Order from chaos in quantum walks on cyclic graphs

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It has been shown classically that combining two chaotic random walks can yield an ordered (periodic) walk. Our aim in this paper is to find a quantum analog for this rather counterintuitive result. We study the chaotic and periodic nature of cyclic quantum walks and focus on a unique situation wherein a periodic quantum walk on 3-cycle graph is generated via a deterministic combination of two chaotic quantum walks on the same graph. We extend our results to even numbered cyclic graphs, specifically a 4-cycle graph too. Our results will be relevant in quantum cryptography and quantum chaos control.

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I. INTRODUCTION

Parrando's paradox describes situations wherein a random or deterministic combination of losing strategies can yield a winning outcome. Parrondo's games seen in random walks have been shown to have significant applications in biological systems, algorithms, and cryptology [1–3]. Studies in recent years have shown Parrando's paradox in classical random walks wherein two chaotic walks were combined to yield an ordered (periodic) walk [4–8]. In these papers the main focus is on combining two chaotic systems to generate order, i.e., $Chaos_1 + Chaos_2 = Order$, a profound and counterintuitive result. This result is far reaching, especially in classical chaos control theory [9], etc. Our aim in this work is to find the quantum analog of this result, and we show it in the context of quantum walks on 3-cycle graphs (see Fig. 1). Quantum walks (QWs) have been used in simulating physical systems [10] as well as in designing quantum algorithms [11,12]. A QW can be described on a one-dimensional lattice, or analogously on a circle with k sites, with the walker starting from the origin [13–15]. Unlike the walker of the classical random walk, in the quantum version the walker is represented by a wave function. However, similar to a classical random walk, QWs consist of a walker and a coin. The coin in a QW, in general, is a qubit. Quantum walks can also be fashioned with coins which are qutrits [16] or even qudits [17] and also entangled coins [18]. In this paper we only focus on qubits as coins. Similar to classical random walks, in QWs if applying the coin operator on the initial coin state yields head the walker shifts to the right or else the shift is to the left. In addition to head or tail, the coin state in the quantum case can be in a superposition of head and tail, in which case the walker moves to a corresponding superposition of left and right lattice sites.

A QW need not be restricted to a line, QWs on k-cycle graphs have been studied in detail in Refs. [13,14]. Intriguingly, in quantum walks on k-cyclic graphs, chaotic behavior has been seen. A quantum walk on a k-cycle graph is termed periodic if it returns periodically to a particular position, say the origin, after a finite number of steps, otherwise it is chaotic. Our aim in this work is to show that two chaotic QWs can be combined to yield a periodic walk. To fulfill our aim we proceed as follows, we first dwell on defining the shift and coin operators in a k-cycle graph and then find out the condition for the walker to be ordered or periodic. Next we deal with specific coin operators which yield chaotic QWs and find out the conditions under which combination of these coins, i.e., Parrondo sequences, will generate periodic QWs. We then show via plots the probability of returning to origin in a 3-cycle graph focusing on our aim of getting periodic QWs via a Parrondo strategy of alternating between coins which yield only chaotic QWs. Next, we show results for a 4-cycle graph wherein similar to the 3-cycle graph we see the combination of chaotic quantum walks leading to periodic quantum walks. Recently there have been reports on designing quantum algorithms for quantum key generation via mixing chaotic signals [19]. We at the end of this paper show how to do that via mixing quantum chaotic walks to generate a secure encryption-decryption mechanism.

II. DISCRETE TIME QUANTUM WALK (DTQW) ON CYCLIC GRAPHS

In the DTQW on a cycle, similar to that on a line, the space of the walker is defined via the tensor product of position and coin space, i.e., $\boldsymbol{H_P} \otimes \boldsymbol{H_c}$, where $\boldsymbol{H_P}$ is the position Hilbert space and $\boldsymbol{H_c}$ is the coin Hilbert space. In this case, coin is a qubit with two states $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$, and then the general unitary coin operator can be written as

$$C_2(\rho, \alpha, \beta) = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} \\ \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} \end{pmatrix}, \quad (1)$$

where $0 \leqslant \rho \leqslant 1$, $0 \leqslant \alpha$, and $\beta \leqslant \pi$.

The walker shifts to the right by one site when the final state is $|1\rangle$ and to the left by one site when the final state is $|0\rangle$. So the shift operator for th0e case when the walker walks

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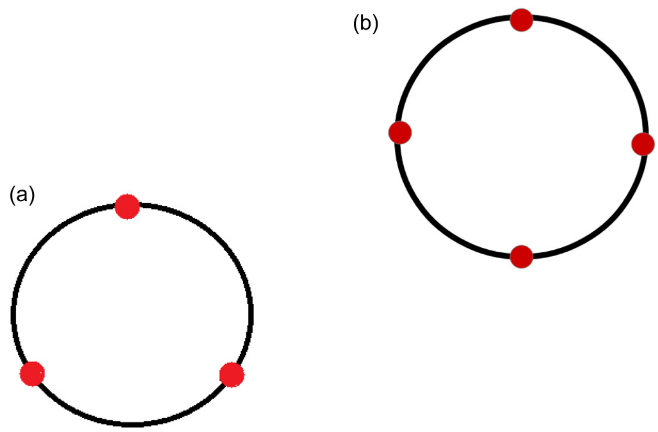

FIG. 1. 3-cycle graph (a) and 4-cycle graph (b).

on a line is given by

$$S = \sum_{s=0}^{1} \sum_{i=-\infty}^{\infty} |i + 2s - 1\rangle\langle i| \otimes |s\rangle\langle s|, \qquad (2)$$

but to perform a QW on a circle of "k" sites the shift operator has to be modified [14] as shown below:

$$S = \sum_{s=0}^{1} \sum_{i=0}^{k-1} |(i + 2s - 1), \quad \text{mod} \quad k\rangle\langle i| \otimes |s\rangle\langle s|. \qquad (3)$$

Using Eq. (3) and the coin operator we can represent the QW by a unitary operator as.

$$U_k = S(I_k \otimes C_2). \qquad (4)$$

The matrix U_k is a 2×2 block circulant matrix (see also Refs. [13,14]). It is represented as

$$U_k = \text{CIRC}_k \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_0, \begin{bmatrix} \sqrt{\rho} & \sqrt{1-\rho}e^{i\alpha} \\ 0 & 0 \end{bmatrix}_1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_2, \ldots, \begin{bmatrix} 0 & 0 \\ \sqrt{1-\rho}e^{i\beta} & -\sqrt{\rho}e^{i(\alpha+\beta)} \end{bmatrix}_{k-1} \right). \qquad (5)$$

The walker repeatedly applies U_k to its initial state to get to its final state. For N steps we get, $U_k^N|\psi_i\rangle = |\psi_f\rangle$. If the walker returns to its initial state after N steps for any arbitrary initial quantum state, then the QW by the walker is said to be periodic; hence, for a periodic QW we have

$$U_k^N|\psi_i\rangle = |\psi_i\rangle. \qquad (6)$$

Now let eigenvectors of U_k be $\{|x_i\rangle\}$ and the corresponding eigenvalue be $\{\lambda_i\}$, as $|\psi_i\rangle$ is arbitrary we can represent it in terms of eigenvectors of U_k as $|\psi_i\rangle = \sum_{j=1}^{2k} \alpha_j |x_j\rangle$. Applying U_k, N times, on an initial state we get

$$U_k^N|\psi_i\rangle = \sum_{j=1}^{2k} \alpha_j \lambda_j^N |\mathbf{x}_j\rangle. \qquad (7)$$

Comparing Eq. (7) with Eq. (6) we get the the condition of periodicity as

$$\lambda_j^N = 1, \ \forall\ 1 \leqslant j \leqslant 2k, \quad \text{or} \quad U_k^N = I_{2k}. \qquad (8)$$

In the case where a particular unitary operator satisfies Eq. (8) and then it gives a periodic result, that operator is said to generate a periodic or ordered QW, while if the QW does not satisfy Eq. (8) it is called chaotic.

Block diagonalizing U_k

By diagonalizing U_k we get its eigenvalues, which then simplifies the problem of finding the periodicity of the QW on cyclic graphs. As U_k is a circulant matrix it is block diagonalized by a tool known as the commensurate Fourier matrix as was also done in Ref. [14]. The commensurate Fourier matrix of M dimension is defined as

$$F^M = (F^M_{m,n}) = \frac{1}{\sqrt{M}} (e^{2\pi i \frac{mn}{M}}).$$

Now as our matrix is of dimension $k \otimes 2$ we define $F = F^{k\otimes 2}$ and $U_{k'} = F U_k F^\dagger$. The matrix $U_{k'}$ has the form

$$U_{k'} = \begin{pmatrix} U_{k,0} & 0 & \ldots & 0 \\ 0 & U_{k,1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & U_{k,k-1} \end{pmatrix}, \qquad (9)$$

where each $U_{k,l}$ is a block 2×2 matrix. The eigenvalues of each such block $U_{k,l}$ is given by

$$\lambda^{\pm}_{k,l} = \frac{1}{2} e^{-2\pi i \frac{l}{k}} \Big[\big(1 - e^{4\pi i \frac{l}{k} + i\delta}\big)\sqrt{\rho} \pm 2\sqrt{e^{4\pi i \frac{l}{k} + i\delta}\left(1 - \rho \sin^2\left[\frac{2\pi l}{k} + \frac{\delta}{2}\right]\right)} \Big]. \qquad (10)$$

Now Eq. (8) is satisfied if both eigenvalues, i.e, $\lambda^{\pm}_{k,l}$, take the form of a de Movire number, i.e., each $\lambda_j = e^{2\pi i \frac{m_j}{n_j}}$, which can be equivalently written as

$$\lambda^{\pm}_{k,l} = e^{2\pi i \frac{m_j}{n_j}} \quad \text{or} \quad \lambda_{k,l} = \frac{\lambda^{+}_{k,l} + \lambda^{-}_{k,l}}{2} = e^{2\pi i \frac{m_j}{n_j}}, \qquad (11)$$

and the least common multiple of $\{n_j\} = N$, with $1 \leqslant j \leqslant 2k$; this gives the periodicity of a QW on a k-cyclic graph to be N. Examples of periodic and chaotic QWs can be seen in Refs. [13,14] which satisfy Eqs. (10) and (11) and are also given below in Fig. 2 for parameters mentioned in the figure caption.

III. PARRONDO STRATEGIES (*ABAB*...) IN DTQW ON 3-CYCLE GRAPHS

In the subsequent analysis in this section we stick to only QWs on 3-cycle graphs [see Fig. 1(a)]. We consider three different unitary operators $A = U_3(\rho_1, \alpha_1, \beta_1) = S[I_3 \times C_2(\rho_1, \alpha_1, \beta_1)]$, $B = U_3(\rho_2, \alpha_2, \beta_2) = S[I_3 \times C_2(\rho_2, \alpha_2, \beta_2)]$, and $C = U_3(\rho, \alpha, \beta) = S[I_3 \times C_2(\rho, \alpha, \beta)]$, with the coin operator C_2 defined in Eq. (2) and U_3 in Eq. (4) with $k = 3$. We note that only a QW obtained from the unitary operator C satisfies Eq. (6) and gives an ordered QW with period N, while unitary operators A and B lead to chaotic QWs. Our aim is to find a suitable combination of A and B coin operators which would give an ordered (periodic) QW. To achieve our goal, we first check the Parrondo sequence *ABAB*... of unitary

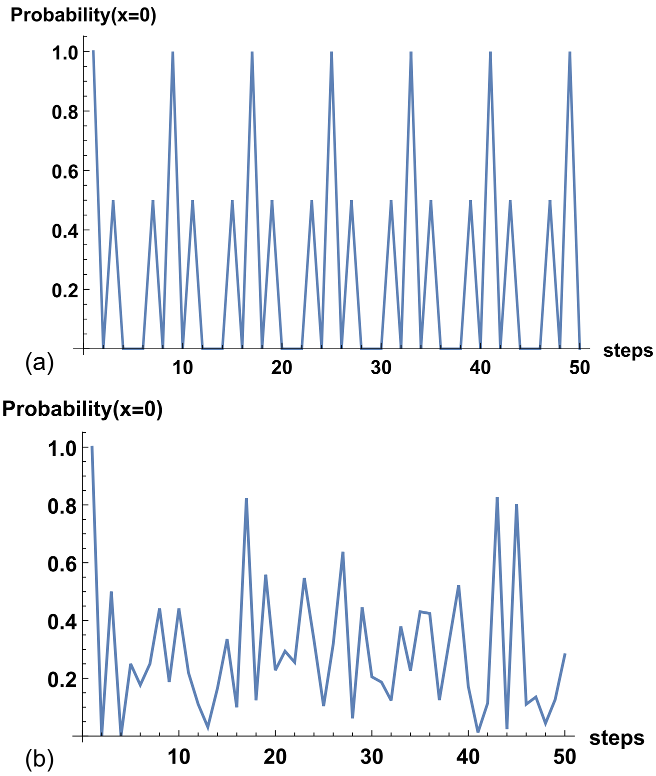

FIG. 2. Probability for finding the walker at its initial site at $x = 0$ for a 4-cycle graph [panel (a), ordered, i.e., periodic with period 8] and a 5-cycle graph [panel (b), chaotic]. Probability is plotted against time steps for the QW with a Hadamard coin $H = C_2(\frac{1}{2}, 0, 0)$. Only even time steps are plotted.

operators and calculate the eigenvalues of the matrix $FABF^{\dagger}$. For the 3-cycle graph, there will be three diagonal blocks $U_{3,0}$, $U_{3,1}$, and $U_{3,2}$ [see Eq. (9)]. The sum of eigenvalues for the diagonal block $U_{3,1}$ is

$$\lambda_{3,1}^{AB} = \frac{\lambda_{3,1}^{AB+} + \lambda_{3,1}^{AB-}}{2} = \big(2e^{i\alpha_2+i\beta_1}\sqrt{1-\rho_1}\sqrt{1-\rho_2} + 2e^{i\alpha_1+i\beta_2}\sqrt{1-\rho_1}\sqrt{1-\rho_2} + 2(-1)^{2/3}\sqrt{\rho_1}\sqrt{\rho_2} - 2(-1)^{1/3}e^{i\alpha_1+i\alpha_2+i\beta_1+i\beta_2}\sqrt{\rho_1}\sqrt{\rho_2}\big)/4. \tag{12}$$

Since our aim is to check if the Parrondo combination $ABAB\ldots$ leads to periodicity. We now evaluate the eigenvalues of the matrix $[FCCF^{\dagger}]$, remembering that the unitary operator C generates a periodic QW. The eigenvalues for a 3-cycle graph with coin $CC\ldots$, and for the same diagonal block, $U_{3,1}$, are given as

$$\lambda_{3,1}^{CC} = \frac{\lambda_{3,1}^{CC+} + \lambda_{3,1}^{CC-}}{2} = \big[4e^{i(\alpha+\beta)} + 2(-1)^{2/3}\rho - 4e^{i(\alpha+\beta)}\rho - 2(-1)^{1/3}e^{2i(\alpha+\beta)}\rho\big]/4. \tag{13}$$

Since repeatedly applying C generates a periodic QW, if repeatedly applying AB were to also generate a periodic QW, then the form of the eigenvalues should match, i.e.,

$$\lambda_{3,1}^{CC} = \lambda_{3,1}^{AB}. \tag{14}$$

Repeating the abovementioned procedure for other diagonal blocks $U_{3,0}$ and $U_{3,2}$, we get exactly similar equations to that shown in Eq. (14). Taking $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ and equating the coefficients of frequencies on both sides, we get two equations as follows:

$$\rho_1\rho_2 = \rho^2, \quad \text{and} \quad \sqrt{1-\rho_1}\sqrt{1-\rho_2} = 1-\rho. \tag{15}$$

The only solution to Eq. (15) is $\rho_1 = \rho_2 = \rho$, which gives the trivial solution $A = B$, as $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. This solution is not desirable as the deterministic combination of A and B gives again a chaotic QW. Following this, similar calculations done for deterministic combinations ABB, $ABBB$, $ABBBB$, and $AABB$ do not generate ordered (periodic) QWs. The solutions for $ABBB$ and $ABBBB$ were trivial as in the case of AB. However, ABB gives a nontrivial solution for a small range of ρ; one could find suitable coin operators A and B for which ABB is periodic but unfortunately we were not able to zero in on any exact values for ρ_1, ρ_2, and ρ. Nonetheless, the deterministic combination $AABB\ldots$ does generates an ordered (periodic) QW as shown below.

IV. PARRONDO STRATEGIES (*AABB*...) IN DTQW ON 3-CYCLE GRAPHS

We start by calculating the eigenvalues of the matrix $FAABBF^{\dagger}$. Similar to the case preceding, for the 3-cycle graph, there will be three diagonal blocks: $U_{3,0}$, $U_{3,1}$, and $U_{3,2}$ [see Eq. (9)]. The sum of eigenvalues for the diagonal block $U_{3,1}$ is

$$\begin{aligned}\lambda_{3,1}^{AABB} &= \frac{\lambda_{3,1}^{AABB+} + \lambda_{3,1}^{AABB-}}{2} = 4e^{i(\alpha_1+\alpha_2+\beta_1+\beta_2)} \\ &+ 2(-1)^{2/3}e^{i(\alpha_2+\beta_2)}\rho_1 - 4e^{i\alpha_1+i\alpha_2+i\beta_1+i\beta_2}\rho_1 \\ &- 2(-1)^{1/3}e^{2i\alpha_1+i\alpha_2+2i\beta_1+i\beta_2}\rho_1 + 2(-1)^{2/3}e^{i\alpha_1+i\beta_1}\rho_2 \\ &- 4e^{i\alpha_1+i\alpha_2+i\beta_1+i\beta_2}\rho_2 - 2(-1)^{1/3}e^{i\alpha_1+2i\alpha_2+i\beta_1+2i\beta_2}\rho_2 \\ &+ 2[-(-1)^{1/3} - (-1)^{2/3}e^{i(\alpha_1+\beta_1)} - (-1)^{2/3}e^{i(\alpha_2+\beta_2)} \\ &+ 2e^{i(\alpha_1+\alpha_2+\beta_1+\beta_2)} + (-1)^{2/3}e^{2i(\alpha_1+\alpha_2+\beta_1+\beta_2)} \\ &+ (-1)^{1/3}e^{i(2\alpha_1+\alpha_2+2\beta_1+\beta_2)} \\ &+ (-1)^{1/3}e^{i(\alpha_1+2\alpha_2+\beta_1+2\beta_2)}]\rho_1\rho_2 - 2(e^{i(\alpha_2+\beta_1)} \\ &+ e^{i(\alpha_1+\beta_2)})[-(-1)^{2/3} + e^{i(\alpha_1+\beta_1)} + e^{i(\alpha_2+\beta_2)} \\ &+ (-1)^{1/3}e^{i(\alpha_1+\alpha_2+\beta_1+\beta_2)}]\sqrt{(1-\rho_1)\rho_1(1-\rho_2)\rho_2}.\end{aligned} \tag{16}$$

Since our aim is to check if the Parrondo combination $AABB\ldots$ leads to periodicity. We now evaluate the eigenvalues of the matrix $[FCCCCF^{\dagger}]$, remembering that the coin operator C generates a periodic QW. The eigenvalues for a 3-cycle graph with coin operators $CCCC\ldots$ being applied successively and for the same diagonal block $U_{3,1}$ are given as

$$\begin{aligned}\lambda_{3,1}^{CCCC} &= \frac{\lambda_{3,1}^{CCCC+} + \lambda_{3,1}^{CCCC-}}{2} = 4e^{2i(\alpha+\beta)} + 8(-1)^{2/3}e^{i(\alpha+\beta)}\rho \\ &- 16e^{2i(\alpha+\beta)}\rho - 8(-1)^{1/3}e^{3i(\alpha+\beta)}\rho - 2(-1)^{1/3}\rho^2 \\ &- 8(-1)^{2/3}e^{i(\alpha+\beta)}\rho^2 + 12e^{2i(\alpha+\beta)}\rho^2 \\ &+ 8(-1)^{1/3}e^{3i(\alpha+\beta)}\rho^2 + 2(-1)^{2/3}e^{4i(\alpha+\beta)}\rho^2.\end{aligned} \tag{17}$$

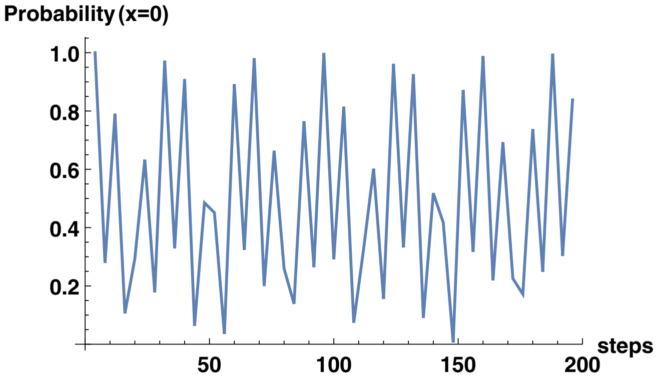

FIG. 3. The QW on a 3-cycle graph by repeatedly applying the unitary $A = U_3(\rho_1, 0, 0)$, which results in a chaotic quantum walk.

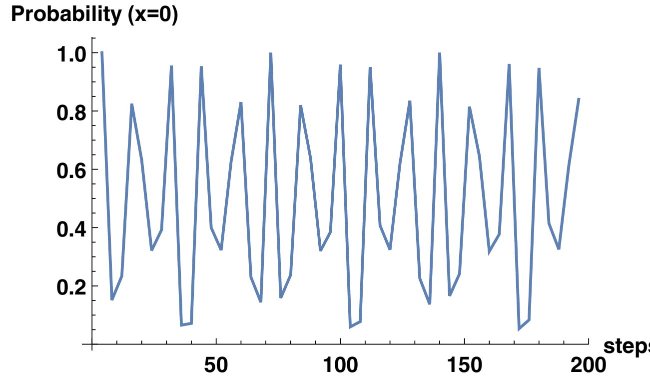

FIG. 4. A chaotic QW on a 3-cycle graph obtained by repeatedly applying the unitary $B = U_3(\rho_2, 0, 0)$.

Since repeatedly applying C gives a periodic QW, if repeatedly applying $AABB$ were to generate a periodic QW, then the form of the eigenvalues should match, i.e.,

$$\lambda_{3,1}^{CCCC} = \lambda_{3,1}^{AABB}. \tag{18}$$

Equations similar to Eq. (18) can be written for other blocks, i.e., $U_{3,0}$ and $U_{3,2}$. Taking $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ and matching frequencies on both sides, we get

$$\begin{aligned} \rho_1 + \rho_2 - 2\rho_1\rho_2 + 2\sqrt{(1-\rho_1)(1-\rho_2)\rho_1\rho_2} &= 4\rho - 4\rho^2, \\ \rho_1 + \rho_2 - \rho_1\rho_2 + 2\sqrt{(1-\rho_1)(1-\rho_2)\rho_1\rho_2} &= 4\rho - 3\rho^2, \\ \rho_1\rho_2 &= \rho^2. \end{aligned} \tag{19}$$

Of the above three equations, only two are independent. The third can be derived from the other two. The solution to Eq. (19) is given by

$$\begin{aligned} \rho_1 &\to 3\rho - 4\rho^2 \pm 2\sqrt{2}\sqrt{\rho^2(1 - 3\rho + 2\rho^2)}, \\ \rho_2 &\to 3\rho - 4\rho^2 \mp 2\sqrt{2}\sqrt{\rho^2(1 - 3\rho + 2\rho^2)}. \end{aligned} \tag{20}$$

This gives the possibility of A and B being chaotic but $AABB$ being periodic.

V. PARRONDO STRATEGIES ($A'A'B'B'\ldots$) IN DTQW ON 4-CYCLE GRAPHS

Similar to the 3-cycle graph, for the 4-cycle graph we consider three different unitary operators: $A' = U_4(\rho_{41}, \alpha_1, \beta_1) = S[I_4 \times C_2(\rho_{41}, \alpha_1, \beta_1)]$, $B' = U_4(\rho_{42}, \alpha_2, \beta_2) = S[I_4 \times C_2(\rho_{42}, \alpha_2, \beta_2)]$, and $C' = U_4(\rho_4, \alpha, \beta) = S[I_4 \times C_2(\rho_4, \alpha, \beta)]$, with the coin operator C_2 defined in Eq. (2) and U_4 in Eq. (4) with $k = 4$. We note that only the QW obtained from the unitary operator C' satisfies Eq. (6) and gives an ordered QW with period N, while unitary operators A' and B' lead to chaotic QWs. We check the Parrondo sequence $A'A'B'B'\ldots$ of unitary operators and calculate the eigenvalues of the matrix $FAABBF^\dagger$. For the 4-cycle graph, there will be four diagonal blocks: $U_{4,0}$, $U_{4,1}$, $U_{4,2}$, and $U_{4,3}$ [see Eq. (9)]. Following exactly the same procedure as was adopted for 3-cycle walks, one gets an identical set of equations, Eqs. (19) and (20), from which we get the condition for unitaries A' and B' generating chaotic quantum walks in the 4-cycle graph, but $A'A'B'B'$ generating a periodic quantum walk. In the Results subsection on the 4-cycle graph (Sec. VI B), we give the details of parameters which lead to the desired outcome of two chaotic quantum walks A' and B' combining in sequence $A'A'B'B'$ to generate a periodic quantum walk in a 4-cycle graph.

VI. RESULTS

A. 3-cycle graph

In Refs. [13,14], a table of examples which satisfy Eq. (6) has been given. From these examples, and considering $k = 3$ with $\rho = \frac{(5-\sqrt{5})}{6} = 0.460\,655$, $\alpha = 0$, and $\beta = 0$, and using Eq. (20) with $\alpha_1 = \alpha_2 = \alpha = 0$ and $\beta_1 = \beta_2 = \beta = 0$, we get

$$\rho_1 = 0.264\,734, \quad \rho_2 = 0.801\,571. \tag{21}$$

Considering unitary operators $A = U_3(\rho_1, 0, 0)$ and $B = U_3(\rho_2, 0, 0)$ while $C = U_3(\rho, 0, 0)$ and repeating the calculations done in Eqs. (14) and (15) shows that one cannot get periodicity, i.e., coins $ABABAB\ldots$ generate a chaotic QW when quantum walks resulting from repeated application of coins $AAA\ldots$ and $BBB\ldots$ are chaotic too. This is shown in Figs. 3, 4, and 5. The initial state of the walker is taken as $|1\rangle \otimes |0\rangle$, and the probability of the walker to be in the initial site 0 versus the number of steps of the walker is plotted. In Fig. 6, we show that the combination of unitary operators $AABB$ does generate an ordered (periodic) QW fulfilling our

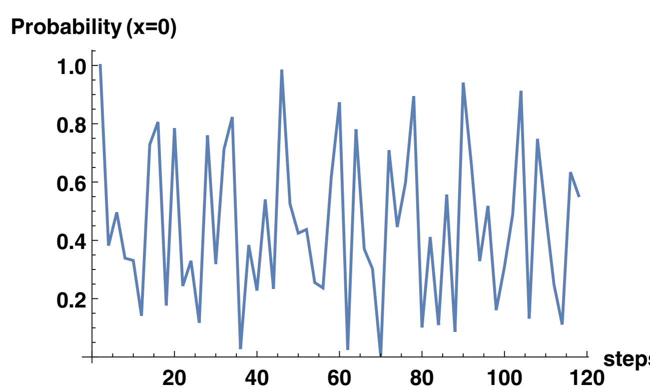

FIG. 5. A chaotic QW on a 3-cycle graph by repeatedly applying the Parrondo sequence $ABAB\ldots$, plotting every second point.

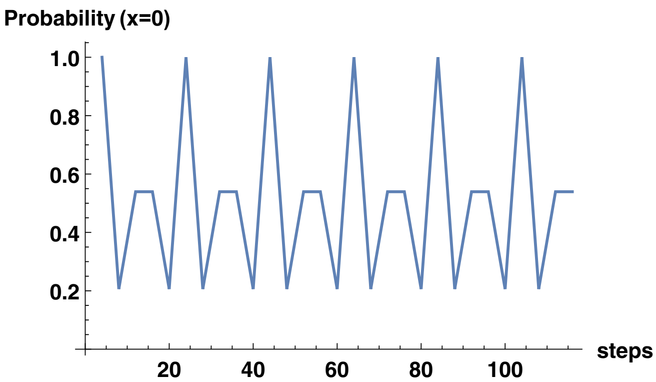

FIG. 6. An ordered QW on a 3-cycle graph by repeatedly applying the Parrondo sequence *AABB*... Every fourth point is plotted. The quantum walk is periodic with a periodicity of 20.

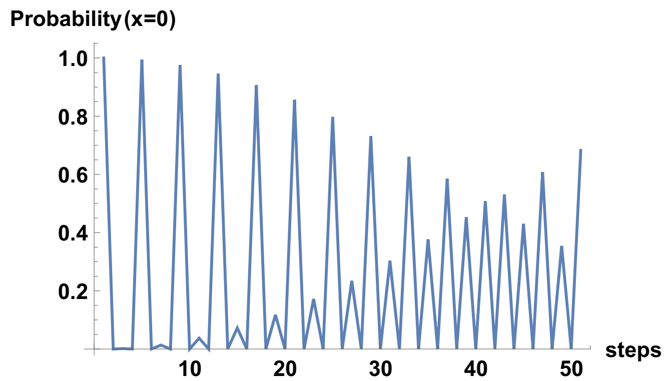

FIG. 8. A QW on a 4-cycle graph by repeatedly applying the unitary $A' = U_4(\rho_{41}, 0, 0)$, which results in chaotic quantum walk.

aim to show that combining two chaotic systems can in certain situations lead to an ordered or periodic outcome. The periodicity of the combination *AABB* is 20. It is to be noted that the combination *CCCC* plotted in Fig. 7 gives a periodicity of 10. The reason for this is because the value of every fourth point in Fig. 6 is equal to the value of every fourth point in Fig. 7; i.e., the probability for finding the walker at $x = 0$, at time steps 4, 8, and 12, etc., is the same, due to which we miss a periodicity of 10 and get a periodicity of 20 for *AABB*... Another method of determining whether the walk is periodic or chaotic is via calculating the Lyapunov exponent. In the Appendix, we give a short recipe for calculating this. For chaotic quatum walks the Lyapunov exponent [20] is positive, while for periodic walks it vanishes. We indeed verify these results for the unitary operators *AAAA*... and *BBBB*..., which give a finite positive value for the Lyapunov exponent and which generate chaotic walks. In the case of unitary operators *AABB*..., we get a vanishing Lyapunov exponent which confirms a periodic quantum walk.

B. 4-cycle graph

Although we have shown results only in the case of a 3-cycle graph, our work can be easily generalized to any arbitrary k-cycle graph. In fact the equations obtained for a 4-cycle graph are identical to those shown in Eqs. (12)–(20). This is no doubt a consequence of the properties of the cyclic graph itself as well as that of the commensurate Fourier matrix F. In Refs. [13,14], tables of examples which satisfy Eq. (6) have been given. From these examples, and considering $k = 4$ with $\rho_4 = \frac{(5-\sqrt{5})}{8} = 0.345\,492$, $\alpha = 0$, and $\beta = 0$, and using Eq. (20) with $\alpha_1 = \alpha_2 = \alpha = 0$ and $\beta_1 = \beta_2 = \beta = 0$, we get

$$\rho_{41} = 0.998\,489, \quad \rho_{42} = 0.119\,545. \tag{22}$$

Considering unitary operators $A' = U_4(\rho_{41}, 0, 0)$ and $B' = U_4(\rho_{42}, 0, 0)$ while $C' = U_4(\rho_4, 0, 0)$ and repeating the calculations done in Eqs. (14) and (15), we show that quantum walks resulting from repeated application of coins $A'A'A'$... and $B'B'B'$... are chaotic. This is shown in Figs. 8 and 9. The initial state of the walker is taken as $|1\rangle \otimes |0\rangle$, and the probability of the walker to be in the initial site 0 versus the number of steps of the walker is plotted. In Fig. 10, we show that the combination of unitary operators $A'A'B'B'$ does generate an ordered (periodic) QW, fulfilling our aim to show that combining two chaotic systems can in certain situations lead to an ordered or periodic outcome. The periodicity of the combination $A'A'B'B'$ is 5. It is to be noted that the combination $C'C'C'C'$ plotted in Fig. 11 gives a periodicity of 10.

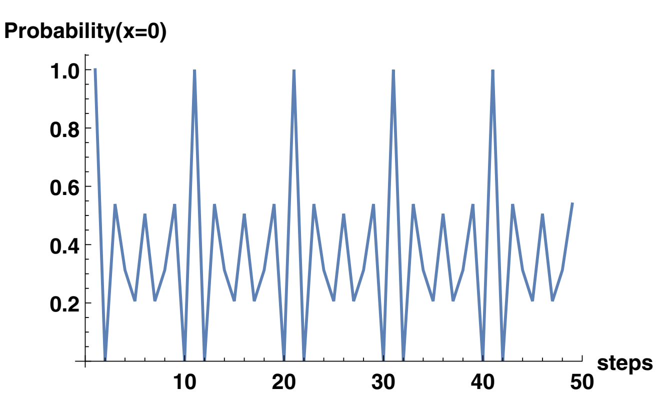

FIG. 7. An ordered QW on a 3-cycle graph by repeatedly applying the unitary C. Every fourth point is plotted. The quantum walk is periodic with a periodicity of 10.

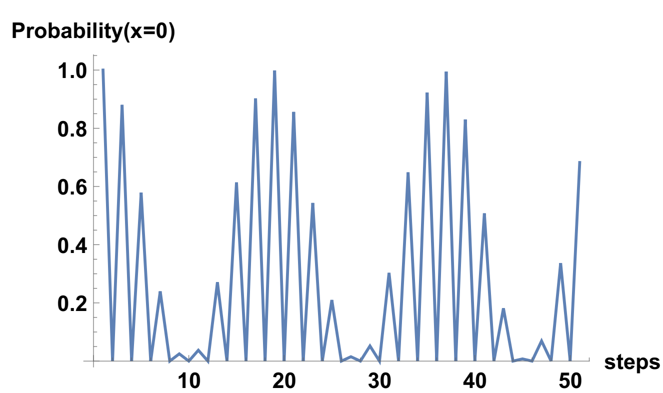

FIG. 9. A chaotic QW on a 4-cycle graph obtained by repeatedly applying the unitary $B' = U_4(\rho_{42}, 0, 0)$.

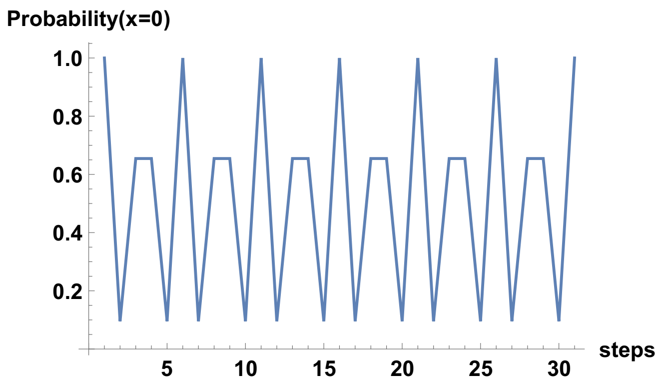

FIG. 10. An ordered QW on a 4-cycle graph by repeatedly applying the Parrondo sequence $A'A'B'B'$ . . . The quantum walk is periodic with a periodicity of 5.

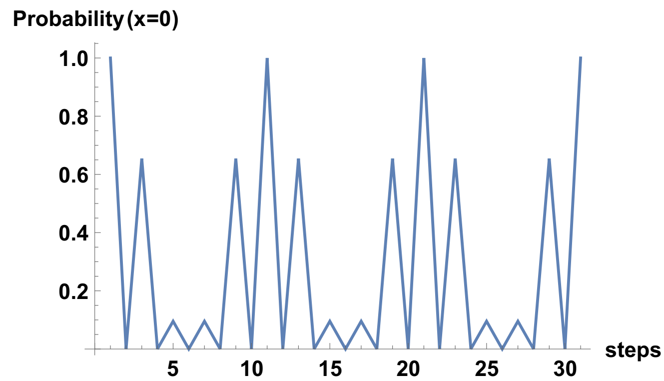

FIG. 11. An ordered QW on a 4-cycle graph by repeatedly applying the unitary C'. The quantum walk is periodic with a periodicity of 10.

VII. SECURE ENCRYPTION-DECRYPTION MECHANISM VIA MIXING CHAOTIC QUANTUM WALKS

In a recent work [21], a quantum cryptographic protocol based on a quantum walk has been proposed. We tweak this proposal by implementing it with chaotic quantum walks. This adds a further layer of security on top of that shown in Ref. [21]. These are the steps.

(i) *Generating the public key*. If Bob has to send a message $m \in \{0, 1, 2\}$ to Alice, then Alice will make a public key using the unitary operator B and the state of the walker $|l\rangle|s\rangle$ as follows $|\psi_{PK}\rangle = BB|l\rangle|s\rangle$. Here, B is the unitary operator that generates the chaotic quantum walk as shown in Fig. 4, $|\psi_{PK}\rangle$ is a public key, and $|l\rangle$ is the walker state on a cyclic graph while $|s\rangle$ is a coin state. We also note that another coin unitary A can generate a chaotic quantum walk too (see Fig. 3). Further, as we have shown in Fig. 6, the Parrondo sequence $AABB$ generates a periodic chaotic quantum walk with a periodicity of 20 meaning $(AABB)^5 = I$, with I being the identity. Alice after generating this chaotic state $|\psi_{PK}\rangle$, which acts as the public key, sends it to Bob.

(ii) *Encrypting the message*. After Bob receives the public key, he encodes the message m as follows: $|\psi(m)\rangle = (T_m \otimes I_c)|\psi_{PK}\rangle$, where $T_m = \sum_{i=0}^{N-1} |i + m, \mod 3\rangle\langle i|$, is akin to the shift operator used in Eq. (3) for the position state, while I_c is the identity operator acting on the coin state.

(iii) *Decrypting the message*. Alice then decrypts the message by applying $D = (AABB)^4 AA$, as $DBB = I$. Alice then performs the measurement $M = \sum_i |i\rangle\langle i| \otimes I_c$ and obtains the message m'. The original message m can then be recovered via $m = m' - l, \mod 3$.

This protocol can be applied using the 4-cycle graph as well, following exactly the same procedure as outlined above for the 3-cycle graph.

VIII. CONCLUSION

The criteria for generating ordered or periodic QWs by combining two chaotic QWs have been established. It is shown that if a chaotic QW is obtained by repeatedly applying coins A or B then one cannot obtain a periodic QW by repeatedly applying AB, $ABBB$, or $ABBBB$. Parrondo's paradox was seen for the deterministic combination $AABB$, i.e., a periodic QW was obtained by repeatedly applying $AABB$ even when coins A and B generated chaotic QWs. Our work shows that it is possible to design a quantum periodic signal from two quantum chaotic signals. This also means that its reverse process is also possible, i.e., breaking a periodic quantum signal into two or more quantum chaotic signals. The periodic probability distribution generated from two chaotic ones seen in discrete time QWs on cyclic graphs is of great interest in designing new quantum algorithms, in quantum cryptology, and in development of quantum chaos control theory [22]. Finally, there have been recent reports on the development of quantum image encryption techniques via chaotic QWs on cyclic graphs [23]. Our work could provide intriguing possibilities in designing better image encryption protocols.

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APPENDIX: CALCULATION OF LYAPUNOV EXPONENT

In Secs. II–V we have used the same definition for a chaotic quantum walk as was also used in Ref. [13], which is if the walker returns to its initial position at $x = 0$ after a finite number of steps, with probability 1 then it is periodic, and if this probability of return to its initial position is never 1 then it is chaotic. However, another definition of whether the walk is chaotic or not can be determined via the Lyapunov exponent [20]. A positive value of the Lyapunov exponent, i.e., $\lambda > 0$, implies the walk is chaotic, while if $\lambda = 0$ then it is periodic or nonchaotic. We herein below give in short the recipe, see also Ref. [20], for calculating the Lyapunov exponent for cyclic quantum walks and then determine it for our case of a 3-cycle quantum walk. In Ref. [20], the chaotic walk is generated via a small change in the initial position of the walker. In our study, on the other hand, chaotic quantum walks are generated via unitary operators. To calculate the Lyapunov coefficient we start with the initial normalized state at time $t = t_0$ on a cyclic graph with three sites, $|\Psi(t_0)\rangle$. Then we let the initial state evolve with

$$|\Psi(t)\rangle = U^{t-t_0}|\Psi(t_0)\rangle,$$

with $t > t_0$. Now two cases arise, one wherein the unitary operators generate a chaotic walk denoted by U_c and the other wherein they generate a periodic walk, say U_p, with period t_p.

Thus, we have for $U = U_p$,

$$|\Psi(t_p)\rangle = U_p^{t_p-t_0}|\Psi(t_0)\rangle = |\Psi(t_0)\rangle,$$

i.e., $U_p^{t_p-t_0} = I$, where I is the identity matrix. On the other hand, for $U = U_c$,

$$|\Psi(t)\rangle = U_c^{t-t_0}|\Psi(t_0)\rangle \neq |\Psi(t_0)\rangle, \forall t.$$

Thus, one can define a "distance" state, i.e., $|\Psi_d\rangle = |\Psi(t)\rangle - |\Psi(t_0)\rangle$, with $|\Psi_d\rangle = 0$ for $U = U_p$ at $t = t_p$ while $|\Psi_d\rangle \neq 0$ for $U = U_c, \forall t$.

This probability distance function $d(t)$ can, similarly to Ref. [20], help us calculate the Lyapunov exponent. The probability distance function can be expanded as

$$\begin{aligned} d(t) &= |\langle\Psi_d \mid \Psi_d\rangle| = (\langle\Psi(t)| - \langle\Psi(t_0)|)(|\Psi(t)\rangle - |\Psi(t_0)\rangle), \\ &= |2 - 2\langle\Psi(t-t_0) \mid \Psi(t_0)\rangle| = f(\lambda, (t-t_0)). \end{aligned} \quad \text{(A1)}$$

In the above equation, λ is the Lyapunov exponent. The distance function $f(\lambda, (t-t_0))$ is bounded by a maximum value of 2 and a minimum value of 0. In the chaotic case, $\lambda > 0$ and the maximum possible value is 2, while for periodic case at $t = t_p$, $\lambda = 0$, and $d = 0$, this implies $f(\lambda, (t-t_0)) = 2(1 - 2^{-\lambda(t-t_0)})$ and for the Lyapunov exponent one obtains

$$\begin{aligned} \lambda &= -\frac{1}{t-t_0}\log_2|\langle\Psi(t) \mid \Psi(t_0)\rangle| \\ &= -\frac{1}{t-t_0}\log_2|\langle\Psi(t) \mid \Psi(0)\rangle| + \log_2|\langle\Psi(t_0) \mid \Psi(0)\rangle|, \end{aligned} \quad \text{(A2)}$$

wherein $|\Psi(0)\rangle$ is the initial state of the walker at time step $t = 0$. As the quantum walk is performed in a circular path, taking a large value of "t" is not suitable. Following this we get the Lyapunov exponent of process $AAAA\ldots$ as 0.012, for process $BBBB\ldots$ as 0.085, and for process $AABB\ldots$ as 0 for $t - t_0 = 20$ time steps. We also verified that the Lyapunov exponent remains positive for both $AAAA\ldots$ as well as $BBBB\ldots$ for any arbitrary value of $t - t_0$, while $\lambda = 0$ for $AABB$... at time steps $t - t_0 = 20, 40, 60, \ldots$ Thus, we indeed see that while the Lyapunov exponents for chaotic quantum walks in the 3-cycle graph are positive values, for the periodic quantum walk, the Lyapunov exponent vanishes. Similar results can be obtained for a 4-cycle graph wherein we also see for the periodic walk that the Lyapunov exponent vanishes, while for the chaotic walk there are finite positive values for the Lyapunov exponent.

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