Quantum discord for multiqubit systems

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We evaluate analytically the quantum discord for a large family of multiqubit states. It is interesting to note that the quantum discord of three qubits and five qubits is the same, as is the quantum discord of two qubits and six qubits. We discover that the quantum discord of this family of states can be classified into three categories. The level surfaces of the quantum discord in the three categories are shown through images. Furthermore, we investigated the dynamic behavior of quantum discord under decoherence. For the odd partite systems, we prove the frozen phenomenon of quantum discord does not exist under the phase flip channel, while it can be found in the even partite systems.

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I. INTRODUCTION

Quantum correlations are essential features of quantum mechanics which distinguish the quantum from the classical world and play very important roles in quantum information processing. The quantum correlated states are shown to be more useful than the classically correlated ones in performing communication and computation tasks. Understanding and quantifying various quantum correlations are the primary goals in quantum information theory. The quantum entanglement and nonlocal correlations can be considered the most fundamental resources in quantum information processing [1–11], which are tightly related to quantum coherence [12–17].

Quantum discord is one of the most famous quantum correlations proposed by Ollivier and Zurek [18] and Henderson and Vedral [19], which quantifies the quantum correlations in bipartite systems without quantum entanglement. It is defined as the minimum difference between the quantum versions of two classically equivalent expressions of mutual information under projective measurements [20,21]. Due to the complexity of the minimization process, the computation of quantum discord is a hard task and analytic results are known only for some restricted families of states [22–27]. For a bipartite state ρ in systems A and B [18,19], the quantum discord $D_{A;B}(\rho)$ is defined by $D_{A;B}(\rho)=\min_{\Pi^A}[S_{B|\Pi^A}(\rho)-S_{B|A}(\rho)]$, where the conditional entropy $S_{B|A}(\rho)=S(\rho)-S(\rho_A)$ with $S(X)=-\mathrm{Tr}\,X\log_2 X$ is the von Neumann entropy of a state X, and ρ_A is the reduced state associated to the system A. $S_{B|\Pi^A}(\rho)=\sum_j p_j^A S(\Pi_j^A\rho\Pi_j^A/p_j^A)$, where Π_j^A is the von Neumann projection operator on subsystem A, and $p_j^A=\mathrm{Tr}(\Pi_j^A\rho\Pi_j^A)$ is the probability with respect to the measurement outcome j.

Very recently, Radhakrishnan *et al.* [28] introduced a generalization of discord for tripartite and multipartite states. One of the main features of this approach is the use of conditional measurements, where each successive measurement is conditionally related to the previous measurements. The $(N-1)$-partite measurement is written as

$$\Pi_{j_1\cdots j_{N-1}}^{A_1\cdots A_{N-1}}=\Pi_{j_1}^{A_1}\otimes\Pi_{j_2|j_1}^{A_2}\cdots\otimes\Pi_{j_{N-1}|j_1\cdots j_{N-2}}^{A_{N-1}},$$

where $\Pi_{j_1|j_2}^{A_2}$ is a projector on subsystem A_2 conditioned on the measurement outcome of A_1. Here, the measurements take place in the order $A_1\to A_2\to\cdots A_{N-1}$. Such conditioned measurements are essential to take into account all the classical correlations that may exist among the subsystems. Viewing the measurements as operations to break the quantum correlations, the optimization over all such measurements allows one to recover the pure quantum contributions. Moreover, there is an obvious asymmetry due to the fixed ordering of the measurements. This asymmetry has similarities with the quantum steering where one also considers measurements on a part of a system, while the aims are somewhat different in that for discord, one minimizes the disturbance due to measurements rather than comparing it to a local hidden state theory. Indeed, in some quantum information processing such as one-way quantum computing, there is a definite ordering of measurements, incompatible with the multipartite discord.

The quantum discord of an N-partite state ρ is defined by

$$D_{A_1;A_2;\ldots;A_N}(\rho)=\min_{\Pi^{A_1\cdots A_{N-1}}}\big[-S_{A_2\cdots A_N|A_1}(\rho)+S_{A_2|\Pi^{A_1}}(\rho)\cdots+S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)\big],\qquad(1)$$

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for the measurement ordering $A_1 \to A_2 \to \cdots A_{N-1}$. Here, we have defined $S_{A_k|\Pi^{A_1\cdots A_{k-1}}}(\rho) = \sum_{j_1\cdots j_{k-1}} p_j^{(k-1)} S_{A_1\cdots A_k}(\Pi_j^{(k-1)}\rho\Pi_j^{(k-1)}/p_j^{(k-1)})$ with $\Pi_j^{(k)} \equiv \Pi_{j_1\cdots j_k}^{A_1\cdots A_k}$, $p_j^{(k)} = \mathrm{Tr}(\Pi_j^{(k)}\rho\Pi_j^{(k)})$.

In general, it is difficult to evaluate the quantum discord (1) due to the complexity of the optimization. We analyze and evaluate this quantum discord for a family of multiqubit states, and graphically show the level surfaces of the quantum discord of this family. Moreover, due to the interaction with the environments, bipartite quantum discord may decrease asymptotically with time [29], and may be also frozen [30–33] and decoherence free for a certain time. We also study the dynamic behavior of quantum discord for a family of three-qubit and four-qubit states under decoherence. We discover that the multiqubit quantum discord of some states cannot be destroyed by decoherence in finite time.

The rest of this article is organized as follows. In Sec. II, we calculate analytically the multiqubit discord for a family of quantum states. We shown that the quantum discord can be classified into three categories. In Sec. III, we investigated the dynamical behavior of the discord for a family of three-qubit and four-qubit states. We discuss and summarize the results in Sec. IV.

II. QUANTUM DISCORD FOR MULTIQUBIT SYSTEMS

Consider the following family of N-qubit states,

$$\rho = \frac{1}{2^N}\left(I + \sum_{j=1}^{3} c_j\sigma_j \otimes \cdots \otimes \sigma_j\right), \quad (2)$$

where σ_j, $j = 1, 2, 3$, are the standard Pauli matrices, and I stands for the corresponding identity operator. The motivation to consider the states (2) is that, for $N = 2$, Eq. (2) reduces to the well-known Bell diagonal states whose famous analytical formulas of quantum discord have been provided by Luo [25], which attracted much attention and resulted in further vital results. For general N, these states are highly symmetric and include some generalized Greenberger-Horne-Zeilinger (GHZ) or W states as special ones.

We consider the family of a three-qubit state, associated with systems A, B, and C,

$$\rho = \frac{1}{8}\left(I + \sum_{j=1}^{3} c_j\sigma_j \otimes \sigma_j \otimes \sigma_j\right). \quad (3)$$

From (1) the quantum discord is given by

$$D_{A;B;C}(\rho) = \min_{\Pi^{AB}} [-S_{BC|A}(\rho) + S_{B|\Pi^A}(\rho) + S_{C|\Pi^{AB}}(\rho)]. \quad (4)$$

Since $\rho_A = \mathrm{Tr}_{BC}(\rho) = \frac{I}{2}$, we have the entropy $S(\rho_A) = 1$. Set $\xi = \sqrt{c_1^2 + c_2^2 + c_3^2}$. One can verify that

$$S(\rho) = -4 \times \frac{1+\xi}{8}\log_2\frac{1+\xi}{8} - 4 \times \frac{1-\xi}{8}\log_2\frac{1-\xi}{8}.$$

Hence,

$$\begin{aligned}-S_{BC|A}(\rho) &= -[S(\rho) - S(\rho_A)]\\ &= \frac{1+\xi}{2}\log_2(1+\xi) + \frac{1-\xi}{2}\log_2(1-\xi) - 2.\end{aligned} \quad (5)$$

Denote $\{\Pi_k = |k\rangle\langle k| : k = 0, 1\}$. The von Neumann measurement on subsystem A is given by $\{A_k = V_A\Pi_k V_A^\dagger : k = 0, 1\}$, where $V_A = t_A I + i\overrightarrow{y_A}\cdot\overrightarrow{\sigma}$ is the unitary operator with $t_A \in \mathbb{R}$, $\overrightarrow{y_A} = (y_{A1}, y_{A2}, y_{A3}) \in \mathbb{R}^3$, and $t_A^2 + y_{A1}^2 + y_{A2}^2 + y_{A3}^2 = 1$. After the measurement A_k, the state ρ is going to become the ensemble $\{\rho_k, p_k\}$ with $\rho_k := \frac{1}{p_k}(A_k \otimes I)\rho(A_k \otimes I)$ and $p_k = \mathrm{Tr}(A_k \otimes I)\rho(A_k \otimes I)$. Then we obtain $p_0 = p_1 = \frac{1}{2}$,

$$\begin{aligned}\rho_0 = \tfrac{1}{4}V_A\Pi_0 V_A^\dagger \otimes (I\otimes I + c_1z_1\sigma_1\otimes\sigma_1 + c_2z_2\sigma_2\otimes\sigma_2\\ + c_3z_3\sigma_3\otimes\sigma_3)\end{aligned} \quad (6)$$

and

$$\begin{aligned}\rho_1 = \tfrac{1}{4}V_A\Pi_1 V_A^\dagger \otimes (I\otimes I - c_1z_1\sigma_1\otimes\sigma_1 - c_2z_2\sigma_2\otimes\sigma_2\\ - c_3z_3\sigma_3\otimes\sigma_3),\end{aligned} \quad (7)$$

with

$$\begin{aligned}z_1 &= 2(-t_Ay_{A2} + y_{A1}y_{A3}),\\ z_2 &= 2(t_Ay_{A1} + y_{A2}y_{A3}),\\ z_3 &= t_A^2 - y_{A1}^2 - y_{A2}^2 + y_{A3}^2.\end{aligned}$$

Thus, we have $\mathrm{Tr}_C(\rho_0) = \frac{1}{2}V_A\Pi_0V_A^\dagger \otimes I$ and $\mathrm{Tr}_C(\rho_1) = \frac{1}{2}V_A\Pi_1V_A^\dagger \otimes I$. The average entropy of subsystem B after measuring Π^A is given by

$$S_{B|\Pi^A}(\rho) = \tfrac{1}{2}\times 1 + \tfrac{1}{2}\times 1 = 1. \quad (8)$$

To evaluate $S_{C|\Pi^{AB}}(\rho)$, one needs to measure subsystem B under the conditions of the outcomes on measuring A. Let

$$\left\{B_k^j = V_{B^j}\Pi_kV_{B^j}^\dagger : k = 0, 1\right\}, \quad j = 0, 1,$$

be the von Neumann measurement on subsystem B when the outcome of the measurement on A is j $(j = 0, 1)$, where $V_{B^j} = t_{B^j}I + i\overrightarrow{y_{B^j}}\cdot\overrightarrow{\sigma}$ is the unitary operator with $t_{B^j} \in \mathbb{R}$, $\overrightarrow{y_{B^j}} = (y_{B^j1}, y_{B^j2}, y_{B^j3}) \in \mathbb{R}^3$, and $t_{B^j}^2 + y_{B^j1}^2 + y_{B^j2}^2 + y_{B^j3}^2 = 1$.

If the measurement outcome on system A is 0, the state after the measurement will be reduced to ρ_0 given in Eq. (6). Notice that subsystems B and C in Eq. (6) are still in a Bell-diagonal state. After performing the measurement $\{B_k^0 : k = 0, 1\}$, the state reduces to

$$\begin{aligned}\rho_{00} &= \tfrac{1}{2}V_A\Pi_0V_A^\dagger \otimes V_{B^0}\Pi_0V_{B^0}^\dagger \otimes (I + c_1z_1l_1\sigma_1\\ &\quad + c_2z_2l_2\sigma_2 + c_3z_3l_3\sigma_3),\\ \rho_{01} &= \tfrac{1}{2}V_A\Pi_0V_A^\dagger \otimes V_{B^0}\Pi_1V_{B^0}^\dagger \otimes (I - c_1z_1l_1\sigma_1\\ &\quad - c_2z_2l_2\sigma_2 - c_3z_3l_3\sigma_3),\end{aligned}$$

with the probability $p_{00} = p_{01} = \frac{1}{4}$, where

$$\begin{aligned}l_1 &= 2(-t_{B^0}y_{B^02} + y_{B^01}y_{B^03}),\\ l_2 &= 2(t_{B^0}y_{B^01} + y_{B^02}y_{B^03}),\\ l_3 &= t_{B^0}^2 - y_{B^01}^2 - y_{B^02}^2 + y_{B^03}^2.\end{aligned}$$

If the measurement outcome on system A is 1, performing the measurement $\{B_k^1 : k = 0, 1\}$ on subsystem B of the state ρ_1, we obtain

$$\rho_{10} = \tfrac{1}{2} V_A \Pi_1 V_A^\dagger \otimes V_{B^1} \Pi_0 V_{B^1}^\dagger \otimes (I - c_1 z_1 m_1 \sigma_1 - c_2 z_2 m_2 \sigma_2 - c_3 z_3 m_3 \sigma_3),$$
$$\rho_{11} = \tfrac{1}{2} V_A \Pi_1 V_A^\dagger \otimes V_{B^1} \Pi_1 V_{B^1}^\dagger \otimes (I + c_1 z_1 m_1 \sigma_1 + c_2 z_2 m_2 \sigma_2 + c_3 z_3 m_3 \sigma_3),$$

with the probability $p_{10} = p_{11} = \frac{1}{4}$, where

$$m_1 = 2(-t_{B^1} y_{B^1 2} + y_{B^1 1} y_{B^1 3}),$$
$$m_2 = 2(t_{B^1} y_{B^1 1} + y_{B^1 2} y_{B^1 3}),$$
$$m_3 = t_{B^1}^2 - y_{B^1 1}^2 - y_{B^1 2}^2 + y_{B^1 3}^2.$$

The state $\rho_{\Pi^{AB}}$ is given by $\rho_{\Pi^{AB}} = p_{00}\rho_{00} + p_{01}\rho_{01} + p_{10}\rho_{10} + p_{11}\rho_{11}$. Set $\alpha = \sqrt{c_1^2 z_1^2 l_1^2 + c_2^2 z_2^2 l_2^2 + c_3^2 z_3^2 l_3^2}$ and $\beta = \sqrt{c_1^2 z_1^2 m_1^2 + c_2^2 z_2^2 m_2^2 + c_3^2 z_3^2 m_3^2}$. Then

$$S_{C|\Pi^{AB}}(\rho) = -\frac{1+\alpha}{4}\log_2(1+\alpha) - \frac{1-\alpha}{4}\log_2(1-\alpha) - \frac{1+\beta}{4}\log_2(1+\beta) - \frac{1-\beta}{4}\log_2(1-\beta) + 1.$$

It can be directly verified that $z_1^2 + z_2^2 + z_3^2 = 1$, $l_1^2 + l_2^2 + l_3^2 = 1$, $m_1^2 + m_2^2 + m_3^2 = 1$. Denote

$$c := \max\{|c_1|, |c_2|, |c_3|\}. \tag{9}$$

Then

$$\alpha \leqslant \sqrt{|c^2|(|z_1|^2|l_1|^2 + |z_2|^2|l_2|^2 + |z_3|^2|l_3|^2)} = c, \tag{10}$$

and

$$\beta \leqslant \sqrt{|c^2|(|z_1|^2|m_1|^2 + |z_2|^2|m_2|^2 + |z_3|^2|m_3|^2)} = c. \tag{11}$$

The equality holds in Eq. (10) for the following cases: (1) If $c = |c_1|$, then $|z_1| = |l_1| = 1$, $z_2 = z_3 = l_2 = l_3 = 0$. For instance, $|t_A| = |y_{A2}| = |t_{B^0}| = |y_{B^0 2}| = \frac{1}{\sqrt{2}}$ and $y_{A1} = y_{A3} = y_{B^0 1} = y_{B^0 3} = 0$. (2) If $c = |c_2|$, then $|z_2| = |l_2| = 1$, $z_1 = z_3 = l_1 = l_3 = 0$. For example, $|t_A| = |y_{A1}| = |t_{B^0}| = |y_{B^0 1}| = \frac{1}{\sqrt{2}}$ and $y_{A2} = y_{A3} = y_{B^0 2} = y_{B^0 3} = 0$. (3) If $c = |c_3|$, then $|z_3| = |l_3| = 1$, $z_1 = z_2 = l_1 = l_2 = 0$, e.g., $y_{A1} = y_{A2} = y_{B^0 1} = y_{B^0 2} = 0$. Similarly, one can prove that the equality holds in Eq. (11) too for the above cases. Therefore, we obtain

$$\min[S_{C|\Pi^{AB}}(\rho)] = -\frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c) + 1. \tag{12}$$

From (5), (8), and (12), we get the quantum discord

$$D_{A;B;C}(\rho) = \min_{\Pi^{AB}}[-S_{BC|A}(\rho) + S_{B|\Pi^A}(\rho) + S_{C|\Pi^{AB}}(\rho)]$$
$$= \frac{1+\xi}{2}\log_2(1+\xi) + \frac{1-\xi}{2}\log_2(1-\xi) - \frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c). \tag{13}$$

We now consider the family of the four-qubit case,

$$\rho = \frac{1}{16}\left(I + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j \otimes \sigma_j \otimes \sigma_j\right), \tag{14}$$

in systems A_1, A_2, A_3, and A_4. The four-qubit quantum discord is given by

$$D_{A_1;A_2;A_3;A_4}(\rho) = \min_{\Pi^{A_1 A_2 A_3}} \left[-S_{A_2A_3A_4|A_1}(\rho) + S_{A_2|\Pi^{A_1}}(\rho) + S_{A_3|\Pi^{A_1A_2}}(\rho) + S_{A_4|\Pi^{A_1A_2A_3}}(\rho)\right]. \tag{15}$$

For (14) we have $\rho_{A_1} = \mathrm{Tr}_{A_2A_3A_4}(\rho) = \frac{I}{2}$ and the entropy of subsystem A_1 is $S(\rho_{A_1}) = 1$. It can be directly verified that

$$S(\rho) = -\tfrac{1}{4}[(1 + c_1 - c_2 - c_3)\log_2(1 + c_1 - c_2 - c_3) + (1 - c_1 + c_2 - c_3)\log_2(1 - c_1 + c_2 - c_3) + (1 - c_1 - c_2 + c_3)\log_2(1 - c_1 - c_2 + c_3) + (1 + c_1 + c_2 + c_3)\log_2(1 + c_1 + c_2 + c_3)] + 4.$$

Therefore,

$$-S_{A_2A_3A_4|A_1}(\rho) = \tfrac{1}{4}[(1 + c_1 - c_2 - c_3)\log_2(1 + c_1 - c_2 - c_3) + (1 - c_1 + c_2 - c_3)\log_2(1 - c_1 + c_2 - c_3) + (1 - c_1 - c_2 + c_3)\log_2(1 - c_1 - c_2 + c_3) + (1 + c_1 + c_2 + c_3)\log_2(1 + c_1 + c_2 + c_3)] - 3. \tag{16}$$

The von Neumann measurement on subsystem A_1 is given by $\{A_{1k} = V_{A_1}\Pi_k V_{A_1}^\dagger : k = 0, 1\}$, where $V_{A_1} = t_{A_1} I + i\overrightarrow{y_{A_1}} \cdot \overrightarrow{\sigma}$, with $t_{A_1} \in \mathbb{R}$, $\overrightarrow{y_{A_1}} = (y_{A_1 1}, y_{A_1 2}, y_{A_1 3}) \in \mathbb{R}^3$, and $t_{A_1}^2 + y_{A_1 1}^2 + y_{A_1 2}^2 + y_{A_1 3}^2 = 1$.

The state $\rho_{\Pi^{A_1}}$ is given by $\rho_{\Pi^{A_1}} = p_0\rho_0 + p_1\rho_1$, where $p_0 = p_1 = \frac{1}{2}$, and

$$\rho_0 = \tfrac{1}{8} V_{A_1}\Pi_0 V_{A_1}^\dagger \otimes (I \otimes I \otimes I + c_1 d_1 \sigma_1 \otimes \sigma_1 \otimes \sigma_1 + c_2 d_2 \sigma_2 \otimes \sigma_2 \otimes \sigma_2 + c_3 d_3 \sigma_3 \otimes \sigma_3 \otimes \sigma_3),$$
$$\rho_1 = \tfrac{1}{8} V_{A_1}\Pi_1 V_{A_1}^\dagger \otimes (I \otimes I \otimes I - c_1 d_1 \sigma_1 \otimes \sigma_1 \otimes \sigma_1 - c_2 d_2 \sigma_2 \otimes \sigma_2 \otimes \sigma_2 - c_3 d_3 \sigma_3 \otimes \sigma_3 \otimes \sigma_3),$$

where

$$d_1 = 2(-t_{A_1} y_{A_1 2} + y_{A_1 1} y_{A_1 3}),$$
$$d_2 = 2(t_{A_1} y_{A_1 1} + y_{A_1 2} y_{A_1 3}),$$
$$d_3 = t_{A_1}^2 - y_{A_1 1}^2 - y_{A_1 2}^2 + y_{A_1 3}^2.$$

Thus, we have $\mathrm{Tr}_{A_3A_4}(\rho_0) = \frac{1}{2} V_{A_1}\Pi_0 V_{A_1}^\dagger \otimes I$ and $\mathrm{Tr}_{A_3A_4}(\rho_1) = \frac{1}{2} V_{A_1}\Pi_1 V_{A_1}^\dagger \otimes I$. The average entropy of subsystem A_2 after the measurement Π^{A_1} is given by $S_{A_2|\Pi^{A_1}}(\rho) = 1$.

To evaluate $S_{A_3|\Pi^{A_1A_2}}(\rho)$ and $S_{A_4|\Pi^{A_1A_2A_3}}(\rho)$, we need to measure subsystem A_2 based on the measurement outcomes on A_1. We obtain

$$\rho_{00} = \tfrac{1}{4} V_{A_1}\Pi_0 V_{A_1}^\dagger \otimes V_{A_2^0}\Pi_0 V_{A_2^0}^\dagger \otimes (I \otimes I + c_1 d_1 e_1 \sigma_1 \otimes \sigma_1 + c_2 d_2 e_2 \sigma_2 \otimes \sigma_2 + c_3 d_3 e_3 \sigma_3 \otimes \sigma_3),$$

$$\rho_{01}=\tfrac{1}{4}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_1V_{A_2^0}^\dagger\otimes(I\otimes I-c_1d_1e_1\sigma_1\otimes\sigma_1 -c_2d_2e_2\sigma_2\otimes\sigma_2-c_3d_3e_3\sigma_3\otimes\sigma_3),$$
$$\rho_{10}=\tfrac{1}{4}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_0V_{A_2^1}^\dagger\otimes(I\otimes I-c_1d_1f_1\sigma_1\otimes\sigma_1 -c_2d_2f_2\sigma_2\otimes\sigma_2-c_3d_3f_3\sigma_3\otimes\sigma_3),$$
$$\rho_{11}=\tfrac{1}{4}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_1V_{A_2^1}^\dagger\otimes(I\otimes I+c_1d_1f_1\sigma_1\otimes\sigma_1 +c_2d_2f_2\sigma_2\otimes\sigma_2+c_3d_3f_3\sigma_3\otimes\sigma_3),$$

where the k in the unitary $\{V_{A_2^k}:k=0,1\}$ is the outcome of the measurement of A_1, and $V_{A_2^k}$ can be written as $V_{A_2^k}=t_{A_2^k}I+i\overrightarrow{y_{A_2^k}}\cdot\overrightarrow{\sigma}$, with $t_{A_2^k}\in\mathbb{R}$, $\overrightarrow{y_{A_2^k}}=(y_{A_2^k1},y_{A_2^k2},y_{A_2^k3})\in\mathbb{R}^3$, and $t_{A_2^k}^2+y_{A_2^k1}^2+y_{A_2^k2}^2+y_{A_2^k3}^2=1$,

$$e_1=2\big(-t_{A_2^0}y_{A_2^02}+y_{A_2^01}y_{A_2^03}\big),$$
$$e_2=2\big(t_{A_2^0}y_{A_2^01}+y_{A_2^02}y_{A_2^03}\big),$$
$$e_3=t_{A_2^0}^2-y_{A_2^01}^2-y_{A_2^02}^2+y_{A_2^03}^2,$$
$$f_1:=2\big(-t_{A_2^1}y_{A_2^12}+y_{A_2^11}y_{A_2^13}\big),$$
$$f_2:=2\big(t_{A_2^1}y_{A_2^11}+y_{A_2^12}y_{A_2^13}\big),$$
$$f_3:=t_{A_2^1}^2-y_{A_2^11}^2-y_{A_2^12}^2+y_{A_2^13}^2.$$

The state $\rho_{\Pi^{A_1A_2}}$ is given by $\rho_{\Pi^{A_1A_2}}=p_{00}\rho_{00}+p_{01}\rho_{01}+p_{10}\rho_{10}+p_{11}\rho_{11}$. Thus, we have $\mathrm{Tr}_{A_4}(\rho_{00})=\frac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_0V_{A_2^0}^\dagger\otimes I$, $\mathrm{Tr}_{A_4}(\rho_{01})=\frac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_1V_{A_2^0}^\dagger\otimes I$, $\mathrm{Tr}_{A_4}(\rho_{10})=\frac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_0V_{A_2^1}^\dagger\otimes I$, $\mathrm{Tr}_{A_4}(\rho_{11})=\frac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_1V_{A_2^1}^\dagger\otimes I$. The average entropy of subsystem A_3 after the measurement $\Pi^{A_1A_2}$ is given by $S_{A_3|\Pi^{A_1A_2}}(\rho)=4\times(\frac{1}{4}\times1)=1$.

To evaluate $S_{A_4|\Pi^{A_1A_2A_3}}(\rho)$, one needs to continue to measure subsystem A_3 based on the measurement outcomes on A_1 and A_2. We obtain

$$\rho_{000}=\tfrac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_0V_{A_2^0}^\dagger\otimes V_{A_3^{00}}\Pi_0V_{A_3^{00}}^\dagger\otimes(I +c_1d_1e_1g_1\sigma_1+c_2d_2e_2g_2\sigma_2+c_3d_3e_3g_3\sigma_3),$$
$$\rho_{001}=\tfrac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_0V_{A_2^0}^\dagger\otimes V_{A_3^{00}}\Pi_1V_{A_3^{00}}^\dagger\otimes(I -c_1d_1e_1g_1\sigma_1-c_2d_2e_2g_2\sigma_2-c_3d_3e_3g_3\sigma_3),$$
$$\rho_{010}=\tfrac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_1V_{A_2^0}^\dagger\otimes V_{A_3^{01}}\Pi_0V_{A_3^{01}}^\dagger\otimes(I -c_1d_1e_1h_1\sigma_1-c_2d_2e_2h_2\sigma_2-c_3d_3e_3h_3\sigma_3),$$
$$\rho_{011}=\tfrac{1}{2}V_{A_1}\Pi_0V_{A_1}^\dagger\otimes V_{A_2^0}\Pi_1V_{A_2^0}^\dagger\otimes V_{A_3^{01}}\Pi_1V_{A_3^{01}}^\dagger\otimes(I +c_1d_1e_1h_1\sigma_1+c_2d_2e_2h_2\sigma_2+c_3d_3e_3h_3\sigma_3),$$
$$\rho_{100}=\tfrac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_0V_{A_2^1}^\dagger\otimes V_{A_3^{10}}\Pi_0V_{A_3^{10}}^\dagger\otimes(I -c_1d_1f_1n_1\sigma_1-c_2d_2f_2n_2\sigma_2-c_3d_3f_3n_3\sigma_3),$$
$$\rho_{101}=\tfrac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_0V_{A_2^1}^\dagger\otimes V_{A_3^{10}}\Pi_1V_{A_3^{10}}^\dagger\otimes(I +c_1d_1f_1n_1\sigma_1+c_2d_2f_2n_2\sigma_2+c_3d_3f_3n_3\sigma_3),$$
$$\rho_{110}=\tfrac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_1V_{A_2^1}^\dagger\otimes V_{A_3^{11}}\Pi_0V_{A_3^{11}}^\dagger\otimes(I +c_1d_1f_1r_1\sigma_1+c_2d_2f_2r_2\sigma_2+c_3d_3f_3r_3\sigma_3),$$
$$\rho_{111}=\tfrac{1}{2}V_{A_1}\Pi_1V_{A_1}^\dagger\otimes V_{A_2^1}\Pi_1V_{A_2^1}^\dagger\otimes V_{A_3^{11}}\Pi_1V_{A_3^{11}}^\dagger\otimes(I -c_1d_1f_1r_1\sigma_1-c_2d_2f_2r_2\sigma_2-c_3d_3f_3r_3\sigma_3),$$

where the k in the unitary $\{V_{A_3^{ku}}:k=0,1;u=0,1\}$ is the outcome of the measurement of A_1, and u is the outcome of the measurement of A_2. Denote

$$\mu_1=\sqrt{c_1^2d_1^2e_1^2g_1^2+c_2^2d_2^2e_2^2g_2^2+c_3^2d_3^2e_3^2g_3^2},$$
$$\mu_2=\sqrt{c_1^2d_1^2e_1^2h_1^2+c_2^2d_2^2e_2^2h_2^2+c_3^2d_3^2e_3^2h_3^2},$$
$$\mu_3=\sqrt{c_1^2d_1^2f_1^2n_1^2+c_2^2d_2^2f_2^2n_2^2+c_3^2d_3^2f_3^2n_3^2},$$
$$\mu_4=\sqrt{c_1^2d_1^2f_1^2r_1^2+c_2^2d_2^2f_2^2r_2^2+c_3^2d_3^2f_3^2r_3^2}.$$

We have

$$\begin{aligned}S_{A_4|\Pi^{A_1A_2A_3}}(\rho)&=-\frac{1+\mu_1}{8}\log_2(1+\mu_1)-\frac{1-\mu_1}{8}\log_2(1-\mu_1)\\&\quad-\frac{1+\mu_2}{8}\log_2(1+\mu_2)-\frac{1-\mu_2}{8}\log_2(1-\mu_2)\\&\quad-\frac{1+\mu_3}{8}\log_2(1+\mu_3)-\frac{1-\mu_3}{8}\log_2(1-\mu_3)\\&\quad-\frac{1+\mu_4}{8}\log_2(1+\mu_4)-\frac{1-\mu_4}{8}\log_2(1-\mu_4)+1.\end{aligned}$$

It can be directly verified that $d_1^2+d_2^2+d_3^2=1$, $e_1^2+e_2^2+e_3^2=1$, $f_1^2+f_2^2+f_3^2=1$, $g_1^2+g_2^2+g_3^2=1$, $h_1^2+h_2^2+h_3^2=1$, $n_1^2+n_2^2+n_3^2=1$, and $r_1^2+r_2^2+r_3^2=1$. Since $\mu_1\leqslant c$, $\mu_2\leqslant c$, $\mu_3\leqslant c$, and $\mu_4\leqslant c$, we obtain

$$\min[S_{A_4|\Pi^{A_1A_2A_3}}(\rho)]=-\frac{1+c}{2}\log_2(1+c)-\frac{1-c}{2}\log_2(1-c)+1. \quad (17)$$

By the definition of the four-qubit quantum discord (15), we get

$$\begin{aligned}D_{A_1;A_2;A_3;A_4}(\rho)&=\frac{1}{4}[(1+c_1-c_2-c_3)\log_2(1+c_1-c_2-c_3)\\&\quad+(1-c_1+c_2-c_3)\log_2(1-c_1+c_2-c_3)\\&\quad+(1-c_1-c_2+c_3)\log_2(1-c_1-c_2+c_3)\\&\quad+(1+c_1+c_2+c_3)\log_2(1+c_1+c_2+c_3)]\\&\quad-\frac{1+c}{2}\log_2(1+c)-\frac{1-c}{2}\log_2(1-c). \quad (18)\end{aligned}$$

From the results of three-qubit and four-qubit states, we can prove the following conclusion for a general N-qubit case.

Theorem 1. For the family of N-qubit states (2), we have the quantum discord:

(1) If $N=2v+1$, $v\in\mathbf{N}^+$,

$$\begin{aligned}D_{A_1;A_2;\ldots;A_{2v+1}}(\rho)&=\frac{1+\xi}{2}\log_2(1+\xi)+\frac{1-\xi}{2}\log_2(1-\xi)\\&\quad-\frac{1+c}{2}\log_2(1+c)-\frac{1-c}{2}\log_2(1-c), \quad (19)\end{aligned}$$

where $\xi=\sqrt{c_1^2+c_2^2+c_3^2}$, $c=\max\{|c_1|,|c_2|,|c_3|\}$.

(2) If $N = 4v - 2$, $v \in \mathbf{N}^+$,

$$D_{A_1;A_2;\ldots;A_{4v-2}}(\rho) = \frac{1}{4}[(1-c_1-c_2-c_3)\log_2(1-c_1-c_2-c_3) + (1-c_1+c_2+c_3)\log_2(1-c_1+c_2+c_3) + (1+c_1-c_2+c_3)\log_2(1+c_1-c_2+c_3) + (1+c_1+c_2-c_3)\log_2(1+c_1+c_2-c_3)] - \frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c). \quad (20)$$

(3) If $N = 4v$, $v \in \mathbf{N}^+$,

$$D_{A_1;A_2;\ldots;A_{4v}}(\rho) = \frac{1}{4}[(1+c_1-c_2-c_3)\log_2(1+c_1-c_2-c_3) + (1-c_1+c_2-c_3)\log_2(1-c_1+c_2-c_3) + (1-c_1-c_2+c_3)\log_2(1-c_1-c_2+c_3) + (1+c_1+c_2+c_3)\log_2(1+c_1+c_2+c_3)] - \frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c). \quad (21)$$

Proof. If $N = 2v + 1$, we have $\mathrm{Tr}_{A_2A_3\cdots A_{2v+1}}(\rho) = \frac{I}{2}$, and $S_{A_1}(\rho) = 1$. Let λ be the eigenvalues of ρ. From the characteristic equation $\det|\rho_{2v+1} - \lambda I| = 0$, we get

$$\left[\frac{(1-2^{2v+1}\lambda)^2 - c_1^2 - c_2^2 - c_3^2}{2^{(4v+2)}}\right]^{2^{2v}} = 0.$$

The 2^{2v} eigenvalues are given by $\frac{1}{2^{2v+1}}(1 - \sqrt{c_1^2+c_2^2+c_3^2})$ and $\frac{1}{2^{2v+1}}(1 + \sqrt{c_1^2+c_2^2+c_3^2})$, respectively. One can verify that

$$-S_{A_2;A_3;\ldots;A_{2v+1}|A_1}(\rho) = \frac{1+\xi}{2}\log_2(1+\xi) + \frac{1-\xi}{2}\log_2(1-\xi) - 2v. \quad (22)$$

We obtain $S_{A_k|\Pi^{A_1A_2\cdots A_{k-1}}}(\rho) = 1$, where $k = 2,\ldots,2v$, namely,

$$S_{A_2|\Pi^{A_1}}(\rho) = S_{A_3|\Pi^{A_1A_2}}(\rho) = \cdots = S_{A_{2v}|\Pi^{A_1A_2\cdots A_{2v-1}}}(\rho) = 1. \quad (23)$$

Hence,

$$\min[S_{A_{2v+1}|\Pi^{A_1A_2\cdots A_{2v}}}(\rho)] = -\frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c) + 1. \quad (24)$$

By the definition (1), we obtain Eq. (19).

If $N = 4v - 2$, we have $\mathrm{Tr}_{A_2A_3\cdots A_{4v-2}}(\rho) = \frac{I}{2}$ and $S_{A_1}(\rho) = 1$. ρ has 2^{4v-4} eigenvalues given by $\frac{1}{2^{4v-2}}(1-c_1-c_2-c_3)$, $\frac{1}{2^{4v-2}}(1-c_1+c_2+c_3)$, $\frac{1}{2^{4v-2}}(1+c_1-c_2+c_3)$, and $\frac{1}{2^{4v-2}}(1+c_1+c_2-c_3)$, respectively. Then we obtain $-S_{A_2;A_3;\ldots;A_4v-2}(\rho) = \frac{1}{4}[(1-c_1-c_2-c_3)\log_2(1-c_1-c_2-c_3) + (1-c_1+c_2+c_3)\log_2(1-c_1+c_2+c_3) + (1+c_1-c_2+c_3)\log_2(1+c_1-c_2+c_3) + (1+c_1+c_2-c_3)\log_2(1+c_1+c_2-c_3)] - 4v + 3$. The entropy after the measurement is the same as (23) and (24). Therefore, we obtain Eq. (20). Equation (21) is similarly proved. ■

From Theorem 1, one has that for five-qubit states,

$$D_{A_1;A_2;A_3;A_4;A_5}(\rho) = \frac{1+\xi}{2}\log_2(1+\xi) + \frac{1-\xi}{2}\log_2(1-\xi) - \frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c), \quad (25)$$

and for six-qubit states,

$$D_{A_1;A_2;A_3;A_4;A_5;A_6}(\rho) = \frac{1}{4}[(1-c_1-c_2-c_3)\log_2(1-c_1-c_2-c_3) + (1-c_1+c_2+c_3)\log_2(1-c_1+c_2+c_3) + (1+c_1-c_2+c_3)\log_2(1+c_1-c_2+c_3) + (1+c_1+c_2-c_3)\log_2(1+c_1+c_2-c_3)] - \frac{1+c}{2}\log_2(1+c) - \frac{1-c}{2}\log_2(1-c). \quad (26)$$

Interestingly, the result (25) is equivalent to the three-qubit quantum discord (13), while (26) is equivalent to the two-qubit quantum discord given by Luo [25].

Figure 1 shows the level surfaces of discord for $D(\rho) = 0.03$, 0.15, and 0.55. The three figures (F_{21}), (F_{22}), and (F_{23}) in the second row of Fig. 1 are for $(4v-2)$-qubit states, which are consistent with the ones given in Ref. [26] for two-qubit states. For small discord, $D(\rho) = 0.03$ and 0.15, the level surfaces are centrally symmetric, consisting of three intersecting "tubes" along the three coordinate axes. For a larger discord value 0.55, these intersecting tubes expand until only a few vertices remained, where (F_{13}) has level surfaces in eight corners, while (F_{23}) and (F_{33}) have only four corners left.

III. DYNAMICS OF QUANTUM DISCORD UNDER LOCAL NONDISSIPATIVE CHANNELS

It has been discovered that for two-qubit states, the quantum discord is invariant under some decoherence channels in a finite time interval [30,34]. To verify if such phenomena still exist in multiqubit systems, we consider that the states ρ (3) and (14) under the phase flip channel, with the Kraus operators $\Gamma_0^{(A_1)} = \mathrm{diag}(\sqrt{1-p/2}, \sqrt{1-p/2}) \otimes I \otimes \cdots \otimes I$, $\Gamma_1^{(A_1)} = \mathrm{diag}(\sqrt{p/2}, -\sqrt{p/2}) \otimes I \otimes \cdots \otimes I, \ldots,$ $\Gamma_0^{(A_N)} = I \otimes \cdots \otimes I \otimes \mathrm{diag}(\sqrt{1-p/2}, \sqrt{1-p/2})$, $\Gamma_1^{(A_N)} = I \otimes \cdots \otimes I \otimes \mathrm{diag}(\sqrt{p/2}, -\sqrt{p/2})$, where $N = 3, 4$, $p = 1 - \exp(-\gamma t)$, and γ is the phase damping rate.

Let $\varepsilon(\cdot)$ represent the operator of decoherence. For the three-qubit state (3) under the phase flip channel, we have

$$\varepsilon(\rho) = \tfrac{1}{8}[I \otimes I \otimes I + (1-p)^3 c_1\sigma_1 \otimes \sigma_1 \otimes \sigma_1 + (1-p)^3 c_2\sigma_2 \otimes \sigma_2 \otimes \sigma_2 + c_3\sigma_3 \otimes \sigma_3 \otimes \sigma_3]. \quad (27)$$

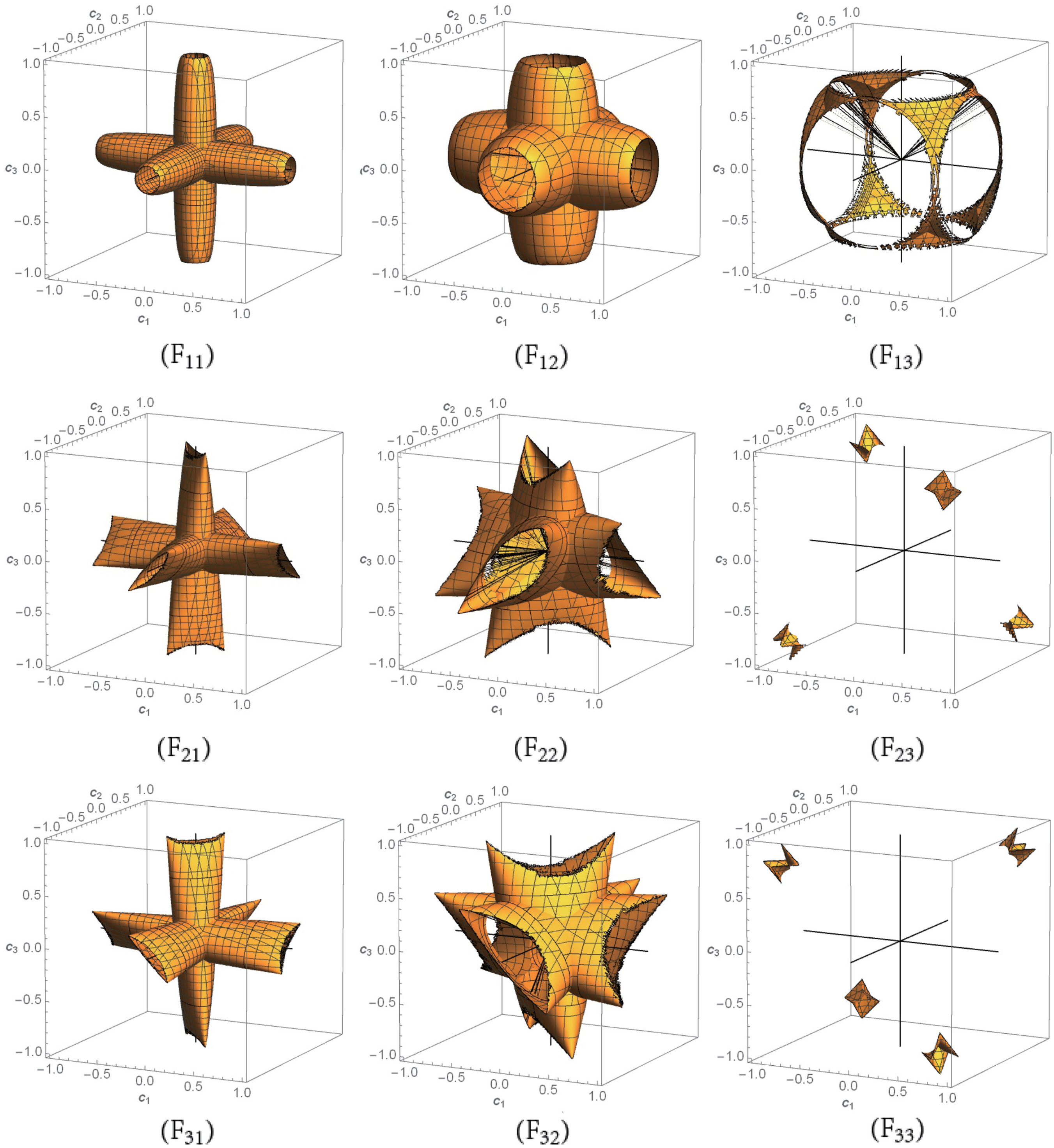

FIG. 1. Level surfaces of constant discord. $N = 2v + 1$ for figures (F_{11}), (F_{12}), and (F_{13}) with $D(\rho) = 0.03$, 0.15, and 0.55, respectively. $N = 4v - 2$ for figures (F_{21}), (F_{22}), and (F_{23}) with $D(\rho) = 0.03$, 0.15, and 0.55, respectively. $N = 4v$ for figures (F_{31}), (F_{32}), and (F_{33}) with $D(\rho) = 0.03$, 0.15, and 0.55, respectively.

From (13), we obtain

$$D_{A_1;A_2;A_3}[\varepsilon(\rho)] = \frac{1+\delta}{2}\log_2(1+\delta) + \frac{1-\delta}{2}\log_2(1-\delta) - \frac{1+\theta}{2}\log_2(1+\theta) - \frac{1-\theta}{2}\log_2(1-\theta), \tag{28}$$

where $\delta = \sqrt{(1-p)^6 c_1^2 + (1-p)^6 c_2^2 + c_3^2}$, $\theta = \max\{|(1-p)^3 c_1|, |(1-p)^3 c_2|, |c_3|\}$. Notice that the derivative of $D_{A_1;A_2;A_3}[\varepsilon(\rho)]$ with respect to p is always less than 0.

Proof. From (28), in general, the derivative of $D_{A_1;A_2;A_3}[\varepsilon(\rho)]$ can be cast as

$$D'_{A_1;A_2;A_3}[\varepsilon(\rho)] = \frac{1}{2}\left[\theta' \log_2\left(\frac{1-\theta}{1+\theta}\right) - \delta' \log_2\left(\frac{1-\delta}{1+\delta}\right)\right].$$

In particular, for

$$\delta = \sqrt{(1-p)^6 c_1^2 + (1-p)^6 c_2^2 + c_3^2} > 0,$$
$$\theta = \max\{|(1-p)^3 c_1|, |(1-p)^3 c_2|, |c_3|\} > 0.$$

This implies that

$$\log_2\left(\frac{1-\theta}{1+\theta}\right) < 0 \quad \text{and} \quad \log_2\left(\frac{1-\delta}{1+\delta}\right) < 0,$$

we have that

$$\delta' = -\frac{3(c_1^2+c_2^2)}{\delta}(1-p)^5,$$

given that $0 < (1-p)^5 < 1$, then $\delta' < 0$.

If $\theta = |c_3|$, then

$$D'_{A_1;A_2;A_3}[\varepsilon(\rho)] = \frac{-\delta'}{2}\log_2\left(\frac{1-\delta}{1+\delta}\right) < 0.$$

And for $\theta = |(1-p)^3 c_1|$, in this case $\theta = (1-p)^3|c_1|$ because $0 < 1-p < 1$, then

$$\theta' = -3(1-p)^2|c_1| < 0.$$

Then

$$D'_{A_1;A_2;A_3}[\varepsilon(\rho)] = \frac{3}{2}\left[\frac{(c_1^2+c_2^2)}{\delta}(1-p)^5\log_2\left(\frac{1-\delta}{1+\delta}\right) - (1-p)^2|c_1|\log_2\left(\frac{1-\theta}{1+\theta}\right)\right].$$

Here, we assume that $D_{A_1;A_2;A_3}[\varepsilon(\rho)]$ is monotonically decreasing, i.e., $D'_{A_1;A_2;A_3}[\varepsilon(\rho)] < 0$. In order to show these, it must satisfy that

$$\frac{(c_1^2+c_2^2)}{\delta}(1-p)^5\log_2\left(\frac{1-\delta}{1+\delta}\right) < (1-p)^2|c_1|\log_2\left(\frac{1-\theta}{1+\theta}\right),$$

by multiplying by the positive numbers $(1-p)$ and δ,

$$(c_1^2+c_2^2)(1-p)^6\log_2\left(\frac{1-\delta}{1+\delta}\right) < \delta\theta\log_2\left(\frac{1-\theta}{1+\theta}\right).$$

Therefore, $D'_{A_1;A_2;A_3}[\varepsilon(\rho)] < 0$, which means that $D_{A_1;A_2;A_3}[\varepsilon(\rho)]$ is monotonically decreasing. ■

Hence, the frozen phenomenon of quantum discord does not exist for three-qubit states under the phase flip channel. Since the quantum discord of three-qubit and $(2v+1)$-qubit states are the same, the odd-qubit systems do not exhibit frozen phenomenon of quantum discord under the phase flip channel. For instance, take $c_1 = \frac{4}{5}$, $c_2 = \frac{c_1}{2}$, and $c_3 = \frac{1}{2}$ in the initial state; the dashed line in Fig. 2 shows the dynamic behavior of the quantum discord under the phase flip channel.

For $N = 4$, the state ρ under the phase flip channel is give by

$$\varepsilon(\rho) = \tfrac{1}{16}[I\otimes I\otimes I\otimes I + (1-p)^4 c_1\sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_1 + (1-p)^4 c_2\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 + c_3\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3]. \tag{29}$$

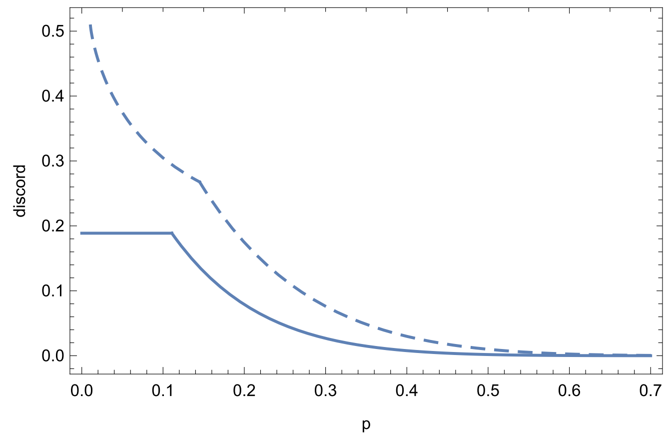

FIG. 2. Quantum discord of the three-qubit state (dashed line) and quantum discord of the four-qubit state (solid line) under a phase flip channel for $c_1 = \frac{4}{5}$, $c_2 = \frac{c_1}{2}$, $c_3 = \frac{1}{2}$.

Noting that c_3 is independent on time, we consider the case that $c_2 = c_1c_3$, $-1 \leqslant c_3 \leqslant 1$. Then we have $|c_2| \leqslant |c_1|$ for any p. From (18) we obtain the quantum discord

$$\begin{aligned} D_{A_1;A_2;A_3;A_4}[\varepsilon(\rho)] &= \frac{1+c_3}{2}\log_2(1+c_3) + \frac{1-c_3}{2}\log_2(1-c_3) \\ &+ \frac{1+(1-p)^4c_1}{2}\log_2[1+(1-p)^4c_1] \\ &+ \frac{1-(1-p)^4c_1}{2}\log_2[1-(1-p)^4c_1] \\ &- \frac{1+\sigma}{2}\log_2(1+\sigma) - \frac{1-\sigma}{2}\log_2(1-\sigma), \end{aligned}$$

where $\sigma = \max\{|(1-p)^4c_1|, |c_3|\}$.

When $\max\{|(1-p)^4c_1|, |c_3|\} = |(1-p)^4c_1|$, we have

$$D_{A_1;A_2;A_3;A_4}[\varepsilon(\rho)] = \frac{1+c_3}{2}\log_2(1+c_3) + \frac{1-c_3}{2}\log_2(1-c_3),$$

and $D_{A_1;A_2;A_3;A_4}[\varepsilon(\rho)]$ is constant under the decoherence channel during the time interval. Otherwise,

$$D_{A_1;A_2;A_3;A_4}[\varepsilon(\rho)] = \frac{1+(1-p)^4c_1}{2}\log_2[1+(1-p)^4c_1] + \frac{1-(1-p)^4c_1}{2}\log_2[1-(1-p)^4c_1],$$

which monotonically decreases to zero.

Therefore, to calculate the quantum discord, we need to determine the magnitude of $|(1-p)^4c_1|$ and $|c_3|$. If for $|c_1| > |c_3|$ there exists $0 \leqslant p_0 \leqslant 1$ such that $\max\{|(1-p)^4c_1|, |c_3|\} = |(1-p)^4c_1|$ for $0 \leqslant p \leqslant p_0$, and $\max\{|(1-p)^4c_1|, |c_3|\} = |c_3|$ for $p_0 \leqslant p \leqslant 1$, then $D_{A_1;A_2;A_3;A_4}[\varepsilon(\rho)]$ remains unchanged first, and then monotonicity goes down to zero. As an example, set $c_1 = \frac{4}{5}$, $c_2 = \frac{c_1}{2}$, and $c_3 = \frac{1}{2}$. The solid line in Fig. 2 shows the dynamic behavior of quantum discord under the phase flip channel. A sudden transition of quantum discord happens at $p = 0.110\,86$. The frozen phenomenon of quantum discord exists for the four-qubit states under the phase flip channel, while for the case of three-qubit states, such a phenomenon does not exist.

In Ref. [34], it has been shown that the frozen phenomenon of quantum discord also exists when the phase noise acts on two-qubit states. Since the quantum discord of two-qubit and $(4v-2)$-qubit states is the same, and the four-qubit quantum discord is equal to that of $(4v)$-qubit states, the even-qubit systems exhibit a frozen phenomenon of quantum discord under the phase flip channel, while the odd-qubit systems do not.

IV. SUMMARY

We have studied the quantum discord for a family of multiqubit states. Analytical formulas have been derived in detail for $(2v+1)$-, $(4v-2)$-, and $(4v)$-qubit states. The level surfaces of quantum discord have been depicted. It has been shown that under the phase flip channel the quantum discord could still remain constant in a certain time interval for the even-qubit systems, but not for odd-qubit systems. Our results may highlight further investigations on multipartite quantum discord and their applications in quantum information processing.

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