Time-domain calculation of forerunners in Drude dispersive media without collisions

Ivan V. Pavlenko, Igor O. Girka,* Oleksandr V. Trush, and Sofiia V. Hnatiuk
Department of Physics and Technology, V. N. Karazin Kharkiv National University, Svobody Sq.4, 61022, Kharkiv, Ukraine

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The forerunners of the step-modulated sine pulse propagating in Drude dispersive media are calculated in the time domain. Two such problems are solved analytically. In the first problem, the plane source of the electric field is inside the medium. And the second problem considers the normal incidence of a linearly polarized wave from a vacuum upon a half-space medium. The obtained forerunners are identified as Sommerfeld precursors but the frequency of the oscillations and their amplitudes are closer to the exact solutions than the predictions of canonical Sommerfeld precursor. Moreover, the obtained periodicity of the forerunners is in perfect agreement with the exact solution. The developed analytics is an important contribution to the time-domain theory of the forerunners. The obtained data will be helpful for an experimental study of the forerunners and for the comparison with the forerunners in other dispersive media or with other waveforms of the incident pulse.

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I. INTRODUCTION

The recent centenary of publishing the Sommerfeld-Brillouin theory of the pulse dynamics in the dispersive media [1–3] revived an interest in this canonical problem. The forerunners as the particular results of this and other theories of the pulse propagation in the dispersive media are still interesting for plasmonic waveguides [4], ground-penetrating radars [5], body-based applications [6], photonic crystals [7] and underwater communications [8]. There are the recommendations for optimal pulse penetrations in remote sensing and medical imaging using the Sommerfeld and Brillouin forerunners [9]. The present day experiments allow us even to measure the speed of the forerunners [10].

Recent reviews of the theory development during the century [11,12] classified and summarized the main obtained results and gave an idea that some important aspects of the pulse dynamics can be described by the modern analytics. The present paper proposes an insight into the pulse dynamics in the time domain, where some results can be obtained more precisely and without complicated integration, which is usually the basis of the frequency-domain consideration.

In his original paper [1,3], Sommerfeld studied the velocity of wave propagation in dielectrics with a Lorentz model of the resonance polarization. He concluded that the signal cannot propagate with a velocity larger than the velocity of light and that the signal front propagates with the velocity of light. Simultaneously, Brillouin described in detail the signal evolution in the Lorentz medium [2,3]. He discussed the calculations of the signal velocity and applied the newly developed saddle-point method of integration. A very weak signal which appears at first at a certain depth in the medium was called a forerunner and was also described by the theory. One can find an extensive review of these fundamental works in Ref. [11]. The extension of the theory due to Haskell and Case [13] for a Drude model of dielectrics (the particular case of the Lorentz model if the resonance frequency of the polarization is equal to zero) was also reviewed in Ref. [11]. This extension is very important for the present study where the same problem is considered but in the time domain.

The second part of the review by the same authors [12] presented the modern asymptotic description of the Sommerfeld-Brillouin theory. The latter is based on the theory of the uniform asymptotic expansions of integrals developed by Oughstun and Sherman [14–16] for the propagated field. The analysis of the saddle-point locations is crucial for the expansions. The review [12] concluded with the problem of the signal transmission into a dispersive half space which, in fact, differs from the problem stated by Sommerfeld where the signal source is inside the medium. The present paper studies both of these problems: the signal source is inside and outside the medium.

Usually, the frequency-domain theory of the forerunners distinguishes the first forerunner (Sommerfeld forerunner or Sommerfeld precursor) and the second forerunner (Brillouin forerunner or Brillouin precursor), which correspond to the high-frequency and low-frequency contributions from the frequency spectrum of the incident pulse, respectively. The simple asymptotic forms for Sommerfeld and Brillouin precursors were obtained in Ref. [17]. The modulated light pulses propagating in a dense Lorentz medium were considered there. The medium was opaque over a broad spectral region including the signal carrier frequency. The different time modulations of the pulse amplitude including the canonical step modulation of the sine signal were studied. The results were obtained by the standard Laplace-Fourier procedures, which were simpler then the uniform saddle-point methods reported, for example, in Ref. [18]. But at short propagation distances these two forerunners overlapped, and the overlapping caused the optical precursor [19].

A more general definition of the precursors than that given by Sommerfeld and Brillouin [1–3] was introduced in Ref. [20]. The forerunner obtained in the time-domain,

*igorgirka@karazin.ua

of course, cannot be classified in a similar way as in the frequency-domain. In the time-domain, there is only a single forerunner as the early time behavior of the signal at a certain coordinate. However, the forerunner can be identified according to the frequency-domain terminology *a posteriori*.

The optical precursors and their interference with the main signal for the small optical depths were studied both experimentally and numerically in Ref. [21]. The authors performed a simple modification of the asymptotic precursor theory to distinguish between the Sommerfeld and the Brillouin precursors. A time-domain analysis of the pulse propagation under similar conditions cannot separate these two precursors from each other (just as the separation cannot be seen from the experimental data).

The hybrid-asymptotic method was applied in Ref. [22] to identify the main signal and the precursors in the case of the step-modulated pulse transmission through a medium with the electromagnetically induced transparency (the EIT media). The authors proposed a robust control of transmissions or delays between the precursors and the main signal. The last one is important in biomedical imaging and optical communication.

The difficulties of the experimental observation of the precursors was discussed in Ref. [23]. As a result, the special media (slow-light systems) with a natural transparency window or a EIT window were proposed for the experiments where the pulse carrier frequency should coincide with the center of the window.

Since the Lorentz model of dielectrics includes a resonance frequency, there is the contribution from the resonance part of the frequency spectrum to the precursors. For some conditions this contribution is important and even dominant. In the case of narrow material resonance, sufficiently small medium plasma frequency, and the pulse carrier frequency being nearly equal to the medium resonance frequency, the precursors were calculated in Ref. [24]. The singular dispersion limit of the precursor propagation was obtained in Ref. [25]. In the study, the damping approached zero and the medium dispersion was concentrated near the resonance frequency. Such effects of the resonance frequency on the precursor evolution cannot appear in our study of the nonresonant media, therefore they are not reviewed in more detail.

The modal analysis of the wave propagation in the dispersive media was reported recently in Ref. [26] where the signal was incident on a dispersive slab. In that case the closed-form expression for the transmitted field was derived through the modal expansion. The time dependence of the transmitted signal was written as the discrete sum over the frequency modes, which can be directly translated to the integral over the frequency domain. Thus, such a modal analysis cannot be considered as a time-domain solution. Also, the authors reported the significantly improved description of the precursor amplitude and oscillation period. We agree that the obtained formula are more accurate than presented by Sommerfeld. However, a more precise formula for the Sommerfeld precursor can be obtained in the time domain.

A first description of the forerunners in the time domain was done in Ref. [27] by the propagator technique. The medium dispersion in this technique was modeled by the time-domain electric susceptibility. But usually the electric susceptibility is not known in the time domain and its evaluation from the frequency-domain is a separate difficult problem. The authors tried to stay in the time domain as much as possible but the time-domain kernels used refer again to the frequency domain.

The effects of the incident signal waveform (including the rise-time effects) on the characteristics of the forerunners are not studied in the present paper. Therefore, a large number of related works is not reviewed here. Also, this introduction cannot cover the many other aspects of the precursor studies, but they can be found, for example, in a new edition of the book by Oughstun [18] and in the reviews [11,12].

Most of the studies discussed above (except the EIT media) deal with the Lorentz dispersive media having one or a few resonance frequencies. But some dispersive media have no resonance frequencies at all. They are, for example, the metals or the isotropic plasmas. Dispersion in such media is usually referred to as the Drude dispersive model due to Drude's theory of metal conductivity [28,29]. Mathematically, the Drude model is obtained from the Lorentz model with the absence of a resonance frequency.

The temporal and frequency evolution of Sommerfeld and Brillouin forerunners in the metals was studied in Ref. [30]. The authors concluded that the precursor fields are important for studying the propagating pulse dynamics and cannot be neglected in both the microwave and optical bands.

The problem of the step-modulated sine pulse propagation in the Drude dispersive media was solved by Haskell and Case [13] in the frequency domain by using the generalized saddle-point integration everywhere except at the signal wavefront. The solution for the wavefront was obtained by a high-frequency expansion technique. The results reproduced completely the Sommerfeld precursor from Refs. [1–3]. The perfect summary of the results obtained by Haskell and Case can be found in Ref. [11]. The exact solution of the same problem but in the time domain was reported recently in Ref. [31]. The last one is the ground for the forerunner study in the present paper.

The method of the asymptotic expansion was applied in Ref. [32] to study the step-modulated sine pulse propagation in a Drude dispersive media. As usual, in the framework of this method, the general solution of the problem was the sum of three different contributions: the Sommerfeld precursor, the Brillouin precursor, and the signal contribution. Each was obtained from different parts of the frequency spectrum of the incident pulse. The analysis in Ref. [32] was carried out for the case when the carrier frequency of the incident pulse was below the plasma frequency of the medium (an opaque medium). This fact does not allow us to compare directly the results from Ref. [32] with the results of the present paper where the medium is transparent (opposite inequality between the frequencies is valid).

The objective of the present paper is to calculate the forerunners of the step-modulated sine pulse propagation in the Drude dispersive media in the framework of a time-domain consideration. Usually [1–3], the forerunner in the Lorentz dispersive media is obtained as the high-frequency limit of the integration and does not depend on the resonance frequency. Therefore we believe that the results obtained in the present paper will describe more precisely the forerunners of the Lorentz dispersive media as well.

The forerunner of the pulse caused by a plane source of the electric field inside the medium is calculated in Sec. II. The forerunners of the linearly polarized wave which is normally incident upon the vacuum-medium interface are calculated in Sec III. The results of the study are discussed and summarized in Sec. IV.

II. ELECTRIC-FIELD SOURCE INSIDE THE MEDIUM

The canonical Sommerfeld problem [1–3] of signal propagation in a dispersive medium is studied in this section for a medium with the Drude model of dielectric dispersion. The sinusoidal plane source of the electric field with a carrier frequency ω_c is placed in an infinite Drude dispersive medium. Let the source plane be the surface $x = 0$ and the source is switched on at time $t = 0$. Thus the problem becomes one-dimensional along coordinate x. Time dependence of the external source electric field is defined as

$$E(0,t) = \sin(\omega_c t)H(t), \tag{1}$$

where $H(t)$ denotes the Heaviside unit step function. The electric-field amplitude is normalized to be unity. In fact, this problem is different from the problem of normal wave incidence from vacuum into the dispersive medium that Sommerfeld intended to solve in Refs. [1,3]. Therefore, the last one will be additionally considered in Sec. III.

The Drude model of the dielectric dispersion is based on the electron motion equation [28,29], which in the case of the negligible collisions can be written as

$$\frac{\partial j(x,t)}{\partial t} = \frac{\omega_p^2}{4\pi}E(x,t), \tag{2}$$

where $j(x,t)$ is the electron current density, $\omega_p^2 \equiv \frac{4\pi n_e e^2}{m_e}$ is the square of the plasma frequency, n_e is the number density of electrons, e is the electron charge, and m_e is the electron mass. Since here the problem is completely considered in the time domain, the conductivity or permittivity derived by Drude in the frequency domain [28,29] are not used as the electric properties of the medium. Instead of this, the motion Eq. (2) is used to derive the time-domain equation for the electric-field evolution. But the relationship between the signal carrier frequency ω_c and the plasma frequency ω_p is used in the form which coincides with the usual representation of the permittivity

$$\epsilon \equiv 1 - \frac{\omega_p^2}{\omega_c^2}. \tag{3}$$

In other words, the developed analytics does not operate with any other frequency (as a variable of the frequency domain) except ω_p and ω_c.

The changes in the medium under the action of the propagating signal is defined by the motion equation (2), but the initial state of the medium is also important for the analytical study. According to Sommerfeld [1,3], initially (before the wavefront arrival at a certain coordinate) the electrons of the medium are at rest and do not oscillate (the thermal motion does not affect the signal propagation in the approximation considered). This allows us to write down the initial condition for the electron current density as $j(x,0) = 0$. Since initially both electric and magnetic fields also do not exist in the medium, the problem of the mathematical physics for the electric field in the medium consists of the Klein-Gordon equation

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} + \frac{\omega_p^2}{c^2}E(x,t), \tag{4}$$

where c is the velocity of light in vacuum. The initial conditions are

$$E(x,0) = 0, \tag{5}$$

$$\left.\frac{\partial E}{\partial t}\right|_{t=0} = 0, \tag{6}$$

and the boundary condition is

$$E(0,t) = \sin(\omega_c t). \tag{7}$$

The set of Eqs. (4)–(7) is Sommerfeld's problem of wave propagation through a Drude dispersive medium (which is a particular case of the Lorentz medium) in the time domain.

The problem (4)–(7) has been solved in Ref. [31]. The wave magnetic field from Ref. [31] is the solution of the same problem in mathematical physics. Therefore,

$$E(x,t) = k\int_x^{ct} \cos[k\sqrt{\epsilon}(g-x)]J_0\left(\frac{\omega_p}{c}\sqrt{c^2t^2-g^2}\right)dg, \tag{8}$$

where $k = \omega_c/c$ is the vacuum wave vector and $J_n(x)$ denotes the Bessel function of the first kind of integer order n. The objective of the present study is to get the early time behavior of the electric field at a certain coordinate x from Eq. (8). Since the lower limit of the integral in Eq. (8) should be close to the upper limit, the following variable replacement is carried out: $z \equiv (1 - \frac{g^2}{c^2t^2})^{1/2}$. As a result, Eq. (8) becomes

$$E(x,t) = \omega_c t\int_0^{\delta} \frac{z}{\sqrt{1-z^2}} \times \cos[\sqrt{\epsilon}(\omega_c t\sqrt{1-z^2} - kx)]J_0(\omega_p t z)dz, \tag{9}$$

where $\delta = (1 - \frac{x^2}{c^2t^2})^{1/2}$.

Now the small parameter of the consideration can be introduced as

$$\delta^2 = 1 - \frac{x^2}{c^2t^2} \ll 1. \tag{10}$$

Then the square roots in the integrand can be expanded as $(1-z^2)^{1/2} \approx 1 - z^2/2$. The approximate expression for the early time behavior of the electric field becomes

$$E^F(x,t) = \omega_c t\int_0^{\delta} z\left(1+\frac{z^2}{2}\right) \times \cos\left\{\sqrt{\epsilon}\left[\omega_c t\left(1-\frac{z^2}{2}\right) - kx\right]\right\}J_0(\omega_p t z)dz. \tag{11}$$

The cosine of the integrand of Eq. (11) is the rapidly oscillating function, therefore the expansion of its argument can lead to a wrong result. The validity of the expansion is verified *a posteriori*. Equation (11) can be integrated by using the table of integrals 1.8.2.8 [33]:

$$E^F(x,t) = \frac{1}{\sqrt{\epsilon}}U_1(x,t) + \frac{1}{\epsilon\omega_c t}U_2(x,t) - \frac{1-\epsilon}{2\epsilon^{3/2}}U_3(x,t), \tag{12}$$

where

$$U_n(x,t) \equiv \sum_{m=0}^{\infty}(-1)^m\left(\frac{\sqrt{\epsilon}}{\sqrt{1-\epsilon}}\delta\right)^{2m+n} J_{2m+n}(\omega_p t\delta). \tag{13}$$

Usually, for the physically interesting cases, $\omega_c t \gg 1$, therefore the second term in Eq. (12) can be neglected. The sum of Eq. (13) can be truncated if

$$\frac{\epsilon}{1-\epsilon}\delta^2 \ll 1. \tag{14}$$

In this approximation, the forerunner becomes

$$E^F(x,t) = \frac{\delta}{\sqrt{1-\epsilon}}J_1(\omega_p t\delta) - \frac{1+\epsilon}{2(1-\epsilon)^{3/2}}\delta^3 J_3(\omega_p t\delta). \tag{15}$$

The first term of Eq. (15) is known as Sommerfeld forerunner or Sommerfeld precursor. It can be rewritten in the variables of Sommerfeld's problem [1–3] $\xi = \frac{\omega_p^2}{2c}x$ and $\tau = t - \frac{x}{c}$ (the retarded time) to make possible the direct comparison:

$$E^S(x,t) = \omega_c\sqrt{\frac{\tau}{\xi}}\frac{\sqrt{\tau\omega_p^2/\xi + 4}}{\tau\omega_p^2/\xi + 2}J_1\left(\sqrt{4\tau\xi + (\omega_p\tau)^2}\right). \tag{16}$$

In the limit $\tau\omega_p^2/\xi \longrightarrow 0$, Eq. (16) coincides with the canonical Sommerfeld forerunner [1–3]. The expression (16) has lower amplitude of the oscillations and higher frequency of the oscillations than the canonical expression for the Sommerfeld precursor. But both the amplitude and frequency of the whole forerunner (15) are higher (that will be seen from the analysis below).

But, in fact, the forerunner can be written from Eq. (12) without the sum truncation if $\frac{\epsilon}{1-\epsilon}\delta^2 < 1$. Since the inequality $\omega_p t\delta \gg 1$ is fulfilled almost always (except a few initial periods), the function $U_n(x,t)$ can be approximated as

$$U_n(x,t) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\omega_c t}}\frac{\epsilon^{n/2}\delta^{(2n-1)/2}}{(1-\epsilon)^{(2n-3)/4}[1-\epsilon(1+\delta^2)]} \times \cos\left(\omega_p t\delta - \frac{\pi}{2}n - \frac{\pi}{4}\right). \tag{17}$$

Then the forerunner becomes

$$E^F(x,t) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\delta}{\omega_c t}}\frac{(1-\epsilon)^{1/4}(1+\delta^2/2)}{1-\epsilon(1+\delta^2)} \times \sin(\omega_p t\delta - \pi/4). \tag{18}$$

Equation (18) becomes even simpler in the approximation (14):

$$E^F(x,t) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\delta}{\omega_c t}}\frac{1}{(1-\epsilon)^{3/4}}\left(1 + \frac{1+\epsilon}{2(1-\epsilon)}\delta^2\right) \times \sin(\omega_p t\delta - \pi/4). \tag{19}$$

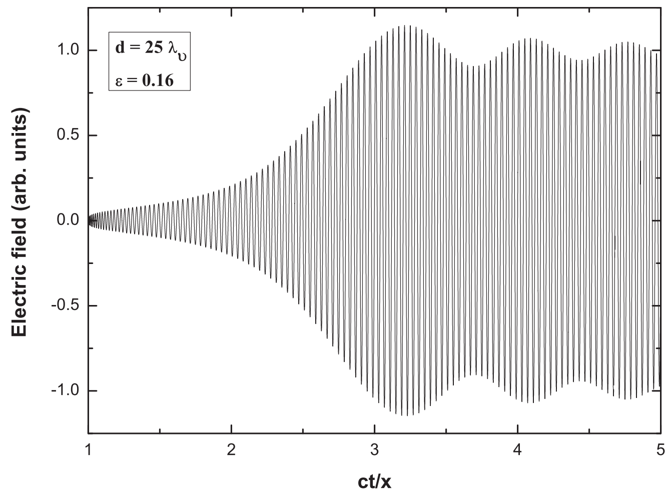

FIG. 1. The temporal behavior of the transmitted step-modulated sine pulse at the distance of 25 vacuum wavelengths from the electric-field source in a medium with $\epsilon = 0.16$. The signal is normalized by the amplitude of the source electric field. The exact solution of the problem from Eq. (8).

Equations (18) or (19) allow us to calculate the amplitude of the forerunner. An instantaneous frequency of the forerunner is $\omega = \frac{d}{dt}(\omega_p t\delta) = \frac{\omega_p}{\delta}$. The instantaneous frequency in the variables of the Sommerfeld's problem [1–3] is $\omega = \sqrt{\frac{\xi}{\tau}}\frac{\tau\omega_p^2/\xi+2}{(\tau\omega_p^2/\xi+4)^{1/2}}$. It means that the canonical Sommerfeld forerunner underestimates the frequency of the oscillations.

The time dynamics of the step-modulated sine pulse at the distance of 25 vacuum wavelengths $\lambda_v \equiv 2\pi c/\omega_c$ is shown in Fig. 1 for a medium with $\epsilon = 0.16$. The same distance is equal to 10 wavelengths in the medium $\lambda_m \equiv 2\pi c/\sqrt{\epsilon}\omega_c$ in this case. The dependence is built as the exact solution of the problem from Eq. (8) and normalized by the source electric-field amplitude. The forerunner (as early time behavior of the electric field) can be seen at small coordinates. The main signal buildup is seen at large coordinates and has been discussed in Ref. [31]. The objective of this study is the early time behavior at the interval $ct/x \in [0; 1.2]$ only, because the applicability of the used approximation (10) is well grounded in this range.

Figure 2 is the increased part of Fig. 1 with some additional curves for comparison. All the data are normalized by the amplitude of the source electric field. Figure 2 shows the forerunner from the exact solution (8) (solid line), Sommerfeld forerunner from Refs. [1,3] (dashed line), the approximate forerunner as the first term of Eq. (15) (dash-dotted line), and the approximate forerunner as both terms of Eq. (15) (dotted line). The envelope of the forerunner oscillations is built as the amplitude of the oscillations from Eq. (19) (thin solid line).

The obtained result (15) for the forerunner is very close to the exact solution of the problem if the both terms are taken into account. The instantaneous frequency of the obtained forerunner is in perfect agreement with the instantaneous frequency of the exact solution and reproduces the result from Refs. [1,3] which is valid in the limit $x \longrightarrow \infty$. In general, the Sommerfeld precursor [1,3] underestimates both the instantaneous frequency and the amplitude of the forerunner.

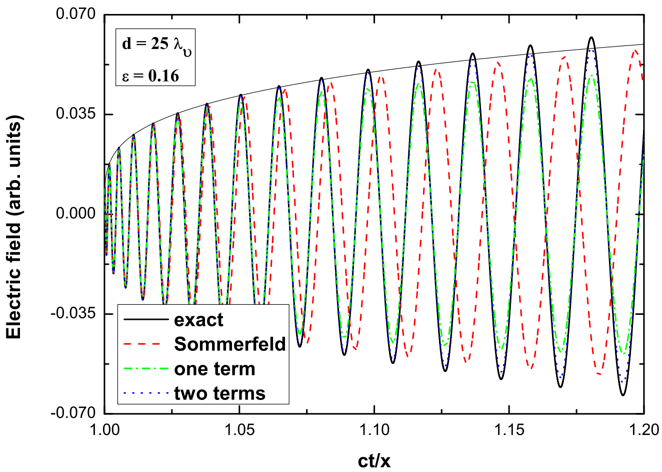

FIG. 2. The forerunner of the step-modulated sine pulse at the distance of 25 vacuum wavelengths from the electric-field source in a medium with $\epsilon = 0.16$: the exact solution of the problem from Eq. (8) (solid line), Sommerfeld precursor from Refs. [1,3] (dashed line), the first term of Eq. (15) (dash-dotted line), and both terms of Eq. (15) (dotted line). The data are normalized by the amplitude of the source electric field. The envelope of the forerunner is shown as the amplitude of the oscillations from Eq. (19) (thin solid line).

The forerunner at the distance of 250 vacuum wavelengths (100 wavelengths in the medium) from the electric-field source is shown in Fig. 3. The lines are built from the same expressions as explained for Fig. 2. The forerunner is the rapidly oscillating function at such a scale. Therefore, the inset in Fig. 3 zooms in on the range of a few oscillation periods to see the details of the dependencies. The dependencies of Fig. 3 confirm the conclusions from Fig. 1.

III. NORMAL INCIDENCE OF THE WAVE UPON THE VACUUM-MEDIUM INTERFACE

The forerunners in a Drude dispersive half space are also studied in the case, when the linearly polarized,

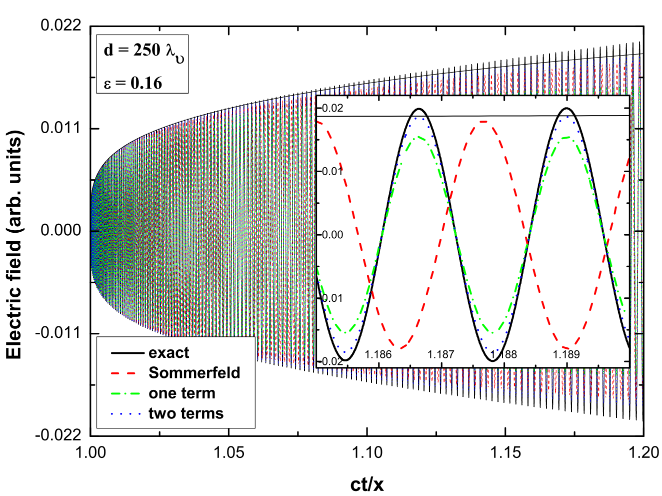

FIG. 3. Same as Fig. 2 at the distance of 250 vacuum wavelengths. The inset in the figure zooms in on a few periods of the oscillations to see the details.

step-modulated sine wave is normally incident from a vacuum half space upon the vacuum-medium interface. Vacuum occupies the half space $x < 0$, and the half space $x > 0$ is the Drude dispersive medium. The electric field of the incident wave is oriented along the y axis and its magnetic field is oriented along the z axis. The incident wave is defined in the vacuum half space by the waveform

$$E_y(x,t), B_z(x,t) = \sin(\omega_c t - kx)H(t)H(-x). \tag{20}$$

The amplitudes of both electric and magnetic fields are normalized to unity. The relationship between the carrier frequency of the incident wave ω_c and the plasma frequency of the medium ω_p is defined, as previously, by ϵ from Eq. (3).

Initially, at time $t = 0$, the electrons of the medium are at rest and do not oscillate. Therefore the wavefront propagates in the medium with the velocity of light. But the electrons of the medium behind the wavefront start to oscillate in the wave electric field. Generated polarization current in the medium changes the electrical permittivity of the medium which was initially equal to the vacuum permittivity and affects the electromagnetic field of propagating wave. As a consequence, the incident wave is partially reflected from the medium interface and partially transmitted into the medium. This phenomenon is the main difference from the problem considered in Sec. II where the electric field is defined by the plane source in the medium (there was no reflection).

The electron current density and the electric and magnetic fields in the medium are described by the Klein-Gordon equation (4) with zero initial conditions (initially, there is no electron motion and the electromagnetic field in the medium). The above-stated problem was solved in Ref. [34]. The objective of this section is to get both electric and magnetic forerunners in the medium from the exact solution of Ref. [34]:

$$E_y(x,t) = -\frac{2}{(1-\epsilon)^{1/2}}\frac{ct-x}{\sqrt{c^2t^2-x^2}}J_1\left(\frac{\omega_p}{c}\sqrt{c^2t^2-x^2}\right) + \frac{2}{1-\epsilon}\left(kI_1 - \frac{\sqrt{\epsilon}}{c}\frac{\partial I_2}{\partial t}\right), \tag{21}$$

$$B_z(x,t) = \frac{2}{(1-\epsilon)^{1/2}}\frac{ct-x}{\sqrt{c^2t^2-x^2}}J_1\left(\frac{\omega_p}{c}\sqrt{c^2t^2-x^2}\right) - \frac{2}{1-\epsilon}\left(\epsilon kI_1 - \frac{\sqrt{\epsilon}}{c}\frac{\partial I_2}{\partial t}\right), \tag{22}$$

where

$$I_1(x,t) \equiv \int_x^{ct}\cos[k\sqrt{\epsilon}(g-x)]J_0\left(\frac{\omega_p}{c}\sqrt{c^2t^2-g^2}\right)dg, \tag{23}$$

$$I_2(x,t) \equiv \int_x^{ct}\sin[k\sqrt{\epsilon}(g-x)]J_0\left(\frac{\omega_p}{c}\sqrt{c^2t^2-g^2}\right)dg. \tag{24}$$

The integral $I_1(x,t)$ is evaluated in the approximation (10) in Sec. II. The forerunners can be calculated by evaluating the expression

$$\frac{1}{c}\frac{\partial}{\partial t}I_2(x,t) = \sin[\sqrt{\epsilon}(\omega_c t - kx)] - \omega_p t\int_x^{ct}\frac{\sin[k\sqrt{\epsilon}(g-x)]}{\sqrt{c^2t^2-g^2}} \times J_1\left(\frac{\omega_p}{c}\sqrt{c^2t^2-g^2}\right)dg \tag{25}$$

in the same approximation. After replacing the variable of integration, the integral

$$\int_0^{\delta} \frac{dz}{\sqrt{1-z^2}} \sin[\sqrt{\epsilon}(\omega_c t\sqrt{1-z^2} - kx)] J_1(\omega_p t z) \qquad (26)$$

is calculated by using the above-reported expansion of the square roots and the table of integrals 1.8.2.8 [33]. As a result,

$$\frac{1}{c}\frac{\partial}{\partial t} I_2(x,t) \approx U_1(x,t) - \frac{1-\epsilon}{2\epsilon} U_3(x,t). \qquad (27)$$

The electric and magnetic forerunners in this section are denoted $E_y^F(x,t)$ and $B_z^F(x,t)$, respectively, to be different from the electric forerunner $E^F(x,t)$ in Sec. II. The approximation $\omega_c t \gg 1$ allows us to neglect the term which is proportional to $U_2(x,t)$ in the expression for $I_1(x,t)$. Then

$$B_z^F(x,t) = \frac{2}{\sqrt{1-\epsilon}} \frac{\sqrt{1-x/ct}}{\sqrt{1+x/ct}} J_1(\omega_p t\delta), \qquad (28)$$

$$E_y^F(x,t) = 2E^F(x,t) - B_z^F(x,t). \qquad (29)$$

One can see that the magnetic forerunner does not depend on the integrals (23) and (24). The electric forerunner in the approximation (14) when the sum (13) can be truncated becomes

$$E_y^F(x,t) = \frac{2}{\sqrt{1-\epsilon}} \frac{x}{ct} \frac{\sqrt{1-x/ct}}{\sqrt{1+x/ct}} J_1(\omega_p t\delta) - \frac{1+\epsilon}{(1-\epsilon)^{3/2}} \delta^3 J_3(\omega_p t\delta). \qquad (30)$$

The first term of Eq. (30) associated with the Sommerfeld precursor underestimates essentially the electric field. This follows from the fact that the amplitude of the electric forerunner in the medium with $\epsilon < 1$ should be larger than the amplitude of the magnetic forerunner (28). Therefore, the second term in Eq. (30) cannot be neglected in the present consideration.

In the approximation $\omega_p t\delta \gg 1$ the magnetic forerunner becomes

$$B_z^F(x,t) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\delta}{\omega_c t}} \frac{(1+\delta^2/4)}{(1-\epsilon)^{3/4}} \sin(\omega_p t\delta - \pi/4), \qquad (31)$$

and the sum truncation is not required to calculate the electric forerunner:

$$E_y^F(x,t) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\delta}{\omega_c t}} \left(-\frac{1+\delta^2/4}{(1-\epsilon)^{3/4}} + \frac{2(1-\epsilon)^{1/4}}{1-\epsilon(1+\delta^2)} (1+\delta^2/2) \right) \sin(\omega_p t\delta - \pi/4) \qquad (32)$$

if $\frac{\epsilon}{1-\epsilon}\delta^2 < 1$, or

$$E_y^F(x,t) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\delta}{\omega_c t}} \frac{1}{(1-\epsilon)^{3/4}} \left(1 + \frac{3+5\epsilon}{4(1-\epsilon)}\delta^2\right) \times \sin(\omega_p t\delta - \pi/4) \qquad (33)$$

if $\frac{\epsilon}{1-\epsilon}\delta^2 \ll 1$.

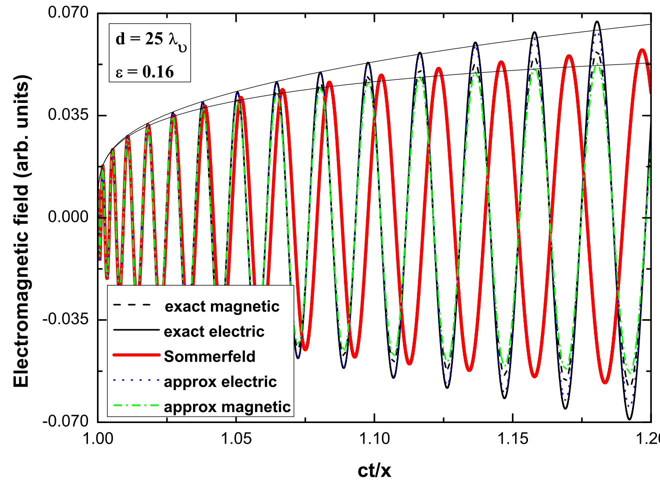

FIG. 4. The forerunners of the transmitted step-modulated sine wave at the distance of 25 vacuum wavelengths from the vacuum-medium interface in a medium with $\epsilon = 0.16$: the exact solution for the electric field from Eq. (21) (solid line), the exact solution for the magnetic field from Eq. (22) (dashed line), Sommerfeld precursor from Refs. [1,3] (thick solid line), the electric forerunner from Eq. (30) (dotted line), and the magnetic forerunner from Eq. (28) (dashed-dotted line). The data are normalized by the amplitude of the incident wave. The envelopes of the forerunners are shown as the amplitudes of the oscillations from Eqs. (33) and (31) (thin solid lines). See Fig. 5 for details.

The magnetic forerunner (28) can be written in the variables of Sommerfeld's problem [1,3] to see the difference from the Sommerfeld forerunner:

$$B_z^F(x,t) = \omega_c \sqrt{\frac{\tau}{\xi}} \frac{2}{\sqrt{\tau\omega_p^2/\xi + 4}} J_1(\sqrt{4\tau\xi + (\omega_p\tau)^2}). \qquad (34)$$

The magnetic and electric forerunners at the distance of 25 vacuum wavelengths (10 wavelengths in the medium) from the vacuum-medium interface are shown in Figs. 4 and 5. The fields are normalized by the amplitude of the incident wave. Figures 4 and 5 show the electric and magnetic

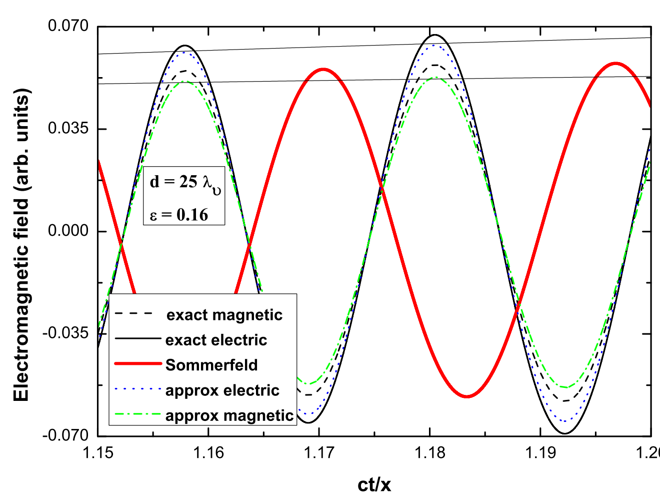

FIG. 5. Same as Fig. 4 but for a few periods of the oscillations to see the details.

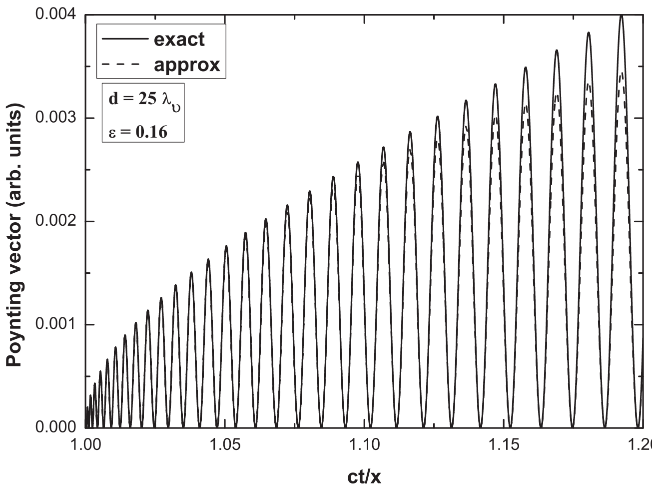

FIG. 6. The Poynting vector of the forerunner at the distance of 25 vacuum wavelengths from the vacuum-medium interface in a medium with $\epsilon = 0.16$. The Poynting vector is normalized by the Poynting vector amplitude of the incident wave. Solid line is obtained from the exact solution of the problem [Eqs. (21) and (22)]. Dashed line is obtained from the magnetic and electric forerunners [Eqs. (28) and (30)].

forerunners as the exact solution of the problem from Eqs. (21) and (22), respectively (solid and dashed lines, respectively), Sommerfeld forerunner from Refs. [1,3] (thick solid line), and the approximate electric and magnetic forerunners from Eqs. (30) and (28), respectively (dotted and dash-dotted lines, respectively). The horizontal axis is built for the range from 1 to 1.2, since this approximation (10) is well applicable in this range. The envelopes of the electric and magnetic oscillations are built as the amplitudes of the oscillations from Eqs. (33) and (31), respectively (thin solid lines). Figure 5 zooms in on a few oscillation periods of Fig. 4 to see the details of the dependencies.

As previously in the problem of Sec. II, the Sommerfeld forerunner underestimates both the instantaneous frequency and the amplitude of the electric forerunner but reproduces surprisingly well the amplitude of the magnetic forerunner.

The Poynting vector ($S_x = cE_yB_z/4\pi$) of the propagating wave at the distance of 25 vacuum wavelengths (10 wavelengths in the medium) from the vacuum-medium interface is shown in Fig. 6. It is normalized by the Poynting vector amplitude of the incident wave. Solid and dashed lines are obtained from exact [Eqs. (21) and (22)] and approximate [Eqs. (28) and (30)] solutions to the problem, respectively. The approximate solution is in good agreement with the exact result but, of course, underestimates the Poynting vector when the parameter $ct = x$ is increased. The data are presented to compare the power magnitude of the forerunner with that of the incident power.

IV. CONCLUSIONS

The forerunners of the step-modulated sine signal in the Drude dispersive media are calculated in time domain. Two problems are considered: the plane source of the electric field is inside the medium, and the linearly polarized wave is incident normally from vacuum upon the half-space medium. The forerunners are obtained from the exact solutions of the problems (8) and (21), (22). The approximate formulas for the forerunners (15) and (28), (30) agree quite well with the exact solutions. The canonical Sommerfeld forerunner [1,3] is shown to underestimate both the frequency of the oscillations and the amplitude of the electric-field oscillations.

All forerunners are obtained as the early time behavior of the signal at a certain coordinate in the approximation $\delta^2 \ll 1$ [Eqs. (12), (28), and (29)]. In the additional approximation $\frac{\epsilon}{1-\epsilon}\delta^2 \ll 1$ they become Eqs. (15), (28), and (30). If a few initial periods of the oscillations are neglected in the approximation $\omega_p t\delta \gg 1$, the forerunners read as (18), (31), and (32) if $\frac{\epsilon}{1-\epsilon}\delta^2 < 1$ or (19), (31), and (33) if $\frac{\epsilon}{1-\epsilon}\delta^2 \ll 1$.

Since the problems are solved completely in the time domain, the contributions to the obtained forerunners from the different parts of the frequency spectrum of the incident pulse cannot be separated from each other. Only the knowledge of frequency-domain theory allows us to identify the obtained forerunner as a Sommerfeld precursor (the high-frequency contribution which propagates almost at the velocity of light). In one sense the time-domain theory of forerunners is simpler because it does not require any preliminary analysis of the frequency spectrum for locating the saddle points and choosing the integration paths.

The forerunners are calculated for the transparent Drude media only (the carrier frequency ω_c is larger than the plasma frequency of the medium ω_p, or $\epsilon > 0$), since the calculations are based on the previously developed analytics for the transparent media [31,34]. It means that we cannot reproduce the data, for example, from Ref. [32] where the considered Drude medium is opaque. The time-domain calculation of the forerunners in the opaque Drude media requires a development of the analytics for the transient propagation. The analytics is more complicated than in the case of the transparent medium but it does not look impossible. At least the authors made essential progress in its development and hope to have promising results soon.

The transmitted signal damping in the medium is usually taken into consideration by including the electron collisions in the motion equation. Unfortunately, this increases the order of the differential Eq. (4) and its solution is not obtained yet in the time domain. Intuitively, the effect of collisions can be included in the presented results through the exponential multiplier $\exp[-2\gamma(t - x/c)]$ to the forerunners [2,17,24], where γ is the damping constant of the medium. But more accurate treatment of the collisions in the time domain requires developing the analytics.

The calculated forerunners are identified as Sommerfeld precursors. But a Sommerfeld precursor in the Lorentz media is the high-frequency contribution of the incident pulse spectrum to the transmitted signal, which does not depend on the resonance frequency. It means that the forerunners calculated here for Drude dispersive media should be close to the forerunners of Lorentz dispersive media if the high-frequency contribution of the incident signal spectrum to the forerunner is dominant (Sommerfeld precursor is dominant in the forerunner). The obtained frequency and amplitudes of the forerunners are closer to the exact solutions than the prediction by canonical Sommerfeld precursor [1,3]. Therefore, the obtained data can be helpful for a direct comparison with the exact results for the forerunners in the Lorentz dispersive media and for an experimental detection of the forerunners. Also, the presented analytics can be used for further development of the time-domain theory.

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