beta$  is projected onto  $|H_\beta\rangle$  and  $|V_\beta\rangle$  polarization states. By the use of Eqs. (16)–(18), we find that the photon counting rates are given by (we have applied  $|b_1|^2 + |b_2|^2 = 1$ )

$$\mathcal{R}_H = I_H \{1 + 2|b_1||b_2| T_H \cos 2\theta \sin(\phi_{\text{in}} + \xi_{H^1}^2)\}, \quad (20a)$$

$$\mathcal{R}_V = I_V \{1 - 2|b_1||b_2| T_V \cos 2\theta \sin(\phi_{\text{in}} + \xi_{V^1}^2)\}, \quad (20b)$$

where  $\phi_{\text{in}} = \phi_\alpha - \phi_\beta + \arg\{b_1\} - \arg\{b_2\}$  is the interferometric phase that is modulated to obtain the interference patterns. It follows from Eqs. (19) and (20) that visibilities measured for  $H$  and  $V$  polarizations are

$$\mathcal{V}_H = 2|b_1||b_2| T_H \cos 2\theta, \quad (21a)$$

$$\mathcal{V}_V = 2|b_1||b_2| T_V \cos 2\theta. \quad (21b)$$

Clearly, when the half-wave plate is set at angle  $\theta = 0$ , the visibilities measured for these polarizations have their maximum values

$$\mathcal{V}_H|_{\theta=0} = 2|b_1||b_2| T_H, \quad \mathcal{V}_V|_{\theta=0} = 2|b_1||b_2| T_V. \quad (22)$$

We note that expressions of  $\mathcal{V}_H$  and  $\mathcal{V}_V$  do not contain  $I_H$ ,  $I_V$ , or  $\mathcal{S}$ . Therefore, measurement in this basis does not yield any information about entanglement. However,  $\mathcal{V}_H|_{\theta=0}$  and  $\mathcal{V}_V|_{\theta=0}$  provide us with a quantitative measure of the photon loss in the propagation mode  $\alpha_1$ . Therefore, it is crucial to measure them in an actual experiment.

We now consider the case when the polarization of the detected photon is diagonal ( $D$ ); i.e., photon  $\beta$  is projected onto state  $|D_\beta\rangle$  [42]. It follows from Eqs. (16)–(18) that the corresponding photon counting rate has the form

$$\begin{aligned} \mathcal{R}_D = & \frac{1}{2} [1 + 2|b_1||b_2| \cos 2\theta \{I_H T_H \sin(\phi_{\text{in}} + \xi_{H^1}^2) \\ & - I_V T_V \sin(\phi_{\text{in}} + \xi_{V^1}^2)\} \\ & + 2\mathcal{S} \sqrt{I_H I_V} |b_1||b_2| \sin 2\theta \{T_V \sin(\phi_{\text{in}} + \xi_{H^1}^2) \\ & + T_H \sin(\phi_{\text{in}} + \xi_{V^1}^2)\}], \end{aligned} \quad (23)$$

where we have applied the formulas  $|b_1|^2 + |b_2|^2 = 1$  and  $I_H + I_V = 1$ . We now set the half-wave-plate angle to be  $\pi/4$ ,

i.e.,  $\cos 2\theta = 0$  and  $\sin 2\theta = 1$ . Under this condition Eq. (23) reduces to [43]

$$\begin{aligned} \mathcal{R}_D|_{\theta=\frac{\pi}{4}} &= \frac{1}{2} [1 + 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 + 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})} \\ &\times \sin(\phi_{\text{in}} + \zeta)], \end{aligned} \quad (24)$$

where  $\phi_{\text{in}} = \phi_\alpha - \phi_\beta + \arg\{b_1\} - \arg\{b_2\}$  and  $\tan \zeta = T_H \sin(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2}) / [T_V + T_H \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})]$ ; the explicit form of  $\zeta$  is not required for our purpose.

Following the same procedure, we find for  $|A_\beta\rangle$  that the photon counting rate is given by [44]

$$\begin{aligned} \mathcal{R}_A|_{\theta=\frac{\pi}{4}} &= \frac{1}{2} [1 - 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 + 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})} \\ &\times \sin(\phi_{\text{in}} + \zeta)]. \end{aligned} \quad (25)$$

It follows from Eqs. (19), (24), and (25) that the single-photon interference patterns recorded for  $D$  and  $A$  polarizations have the same visibility:

$$\begin{aligned} \mathcal{V}_D|_{\theta=\frac{\pi}{4}} &= \mathcal{V}_A|_{\theta=\frac{\pi}{4}} = 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 + 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})}. \end{aligned} \quad (26)$$

We note that the concurrence ( $\mathcal{C}(\hat{\rho}) = 2\mathcal{S}\sqrt{I_H I_V}$ ) appears in the formulas of  $\mathcal{V}_D|_{\theta=\frac{\pi}{4}}$  and  $\mathcal{V}_A|_{\theta=\frac{\pi}{4}}$ . This fact implies that the single-photon interference patterns recorded for diagonal and antidiagonal polarizations contain information about the amount of entanglement in the two-photon mixed state.

Calculations for right-circular ( $R$ ) and left-circular ( $L$ ) polarizations are very similar to those for diagonal and antidiagonal polarizations. When the half-wave-plate angle is set such that  $\theta = \pi/4$ , the corresponding photon counting rates become

$$\begin{aligned} \mathcal{R}_R|_{\theta=\frac{\pi}{4}} &= \frac{1}{2} [1 - 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 - 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})} \\ &\times \sin(\phi_{\text{in}} - \zeta')], \end{aligned} \quad (27a)$$

$$\begin{aligned} \mathcal{R}_L|_{\theta=\frac{\pi}{4}} &= \frac{1}{2} [1 + 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 - 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})} \\ &\times \sin(\phi_{\text{in}} - \zeta')], \end{aligned} \quad (27b)$$

where  $\phi_{\text{in}} = \phi_\alpha - \phi_\beta + \arg\{b_1\} - \arg\{b_2\}$  and  $\tan \zeta' = T_H \sin(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2}) / [T_V - T_H \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})]$ ; the explicit form of  $\zeta'$  is not required for our purpose. Clearly, visibilities measured for  $R$  and  $L$  polarizations are given by

$$\begin{aligned} \mathcal{V}_R|_{\theta=\frac{\pi}{4}} &= \mathcal{V}_L|_{\theta=\frac{\pi}{4}} = 2\mathcal{S}\sqrt{I_H I_V} |b_1| |b_2| \\ &\times \sqrt{T_H^2 + T_V^2 - 2T_H T_V \cos(\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2})}. \end{aligned} \quad (28)$$

The presence of the concurrence ( $\mathcal{C}(\hat{\rho}) = 2\mathcal{S}\sqrt{I_H I_V}$ ) in Eqs. (27) and (28) shows that the single-photon interference patterns recorded for right-circular and left-circular polariza-

tions contain information about the amount of entanglement in the two-photon mixed state.

We note that the visibilities measured for diagonal, antidiagonal, right-circular, and left-circular polarizations are linearly proportional to the concurrence of the two-photon state ( $\hat{\rho}$ ).

### E. Test of the PPT criterion

We now show that if  $\mathcal{S}\sqrt{I_H I_V} \neq 0$ , visibilities measured for diagonal ( $D$ ) and right-circular ( $R$ ) polarizations can never be simultaneously zero when the HWP angle is set at  $\theta = \pi/4$ . It follows from Eqs. (26) and (28) that

$$\begin{aligned} (\mathcal{V}_D|_{\theta=\frac{\pi}{4}})^2 + (\mathcal{V}_R|_{\theta=\frac{\pi}{4}})^2 &= 8|b_1|^2 |b_2|^2 (T_H^2 + T_V^2) \\ &\times (\mathcal{S}\sqrt{I_H I_V})^2. \end{aligned} \quad (29)$$

Since  $|b_1|$ ,  $|b_2|$ ,  $T_H$ , and  $T_V$  must be nonzero quantities,  $\mathcal{V}_D|_{\theta=\frac{\pi}{4}}$  and  $\mathcal{V}_R|_{\theta=\frac{\pi}{4}}$  can be simultaneously equal to zero if and only if  $\mathcal{S}\sqrt{I_H I_V} = 0$ . According to the PPT criterion (Sec. III), the condition  $\mathcal{S}\sqrt{I_H I_V} = 0$  implies that the two-photon mixed state [Eq. (1)] is separable. Since  $\mathcal{V}_D|_{\theta=\frac{\pi}{4}} = \mathcal{V}_A|_{\theta=\frac{\pi}{4}}$  and  $\mathcal{V}_R|_{\theta=\frac{\pi}{4}} = \mathcal{V}_L|_{\theta=\frac{\pi}{4}}$ , it follows from the PPT criterion that when the HWP angle is set at  $\theta = \pi/4$ , a nonzero value of the visibility (of the single-photon interference patterns) obtained for any one of polarizations  $D$ ,  $A$ ,  $R$ , and  $L$  confirms that the two-photon mixed state,  $\hat{\rho}$  [Eq. (1)], is entangled. The state is separable (not entangled) if and only if visibilities measured for *all* of these polarizations are zero.

We illustrate the test of the PPT criterion by numerical examples in Sec. IV G.

### F. Determining the concurrence

It follows from the results of Sec. IV D that the concurrence,  $\mathcal{C}(\hat{\rho})$ , of the two-photon mixed state can be determined from the single-photon patterns. By the use of Eqs. (5), (22), and (29), we find that

$$\mathcal{C}(\hat{\rho}) = \sqrt{\frac{(\mathcal{V}_D|_{\theta=\frac{\pi}{4}})^2 + (\mathcal{V}_R|_{\theta=\frac{\pi}{4}})^2}{(\mathcal{V}_H|_{\theta=0})^2 + (\mathcal{V}_V|_{\theta=0})^2}}, \quad (30)$$

where  $D$  and  $R$  can be replaced by  $A$  and  $L$ , respectively.

It follows from Eq. (30) that in order to determine the concurrence, one needs to measure visibilities not only for  $|D_\beta\rangle$  (or  $|A_\beta\rangle$ ) and  $|R_\beta\rangle$  (or  $|L_\beta\rangle$ ) but also for  $|H_\beta\rangle$  and  $|V_\beta\rangle$ . However, we recall that although measurements corresponding to  $|D_\beta\rangle$ ,  $|A_\beta\rangle$ ,  $|R_\beta\rangle$ , and  $|L_\beta\rangle$  yield information about entanglement, measurements corresponding to  $|H_\beta\rangle$  and  $|V_\beta\rangle$  do not. Therefore, it is natural to ask why measuring visibility for  $|H_\beta\rangle$  and  $|V_\beta\rangle$  is necessary to determine the concurrence.

Actually, under the ideal conditions ( $|b_1| = |b_2| = 1/\sqrt{2}$  and  $T_H = T_V = 1$ ), it is not required to measure the visibility for  $|H_\beta\rangle$  and  $|V_\beta\rangle$ . It can be readily checked that in this case the denominator on the right-hand side of Eq. (30) is equal to 1. However, no experimental situation is perfectly ideal. In particular, it is extremely challenging to achieve the condition  $T_H = T_V = 1$  due to photon losses and imperfect alignment. Furthermore, emission probabilities at the

TABLE I. Two-photon mixed states used for illustration. The symbol “–” implies not applicable. Parameter  $\phi$  [see Eq. (1)] is not displayed because it plays no role in determining the amount of entanglement.

State	$I_H$	$I_V$	$\mathcal{I}$	Concurrence
$\hat{\rho}_1$	1	0	–	0
$\hat{\rho}_2$	0.5	0.5	0	0
$\hat{\rho}_3$	0.5	0.5	0.32	0.32
$\hat{\rho}_4$	0.5	0.5	0.5	0.5
$\hat{\rho}_5$	0.5	0.5	1	1

two sources ( $Q_1$  and  $Q_2$ ) may not be equal; i.e., the condition  $|b_1| = |b_2|$  may not always apply. The measurement of visibility when photon  $\beta$  is projected onto states  $|H_\beta\rangle$  and  $|V_\beta\rangle$  allows us to take care of these experimental imperfections. In fact, Eq. (30) shows that in spite of all such imperfections being present, one is able to determine the concurrence by the use of our method. This fact has been verified experimentally [38].

### G. Numerical illustration of results

We choose five density operators whose parameters are listed in Table I. We determine the values of concurrence of these states by the use of Eq. (5) and find that states  $\hat{\rho}_1$  and  $\hat{\rho}_2$  are separable, whereas state  $\hat{\rho}_5$  is maximally entangled (a Bell state). Note that the density operator  $\hat{\rho}_1$  represents a pure state and  $\hat{\rho}_2$  represents a fully mixed state. States  $\hat{\rho}_3$  and  $\hat{\rho}_4$  are neither maximally entangled nor separable.

For testing the PPT criterion with these states, we simulate an experimental situation in which experimental imperfections are present. In order to simulate the experimental imperfections, we assume that probabilities of emission at the two sources are not equal ( $|b_1| \neq |b_2|$ ) and that there are photon losses in beam  $\alpha_1$  due to imperfect alignment ( $T_H \neq 1$ ,  $T_V \neq 1$ ). The parameters are chosen as follows:  $|b_1|^2 = 0.55$ ,  $|b_2|^2 = 0.45$ ,  $T_H = 0.9$ ,  $T_V = 0.85$ , and  $\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2} = \pi/4$ . By the use of Eqs. (26) and (28), we compute the visibility of the single-photon patterns recorded for  $|D_\beta\rangle$ ,  $|A_\beta\rangle$ ,  $|R_\beta\rangle$ , and  $|L_\beta\rangle$ . The values of visibility are listed in Table II.

We find that for separable states ( $\hat{\rho}_1$  and  $\hat{\rho}_2$ ) the visibilities obtained for  $|D_\beta\rangle$ ,  $|A_\beta\rangle$ ,  $|R_\beta\rangle$ , and  $|L_\beta\rangle$  are all equal to zero.

TABLE II. Test of the PPT criterion for five different two-photon mixed states (Table I). Choice of parameters:  $|b_1|^2 = 0.55$ ,  $|b_2|^2 = 0.45$ ,  $T_H = 0.9$ ,  $T_V = 0.85$ , and  $\xi_{V_1}^{H^2} - \xi_{H_1}^{V^2} = \pi/4$ . For separable (not-entangled) states, visibilities measured for  $|D_\beta\rangle$ ,  $|A_\beta\rangle$ ,  $|R_\beta\rangle$ , and  $|L_\beta\rangle$  are all zero.

State	$\mathcal{V}_D _{\theta=\pi/4}$	$\mathcal{V}_A _{\theta=\pi/4}$	$\mathcal{V}_R _{\theta=\pi/4}$	$\mathcal{V}_L _{\theta=\pi/4}$	PPT criterion
$\hat{\rho}_1$	0	0	0	0	separable
$\hat{\rho}_2$	0	0	0	0	separable
$\hat{\rho}_3$	0.76	0.76	0.31	0.31	entangled
$\hat{\rho}_4$	0.40	0.40	0.17	0.17	entangled
$\hat{\rho}_5$	0.80	0.80	0.33	0.33	entangled

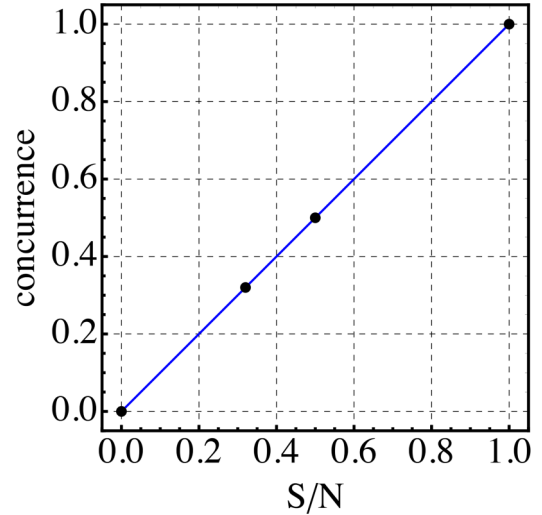


FIG. 3. Determining concurrence from single-photon visibility. Experimental imperfections are simulated by choosing  $|b_1|^2 = 0.55$ ,  $|b_2|^2 = 0.45$ ,  $T_H = 0.9$ , and  $T_V = 0.85$ . Simulated data points (solid circles) represent computed values of the concurrence and of  $S/N$  for five quantum states given by Table I. (Data points for  $\hat{\rho}_1$  and  $\hat{\rho}_2$  coincide.) All simulated data points lie on the straight line predicted by Eq. (30) showing that the concurrence is equal to  $S/N$  despite the presence of experimental imperfections.

Nonzero values of visibility for one of these polarizations confirm entanglement in the two-photon state.

We now illustrate how the concurrence of the two-photon mixed state can be determined from the single-photon interference patterns even when experimental imperfections are present. For simplicity of notation, we denote the quantity  $\sqrt{(\mathcal{V}_D|_{\theta=\pi/4})^2 + (\mathcal{V}_R|_{\theta=\pi/4})^2}$  by  $S$  and the quantity  $\sqrt{[(\mathcal{V}_H|_{\theta=0})^2 + (\mathcal{V}_V|_{\theta=0})^2]/2}$  by  $N$ . In this notation, the right-hand side of Eq. (30) becomes  $S/N$ .

We choose the same experimental parameters given above. For the five states given by Table I, we compute the values of the concurrence in two ways: (i) by the use of Eq. (5) and (ii) by determining the values of  $S$  and  $N$  from Eqs. (22), (26), and (28). In Fig. 3, we plot the obtained values of the concurrence against  $S/N$  and find that they lie exactly on the straight line predicted by Eq. (30). Since both  $S$  and  $N$  can be measured experimentally, Fig. 3 illustrates that the concurrence of a two-photon mixed state can be experimentally determined from single-photon interference patterns.

## V. SUMMARY AND CONCLUSIONS

Entanglement certification and measurement is an active and widely studied field of research which is directly connected to the fundamental topic of quantum state measurement and has immediate applications to quantum information science. It is a common perception in this field that in order to certify entanglement in a two-particle mixed state one must detect both particles. We have shown that it is possible to verify and measure entanglement in a two-particle *mixed state* without detecting one of the particles and without any

postselection. Our method has also been successfully tested experimentally [38].

To demonstrate our method, we have chosen a mixed state that can be obtained by generalizing two Bell states (see Sec. III). It is straightforward to show that our method also applies to the mixed state which can be obtained by generalizing the other two Bell states [45]. Therefore, our method currently covers all four Bell states and any two-dimensional mixed state that is obtained by generalizing them.

Our method is based on the concept of path identity [29,30]. This concept has recently drawn considerable attention after it has found important applications to imaging [32,33], spectroscopy [34], microwave superconducting cavities [46], polarization control [35], optical coherence tomography [47,48], the measurement of momentum correlation [49,50], the generation of entangled states [51–55], and fundamental tests of quantum mechanics [36,37]. Like many of these applications, our entanglement verification method requires detection of only one of the particles of a two-particle system. Therefore, our method will be practically useful to test entanglement of a two-particle mixed state when a detector for one of the particles is not available.

The theoretical analysis is based on quantum field theory, which shows that the method can, in principle, be applied to nonphotonic quantum states. Finally, our results open up a distinct avenue in verifying and measuring entanglement. They also inspire further questions. For example, one may now ask how to generalize the method so that it applies to many-particle high-dimensional entangled states.

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#### APPENDIX A: DENSITY OPERATOR REPRESENTED BY EQS. (8) AND (9)

Here, we derive the density operator given by Eqs. (8) and (9). In order to derive the density operator, we first recall two important facts: (1) the sources  $Q_1$  and  $Q_2$  emit in such a way that they jointly produce only one photon pair at a time, i.e., the density operator must represent a state that is occupied by only two photons, and (2) the sources cannot produce biphoton states of the form  $|H_\alpha, V_\beta\rangle$  and  $|V_\alpha, H_\beta\rangle$ . Therefore, the most general form that the density operator can take is

given by

$$\hat{\rho}'_{\alpha\beta} = \sum_{j,k}^{1,2} \sum_{\mu,\nu}^{H,V} b_j b_k^* \sqrt{I_\mu^j I_\nu^k} P_{\mu\nu}^{jk} \exp(i\xi_{\mu\nu}^{jk}) |\mu_\alpha^j, \mu_\beta^j\rangle \langle \nu_\alpha^k, \nu_\beta^k|, \quad (\text{A1})$$

where  $j = 1, 2$  and  $k = 1, 2$  represent the sources;  $\mu = H, V$  and  $\nu = H, V$ ;  $|b_j|^2$  is the probability of the photon pair being emitted by source  $Q_j$ ;  $|b_1|^2 + |b_2|^2 = 1$ ;  $I_\mu^j$  is the probability with which source  $Q_j$  emits the photon pair  $|\mu_\alpha, \mu_\beta\rangle$ ;  $P_{\mu\nu}^{jk}$  is a non-negative real quantity; and  $\xi_{\mu\nu}^{jk}$  is a phase (real quantity). Since  $Q_1$  and  $Q_2$  are identical sources,  $I_\mu^j$  does not depend on  $j$ ; we therefore drop this superscript and obtain the form [Eq. (8) in main text]

$$\hat{\rho}'_{\alpha\beta} = \sum_{j,k}^{1,2} \sum_{\mu,\nu}^{H,V} b_j b_k^* \sqrt{I_\mu I_\nu} P_{\mu\nu}^{jk} \exp(i\xi_{\mu\nu}^{jk}) |\mu_\alpha^j, \mu_\beta^j\rangle \langle \nu_\alpha^k, \nu_\beta^k|. \quad (\text{A2})$$

One can readily check that this density operator has unit trace. Below we derive the conditions that the coefficients associated with the density operator must obey.

If  $Q_2$  does not emit (i.e.,  $|b_2| = 0$  and  $|b_1| = 1$ ), the density operator,  $\hat{\rho}'_{\alpha\beta}$ , must reduce to the state of light generated by  $Q_1$  alone [i.e., state  $\hat{\rho}_1$  given by Eq. (6)]. Likewise, if  $Q_1$  does not emit (i.e.,  $|b_1| = 0$  and  $|b_2| = 1$ ), the density operator,  $\hat{\rho}'_{\alpha\beta}$ , must reduce to the state of light generated by  $Q_2$  alone [i.e., state  $\hat{\rho}_2$  given by Eq. (6)]. Using these two facts, we immediately obtain

$$\xi_{\mu\nu}^{vj} = \phi_{\mu\nu} \quad \text{and} \quad P_{\mu\nu}^{vj} = \mathcal{I}_{\mu\nu}, \quad j = 1, 2. \quad (\text{A3})$$

Furthermore, the density operator must be Hermitian. We therefore have

$$\xi_{\mu\nu}^{vk} = -\xi_{\nu\mu}^{\mu k} \quad \text{and} \quad P_{\mu\nu}^{vk} = P_{\nu\mu}^{\mu k}, \quad \forall j, k, \mu, \nu. \quad (\text{A4})$$

This completes the derivation of the density operator given by Eqs. (8) and (9).

#### APPENDIX B: DERIVATION OF EQ. (11)

In this Appendix, we derive Eq. (11) by the use of Eqs. (8)–(10). Equations (8) and (9) have already been redisplayed in Appendix A. For the convenience of readers, we display Eq. (10) once again below:

$$P_{\mu^1}^{\mu^2} = P_{\mu^2}^{\mu^1} = 1 \quad \text{for} \quad \mu = H, V. \quad (\text{B1})$$

We note that the matrix elements of the density operator [Eq. (8)] contain four kinds of parameters:  $b_j$ ,  $I_\mu$ ,  $P_{\mu\nu}^{jk}$ , and  $\xi_{\mu\nu}^{jk}$ . Each of them has a distinct physical meaning, and their values do *not* depend on each other. For example, the value of  $P_{\mu\nu}^{jk}$  does not change if the values of  $b_j$ ,  $I_\mu$ , and  $\xi_{\mu\nu}^{jk}$  are changed. This fact allows for a very simple derivation of Eq. (11). We set  $b_1 = b_2 = 1/\sqrt{2}$ ,  $I_H = I_V = 1/2$ , and  $\xi_{\mu\nu}^{jk} = 0$  and represent the density operator in the following matrix form by

combining Eqs. (8)–(10):

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{\mathcal{J}}{4} & \frac{1}{4} & 0 & 0 & \frac{P_{H^1}^{V^2}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mathcal{J}}{4} & 0 & 0 & \frac{1}{4} & \frac{P_{V^1}^{H^2}}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{P_{V^1}^{H^2}}{4} & \frac{1}{4} & 0 & 0 & \frac{\mathcal{J}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{P_{H^1}^{V^2}}{4} & 0 & 0 & \frac{1}{4} & \frac{\mathcal{J}}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \quad (\text{B2})$$

Since a density matrix must be positive semidefinite, all principal minors of the matrix given by Eq. (B2) must be

non-negative [56]. It now immediately follows that

$$P_{H^1}^{V^2} = P_{V^1}^{H^2} = \mathcal{J}. \quad (\text{B3})$$

These relationships given by Eqs. (10), (9b), and (B3) are jointly represented by Eq. (11):

$$P_{\mu^j}^{V^k} = \mathcal{J}_{\mu\nu}, \quad \forall j, k, \mu, \nu. \quad (\text{B4})$$

### APPENDIX C: EXPLICIT FORM OF THE FINAL DENSITY OPERATOR

Here we provide the explicit form of the density operator,  $\hat{\rho}_{\alpha\beta}^{(f)}$ , representing the photon pair in our system. We substitute Eq. (15) into Eq. (12) and find that

$$\begin{aligned} \hat{\rho}_{\alpha\beta}^{(f)} = & \sum_{\mu,\nu}^{H,V} |b_1|^2 \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp(i\phi_{\mu\nu}) |\mu_\alpha^1, \mu_\beta^1\rangle \langle \nu_\alpha^1, \nu_\beta^1| + \sum_{\mu,\nu}^{H,V} b_1 b_2^* \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp[i(\xi_{\mu^1}^{V^2} + \phi_\alpha)] R_\nu |\mu_\alpha^1, \mu_\beta^1\rangle \langle \nu_0, \nu_\beta^2| \\ & + \sum_{\mu,\nu}^{H,V} b_1^* b_2 \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp[i(\xi_{\mu^2}^{V^1} - \phi_\alpha)] R_\mu |\mu_0, \mu_\beta^2\rangle \langle \nu_\alpha^1, \nu_\beta^1| \\ & + \sum_{\mu,\nu}^{H,V} |b_2|^2 \sqrt{I_\mu I_\nu} \mathcal{J}_{\mu\nu} \exp(i\phi_{\mu\nu}) \left[ R_\mu R_\nu |\mu_0, \mu_\beta^2\rangle \langle \nu_0, \nu_\beta^2| + R_\nu \sum_{\lambda}^{H,V} \Lambda_{\mu\lambda}(\theta) |\lambda_\alpha^1, \mu_\beta^2\rangle \langle \nu_0, \nu_\beta^2| \right. \\ & \left. + R_\mu \sum_{\lambda}^{H,V} \Lambda_{\nu\lambda}(\theta) |\mu_0, \mu_\beta^2\rangle \langle \lambda_\alpha^1, \nu_\beta^2| + \sum_{\lambda}^{H,V} \sum_{\epsilon}^{H,V} \Lambda_{\mu\lambda}(\theta) \Lambda_{\nu\epsilon}(\theta) |\lambda_\alpha^1, \mu_\beta^2\rangle \langle \epsilon_\alpha^1, \nu_\beta^2| \right]. \quad (\text{C1}) \end{aligned}$$

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- [41] It is important to note that our entanglement verification scheme can also be applied to states  $\hat{\rho} = I_1|H_\alpha V_\beta\rangle\langle H_\alpha V_\beta| + I_2|V_\alpha H_\beta\rangle\langle V_\alpha H_\beta| + (\mathcal{S}\sqrt{I_1 I_2}e^{-i\phi}|H_\alpha V_\beta\rangle\langle V_\alpha H_\beta| + \text{H.c.})$ .
- [42] Note that  $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ .
- [43] We have applied the trigonometric identity  $p \sin x + q \sin(x + \delta) = \sqrt{p^2 + q^2 + 2pq \cos \delta} \sin(x + \zeta'')$ , where  $\tan \zeta'' = q \sin \delta / (p + q \cos \delta)$ .
- [44] Note that  $|A\rangle = (|V\rangle - |H\rangle)/\sqrt{2}$ .
- [45] This mixed state has the form  $\hat{\rho} = I_1|H_\alpha, V_\beta\rangle\langle H_\alpha, V_\beta| + I_2|V_\alpha, H_\beta\rangle\langle V_\alpha, H_\beta| + (e^{-i\phi}\mathcal{S}\sqrt{I_1 I_2}|H_\alpha, V_\beta\rangle\langle V_\alpha, H_\beta| + \text{H.c.})$ . It is obtained by generalizing the two Bell states  $|\Psi^+\rangle = (|H_\alpha, V_\beta\rangle + |V_\alpha, H_\beta\rangle)/\sqrt{2}$  and  $|\Psi^-\rangle = (|H_\alpha, V_\beta\rangle - |V_\alpha, H_\beta\rangle)/\sqrt{2}$ .
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