Dissipative soliton interaction in Kerr resonators with high-order dispersion

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We consider an optical resonator containing a photonic crystal fiber and driven coherently by an injected beam. This device is described by a generalized Lugiato-Lefever equation with fourth-order dispersion. We use an asymptotic approach to derive interaction equations governing the slow time evolution of the coordinates of two interacting dissipative solitons. We show that Cherenkov radiation induced by positive fourth-order dispersion leads to a strong increase of the interaction force between the solitons. As a consequence, a large number of equidistant soliton bound states in the phase space of the interaction equations can be stabilized. We show that the presence of even small spectral filtering not only dampens the Cherenkov radiation at the soliton tails and reduces the interaction strength, but can also affect the bound state stability.

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I. INTRODUCTION

Optical frequency combs generated by microcavity resonators have revolutionized many fields of science and technology, such as high-precision spectroscopy, metrology, and photonic analog-to-digital conversion [1]. A particular interest is paid to the soliton frequency combs associated with the formation in the time domain of the so-called temporal cavity solitons—nonlinear localized structures of light, which preserve their shape in the course of propagation. Temporal dissipative solitons, often called cavity solitons, were reported experimentally in mode-locked lasers, microcavity resonators [2,3], and in coherently driven fiber cavities [4].

In this work, we consider a photonic crystal fiber cavity driven by a coherent injected beam. When operating close to the zero dispersion wavelength, high-order chromatic dispersion effects could play an important role in the dynamics of the system. Taking into account these effects together with spectral filtering, the dimensionless model equation in the mean-field limit reads

$$\partial_t U = S - (1 + i\theta)U + iU|U|^2 + (\delta + i)\partial_\tau^2 U + \beta_3 \partial_\tau^3 U + i\beta_4 \partial_\tau^4 U, \tag{1}$$

where $U(\tau, t)$ is the complex electric field envelope, τ is time, and t is the slow time variable describing the number of round trips in the cavity. The parameter S measures the injection rate, θ describes frequency detuning, second-order dispersion and Kerr nonlinearity coefficients are normalized to unity, β_3 and β_4 are the third- and fourth-order dispersion coefficients, respectively, and $0 < \delta \ll 1$ is the small spectral filtering coefficient (or, in time domain, dispersion of the losses) [5]. The optical losses are determined by the mirror transmission and the intrinsic material absorption. This losses are normalized to unity.

In the absence of high-order dispersion and spectral filtering, we recover from Eq. (1) the Lugiato-Lefever equation [6] which is a paradigmatic model to study temporal cavity solitons (see overview in [7,8]). It is widely applied to describe two important physical systems: a passive ring fiber cavity with coherent optical injection and a driven optical microcavity used for frequency comb generation [9–12]. The inclusion of the fourth-order dispersion allows the modulational instability to have a finite domain of existence delimited by two pump power values [13]. As a consequence, the upper homogeneous steady-state solution becomes modulationally stable and dark dissipative solitons sitting in this solution can appear [14]. In the presence of third-order dispersion, bright and dark dissipative solitons become asymmetric and acquire an additional group velocity shift associated with this asymmetry [15–18].

Being well separated from one another, dissipative solitons can interact via their exponentially decaying tails and form bound states characterized by fixed distances between the solitons. This weak interaction can be strongly affected by different perturbations, such as periodic modulation [5,19,20] and high-order dispersions [21], which lead to the appearance of the so-called soliton Cherenkov radiation at the soliton tails [22]. Single soliton self-locking by Cherenkov radiation in a microring resonator with high-order dispersions was studied in [23]. Soliton interaction in the presence of high-order dispersions was studied in several works in one-dimensional (1D) [17,18,20,21,24,25] and 2D settings [26]. However, they were either focused on the asymmetric soliton interaction in the presence of third-order dispersion or based mainly on the numerical calculation of the soliton interaction potential. Unlike these works, here we present an analytical theory of the interaction of two dissipative solitons of the Lugiato-Lefever equation with the fourth-order dispersion term based on the asymptotic approach developed in [27,28]. In this approach,

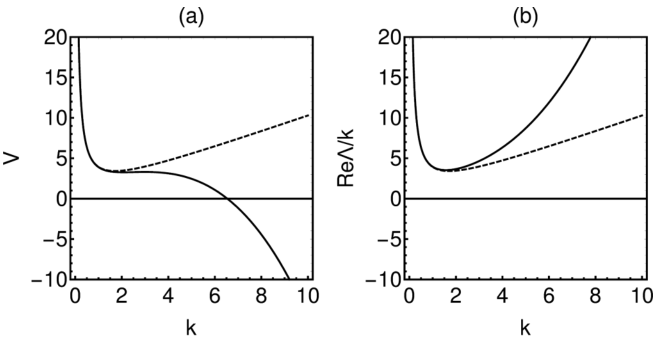

FIG. 1. Phase velocity V of small dispersive waves with (a) positive and (b) negative fourth-order dispersion coefficient β_4, and $\beta_3 = 0$. Solid line corresponds to (a) $\beta_4 = 0.025$ and (b) $\beta_4 = -0.025$. Dashed line corresponds to $\beta_4 = 0$. The parameter values are $S = 1.8$, $\theta = 3.5$, and $\delta = 0.02$.

only a single complex number has to be calculated numerically that is the product of the Cherenkov radiation amplitudes for the soliton itself and the neutral mode of the adjoint operator obtained by linearization of the model equation on the soliton solution. Note that the asymptotic method for estimation of the Cherenkov radiation amplitude was discussed in [22,29]. Furthermore, we show that similarly to the case of the interacting oscillatory solitons [5], a small spectral filtering effect can strongly affect the interaction force and the stability properties of the bound soliton states.

II. SINGLE PEAK DISSIPATIVE SOLITON

Without high-order dispersion and spectral filtering terms, $\beta_3 = \beta_4 = \delta = 0$, Eq. (1) supports single or multipeak dissipative solitons characterized by damped oscillatory tails [30]. Stable dissipative solitons have been found in a strongly nonlinear regime, where the modulational instability is subcritical, i.e., for $\theta > 41/30$. More precisely, they have been found in the pinning region, where the lower stationary homogeneous solution coexists with a periodic one. The number of dissipative solitons and their distribution in the cavity are determined by the initial conditions, while their maximum peak power remains constant for fixed values of the system parameters [30]. Note that the stability and bifurcations of the soliton solutions of the Lugiato-Lefever model with small dissipation were studied analytically in a number of earlier works; see, e.g., [31–34].

For $\theta > 41/30$, Eq. (1) supports a single peak dissipative soliton solution in the form $U(t,\tau) = U_0 + u_0(\tau)$, where $I_0 = |U_0|^2 = \text{const}$ is the intensity of the stationary homogeneous solution of Eq. (1) and $u_0(\tau)$ decays exponentially at $\tau \to \pm\infty$. This solution persists also at sufficiently small β_3, β_4, and δ. It remains motionless for $\beta_3 = 0$ and becomes uniformly moving otherwise, $U(t,\tau) = U_0 + u_0(\tau - vt)$. Asymptotic analytic theory of the asymmetric dissipative soliton interaction via Cherenkov radiation induced by the third-order dispersion coefficient β_3 was developed in [18]. Below we assume that only small fourth-order dispersion is present, $\beta_3 = 0$ and $|\beta_4| \ll 1$. In this case, we consider only soliton solutions, which are invariant under the symmetry property of Eq. (1), $\tau \to -\tau$. For these solutions, the soliton velocity is equal to zero, $v = 0$. Note that traveling localized solutions were reported in the undamped Lugiato-Lefever model [35] as well as in the parametrically driven damped nonlinear Schrödinger equation [36].

The dispersion relation for the small amplitude waves is determined by substituting $U(t,\tau) = U_0 + A_0 e^{ik\tau - i\Lambda t}$ into Eq. (1) and linearizing the resulting equation at $U = U_0$. This yields

$$\Lambda = -2I_0^2 + i\sqrt{(1+\delta k^2)^2 - I_0^2} + k^2 - \beta_4 k^4.$$

The phase velocity of the dispersive waves $V = \mathrm{Re}(\Lambda)/k$ is shown in Fig. 1(a) for positive [1(a)] and negative [1(b)] β_4, as a function of the wave number k. Cherenkov radiation appears when the phase velocity V coincides with the zero soliton velocity, as shown in Fig. 1(a). It is seen from this figure that the Cherenkov radiation emitted from the soliton tail occurs only when β_4 is positive. Therefore, below we consider only the case of positive fourth-order dispersion coefficient $0 < \beta_4 \ll 1$ when the Cherenkov radiation is present. For negative β_4, the soliton interaction is only weakly affected by the small fourth-order dispersion term.

Linear stability of the dissipative soliton solution $u_0(\tau)$ is determined by calculating the eigenvalue spectrum λ of the operator,

$$\hat{L}(\mathbf{u}_0) = \hat{L}_0 + \hat{L}_1(\mathbf{u}_0), \tag{2}$$

obtained by linearization of Eq. (1) around the soliton solution. Here, $\mathbf{u}_0 = \binom{u_0}{u_0^*}$, $\hat{L}_0 = \hat{L}(0)$ is the linear differential operator evaluated at the stationary homogeneous solution $\mathbf{U} = \mathrm{U}_0$,

$$\hat{L}_0 = \begin{pmatrix} -1 - i\theta + 2iI_0^2 + (i+\delta)\partial_\tau^2 + i\beta_4\partial_\tau^4 & iU_0^2 \\ -iU_0^{*2} & -1 + i\theta - 2iI_0^2 - (i-\delta)\partial_\tau^4 - i\beta_4\partial_\tau^4 \end{pmatrix}$$

and

$$\hat{L}_1(\mathbf{u}_0) = \begin{pmatrix} 2iU_0^* u_0 + 2iU_0 u_0^* + 2i|u_0|^2 & 2iU_0 u_0 + iu_0^2 \\ -2iU_0^* u_0^* - iu_0^{*2} & -2iU_0^* u_0 + 2iU_0 u_0^* - 2i|u_0|^2 \end{pmatrix}.$$

We have calculated numerically the soliton solution and the eigenvalue spectrum λ of the operator $\hat{L}(\mathbf{u}_0)$ by discretizing Eq. (1) on a uniform grid of 2000 points on the interval $\tau \in [0, 80]$ with periodic boundary conditions. The result is shown in Fig. 2 for $\beta_3 = \beta_4 = \delta = 0$. The continuous spectrum lies on the line $\mathrm{Re}(\lambda) = -1$, while the discrete spectrum of the

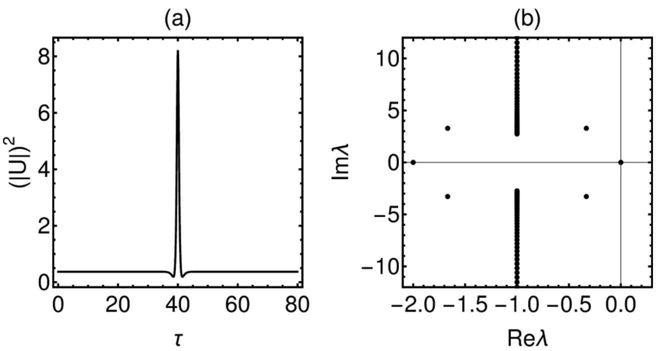

FIG. 2. (a) Soliton solution of the Lugiato-Lefever equation (1) with $\beta_3 = \beta_4 = \delta = 0$ and (b) eigenvalue spectrum obtained by numerical linear stability analysis of this solution. Other parameters are the same as in Fig. 1.

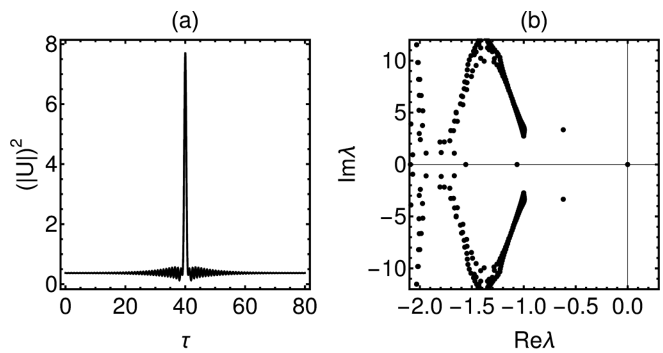

FIG. 3. (a) Soliton solution of the Lugiato-Lefever equation (1) with $\beta_4 = 0.025$ and $\delta = 0.02$; (b) eigenvalue spectrum obtained by numerical linear stability analysis of this solution. Other parameters are the same as in Fig. 1.

soliton is symmetric with respect to this line [37]. For the parameter values of Fig. 2 apart from two real eigenvalues, i.e., the zero eigenvalue $\lambda = 0$, associated with the translational symmetry of the Lugiato-Lefever equation, and the symmetric one $\lambda = -2$, the soliton has two symmetric pairs of complex conjugated eigenvalues. The right pair of these complex eigenvalues is responsible for an Andronov-Hopf bifurcation taking place with the increase of the injection parameter S. The decay rates of the soliton tails depend on the eigenvalues μ satisfying the characteristic equation

$$\beta_4^2\mu^8 + 2\beta_4\mu^6 + [1+\delta^2 + 2\beta_4(2I_0 - \theta)]\mu^4 + 2(2I - \theta - \delta)\mu^2 + \left[1 - I_0^2 + (2I_0 - \theta)^2\right] = 0 \quad (3)$$

obtained by linearization of Eq. (1) with $\partial_t U = 0$ at the homogeneous steady-state solution $U = U_0$.

In the case when the high-order dispersion and spectral filtering are absent, $\beta_3 = \beta_4 = \delta = 0$, Eq. (3) gives two pairs of complex conjugate eigenvalues,

$$\mu_{1,2}^{(0)} = \pm\sqrt{\theta - 2I_0 + i\sqrt{1 - I_0^2}} \quad (4)$$

and $\mu_{1,2}^{(0)*}$, which determine the decay and oscillation rates of the soliton tails. For example, for $S = 2.0$ and $\theta = 3.5$, we have $\mu_{1,2}^{(0)} = \pm(1.6837 + 0.275817i)$, which means that in the absence of high-order dispersions, the soliton tail oscillations are strongly damped. This might explain the fact that without soliton Cherenkov radiation, it is hardly possible to observe experimentally the formation of bound states with large distances between the solitons [4].

For nonzero but sufficiently small fourth-order dispersion coefficient, $0 < \beta_4 \ll 1$, the eigenvalues (4) of Eq. (3) are only slightly perturbed. However, in addition to (4), two more pairs of complex conjugate eigenvalues, $\mu_{3,4}$ and $\mu_{3,4}^*$, appear. For zero spectral filtering coefficient, $\delta = 0$, they are given by

$$\mu_{3,4} = \mp i\sqrt{\frac{1}{2\beta_4}\left[1 + \sqrt{1 - 4\beta_4\left(2I_0 - \theta + i\sqrt{1 - I_0^2}\right)}\right]}. \quad (5)$$

It is seen that real (imaginary) parts of $\mu_{3,4}$ in Eq. (5) vanish (diverge) in the limit $\beta_4 \to 0$. When the spectral filtering coefficient is nonzero, $\delta > 0$, analytical expressions for the eigenvalues $\mu_{3,4}$ become very cumbersome. However, in the limit $\beta_4 = O(\delta) \ll 1$, we get

$$\mu_{3,4} = \mp\sqrt{\beta_4}\left[\frac{\sqrt{(1+\delta/\beta_4)^2 - I_0^2}}{2} + i\left(\frac{1}{\beta_4} + \frac{\theta - 2I_0}{2}\right) + O(\delta)\right]. \quad (6)$$

Due to the presence of the eigenvalues $\mu_{3,4}$ and $\mu_{3,4}^*$, the tails of the soliton of Eq. (1) with $\beta_3 = 0$ and $0 < \beta_4 \ll 1$ become weakly decaying and fast oscillating, which favors the formation of soliton bound states, and can be referred to as the soliton Cherenkov radiation [22]. Note that when β_4 is sufficiently small, the term δ/β_4 describing, in Eq. (6), the contribution of spectral filtering into the real part of $\mu_{3,4}$ can lead to a considerable increase of the decay rate of the soliton tails without significant change of their oscillation frequency. For example, for $S = 2.0$, $\theta = 3.5$, $\beta_4 = 0.025$, and $\delta = 0.02$, we get $\mu_3 = -0.123 - 6.529i$, while for the same parameter set and $\delta = 0$, one obtains $\mu_3 = -0.063 - 6.528i$. The numerically calculated intensity profile of the soliton solution of Eq. (1) with small fourth-order dispersion coefficient $\beta_4 = 0.025$ is depicted in Fig. 3, together with the corresponding eigenvalue spectrum of the operator $\hat{L}(\mathbf{u}_0)$ defined by Eq. (2).

Note that the proof of the reflectional symmetry property of the discrete soliton spectrum with respect to the $\mathrm{Re}\lambda = -1$ line given in [37] is trivially generalized to the case when even high-order dispersions are present. Nevertheless, the soliton spectrum shown in Fig. 3 does not possess this symmetry property due to the presence of nonzero spectral filtering coefficient $\delta = 0.02$. Furthermore, as seen from Fig. 3, for $\delta = 0.02$, real parts of the complex conjugate eigenvalues, responsible for the Andronov-Hopf bifurcation of the soliton, are shifted to the left from the imaginary axis as compared to those shown in Fig. 2 obtained for $\delta = 0$.

Sufficiently far away from the soliton core, its trailing tail can be represented in the form

$$u_0(\tau) \sim a_3 e^{\mu_3\tau} + a_4 e^{\mu_3^*\tau} \quad \text{when } \tau \to +\infty, \quad (7)$$

where the Cherenkov radiation amplitude a_3 is exponentially small in the limit $\beta_4 \to 0$ [22,29], $a_4 = p_a a_3^*$, and, for $\beta_4 =$

$O(\delta) \ll 1$, we get

$$p_a = i\frac{1-\sqrt{1-I_0^2}}{A_0^{*2}\left(\frac{\delta}{\beta_4\sqrt{1-I_0^2}}+1\right)} + O(\delta), \tag{8}$$

where p_a is independent of β_4 at $\delta = 0$. Numerically for $S = 2.0$, $\theta = 3.5$, $\delta = 0.02$, and $\beta_4 = 0.025$, we obtain $p_a \approx 0.0571 + 0.0833i$.

III. INTERACTION BETWEEN DISSIPATIVE SOLITONS

The study of weak dissipative soliton interaction in optical systems and, in particular, in the Lugiato-Lefever equation has a relatively long history [19,27,28,38–47]. Two or more solitons will interact through their overlapping oscillatory tails when they are sufficiently close to one another. In what follows, we investigate the interaction between two dissipative solitons. We consider the limit of weak overlap when the solitons are well separated from each other and derive the interaction equations describing the slow time evolution of the soliton coordinates denoted by $\tau_{1,2}$. To this end, let us first rewrite Eq. (1) in a general form,

$$\partial_t \mathbf{U} = \hat{F}\mathbf{U}, \tag{9}$$

where $\mathbf{U} = \binom{U}{U^*}$, $\hat{F}\mathbf{U} = \binom{\hat{f}U}{\hat{f}^*U^*}$, and $\hat{f}$ is the differential operator defined by the right-hand side of Eq. (1). We look for the solution of Eq. (9) in the form

$$U(\tau,t) = U_0 + u_1 + u_2 + \Delta u(\tau,t). \tag{10}$$

Here, $u_{1,2} = u_0(\tau - \tau_{1,2})$ are two unperturbed soliton solutions, with slowly evolving-in-time coordinates $\tau_{1,2}(\varepsilon t)$, $\Delta u(\tau,t) = O(\varepsilon)$ is a small correction to the superposition of two solitons, and small parameter ε describes the weakness of the overlap of the two solitons. Substituting Eq. (10) into the model equation (9) and collecting first-order terms in ε, we obtain the following linear inhomogeneous equation for $\mathbf{\Delta u} = \binom{\Delta u}{\Delta u^*}$:

$$\hat{L}(\mathbf{u}_1+\mathbf{u}_2)\mathbf{\Delta u} = -\partial_x\mathbf{u}_1\partial_t\tau_1 - \partial_x\mathbf{u}_2\partial_t\tau_2 - \hat{F}(\mathbf{u}_1+\mathbf{u}_2), \tag{11}$$

where the linear operator $\hat{L}(\mathbf{u})$ is defined by Eq. (2). Due to the translational invariance of Eq. (1), this linear operator evaluated at the soliton solution $\mathbf{u}_0$ has zero eigenvalue corresponding to the so-called translational neutral (or Goldstone) mode $\mathbf{v}_0 = \binom{v_0}{v_0^*}$, with $v_0 = du_0/d\tau$, $\hat{L}(\mathbf{u}_0)\mathbf{v}_0 = 0$. The adjoint linear operator $\hat{L}^\dagger(\mathbf{u})$ obtained from $\hat{L}(\mathbf{u})$ by transposition and complex conjugation also has zero eigenvalue with the eigenfunction $\mathbf{w}_0 = \binom{w_0}{w_0^*}$, which is referred to below as the "adjoint neutral mode," $\hat{L}^\dagger(\mathbf{u}_0)\mathbf{w}_0 = 0$. Below, we will assume that $\mathbf{w}_0$ satisfies the normalization condition $\langle \mathbf{w}_0\cdot\mathbf{u}_0\rangle = \int_{-\infty}^{\infty}(\mathbf{w}_0\cdot\mathbf{u}_0)d\tau = 2\int_{-\infty}^{\infty}\mathrm{Re}(w_0^*u_0)d\tau = 1$. Far away from the soliton core, the asymptotic behavior of the adjoint neutral mode is given by

$$w_0(\tau) \sim b_3 e^{\mu_3^*\tau} + b_4 e^{\mu_3\tau}, \quad \tau \to +\infty, \tag{12}$$

with $b_4 = p_b b_3^*$, where the asymptotic expression for p_b coincides with that of p_a given by Eq. (8).

When the two interacting solitons are located sufficiently far away from one another, the solvability conditions of Eq. (11) can be written as

$$\partial_t\tau_{1,2} \approx G_{1,2}, \quad G_{1,2} = \langle \mathbf{w}_{1,2}\cdot\hat{F}(\mathbf{u}_1+\mathbf{u}_2)\rangle, \tag{13}$$

where we approximated the adjoint neutral modes of the operator $\hat{L}^\dagger(\mathbf{u}_1+\mathbf{u}_2)$ by the adjoint neutral modes $\mathbf{w}_{1,2} = \mathbf{w}_0(\tau-\tau_{1,2})$ of the operators $\hat{L}^\dagger(\mathbf{u}_{1,2})$.

In order to derive the soliton interaction equations, we need to calculate $G_{1,2}$ in Eq. (13). To this end, we split the integral in Eq. (13) into two parts and, using the relations $\hat{L}^\dagger(\mathbf{u}_{1,2})\mathbf{w}_{1,2} = 0$ together with the fact that $\mathbf{u}_1$ and $\mathbf{w}_1$ ($\mathbf{u}_2$ and $\mathbf{w}_2$) are small for $\tau \in [0,+\infty)$ ($\tau \in (-\infty,0]$), where the origin of coordinates $\tau = 0$ corresponds to the central point between two solitons, $(\tau_2+\tau_1)/2 = 0$, we get

$$\begin{aligned}G_{1,2} &= \langle \mathbf{w}_{1,2}\cdot\hat{F}(\mathbf{u}_1+\mathbf{u}_2)\rangle_{1,2} + \langle \mathbf{w}_{1,2}\cdot\hat{F}(\mathbf{u}_1+\mathbf{u}_2)\rangle_{2,1}\\ &\approx \langle \mathbf{w}_{1,2}\cdot\hat{F}(\mathbf{u}_1+\mathbf{u}_2)\rangle_{1,2}\\ &\approx \langle \mathbf{w}_{1,2}\cdot\hat{L}(\mathbf{u}_{1,2})\mathbf{u}_{2,1}\rangle_{1,2} - \left\langle \hat{L}^\dagger(\mathbf{u}_{1,2})\mathbf{w}_{1,2}\cdot\mathbf{u}_{2,1}\right\rangle_{1,2}\\ &= (\delta+i)\left[\left\langle w_{1,2}\partial_\tau^2 u_{2,1}\right\rangle_{1,2} - \left\langle u_{2,1}\partial_\tau^4 w_{1,2}\right\rangle_{1,2}\right]\\ &\quad + i\beta_4\left[\left\langle w_{1,2}\partial_\tau^4 u_{2,1}\right\rangle_{1,2} - \left\langle u_{2,1}\partial_\tau^4 w_{1,2}\right\rangle_{1,2}\right] + \text{c.c.},\end{aligned} \tag{14}$$

with $\langle \mathbf{w}\cdot\mathbf{u}\rangle_1 = \int_{-\infty}^{0}(\mathbf{w}\cdot\mathbf{u})d\tau$, $\langle \mathbf{w}\cdot\mathbf{u}\rangle_2 = \int_0^{\infty}(\mathbf{w}\cdot\mathbf{u})d\tau$, and $\hat{L}^\dagger(\mathbf{u}_{1,2})\mathbf{w}_{1,2} = 0$.

Next, performing integration by parts and using the symmetry properties of the soliton and its neutral modes, $u_0(\tau) = u_0(-\tau)$, $\partial_\tau u_0(\tau) = -\partial_\tau u_0(-\tau)$, $w_0(\tau) = -w_0(-\tau)$, and $\partial_\tau w_0(\tau) = \partial_\tau w_0(\tau)$, we get

$$\begin{aligned}G_{1,2} &\approx \pm\big[(\delta+i)(w_{1,2}^*\partial_\tau u_{2,1} - u_{2,1}\partial_\tau w_{1,2}^*)\\ &\quad + i\beta_4\big(w_{1,2}^*\partial_\tau^3 u_{2,1} - u_{2,1}\partial_\tau^3 w_{1,2}^* - \partial_\tau w_{1,2}^*\partial_\tau^2 u_{2,1}\\ &\quad + \partial_\tau^2 w_{1,2}^*\partial_\tau u_{2,1}\big)\big]_{\tau=0} + \text{c.c.}\\ &= \pm\big\{(\delta+i)\partial_\tau(w_0^*u_0) - i\beta_4\big[w_0^*\partial_\tau^3 u_0 + u_0\partial_\tau^3 w_0^*\\ &\quad + \partial_\tau(\partial_\tau w_0^*\partial_\tau u_0)\big]\big\}_{\tau=(\tau_2-\tau_1)/2} + \text{c.c.}\end{aligned} \tag{15}$$

Finally, substituting into Eq. (15) the asymptotic relations (7) and (12), we obtain

$$\frac{d(\tau_2-\tau_1)}{dt} \approx -\frac{12}{\sqrt{\beta_4}}e^{-\gamma(\tau_2-\tau_1)}\mathrm{Re}\left[\left(1-i\frac{\delta}{3}\right)\left(a_3b_3^*e^{-i\Omega(\tau_2-\tau_1)} - p_ap_b^*a_3^*b_3e^{i\Omega(\tau_2-\tau_1)}\right)\right], \tag{16}$$

$$\frac{d(\tau_2+\tau_1)}{dt} = 0, \tag{17}$$

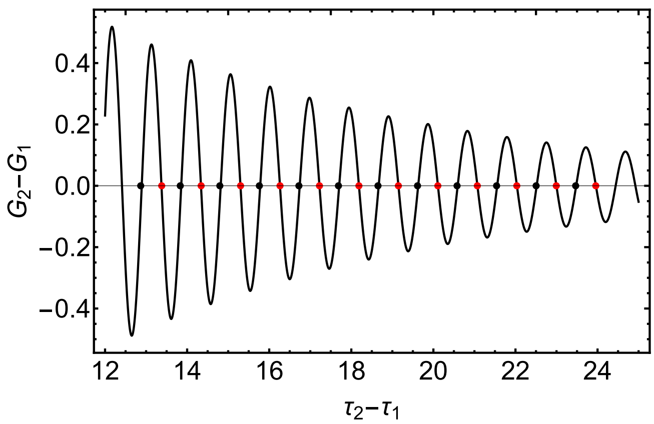

FIG. 4. Right-hand side of Eq. (16) as a function of the soliton separation $\tau_2-\tau_1$. Black (red) dots indicate the separations of the two solitons in stable (unstable) bound states calculated numerically. Parameter values: $S=2.0$, $\theta=3.5$, $\delta=0.02$, and $\beta_4=0.025$.

where $\gamma=\mathrm{Re}(\mu_3)\approx(\sqrt{\beta_4}/2)\sqrt{(1+\delta/\beta_4)-I_0^2}$, $\Omega=-\mathrm{Im}(\mu_3)\approx 1/\sqrt{\beta_4}+\sqrt{\beta_4}(\theta-2I_0)$, and the Cherenkov radiation coefficients a_3 and b_3 are exponentially small in the limit $\beta_4\to 0$. For $S=2.0$, $\theta=3.5$, $d=0.02$, and $\beta_4=0.025$ numerically, we get $a_3\approx-0.158+0.149i$ and $b_3\approx 0.017+0.136i$. Finally, neglecting $O(\delta)$ terms and taking into account that in the leading order in δ, we have $p_a=p_b\equiv p$, Eq. (16) can be rewritten in the form

$$\frac{d(\tau_2-\tau_1)}{dt}\approx\frac{12}{\sqrt{\beta_4}}e^{-\gamma(\tau_2-\tau_1)}|a_3b_3|(|p|^2-1)\times\cos[\Omega(\tau_2-\tau_1)+\arg(b_3/a_3)]. \quad (18)$$

The right-hand side of Eq. (18) is plotted in Fig. 4, where the intersections of the black solid line with axis of abscissas correspond to the soliton bound states. Examples of stable and unstable soliton bound states calculated numerically are shown in Figs. 5 and 6, respectively, together with the most unstable eigenvalues of the operator $\hat{L}$ evaluated on the bound state solutions.

Finally, in Fig. 7, we present the same soliton bound state as the one shown in Fig. 5, but calculated for $\delta=0$. It is seen that the eigenvalue spectrum of this state contains many discrete eigenvalues, which split from the continuous spectrum, and that it is oscillatory unstable due to the presence of two complex conjugate eigenvalues with positive real parts. Therefore, we can conclude that in the absence of spectral filtering, the one-dimensional asymptotic equations (16)–(18) can be insufficient to describe the soliton interaction. The derivation of the interaction equations taking into account an Andronov-Hopf bifurcation of the soliton bound states in the presence of fourth-order dispersion is beyond the scope of this study. A related problem concerning the effect of oscillatory instability on the soliton interaction was studied in [5].

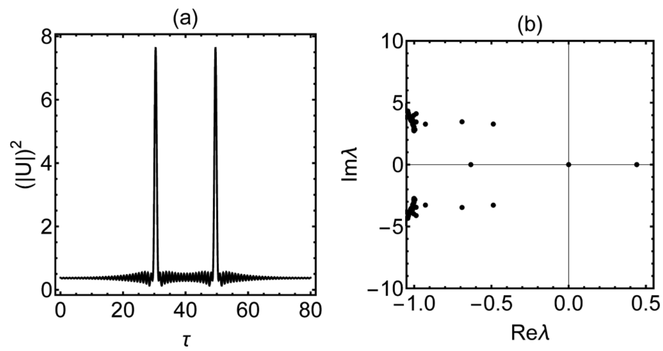

FIG. 6. Unstable bound state of two dissipative solitons corresponding to a red point in Fig. 4, $\delta=0.02$. (a) Intensity distribution and (b) eigenvalue spectrum. Other parameter values are the same as for Fig. 4.

IV. CONCLUSIONS

We have considered an all-fiber photonic crystal cavity coherently driven by an injected field. The intracavity field inside the fiber experiences self-phase modulation, dispersion, optical injection, and optical losses. Its space-time evolution can be described by the Lugiato-Lefever equation with high-order dispersion, where, in addition, we have taken into account the small spectral filtering term. We have first discussed the properties of a single dissipative soliton and derived asymptotic expressions for the soliton Cherenkov

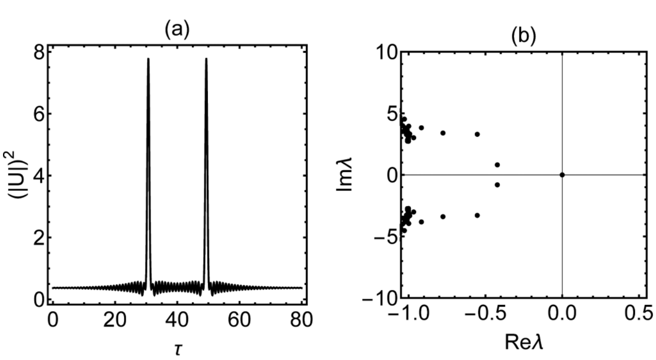

FIG. 5. A stable bound state of two dissipative solitons corresponding to a black point in Fig. 4, $\delta=0.02$. (a) Intensity distribution and (b) eigenvalue spectrum. Other parameter values are the same as for Fig. 4.

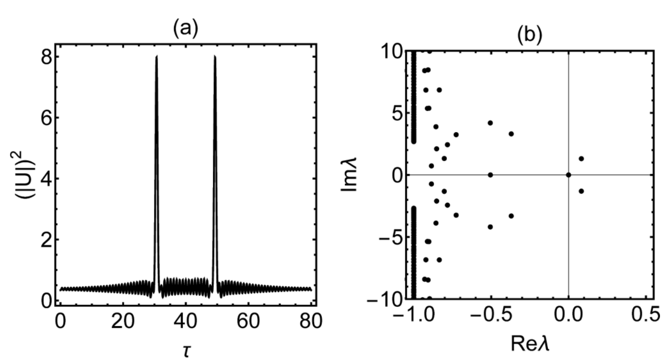

FIG. 7. The same bound state as shown in Fig. 5, but calculated for $\delta=0$. (a) Intensity distribution and (b) eigenvalue spectrum. The bound state is unstable with respect to an Andronov-Hopf bifurcation.

radiation amplitudes. We have focused our analysis on the regime, where the fourth-order dispersion and the spectral filtering coefficients are small, $0 < \beta_4, \delta \ll 1$. Second, we have investigated the interaction between two dissipative solitons in the case when they are well separated from each other. Assuming a weak overlap of soliton tails, we have established analytically the interaction law [Eqs. (17) and (18)] governing the slow time evolution of the coordinates of two interacting solitons. We have shown that although the Cherenkov radiation due to the small fourth-order dispersion can strongly enhance the soliton interaction and thus lead to the formation of a large number of soliton bound states, in the absence of spectral filtering these states can be unstable with respect to an oscillatory instability even when a single soliton is well below the Andronov-Hopf bifurcation threshold. This means that taking into consideration, in the interaction equations, additional degrees of freedom responsible for the Andronov-Hopf bifurcation (as was done in Ref. [5]) can be necessary to describe the soliton interaction in the generalized Lugiato-Lefever model (1) with zero spectral filtering coefficient, $\delta = 0$. On the other hand, the inclusion of small but sufficiently large spectral filtering, $0 < \delta \ll 1$, allows one to suppress the oscillatory instability and to keep the interaction equation one dimensional.

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