Directional population control beyond the exceptional point in a non-Hermitian system

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In the context of dynamical control, non-Hermitian systems offer unique opportunities and challenges. In the simplest case of two interacting levels with gain and/or loss, the existence of an exceptional point (EP) of degeneracy in the chiral complex eigenvalue landscape fundamentally changes how the system responds to adiabatic changes in the coupling γ and energy separation δ of the bare levels. In particular, previous studies have shown that selective population transfer can be achieved by adiabatically steering the system around closed paths that encircle the EP with the proper helicity in the γ vs δ parameter space. Here we show that efficient state-selective population transfer can, in some cases, be achieved even with control loops that do not enclose the EP.

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I. INTRODUCTION

In the presence of exponential loss and/or gain, pseudo-two-level quantum systems exhibit complex eigenenergy surfaces whose real and imaginary parts are chiral functions of the coupling strength γ and energy separation δ between the bare states [1]. There has been substantial interest in exploring methods for robust quantum control in such non-Hermitian models. For example, controlled population transfer can be realized by adiabatically steering the system around a closed control loop in the (γ, δ) parameter space [2,3]. Given the change in state in this context, the system evolution cannot be truly adiabatic. Nevertheless, the term is commonly used to describe transformations that would be adiabatic for a similar system governed only by the real part of the Hamiltonian. Previous work has suggested that population transfer can only occur through such a transformation if the control loop encloses an exceptional point (EP), i.e., a point of energy degeneracy that exists at nonzero coupling in the eigenenergy landscape [4,5]. Further studies predicted that the helicity of the control path, as well as its starting or ending point, can impact the population transfer probability and serve as additional control knobs [2,6–8]. Experiments exploring these and related phenomena have been performed using microwave cavities [9], optomechanical cavities [3,10], molecules [11–13], and other systems [14–18]. Recently, the nonadiabatic transition probability in a two-level system steered directly through two exceptional points was determined [19]. Here we present results of simulations based on numerical integration of the time-dependent Schrödinger equation (TDSE), showing that encircling an EP is not a necessary condition for control of directional population transfer in a two-level non-Hermitian system.

In the following sections we examine the adiabatic control problem for simple rectangular paths in the (γ, δ) parameter space for a generic two-level system in the presence of exponential decay (and/or gain). Our numerical simulations clearly show that directional population transfer can still occur in adiabatic transformations that do not enclose the EP. We also present an analytical model that identifies the mechanism responsible and predicts how far beyond the control loop boundary the EP can lie while still affording full directional population control.

II. TWO-LEVEL SYSTEM WITH DECAY

Consider a pair of uncoupled states with energies E_1 and E_2 that spontaneously decay at rates of $2\Gamma_1$ and $2\Gamma_2$, respectively, to some unspecified levels. We assume that a coupling γ between the two states can be externally applied and that γ and the energy splitting $\delta = E_2 - E_1$ between the bare states can be continuously varied. As a concrete example, this situation might be realized by driving two opposite-parity atomic states with a nearly resonant oscillating field. In a dressed-atom picture, δ is determined by the detuning of the field frequency from resonance and γ can be changed by varying the field strength. The system can be described by an effective non-Hermitian Hamiltonian

$$H = \begin{pmatrix} -E_1 - i\Gamma_1 & \gamma^* \\ \gamma & -E_2 - i\Gamma_2 \end{pmatrix} \quad (1)$$

in atomic units.

Without loss of generality, we can further simplify the Hamiltonian. First, we translate the complex energy origin, subtracting $-E_1 - i\Gamma_1$ from the diagonal matrix elements. Next, we define γ to be real and positive. Finally, we rescale all terms in the Hamiltonian by dividing them by the difference in the decay rates of the two bare states. The Hamiltonian is then a function of only two variables γ and δ and takes the convenient form

$$H' = \begin{pmatrix} 0 & \gamma \\ \gamma & -\delta - i \end{pmatrix}, \quad (2)$$

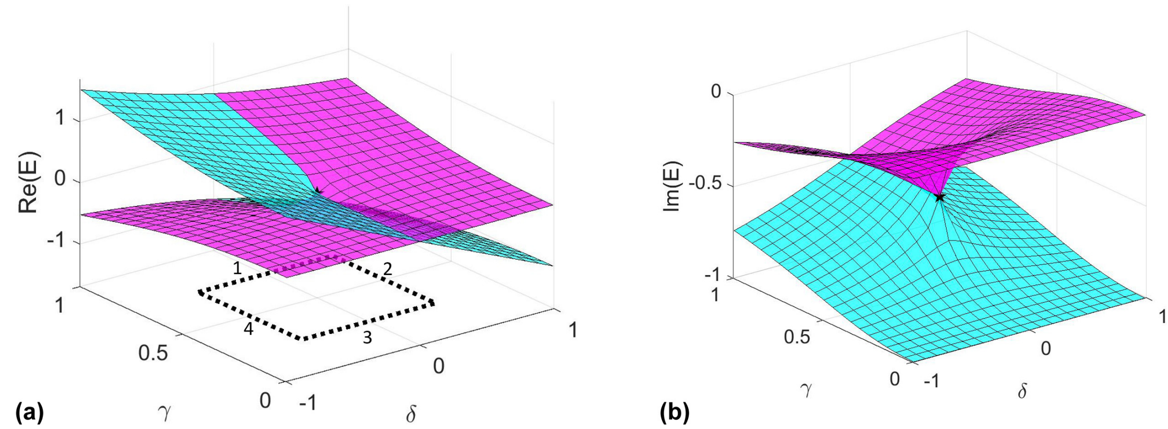

FIG. 1. (a) Real and (b) imaginary parts of the eigenvalue surfaces E for the coupled pseudo-two-level system with decay. Magenta (dark gray) and cyan (light gray) correspond to the states with the smallest and largest decay rates, respectively. The black star marks the EP ($\delta = 0, \gamma = \frac{1}{2}$). The dashed line beneath the real part of the eigensurfaces illustrates the form of the closed path transformations we consider in detail (see the text), with two legs (labeled 1 and 3) at constant γ and two legs (labeled 2 and 4) at constant δ.

where one of the uncoupled basis states does not decay and the other decays with a characteristic lifetime $\tau = \frac{1}{2}$.

Figures 1(a) and 1(b) show the real and imaginary parts of the eigenvalue surfaces E as a function of δ and γ. At each point, the surface colors identify the eigenstates with the smallest [magenta (dark gray)] and largest [cyan (light gray)] rates of decay, i.e., the magenta (dark gray) eigenstate has the eigenvalue with the least-negative imaginary part. The surfaces exhibit a point of degeneracy, the EP, at ($\delta = 0, \gamma = \frac{1}{2}$). For ($\delta = 0, \gamma > \frac{1}{2}$), the real parts of the eigenvalues form an avoided level crossing, whereas for ($\delta = 0, \gamma \leqslant \frac{1}{2}$) they cross with zero gap. Conversely, the imaginary parts of the eigenvalues exhibit a line of degeneracy ($\delta = 0, \gamma \geqslant \frac{1}{2}$).

When attempting to understand how an initial state of the non-Hermitian system will be transformed during a closed control loop, it is tempting to assume that the principal effect of the imaginary part of the Hamiltonian is a steady leak of population from the two levels. Under that assumption, one might neglect the difference in the decay rates of the two states (i.e., ignore the imaginary part of the effective Hamiltonian) and consider only the evolution on the real part of the eigenvalue surfaces. In that case, if the system is initially in one of the two eigenstates, it will remain in an eigenstate throughout a perfectly adiabatic transformation. The character of the initial state will vary smoothly with time as γ and/or δ are changed, with the system evolution well described by a path that does not leave the eigenenergy surface. Inspection of Fig. 1(a) would then predict that because of the line of degeneracy in the surface, any closed adiabatic path (regardless of its shape or the direction of travel) that encircles the EP (an odd integer number of times) will result in complete population transfer from one eigenstate to the other. Any adiabatic path that does not encircle the EP (or encircles it an even integer number of times) results in no net population transfer.

However, not only does the presence of an imaginary part of the Hamiltonian lead to decay of the system as a whole, it can fundamentally change the system evolution during the control loop and accordingly the final result of the transformation. The primary issue is related to the asymmetric decay of the two eigenstates and the fact that no dynamical process of finite duration can be truly adiabatic. While nonadiabatic (de)excitations resulting from time-varying external controls can be reduced to negligible levels in Hermitian systems, they can be dramatically amplified by the unequal decay rates from the constituent eigenstates during a slow transformation. Increasing the loop time can reduce any nonadiabatic effects associated with the time-dependent controls, but simultaneously enhances the impact of the differential decay. Indeed, it has been shown, contrary to the naive picture presented in the preceding paragraph, that the population transfer in the non-Hermitian system is actually chiral. Depending on the helicity of the control loop, population transfer is effective for only one of the two eigenstates, with the other essentially unaffected by the process [3]. Also in contrast to the Hermitian picture, the transfer probability for a given closed control loop also depends on the starting point (δ_0, γ_0) of the transformation [8]. Here we demonstrate via simulations based on numerical integration of the TDSE and explain through an analytic model another important result stemming from the difference in decay rates of the bare states, namely, closed control loops need not enclose the EP to induce selective population transfer.

For convenience, we focus on rectangular control loops that can be separated into four distinct segments (or legs). As shown in Fig. 1(a), δ varies along legs 1 and 3 (with constant $\gamma = \gamma_{\max}$ and $\gamma_{\min}$, respectively) and γ varies along legs 2 and 4 (with constant $\delta = \delta_{\max}$ and $-\delta_{\max}$, respectively). Moreover, for ease of illustration, we exclusively consider population transfer from systems that are initially prepared in one of the two eigenstates. All closed loop transformations start from ($\delta_0 = 0, \gamma_0 = \gamma_{\max}$). Because our primary goal is to understand how the *relative* populations are affected by the system transformations, when plotting the state populations (or population transfer probability) at a particular time, we normalize to the total population remaining in the system at that time. It is worth noting that the reduced Hamiltonian has the same form if one or both of the bare states experience exponential gain rather than decay (as, for example, with optical modes in a cavity with gain), so our consideration of

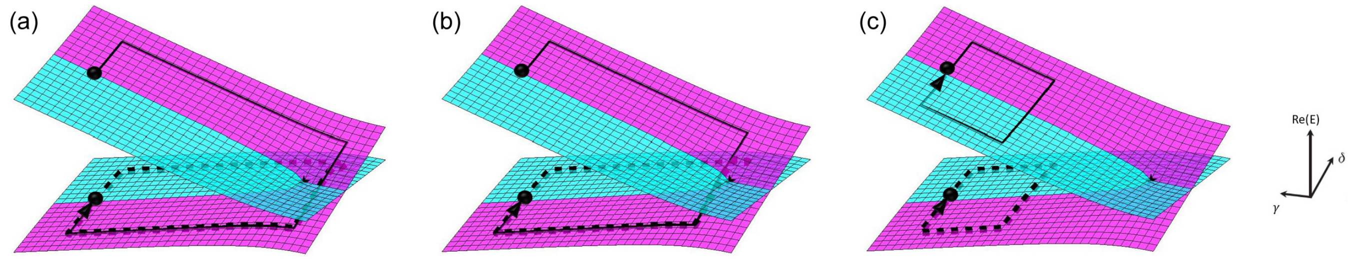

FIG. 2. Energy expectation value of the two-level system, shown as a path in the real part of the energy eigenvalue landscape, computed via numerical integration of the TDSE for clockwise rectangular control loops. The three plots show control loops with different values of $\gamma_{\min}$: (a) 0.3, (b) 0.6, and (c) 2. The surface coloring designates the eigenvalue with the slowest [magenta (dark gray)] and fastest [cyan (light gray)] decay, analogous to Fig. 1, with a black star positioned at the EP ($\delta = 0, \gamma = \frac{1}{2}$). The thin solid (thick dashed) black lines show the energy expectation value vs (δ, γ) during the transformation, for initial system preparation in the upper (lower) eigenstate. For all cases in the figure (i) the start and stop point is ($\delta_0 = 0, \gamma_0 = 3$), (ii) the detuning is varied over the same range $-1 \leqslant \delta \leqslant 1$, and (iii) the duration of each leg of the control loop is 20τ, for a total transformation time of 80τ. In (a) the clockwise control loop encloses the EP and the initial population in the upper state is transferred to the lower state, while the initial population in the lower state remains in the lower state. In (c) the control loop does not enclose the EP and there is no population transfer from either initial state. Interestingly, complete and directional population transfer is observed in (b), despite the fact that the EP is not enclosed within the control loop.

times long compared to τ does not imply negligible system population.

Figure 2 illustrates the principal effect we explore in this paper. In Fig. 2(a) the clockwise control loop encloses the EP and initial population in the upper state is transferred to the lower state, while initial population in the lower state remains in the lower state. Following the same path in a counterclockwise direction (not shown) induces population transfer from the lower to the upper state, but not from upper to lower. In Fig. 2(c) the control loop does not enclose the exceptional point and there is no population transfer from either initial state. Figure 2(b) shows that efficient chiral population transfer can still occur, even for closed control loops that do not enclose an EP, and under conditions where the evolution would be purely adiabatic (with negligible net population transfer from either initial state) in a Hermitian analog system. Adiabaticity of identical transformations in a Hermitian analog system whose eigenenergy surface closely matches the real part of the surface in our non-Hermitian system[1] has been directly confirmed via numerical integration of the TDSE. This observation begs the following questions. If not enclosure of the EP, what characteristics of a closed control loop determine whether population transfer occurs? For the rectangular loops that we consider, can we predict the range of $\gamma_{\min}$ values for which the transformation leaves the initial state unchanged?

To gain additional insight toward answering these questions, we have calculated the population transfer probability vs $\gamma_{\min}$ for families of clockwise and counterclockwise control loops [with the same value of $\gamma_{\max} = 3$, the same starting point ($\delta = 0, \gamma = 3$), and the same total loop time of 80τ] for three different values of $\delta_{\max}$. As shown in Fig. 3, population initially in the upper (lower) eigenstates is largely unaffected by counterclockwise (clockwise) control loops for any values of $\gamma_{\min}$ and $\delta_{\max}$. However, for $\gamma_{\min}$ below some threshold (greater than 0.5), clockwise (counterclockwise) transformations result in efficient transfer from the upper (lower) eigenstate. We define the critical coupling γ_c as the smallest value of $\gamma_{\min}$ for which the population transfer probability equals 0.5. As Fig. 3 clearly shows, γ_c depends on the detuning range in the control loop. In particular, $\gamma_c > 0.5$ and increases monotonically, but nonlinearly, as a function of $\delta_{\max}$. For $\gamma_{\min} > \gamma_c$, the transfer probability exhibits oscillations whose amplitude decays with increasing $\gamma_{\min}$. The amplitude and rate of decay of those oscillations also depend on $\delta_{\max}$.

We first consider the dependence of γ_c on $\delta_{\max}$, i.e., on the width of the control loop. We note that Fig. 2 suggests that, in general, population transfer is only significant during leg 3, as δ varies with $\gamma = \gamma_{\min}$. Therefore, to understand the principal aspects of the population transfer dynamics, we can focus on the evolution during leg 3. Along that path, for $\gamma_{\min} > 0.5$, the system traverses an avoid crossing in the real part of the eigenvalue surface (Fig. 4). Accordingly, if the Hamiltonian were Hermitian, the population transfer probability would be well described by the standard Landau-Zener formula [20]. As such, one might expect that the key parameter in determining the population transfer probability would be the rate at which the system passes through the avoided crossing. However, Fig. 5 illustrates that this expectation generally fails in the non-Hermitian case, even for adiabatic transformations where the detuning is scanned sufficiently slowly that there is negligible population transfer in an analogous Hermitian system with the same avoided crossing characteristics.

Figure 5 shows the population transfer probability as the detuning is adiabatically scanned along leg 3, for fixed values of the detuning end points, but for different scan rates and different coupling strengths (i.e., different energy gaps at the center of the avoided crossing). While the details of the population transfer along the path depend on the scan rate and coupling strength, the value of the detuning (i.e., the position along the path) at which 50% transfer occurs is nearly independent of the scan rate (within the adiabatic regime). Apparently, it is the range of δ, and not the rate at which δ is scanned, that determines γ_c for the adiabatic passage. As

[1]The Hermitian analog Hamiltonian is obtained by replacing the diagonal elements of the non-Hermitian Hamiltonian in Eq. (2) with their real parts and replacing the off-diagonal elements with $\sqrt{\gamma'^2}$, where γ'^2 is the greater of $\gamma^2 - \frac{1}{4}$ and 0.

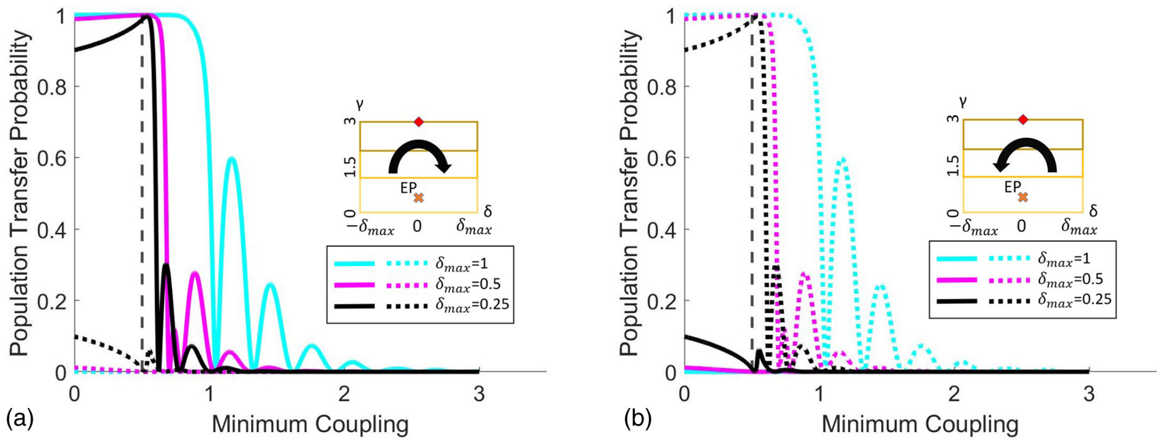

FIG. 3. Population transfer probability computed via numerical integration of the TDSE as a continuous function of $\gamma_{\min}$ for closed rectangular control loops analogous to those in Fig. 2 but for several different values of $\delta_{\max}$. Black, magenta (dark gray), and cyan (light gray) curves correspond to $\delta_{\max} = 0.25$, 0.5, and 1, respectively. Results are plotted for (a) clockwise and (b) counterclockwise paths. The thick solid (dotted) lines give the transfer probabilities when the system is initially prepared in the upper (lower) eigenstate at the start of the control loop ($\delta = 0, \gamma = 3$). The thin black vertical dashed line at $\gamma_{\min} = \frac{1}{2}$ marks the position of the EP. Loops with $\gamma_{\min} < \frac{1}{2}$ enclose the exceptional point, whereas those with $\gamma_{\min} > \frac{1}{2}$ do not. The duration of each leg of each control loop is 20τ, for a total transformation time of 80τ. The insets show examples of three different control loops, each with the same values of $\delta_{\max}$ and $\gamma_{\max}$, but different values of $\gamma_{\min}$.

we show below, this is because the differential decay rate of the bare states plays a dominant role in the population transfer.

III. LANDAU-ZENER TRANSITION IN NON-HERMITIAN SYSTEMS

As noted above, the well-known Landau-Zener formula gives the probability of population transfer at an avoided crossing in a two-level Hermitian system as the energy difference between two bare states is scanned through degeneracy at a constant rate [20]. The extension of the problem to non-Hermitian systems has also been studied in detail [21,22]. We take an alternative approach, using an approximate model that allows us to develop an analytic expression for the critical value $\delta_{\max} = \delta_c$ at which the population transfer probability is 0.5 for a given value of γ. We assume that the detuning range is sufficiently large that the system evolution along leg 3 can be divided into three regions (Fig. 4). In regions A and C, $\delta \gg \gamma$, so the eigenstates are approximately equivalent to the bare states, one of which does not decay and the other decaying with a lifetime $\tau = \frac{1}{2}$. Note that energy ordering of the slow and fast decaying states is opposite for regions A and C. In region B, $\delta < \gamma$, the two eigenstates are nearly equal admixtures of the two bare states, and they decay at approximately the same rate. Therefore, there is negligible

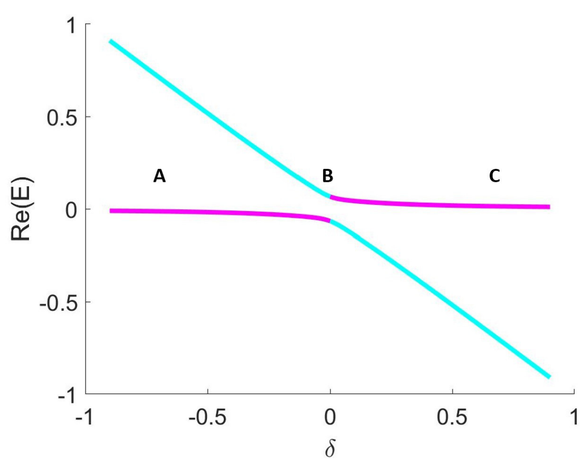

FIG. 4. Avoided level crossing along leg 3 of the control loop for a coupling strength γ slightly greater than $\frac{1}{2}$. Far from the avoided crossing, the eigenstates are nearly equivalent to the bare states. Magenta (dark gray) and cyan (light gray) denote the more slowly and rapidly decaying eigenstates, respectively. The letters label three principal regions of population evolution during a detuning scan through the avoided crossing.

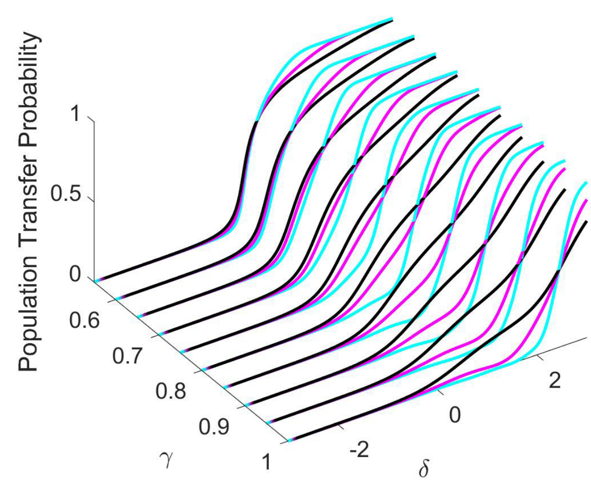

FIG. 5. Population transfer probability between two eigenstates calculated via numerical integration of the TDSE as the detuning δ is scanned from -3 to 3 at different couplings γ with constant detuning scan rates of $d\delta/dt = 0.25$ [cyan (light gray)], 0.5 [magenta (dark gray)], and 1(black).

relative decay. Accordingly, the population transfer between the two states in region B is accurately described by a Hermitian Landau-Zener formula, assuming a coupling $\sqrt{\gamma^2 - \frac{1}{4}}$ that exhibits approximately the same energy gap as the non-Hermitian system at $\delta = 0$.

Inspection of Figs. 2 and 3 shows that for the control loops we have considered, significant population transfer occurs only when the system enters leg 3 with essentially all population in the more slowly decaying eigenstate. Therefore, to predict the value of δ_c for a given γ along leg 3, we only need to examine two cases: all population initially in the lower (upper) state traversing the avoided crossing from left to right (right to left) as shown in Fig. 4. Since these two cases are equivalent, we focus on the left to right transformation, progressing through regions from A to B to C.

Within the model, the eigenstates are approximately equivalent to the bare states in region A. So if all population is initially in the nondecaying state, it will remain there throughout region A and at the start of region B. The Landau-Zener formula [20] then predicts the populations in the nondecaying (upper) state

$$P_{ND} = \exp\left[-2\pi\left(\gamma^2 - \frac{1}{4}\right)\Big/\frac{d\delta}{dt}\right] \tag{3}$$

and decaying (lower) state

$$P_D = 1 - \exp\left[-2\pi\left(\gamma^2 - \frac{1}{4}\right)\Big/\frac{d\delta}{dt}\right] \tag{4}$$

at the beginning of region C. Of course, in the adiabatic regime $(\gamma^2 - \frac{1}{4})/\frac{d\delta}{dt} \gg 1$, so $P_D \simeq 1$. However, P_{ND} is nonzero provided $d\delta/dt > 0$.

In region C, the population in the nondecaying level does not change, but the other decays exponentially with a time constant τ,

$$P_D(t) \simeq \exp\left(-\frac{t}{\tau}\right), \tag{5}$$

where we have defined $t = 0$ at the start of region C. For a constant scan rate, we can substitute $t = \delta/\frac{d\delta}{dt}$ and $\tau = \frac{1}{2}$ to obtain

$$P_D(\delta) \simeq \exp\left(-2\delta\Big/\frac{d\delta}{dt}\right). \tag{6}$$

At the end of region C, $\delta = \delta_{\max}$. By definition, if $\delta_{\max} = \delta_c$, then there is 50% relative population transfer during the transformation and we have $P_{ND} = P_D(\delta_c)$. Accordingly, we find $\delta_c = \pi(\gamma^2 - \frac{1}{4})$, independently of the scanning rate.

Figure 6 compares the approximate analytic prediction for δ_c with simulation results based on population transfer along leg 3 using the non-Hermitian Hamiltonian. The agreement is excellent.

IV. EXTENSION TO CLOSED CONTROL LOOPS

We can readily extend the model of population transfer during just leg 3 to the full control loop, starting and ending at $(\delta = 0, \gamma_{\max})$. To predict γ_c for the closed loop, we again only need focus on situations where non-negligible population transfer occurs during leg 3, i.e., clockwise paths starting from the upper eigenstate and counterclockwise paths starting from the lower eigenstate (Fig. 2). During the first $\frac{3}{8}$ of the loop, the system population remains in the initial, slow decaying state. Any small level of population transferred to the other eigenstate (due to imperfect adiabaticity) rapidly decays. Thus, all population is in the slow decaying state when the system enters leg 3. Population transfer can then occur during leg 3 as described in the preceding section, with $\gamma_c = \sqrt{\delta_{\max}/\pi + \frac{1}{4}}$. The plot in Fig. 7(b) shows this analytic prediction for γ_c, along with simulation results based on population transfer during the first $\frac{5}{8}$ of the control loop with the full non-Hermitian Hamiltonian. The agreement is again excellent.

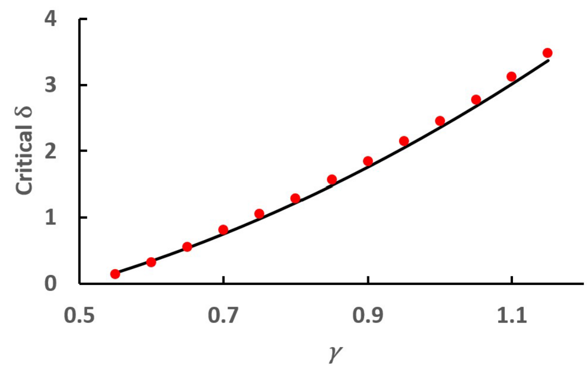

FIG. 6. Critical detuning δ_c vs coupling γ as determined from TDSE simulations of population transfer along leg 3 using the non-Hermitian Hamiltonian (filled circles) and the analytic approximation $\delta_c = \pi(\gamma^2 - \frac{1}{4})$ (solid line).

Continuing on the remaining $\frac{3}{8}$ of the control loop after leg 3, small levels of probability amplitude transfer between the two eigenstates (again due to imperfect adiabaticity) can interfere with the non-negligible population in the two eigenstates for a substantial effect. The oscillations in the population transfer for the full loop, visible in Fig. 3 for $\gamma > \gamma_c$, are the result of that interference. Those interferences also cause a substantial steplike increase in the value of γ_c with increasing $\delta_{\max}$, as shown in Fig. 7(a).

It is worth noting that using our operational definition based on the evolution of a Hermitian analog system, the adiabaticity of the closed-loop transformation improves with increasing distance of the (excluded) EP from the path. This is because the energy gap at the avoided crossing along the minimum coupling leg increases the further the EP is from the path. Thus, adiabatic behavior can be achieved with reduced transformation times. In addition, as shown by the magenta (dark gray) and especially the black curves Fig. 3, we find that in some cases the effectiveness of the population swap actually improves for loops that do not enclose the EP, with the population transfer probability increasing as $\gamma_{\min}$ is tuned from the EP toward γ_c.

V. SUMMARY AND OUTLOOK

We have shown that encircling an EP is not a necessary condition for achieving directional population control via closed-loop adiabatic transformations in a non-Hermitian

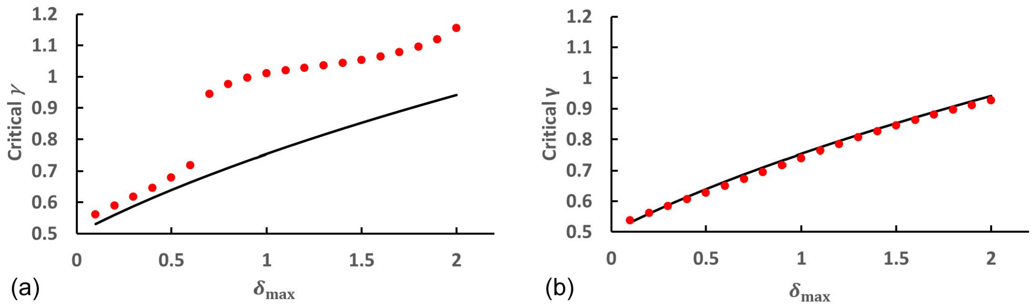

FIG. 7. Critical coupling γ_c (filled circles) for which 50% relative population transfer occurs vs maximum detuning, according to TDSE simulations with the non-Hermitian Hamiltonian over (a) the full control loop and (b) the first $\frac{5}{8}$ of the control loop. The solid curve is the expression $\gamma_c = \sqrt{\delta_{\max}/\pi + \frac{1}{4}}$ derived in the text.

system. We present an analytic model that predicts the conditions needed to achieve control outside the EP in a two-level system, predicting the minimum distance between the control path and the EP for a class of rectangular control loops in the two-parameter (bare level detuning and coupling strength) energy landscape. Experimental verification of the predictions may require a system in which one or both of the uncoupled states experiences exponential gain, rather than loss, to maintain non-negligible population in the system during the long adiabatic transformation times.

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