Many-body anticommutator and its applications in the normalization of completely symmetric states with Majorana's stellar representations

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Understanding the quantum properties of many-body states is important in the fields of both quantum information and condensed-matter physics. For this purpose, we generalize the basic concept of the anticommutator from two operators to the many-body case. Some key properties are given for the many-body anticommutators, and examples are provided for Pauli operators and density matrices. Using these results and techniques from the symmetry group, in a straightforward way, we give the normalization of the completely symmetric states with Majorana's stellar representations. Our developed method will help to pave the way in the study of many-body symmetric systems.

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I. INTRODUCTION

The completely symmetric state can describe lots of quantum systems [1–3], especially the spin and boson system. Boson systems such as spin-J particles coupled by $n = 2J$ spin-$\frac{1}{2}$ particles (or pseudospin) have been widely studied in recent years [4–7]. The dynamics of Bose-Einstein condensates is observed with spin-1, spin-2, and even spin-3 atomic gases in experiments [8–13]. Theoretical research on the entanglement [1–3,14–18], the dynamics of the geometric phase [18–23], the metastable decay of currents [24], and the spin knots [25] in the boson system also has attracted a lot of interest.

How to present the evolution of these high-dimensional quantum systems clearly is always a challenge. For two-level systems, the identity matrix $\mathbb{I}$ and the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ constitute a complete basis of the space of 2×2 Hermitian matrices over the real numbers. Especially, the density matrix of two-level systems can be expressed as $\rho = \frac{1}{2}(\mathbb{I} + \boldsymbol{u} \cdot \boldsymbol{\sigma})$. To simplify the writing of the derivation process, we rewrite the identity matrix $\mathbb{I}$ as 1 in this paper, so the density matrix is rewritten as

$$\rho = \tfrac{1}{2}(1 + \boldsymbol{u} \cdot \boldsymbol{\sigma}). \quad (1)$$

Since the eigenvalues of ρ are positive, i.e., $\frac{1}{2}(1 \pm |\boldsymbol{u}|) \geqslant 0$, the coefficient vector $\boldsymbol{u}$ satisfies the condition $|\boldsymbol{u}| \leqslant 1$, with equality if and only if the state of the system is a pure state [26]. So the vector $\boldsymbol{u}$ can be represented by a point on or within the unit sphere, and the vector $\boldsymbol{u}$ is referred to as a Bloch vector when the unit sphere here is referred to as a Bloch sphere, after the physicist Felix Bloch.

Similarly, the Italian physicist Majorana proposed the representation of high-dimensional quantum systems in 1932, in which a quantum state of a spin-J system is viewed as a constellation of $2J$ stars on the Bloch sphere [27].

Majorana's stellar representation (MSR) is an intuitive way to study a spin-J quantum system, and it has also been widely used in a general completely symmetric state system in recent years [1,2,7,14,17–23,27–30]. For the completely symmetric state system coupled by n particles, if we know the n states $|\psi_i\rangle$, $i = 1, 2, \ldots, n$, we can express the wave function of the system as

$$|\Psi_{(n)}\rangle = \frac{1}{\mathcal{N}_n} \sum_{P \in S_n} |\psi_{P(1)}\rangle \otimes |\psi_{P(2)}\rangle \otimes \cdots \otimes |\psi_{P(n)}\rangle, \quad (2)$$

where $\mathcal{N}_n$ is the normalization constant, S_n denotes the nth permutation group, and $\sum_{P\in S_n}$ sums all n permutations with $P = \begin{pmatrix} 1 & 2 & \cdots & n \\ P(1) & P(2) & \cdots & P(n) \end{pmatrix}$ [31].

Although MSR of the completely symmetric state system has been used for many years, the origin of the form is much more empirical than it is intuitive [23]. The state $|\Psi_{(n)}\rangle$ is a multipartite entangled state [17,18,32]. How to study its properties is an essential issue. Even the normalization constant is not very direct to derive when one often can guess the form of it [23]. In this paper, we first introduce the anticommutator in a many-body system. By calculating the many-body anticommutator of density matrices, we can get the form of the normalization constant of the completely symmetric state in MSR in a straightforward way [23].

The structure of this paper is as follows. In Sec. II, we promote the concept of the anticommutator to a many-body system and calculate some results of many-body anticommutators in MSR. Then in Sec. III, we apply the results to normalize the the completely symmetric state, and in this way

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we get the form of the the completely symmetric state in MSR. In Sec. IV, we provide a summary and outlook.

II. MANY-BODY ANTICOMMUTATOR

A. Definition and properties

Considering two operators A_1 and A_2, we know that the anticommutator of them is defined as $\{A_1, A_2\} = A_1A_2 + A_2A_1$. Now we expand the anticommutator into many-body cases and define

$$\{A_1, A_2, \dots, A_n\} = \sum_{P\in S_n} A_{P(1)}A_{P(2)}\cdots A_{P(n)}, \tag{3}$$

where $\sum\limits_{P\in S_n}$ goes through all n permutations as in Eq. (2). Especially, when $n = 1$, we have $\{A_1\} = A_1$.

From this definition we can see the following properties.

Property i. If $A_1, A_2, \dots, A_n$ are linear operators, then the many-body anticommutator is an n-fold linear operator.

Property i is easy to obtain since for $\forall a, b \in \mathbb{C}$ and $\forall i \in \{1, 2, \dots, n\}$ we have

$$\begin{aligned}&\{A_1, \dots, aA_i + bB_i, \dots, A_n\}\\ &= \sum_{P\in S_n} A_{P(1)}\cdots(aA_i + bB_i)\cdots A_{P(n)}\\ &= \sum_{P\in S_n}(aA_{P(1)}\cdots A_i\cdots A_{P(n)} + bA_{P(1)}\cdots B_i\cdots A_{P(n)})\\ &= a\{A_1, \dots, A_i, \dots, A_n\} + b\{A_1, \dots, B_i, \dots, A_n\}.\end{aligned} \tag{4}$$

Property ii. The many-body anticommutator operation is symmetric about any pair of A_i and A_j.

To get Property ii, we first build a new n permutation P' which only swaps i and j in P, and in this way we have

$$\begin{aligned}&\{A_1, \dots, A_i, \dots, A_j, \dots, A_n\}\\ &= \sum_{P\in S_n} A_{P(1)}\cdots A_{P(i)}\cdots A_{P(j)}\cdots A_{P(n)}\\ &= \sum_{P'\in S_n} A_{P'(1)}\cdots A_{P'(j)}\cdots A_{P'(i)}\cdots A_{P'(n)}\\ &= \{A_1, \dots, A_j, \dots, A_i, \dots, A_n\}.\end{aligned} \tag{5}$$

With the above process, we can get another equivalent expression of Property ii as

$$\{A_1, A_2, \dots, A_n\} \quad = \{A_{P(1)}, A_{P(2)}, \dots, A_{P(n)}\}, \tag{6}$$

where $P(1)P(2)\cdots P(n)$ is the arbitrary permutation of all elements in index set $T_n = \{1, 2, \dots, n\}$.

Property iii. When dividing the n operators $A_1, A_2, \dots, A_n$ into q groups of fixed size as $k_1, k_2, \dots, k_q$, the anticommutators can be divided as

$$\begin{aligned}&\{A_1, A_2, \dots, A_n\}\\ &= \sum_{T_n'}\left\{A_{i_1^{(1)}}, \dots, A_{i_{k_1}^{(1)}}\right\}\cdots\left\{A_{i_1^{(q)}}, \dots, A_{i_{k_q}^{(q)}}\right\},\end{aligned} \tag{7}$$

where $\{i_1^{(1)}, \dots, i_{k_1}^{(1)}, \dots, i_1^{(q)}, \dots, i_{k_q}^{(q)}\} = T_n$ and $\sum_{T_n'}$ sums all n permutations but requires $i_k^{(m)} < i_{k+1}^{(m)}$.

Property iii is also easy to obtain with the definition of the many-body anticommutator. After setting the q groups of fixed sizes, we have

$$\begin{aligned}&\{A_1, A_2, \dots, A_n\}\\ &= \sum_{T_n}\left[A_{i_1^{(1)}}\cdots A_{i_{k_1}^{(1)}}\right]\cdots\left[A_{i_1^{(q)}}\cdots A_{i_{k_q}^{(q)}}\right]\\ &= \sum_{T_n^{(1)}}\cdots\sum_{T_n^{(q)}}\left[A_{i_1^{(1)}}\cdots A_{i_{k_1}^{(1)}}\right]\cdots\left[A_{i_1^{(q)}}\cdots A_{i_{k_q}^{(q)}}\right],\end{aligned} \tag{8}$$

where $T_n^{(1)}, \dots, T_n^{(q)}$ are determined by being chosen from T_n one by one. Accordingly, $\sum_{T_n^{(1)}}\cdots\sum_{T_n^{(q)}}$ represents the summation based on all the fixed $\{T_n^{(1)}, \dots, T_n^{(q)}\}$. Then after setting $i_k^{(m)} < i_{k+1}^{(m)}$, we can get the result in Eq. (7).

Property iv. When some of the operators are the identity operator, the anticommutators can be simplified as

$$\{A_1, A_2, \dots, A_k, 1, \dots, 1\} = \frac{n!}{k!}\{A_1, A_2, \dots, A_k\}. \tag{9}$$

For Property iv, the sum of coefficients of the left-hand side is $n!$ since the coefficient of each of the $n!$ terms is 1. Due to the completely symmetric property, it should equal $\{A_1, A_2, \dots, A_k\}$ multiplied by a coefficient, which contains $k!$ terms. So the coefficient of each term of the right-hand side is $\frac{n!}{k!}$.

B. Examples: Some results of many-body anticommutators in MSR

With the promotion of the definition, we can calculate some useful results in MSR with the many-body anticommutator.

Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be a series of Bloch vectors. For two arbitrary vectors, we know that $(\boldsymbol{u}_i\cdot\boldsymbol{\sigma})(\boldsymbol{u}_j\cdot\boldsymbol{\sigma}) = (\boldsymbol{u}_i\cdot\boldsymbol{u}_j)\mathbb{I} + \boldsymbol{i}(\boldsymbol{u}_i\times\boldsymbol{u}_j)\cdot\boldsymbol{\sigma} = \boldsymbol{u}_i\cdot\boldsymbol{u}_j + \boldsymbol{i}(\boldsymbol{u}_i\times\boldsymbol{u}_j)\cdot\boldsymbol{\sigma}$, where $\mathbb{I}$ is omitted for simplicity of writing as stated above. Then it is easy to obtain

$$\{\boldsymbol{u}_i\cdot\boldsymbol{\sigma}, \boldsymbol{u}_j\cdot\boldsymbol{\sigma}\} = 2\boldsymbol{u}_i\cdot\boldsymbol{u}_j. \tag{10}$$

Now we consider general n. When n is even, with the results in Eq. (7), i.e., Property iii, we have

$$\begin{aligned}&\{\boldsymbol{u}_1\cdot\boldsymbol{\sigma}, \boldsymbol{u}_2\cdot\boldsymbol{\sigma}, \dots, \boldsymbol{u}_n\cdot\boldsymbol{\sigma}\}\\ &= \sum_{T_n'}\{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_1}\cdot\boldsymbol{\sigma}\}\cdots\{\boldsymbol{u}_{i_{\frac{n}{2}}}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_{\frac{n}{2}}}\cdot\boldsymbol{\sigma}\}\\ &= \left(\frac{n}{2}\right)!\sum_{T_n''}\{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_1}\cdot\boldsymbol{\sigma}\}\cdots\{\boldsymbol{u}_{i_{\frac{n}{2}}}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_{\frac{n}{2}}}\cdot\boldsymbol{\sigma}\}\\ &= \left(\frac{n}{2}\right)!\sum_{T_n''}(2\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})\cdots\left(2\boldsymbol{u}_{i_{\frac{n}{2}}}\cdot\boldsymbol{u}_{j_{\frac{n}{2}}}\right)\\ &= \left(\frac{n}{2}\right)!2^{\frac{n}{2}}D_{T_n}^{(\frac{n}{2})}\\ &= n!!D_{T_n}^{(\frac{n}{2})},\end{aligned} \tag{11}$$

where $\sum_{T_n''}$ requires not only $i_k^{(m)} < i_{k+1}^{(m)}$ but also $i_1^{(m)} < i_1^{(m+1)}$, with which we can extract the factor $(\frac{n}{2})!$, and $D_T^{(k)}$ is defined as

$$D_T^{(k)} = \begin{cases} \sum_{T''}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k}), & k>0\\ 1, & k=0, \end{cases} \tag{12}$$

in which $i_1, j_1, \dots, i_k, j_k \in T$ are distinct numbers [23]. For example, for the elements in index set T_n, we have

$$\begin{aligned} D_{T_2}^{(1)} &= \boldsymbol{u}_1\cdot\boldsymbol{u}_2,\\ D_{T_3}^{(1)} &= \sum_{ij\in\{12,13,23\}} \boldsymbol{u}_i\cdot\boldsymbol{u}_j,\\ D_{T_4}^{(1)} &= \sum_{ij\in\{12,13,14,23,24,34\}} \boldsymbol{u}_i\cdot\boldsymbol{u}_j,\\ D_{T_4}^{(2)} &= \sum_{i_1j_1i_2j_2\in\{1234,1324,1423\}} (\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2}),\\ &\cdots, \end{aligned} \tag{13}$$

When we define $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})$ as a $(2k)$-order term, it is obvious that $D_{T_n}^{(k)}$ is the sum of $N(D_{T_n}^{(k)})$ different $(2k)$-order terms, where

$$N\big(D_{T_n}^{(k)}\big) = \frac{n!}{(2k)!!(n-2k)!}. \tag{14}$$

In the same way, when n is odd, it is similar that

$$\begin{aligned} &\{\boldsymbol{u}_1\cdot\boldsymbol{\sigma}, \boldsymbol{u}_2\cdot\boldsymbol{\sigma}, \dots, \boldsymbol{u}_n\cdot\boldsymbol{\sigma}\}\\ &= \sum_{T_n'} \{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_1}\cdot\boldsymbol{\sigma}\}\cdots\{\boldsymbol{u}_{i_{\frac{n-1}{2}}}\cdot\boldsymbol{\sigma}, \boldsymbol{u}_{j_{\frac{n-1}{2}}}\cdot\boldsymbol{\sigma}\}(\boldsymbol{u}_{i_{\frac{n+1}{2}}}\cdot\boldsymbol{\sigma})\\ &= \left(\frac{n-1}{2}\right)!2^{\frac{n-1}{2}} \sum_{T_n''}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})\\ &\quad\cdots(\boldsymbol{u}_{i_{\frac{n-1}{2}}}\cdot\boldsymbol{u}_{j_{\frac{n-1}{2}}})(\boldsymbol{u}_{j_{\frac{n+1}{2}}}\cdot\boldsymbol{\sigma})\\ &= (n-1)!!\boldsymbol{V}_{T_n}^{(\frac{n-1}{2})}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{15}$$

where $\boldsymbol{V}_T^{(k)}$ is defined as

$$\boldsymbol{V}_T^{(k)} = \sum_{T''}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\boldsymbol{u}_{j_{k+1}}, \tag{16}$$

in which $i_1, j_1, \dots, i_k, j_k, j_{k+1} \in T$ are also distinct.

Then we calculate the many-body anticommutator on density matrices $\rho_1, \rho_2, \dots, \rho_n$. Applying Eq. (1) and the properties of the anticommutator, we have

$$\begin{aligned} &\{\rho_1, \rho_2, \dots, \rho_n\}\\ &= \frac{1}{2^n}\{1+\boldsymbol{u}_1\cdot\boldsymbol{\sigma}, \dots, 1+\boldsymbol{u}_n\cdot\boldsymbol{\sigma}\}\\ &= \frac{1}{2^n}\sum_{k=0}^{n}\sum_{\{i_1,\cdots,i_k\}\subset T_n}\{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \dots, \boldsymbol{u}_{i_k}\cdot\boldsymbol{\sigma}, 1, \dots, 1\}\\ &= \frac{1}{2^n}\left(n! + \sum_{k=1}^{n}\frac{n!}{k!}\sum_{\{i_1,\dots,i_k\}\subset T_n}\{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \dots, \boldsymbol{u}_{i_k}\cdot\boldsymbol{\sigma}\}\right). \end{aligned} \tag{17}$$

Then applying the result of the many-body anticommutator on Pauli matrices, we have

$$\begin{aligned} &\sum_{\{i_1,\dots,i_k\}\subset T_n}\{\boldsymbol{u}_{i_1}\cdot\boldsymbol{\sigma}, \dots, \boldsymbol{u}_{i_k}\cdot\boldsymbol{\sigma}\}\\ &= \begin{cases} k!!\sum_{\{i_1,\dots,i_k\}\subset T_n} D_{\{i_1,\dots,i_k\}}^{(\frac{k}{2})}, & k \text{ is even}\\ (k-1)!!\sum_{\{i_1,\dots,i_k\}\subset T_n} \boldsymbol{V}_{\{i_1,\dots,i_k\}}^{(\frac{k-1}{2})}\cdot\boldsymbol{\sigma}, & k \text{ is odd} \end{cases}\\ &= \begin{cases} k!!D_{T_n}^{(\frac{k}{2})}, & k \text{ is even}\\ (k-1)!!\boldsymbol{V}_{T_n}^{(\frac{k-1}{2})}\cdot\boldsymbol{\sigma}, & k \text{ is odd.} \end{cases} \end{aligned} \tag{18}$$

In this way, we can get

$$\begin{aligned} &\{\rho_1, \rho_2, \dots, \rho_n\}\\ &= \frac{1}{2^n}\left[\sum_{j=0}^{[\frac{n}{2}]}\frac{n!(2j)!!}{(2j)!}D_{T_n}^{(j)} + \sum_{j=0}^{[\frac{n-1}{2}]}\frac{n!(2j)!!}{(2j+1)!}\boldsymbol{V}_{T_n}^{(j)}\cdot\boldsymbol{\sigma}\right]\\ &= \frac{n!}{2^n}\left[\sum_{j=0}^{[\frac{n}{2}]}\frac{D_{T_n}^{(j)}}{(2j-1)!!} + \sum_{j=0}^{[\frac{n-1}{2}]}\frac{\boldsymbol{V}_{T_n}^{(j)}}{(2j+1)!!}\cdot\boldsymbol{\sigma}\right], \end{aligned} \tag{19}$$

and since $\mathrm{Tr}(\boldsymbol{A}\cdot\boldsymbol{\sigma}) = 0$ for arbitrary vector $\boldsymbol{A}$, we have

$$\mathrm{Tr}\{\rho_1, \rho_2, \dots, \rho_n\} = \frac{n!}{2^{n-1}}\sum_{j=0}^{[\frac{n}{2}]}\frac{D_{T_n}^{(j)}}{(2j-1)!!}, \tag{20}$$

where $(-1)!! = 1$.

III. APPLICATIONS TO THE COMPLETELY SYMMETRIC STATES

With the conclusion in the previous section, we can get more useful results in MSR. In this section we calculate the normalization constant of completely symmetric states with MSR.

From Eq. (2), we know that the normalization constant $\mathcal{N}_n$ satisfies

$$\mathcal{N}_n^2 \quad = \sum_{P,\tilde{P}\in S_n}\prod_{i=1}^{n}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle, \tag{21}$$

where both P and $\tilde{P}$ go through all n permutations. It is obvious that the value of $\prod_{i=1}^{n}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle$ will not change when the factors $\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle, i = 1, 2, \dots, n$ exchange with each other. Then for every fixed $\tilde{P}$ we can adjust the factors with permutation $\tilde{P}^{-1}$ and get

$$\begin{aligned} \sum_{P\in S_n}\prod_{i=1}^{n}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle &= \sum_{P\in S_n}\prod_{i=1}^{n}\langle\psi_i|\psi_{P(\tilde{P}^{-1}(i))}\rangle\\ &= \sum_{P\in S_n}\prod_{i=1}^{n}\langle\psi_i|\psi_{P(i)}\rangle. \end{aligned} \tag{22}$$

Thus

$$\mathcal{N}_n^2 = n!\sum_{P\in S_n}\prod_{i=1}^{n}\langle\psi_i|\psi_{P(i)}\rangle, \tag{23}$$

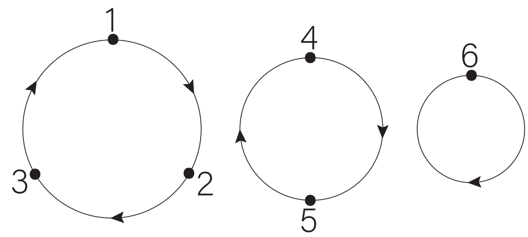

FIG. 1. The cycle diagram of $P = \binom{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6}{2\ \ 3\ \ 1\ \ 5\ \ 4\ \ 6}$.

where $\prod_{i=1}^{n}\langle\psi_i|\psi_{P(i)}\rangle$ are different from each other for arbitrary fixed P when the states $|\psi_i\rangle, i = 1, \ldots, n$ are distinguishable.

On the other hand, with the definition of the l cycle as the structure of $P(i_1) = i_2$, $P(i_1) = i_2$, ..., $P(i_l) = i_1$, every permutation P in Eq. (23) is either an n cycle or a product of some disjoint cycles [31]. Taking $P = \binom{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6}{2\ \ 3\ \ 1\ \ 5\ \ 4\ \ 6} \in S_6$ for an example, it is constructed by three cycles (1, 2, 3)(4, 5)(6), and the corresponding cycle diagram is given in Fig. 1. It is obvious that any permutation P can be reconstructed as a combination of several cycles $(i_1^{(1)}, \ldots, i_{l_1}^{(1)}) \cdots (i_1^{(q)}, \ldots, i_{l_q}^{(q)})$, where $1 \leqslant q \leqslant n$. Correspondingly, with the cycles we have

$$
\begin{aligned}
\prod_{i=1}^{n}&\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle \\
&= \prod_{j=1}^{q}\left(\langle\psi_{i_1^{(j)}}|\psi_{i_2^{(j)}}\rangle\langle\psi_{i_2^{(j)}}|\psi_{i_3^{(j)}}\rangle\cdots\langle\psi_{i_{l_j}^{(j)}}|\psi_{i_1^{(j)}}\rangle\right) \\
&= \prod_{j=1}^{q}\mathrm{Tr}\left(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\cdots\rho_{i_{l_j}^{(j)}}\right). \qquad (24)
\end{aligned}
$$

So when the states $|\psi_i\rangle$ are distinguishable (all of the discussion below assumes this condition), there are $n!$ different terms as $\prod_{j=1}^{q}\mathrm{Tr}(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\cdots\rho_{i_{l_j}^{(j)}})$.

Then to distinguish the $n!$ different terms as the final result in Eq. (24) and sum all of them, we introduce Young diagrams to describe the grouping. For example, when $n = 4$, there are five structures of the grouping described by the Young diagrams in Fig. 2, which are associated with the partitions of integer $n = 4$. In general, the grouping is associated with

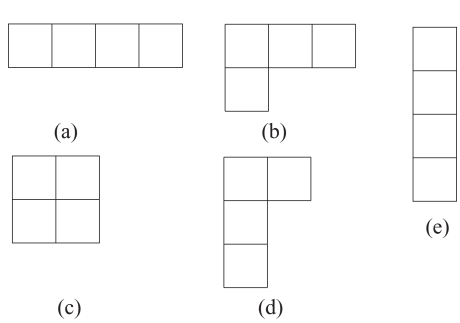

FIG. 2. The Young diagrams associated with the partitions of $n = 4$, where the rows stand for l_i, $i = 1, \ldots, q$, with $l_i \geqslant l_{i+1}$, as (a) 4, (b) 3 + 1, (c) 2 + 2, (d) 2 + 1 + 1, and (e) 1 + 1 + 1 + 1.

TABLE I. The n_T, n_I, and K_I associated with the Young diagrams when $n = 4$.

	n_T	n_I	K_I
Fig. 2(a)	6	24	6
Fig. 2(b)	8	24	8
Fig. 2(c)	3	12	6
Fig. 2(d)	6	12	12
Fig. 2(e)	1	1	24

the partitions of integer n as

$$
n \quad = \sum_{i=1}^{q} l_i \; = \sum_{\alpha=1}^{\infty}\alpha\cdot n_\alpha, \qquad (25)
$$

where n_α is the times that α repeats in the partition.

It is obvious that $\prod_{j=1}^{q}\mathrm{Tr}(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\cdots\rho_{i_{l_j}^{(j)}})$ are different when corresponding with a different Young diagram. However, when we fill the same Young diagram with indices $(i_1^{(1)}, \ldots, i_{l_1}^{(1)}) \cdots (i_1^{(q)}, \ldots, i_{l_q}^{(q)})$ and get different Young tableaux, the corresponding results $\prod_{j=1}^{q}\mathrm{Tr}(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\cdots\rho_{i_{l_j}^{(j)}})$ will repeat. This is easy to prove since the Young tableaux are equivalent to the cycle diagrams, in which the rotation of indices in the same circle and the exchange between circles of the same size will not change the value of the terms $\prod_{j=1}^{q}\mathrm{Tr}(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\cdots\rho_{i_{l_j}^{(j)}})$. Then, in this way, we can get the number of unequal terms for every Young diagram in Fig. 2, and the results are given in Table I as n_T. In general, we can get the number of unequal terms in a Young diagram as

$$
n_T = \frac{n!}{\prod_{\alpha=1}^{\infty}\alpha^{n_\alpha}n_\alpha!}. \qquad (26)
$$

Then we sum all the unequal terms to get $\mathcal{N}_n^2$. With symmetry, we can assume

$$
\mathcal{N}_n^2 = \sum_I K_I \sum_{P_I}\prod_{j=1}^{q}\mathrm{Tr}\{\rho_{i_1^{(j)}}, \cdots, \rho_{i_{l_j}^{(j)}}\}, \qquad (27)
$$

where I stands for all kinds of Young diagrams and $P_I = \{i_1^{(1)}, \ldots, i_{l_1}^{(1)}\}, \ldots, \{i_1^{(q)}, \ldots, i_{l_q}^{(q)}\}$ stands for all the possible sequences as the Young tableaux that are associated with the Young diagrams, in which $i_k^{(j)} \leqslant i_{k+1}^{(j)}$, $l_j \geqslant l_{j+1}$, and, if $l_j = l_{j+1}$, i.e., if the lines are of equal width in the Young diagram, there should be $i_1^j \leqslant i_1^{j+1}$ to exclude the duplicates. Furthermore, the coefficient K_I should be determined. For each Young diagram in Fig. 2, we list all the terms of $\sum_{P_I}\prod_{j=1}^{q}\mathrm{Tr}\{\rho_{i_1^{(j)}}, \ldots, \rho_{i_{l_j}^{(j)}}\}$ in Table II, and in this way we can get the number of terms as

$$
\begin{aligned}
n_{P_I} &= \frac{1}{\prod_{\alpha=1}^{\infty}n_\alpha!}C_n^{l_1}C_{n-l_1}^{l_2}\cdots C_{n-\sum_{j=1}^{q-1}l_j}^{l_q} \\
&= \frac{n!}{\prod_{\alpha=1}^{\infty}n_\alpha!(\alpha!)^{n_\alpha}}. \qquad (28)
\end{aligned}
$$

TABLE II. The terms and corresponding n_{P_I} and n_I associated with the Young diagrams when $n = 4$.

	Terms in $\sum_{P_I}\prod_{j=1}^{q}\mathrm{Tr}\{\rho_{i_1^{(j)}},\ldots,\rho_{i_{l_j}^{(j)}}\}$	n_{P_I}	n_I
Fig. 2(a)	$\{\rho_1,\rho_2,\rho_3,\rho_4\}$	1	24
Fig. 2(b)	$\{\rho_1,\rho_2,\rho_3\}\{\rho_4\}$;$\{\rho_1,\rho_2,\rho_4\}\{\rho_3\}$; $\{\rho_1,\rho_3,\rho_4\}\{\rho_2\}$;$\{\rho_2,\rho_3,\rho_4\}\{\rho_1\}$	4	24
Fig. 2(c)	$\{\rho_1,\rho_2\}\{\rho_3,\rho_4\}$;$\{\rho_1,\rho_3\}\{\rho_2,\rho_4\}$;$\{\rho_1,\rho_4\}\{\rho_2,\rho_3\}$	3	12
Fig. 2(d)	$\{\rho_1,\rho_2\}\{\rho_3\}\{\rho_4\}$;$\{\rho_1,\rho_3\}\{\rho_2\}\{\rho_4\}$; $\{\rho_1,\rho_4\}\{\rho_2\}\{\rho_3\}$; $\{\rho_2,\rho_3\}\{\rho_1\}\{\rho_4\}$; $\{\rho_2,\rho_4\}\{\rho_1\}\{\rho_3\}$;$\{\rho_3,\rho_4\}\{\rho_1\}\{\rho_2\}$	6	12
Fig. 2(e)	$\{\rho_1\}\{\rho_2\}\{\rho_3\}\{\rho_4\}$	1	1

Then after expanding all the $\{\rho_{i_1^{(j)}},\ldots,\rho_{i_{l_j}^{(j)}}\}$, the terms are counted as

$$n_I = \frac{n!}{\prod_{\alpha=1}^{\infty} n_\alpha!}. \tag{29}$$

In this way, K_I is determined as

$$K_I = \frac{n! n_T}{n_I} = \frac{n!}{\prod_{\alpha=1}^{\infty}\alpha^{n_\alpha}} = \frac{n!}{\prod_{j=1}^{q} l_j}. \tag{30}$$

Furthermore, applying the result in Eq. (20), we have

$$\mathcal{N}_n^2 = n!\sum_I\sum_{P_I}\prod_{j=1}^{q}\left[\frac{(l_j-1)!}{2^{l_j-1}}\sum_{k_j=0}^{[\frac{l_j}{2}]}\frac{D^{(k_j)}_{\{i_1^{(j)},\ldots,i_{l_j}^{(j)}\}}}{(2k_j-1)!!}\right]. \tag{31}$$

Then since $T_n = \{i_1^{(1)},\ldots,i_{l_1}^{(1)},\ldots,i_1^{(q)},\ldots,i_{l_q}^{(q)}\}$, we can get that the product of the $(2k_j)$-order term in $D^{(k_j)}_{\{i_1^{(j)},\ldots,i_{l_j}^{(j)}\}}$ is a $(2\sum_{j=1}^{q}k_j)$-order term in $D_{T_n}^{(\sum_{j=1}^{q}k_j)}$. In this way, we introduce $J=\sum_{j=1}^{q}k_j$, and it is obvious that $0\leqslant J\leqslant[\frac{n}{2}]$. Then with the symmetry of all of the sequence in P_I we can assume

$$\mathcal{N}_n^2 = n!\sum_{J=0}^{[n/2]} a_{n,J} D_{T_n}^{(J)}. \tag{32}$$

To calculate $a_{n,J}$, we first introduce a counting function $N(\cdots)$ like $N(D_T^{(k)})$ in Eq. (14) to count the $(2k)$-order terms, and it is obvious that $N(\lambda D_T^{(k)}) = \lambda N(D_T^{(k)})$ for all $\lambda\in\mathbb{R}$ and $N(D_{T_1}^{(k_1)}D_{T_2}^{(k_2)}) = N(D_{T_1}^{(k_1)}) + N(D_{T_2}^{(k_2)})$. In this way, we can get

$$N\left(\frac{D^{(k_j)}_{\{i_1^{(j)},\ldots,i_{l_j}^{(j)}\}}}{(2k_j-1)!!}\right) = \frac{1}{(2k_j-1)!!}N\left(D^{(k_j)}_{\{i_1^{(j)},\ldots,i_{l_j}^{(j)}\}}\right) = C_{l_j}^{2k_i}, \tag{33}$$

where $C_{l_j}^{2k_j}$ is the general binomial coefficient, which equals the binomial coefficient $C_{l_j}^{2k_j}$ when $l_j\geqslant 2k_j$ and equals zero when $l_j<2k_j$. Then for $\sum_{j=1}^{q}k_j = J$, we count the $(2J)$-order terms in Eqs. (31) and (32), respectively, and get

$$n!\sum_I\sum_{P_I}\prod_{j=1}^{q}\frac{(l_j-1)!}{2^{l_j-1}}\sum_{\sum_{j=1}^{q}k_j=J}C_{l_j}^{2k_j} = n! a_{n,J}\frac{n!}{(n-2J)!(2J)!!}. \tag{34}$$

From the Appendix we see $a_{n,J} = \frac{(n+1)!}{2^n}\frac{1}{(2J+1)!!}$. So the normalization constant is determined as

$$\mathcal{N}_n^2 = \frac{n!(n+1)!}{2^n}\sum_{J=0}^{[\frac{n}{2}]}\frac{1}{(2J+1)!!}D_{T_n}^{(J)}. \tag{35}$$

IV. CONCLUSION

In this paper, we first give the definition of the many-body anticommutator by expanding the two-body anticommutator, and then we list several properties of the anticommutator. With these properties, we calculate the anticommutator of Pauli matrices and density matrices as examples, which are useful results to study completely symmetric state systems with MSR.

Then we calculate the normalization constant of the completely symmetric state. We view every permutation as a disjoint union of cycles, and each cycle shows up as the trace of the product of several density matrices. Summing all permutations, we can see that every term is in an anticommutator. Then with the result of the anticommutator and the method of partition and power series, we get the final answer.

The anticommutation relationship between the operators is the manifestation of symmetry in the many-body system. In addition to the above examples and results, we believe that the promotion of the anticommutator as the many-body anticommutator will be useful in further analysis of the completely symmetric system.

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APPENDIX: THE CALCULATION OF $a_{n,J}$

With Eq. (34), we define $A_{n,J} = \frac{a_{n,J}}{(n-2J)!(2J)!!}$ and have

$$A_{n,J} = \frac{1}{n!}\sum_{I}\sum_{P_I}\prod_{j=1}^{q}\frac{(l_j-1)!}{2^{l_j-1}}\sum_{\sum_{j=1}^{q}k_j=J}C_{l_j}^{2k_j}, \quad \text{(A1)}$$

which can be associated with the coefficients in the series expansion of the binary generating function.

To find the binary generating function, first we determine $\prod_{j=1}^{q}\sum_{\sum_{j=1}^{q}k_j=J}C_{l_j}^{2k_j}$. It is exactly the $(2J)$th coefficient of the power series about variable x in

$$\begin{aligned}&\prod_{j=1}^{q}\sum_{k_j=0}^{\infty}C_{l_j}^{2k_j}x^{2k_j}\\ &=\prod_{j=1}^{q}\frac{1}{2}\left[\sum_{i=0}^{\infty}C_{l_j}^{i}x^{i}+\sum_{i=0}^{\infty}C_{l_j}^{i}(-x)^{i}\right]\\ &=\prod_{j=1}^{q}\frac{1}{2}\left[(1+x)^{l_j}+(1-x)^{l_j}\right], \quad \text{(A2)}\end{aligned}$$

since each $\{k_1,\dots,k_q\}$ satisfies $\sum_{j=1}^{q}k_j=J$ corresponding to $x^{2k_1}\cdots x^{2k_q}=x^{2J}$ that arises in the product.

Then we have

$$\begin{aligned}&\sum_{J=0}^{\infty}A_{n,J}x^{2J}\\ &=\frac{1}{n!}\sum_{I}\sum_{P_I}\prod_{j=1}^{q}(l_j-1)!\frac{(1+x)^{l_j}+(1-x)^{l_j}}{2^{l_j}}\\ &=\frac{1}{n!}\sum_{I}\sum_{P_I}\prod_{\alpha=1}^{\infty}\left[(\alpha-1)!\frac{(1+x)^{\alpha}+(1-x)^{\alpha}}{2^{\alpha}}\right]^{n_\alpha}. \quad \text{(A3)}\end{aligned}$$

Then for all P_I in a Young diagram, $\{n_1,n_2,\dots,n_\alpha,\dots\}$ is fixed, which means that the terms repeat n_{P_I} times as shown in Eq. (28). In this way, Eq. (A3) can be simplified as

$$\begin{aligned}\sum_{J=0}^{\infty}A_{n,J}x^{2J}&=\sum_{I}\prod_{\alpha=1}^{\infty}\frac{1}{n_\alpha!}\left[\frac{(1+x)^{\alpha}+(1-x)^{\alpha}}{2^{\alpha}\cdot\alpha}\right]^{n_\alpha}\\ &=\sum_{\sum_{\alpha=1}^{\infty}\alpha\cdot n_\alpha=n}\prod_{\alpha=1}^{\infty}\frac{1}{n_\alpha!}\left[\frac{(1+x)^{\alpha}+(1-x)^{\alpha}}{2^{\alpha}\cdot\alpha}\right]^{n_\alpha}.\end{aligned} \quad \text{(A4)}$$

Again, Eq. (A4) is exactly the nth power-series coefficient about variable y of

$$\begin{aligned}&\sum_{n_\alpha=0}^{\infty}\prod_{\alpha=1}^{\infty}\frac{1}{n_\alpha!}\left[\frac{(1+x)^{\alpha}+(1-x)^{\alpha}}{\alpha\cdot 2^{\alpha}}\right]^{n_\alpha}y^{n_\alpha\alpha}\\ &=\prod_{\alpha=1}^{\infty}\exp\left[\frac{(1+x)^{\alpha}+(1-x)^{\alpha}}{\alpha}\left(\frac{y}{2}\right)^{\alpha}\right]\\ &=\exp\left[\sum_{\alpha=1}^{\infty}\frac{1}{\alpha}\left(\frac{(1+x)y}{2}\right)^{\alpha}+\sum_{\alpha=1}^{\infty}\frac{1}{\alpha}\left(\frac{(1-x)y}{2}\right)^{\alpha}\right]\\ &=\exp\left[-\ln\left(1-\frac{(1+x)y}{2}\right)-\ln\left(1-\frac{(1-x)y}{2}\right)\right]\\ &=\frac{4}{(2-y-xy)(2-y+xy)}, \quad \text{(A5)}\end{aligned}$$

since each $\{n_1,n_2,\dots,n_\alpha,\dots\}$ corresponds to the term $y^{n_1}(y^2)^{n_2}\cdots(y^\alpha)^{n_\alpha}\cdots=y^n$ in the product and it is the binary generating function of $A_{n,J}$ that we are finding.

So the last step is to expand the binary generating function into a series expression as

$$\begin{aligned}&\frac{4}{(2-y-xy)(2-y+xy)}\\ &=\frac{1}{xy}\left[\left(\frac{1}{1-(1+x)y/2}-1\right)-\left(\frac{1}{1-(1-x)y/2}-1\right)\right]\\ &=\frac{1}{xy}\sum_{n=0}^{\infty}\left[(1+x)^{n+1}-(1-x)^{n+1}\right]\left(\frac{y}{2}\right)^{n+1}\\ &=\frac{2}{xy}\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}C_{n+1}^{2J+1}x^{2J+1}\left(\frac{y}{2}\right)^{n+1}\\ &=\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\frac{1}{2^n}C_{n+1}^{2J+1}x^{2J}y^{n}. \quad \text{(A6)}\end{aligned}$$

In this way,

$$\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}A_{n,J}x^{2J}y^{n}=\sum_{n=0}^{\infty}\sum_{J=0}^{\infty}\frac{1}{2^n}C_{n+1}^{2J+1}x^{2J}y^{n}. \quad \text{(A7)}$$

So we get $A_{n,J}=\frac{1}{2^n}C_{n+1}^{2J+1}$, with which $a_{n,J}$ is determined as

$$a_{n,J}=\begin{cases}\frac{(n+1)!}{2^n}\frac{1}{(2J+1)!!}, & 0\leqslant J\leqslant\left[\frac{n}{2}\right]\\ 0, & \text{otherwise.}\end{cases} \quad \text{(A8)}$$

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