

**Entanglement of a bipartite channel**Gilad Gour<sup>\*</sup> and Carlo Maria Scandolo<sup>†</sup>*Department of Mathematics and Statistics and Institute for Quantum Science and Technology,  
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The most general quantum object that can be shared between two distant parties is a bipartite channel, as it is the basic element to construct all quantum circuits. In general, bipartite channels can produce entangled states, and can be used to simulate quantum operations that are not local. While much effort over the last two decades has been devoted to the study of entanglement of bipartite states, very little is known about the entanglement of bipartite channels. In this work, we rigorously study the entanglement of bipartite channels as a resource theory of quantum processes. We present an infinite and complete family of measures of dynamical entanglement, which gives necessary and sufficient conditions for convertibility under local operations and classical communication. Then we focus on the dynamical resource theory where free operations are positive partial transpose (PPT) superchannels, but we do not assume that they are realized by PPT pre- and postprocessing. This leads to a greater mathematical simplicity that allows us to express all resource protocols and the relevant resource measures in terms of semidefinite programs. Along the way, we generalize the negativity from states to channels, and introduce the max-logarithmic negativity, which has an operational interpretation as the exact asymptotic entanglement cost of a bipartite channel. Finally, we use the non-positive partial transpose (NPT) resource theory to derive a no-go result: it is impossible to distill entanglement out of bipartite PPT channels under any sets of free superchannels that can be used in entanglement theory. This allows us to generalize one of the long-standing open problems in quantum information—the NPT bound entanglement problem—from bipartite states to bipartite channels. It further leads us to the discovery of bound entangled positive operator valued measures.

DOI: [10.1103/PhysRevA.103.062422](https://doi.org/10.1103/PhysRevA.103.062422)**I. INTRODUCTION**

Quantum entanglement [1,2] is universally regarded as the most important aspect of quantum theory, making it radically different from classical theory. Schrödinger himself summarized this phenomenon as the fact that [3] “[...] the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of all its *parts*.” Indeed, entanglement is a necessary ingredient for the nonlocal phenomena observed in quantum theory [4–8]. The development of quantum information theory has brought a new perspective on quantum entanglement, seen as a resource in many protocols that cannot be implemented in classical theory. Think, for instance, of the paradigmatic examples of quantum teleportation [9], dense coding [10], and quantum key distribution [11]. The idea of entanglement concretely helping in information-theoretic tasks can be made precise and rigorous using the framework of *resource theories* [12–22].

Resource theories have been studied in great detail when the resources involved are *states* (also known as static resources) [19]. In this case, one wants to study the conversion between states. This is the usual setting in which a rigorous theory of entanglement can be put forward. The physical situ-

ation is when there are two separated parties, and, because of their spatial separation, they are restricted to performing local operations (LOs), and exchanging classical communication (CC) [1,2,23–25]. These free operations are called LOCC. In this setting, free states are those that can be prepared from scratch using an LOCC protocol; they are *separable states*. Then one studies the conversion between bipartite states when the two parties initially share a state, which they are tasked to manipulate and transform into a target state using LOCC channels. For pure entangled states, this conversion is fairly easy to study [26], and for them the distillation of maximal entangled states and the cost coincide. This is not the case for entangled mixed states, for which the distillation can be zero, yet the cost is strictly nonzero [27,28]. In other words, for some states, we need to spend maximally entangled states to create them, but, once created, we cannot get back any maximal entanglement. This phenomenon is called *bound entanglement*.

Despite being the natural choice dictated by the physical setting for entanglement, working with LOCC protocols is, in general, not easy [29]. For this reason, other choices of free operations have been considered, which are structurally and mathematically simpler to deal with. The first class is that of *separable operations* (SEPs) [30–32], which are the operations that send separable states to separable states, even when tensored with the identity channel. In resource-theoretic terminology they are *completely resource nongenerating op-*

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erations, i.e., the largest set of free operations (in the sense of inclusion) transforming free states into free states, in a complete sense [19]. LOCC channels (and even their topological closure [29]) have been shown to be a *strict* subset of separable operations [30,33,34].

We can also consider positive partial transpose (PPT) operations [32,35]. The definition of these operations is inspired by the Peres-Horodecki criterion [36,37] for the separability of bipartite states, based on partial transpose: a state is separable only if its partial transpose is still positive semidefinite. In this resource theory, free states are states with positive-semidefinite partial transpose (*PPT states*). They coincide with separable states for bipartite systems of dimension  $2 \otimes 2$  and  $2 \otimes 3$ , but in general there are also nonseparable PPT states [27]. This is indeed the case for all known bound entangled states [28]. In this non-positive partial transpose (NPT) resource theory, the free operations are the channels that send PPT states to PPT states even when tensored with the identity channel. They are called PPT operations. Clearly both LOCC and separable operations are subsets of PPT operations.

Despite not being so physically motivated, separable operations and PPT operations are helpful for their greater mathematical simplicity, and because they allow us to prove no-go results: if a state conversion is *not* possible under separable or PPT operations, then it is *not* possible under LOCC as well. Similarly, PPT and separable operations can provide upper and lower bounds for conversions with LOCC channels.

If one looks closely at the first examples where entanglement proved to be a resource (e.g., quantum teleportation and dense coding), one notices they involve the conversion of a state into a particular channel, i.e., a static resource into a dynamical one [38,39]. Therefore, the need to go beyond conversion between static resources is built into the very notion of entanglement as a resource. This is supported by the fact that in physics everything, including a state, can be viewed as a *dynamical resource* [40–42]. Therefore, it is really *necessary* to phrase entanglement theory as a resource theory of *quantum processes*. In these theories the agent converts different dynamical resources by means of a restricted set of *superchannels*.

In light of this, in this article we expand the results originally announced in Ref. [43], formulating a rigorous treatment of the resource theory of entanglement as a resource theory of processes (an independent work in this respect is Ref. [44]).

The generic resource will be *abipartite channel* [45,46] rather than a bipartite state. A bipartite channel, represented in Fig. 1, is a channel with two inputs and two outputs. We assume there is a spatial separation between the two inputs (and also between the two outputs). This spatial separation is associated with the presence of two spacelike separated parties, Alice and Bob, as for bipartite states. The novelty coming from the fact that we are considering channels rather than states is that we also have a time separation between the input side of the channel and its outputs. This makes bipartite channels the most general resource for the study of entanglement and, at the same time, the most versatile. Indeed, if we trivialize (i.e., make one dimensional) the two inputs of a bipartite channel, we recover the theory of entanglement for bipartite states. On the other hand, if we consider classical outputs, we obtain the “dual” resource theory of entanglement

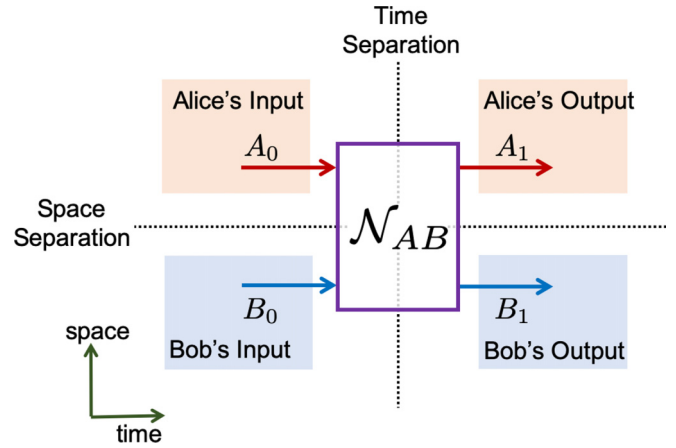


FIG. 1. The four regions of a bipartite channel. Note the space separation between the two parties, Alice and Bob. Unlike for bipartite states, we can also distinguish a temporal separation between the input and the output of each party.

for positive operator valued measures (POVMs). We can also consider other scenarios. For instance, if we trivialize Alice’s output and Bob’s input, we get a one-way channel from Alice to Bob, a situation studied in Ref. [47].

In this article, we apply the resource-theoretic constructions introduced in Ref. [48] to the resource theory of entanglement for bipartite channels. In particular, we focus on PPT and separable superchannels, for their greater mathematical simplicity, in the same spirit as one considers PPT and separable channels to study the entanglement of states. Our approach differs from Ref. [47] in a twofold way. First, we study the most general resource: bipartite channels, instead of just states and one-way channels. This allows us to generalize the notion of  $\kappa$ -entanglement [47], which we call *max-logarithmic negativity*, in two distinct ways. Second, we do not require PPT superchannels to have PPT pre- and postprocessing [49]. This leads to a great simplification in the mathematical treatment and the derivation of results, as all conditions on resource conversion can be expressed in terms of semidefinite programs (SDPs).

We conclude the article by analyzing bound entanglement for bipartite channels, showing that no entanglement can be distilled from PPT channels. We also provide the example of a bound entangled POVM.

The article is organized as follows. After some background information on superchannels and their Choi matrices presented in Sec. II, the resource theory of entanglement for bipartite channels is introduced in Sec. III, where we define the basic resource-theoretic protocols. In Sec. IV we analyze the simplest resource theory for entanglement from a mathematical point of view: the NPT resource theory. We show that all resource conversion tasks can be expressed in terms of SDPs, and, in particular, we provide an operational interpretation for the max-logarithmic negativity. Separable superchannels are introduced in Sec. V. We conclude the article with a study of bound entanglement for bipartite channels in Sec. VI. Conclusions are drawn in Sec. VII.

**II. PRELIMINARIES**

This section contains some basic notions that are extensively used throughout this article. First we specify the notation we use, and then we move to give a brief presentation of the main properties of supermaps and superchannels. We conclude the section with an overview of quantum combs.

**A. Notation**

Physical systems and their corresponding Hilbert spaces will be denoted by  $A, B, C$ , etc., where the notation  $AB$  means  $A \otimes B$ . Dimensions will be denoted with vertical bars, so the dimension of system  $A$  will be denoted by  $|A|$ . The tilde symbol will be reserved to indicate a replica of a system. For example,  $\tilde{A}$  denotes a replica of  $A$ , i.e.,  $|\tilde{A}| = |A|$ . Density matrices acting on Hilbert spaces will be denoted by lowercase greek letters  $\rho, \sigma$ , etc., except for the maximally mixed state (i.e., the uniform state), which will be denoted by  $u_A := \frac{1}{|A|}I_A$ .

The set of all bounded operators acting on system  $A$  is denoted by  $\mathfrak{B}(A)$ , the set of all Hermitian matrices acting on  $A$  by  $\text{Herm}(A)$ , and the set of all density matrices acting on system  $A$  by  $\mathfrak{D}(A)$ . We use calligraphic letters  $\mathcal{D}, \mathcal{E}, \mathcal{F}$ , etc., to denote quantum maps, reserving  $\mathcal{T}$  to represent the transpose map. The identity map on a system  $A$  will be denoted by  $\text{id}_A$ . The set of all linear maps from  $\mathfrak{B}(A)$  to  $\mathfrak{B}(B)$  is denoted by  $\mathfrak{L}(A \rightarrow B)$ , the set of all completely positive (CP) maps by  $\text{CP}(A \rightarrow B)$ , and the set of quantum channels, which are completely positive and trace-preserving (CPTP), by  $\text{CPTP}(A \rightarrow B)$ .  $\text{Herm}(A \rightarrow B)$  will denote the real vector space of all Hermitian-preserving maps in  $\mathfrak{L}(A \rightarrow B)$ . We will write  $\mathcal{N} \geq 0$  to mean that the map  $\mathcal{N} \in \text{Herm}(A \rightarrow B)$  is completely positive.

Unless otherwise specified, we will associate two subsystems  $A_0$  and  $A_1$  with every physical system  $A$ , referring, respectively, to the input and output of the resource. Hence, any physical system will be comprised of two subsystems  $A = (A_0, A_1)$ , even those representing a static resource, in which case we simply have  $|A_0| = 1$ . For simplicity, we will denote a channel with a subscript  $A$ , e.g.,  $\mathcal{N}_A$ , to mean that it is an element of  $\text{CPTP}(A_0 \rightarrow A_1)$ . Similarly, a bipartite channel in  $\text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  will be denoted by  $\mathcal{N}_{AB}$ . This notation makes the analogy with bipartite states more transparent.

In this setting, when we consider  $A = (A_0, A_1)$ ,  $B = (B_0, B_1)$ , etc., comprised of input and output subsystems, the symbol  $\mathfrak{L}(A \rightarrow B)$  refers to all linear maps from the vector space  $\mathfrak{L}(A_0 \rightarrow A_1)$  to the vector space  $\mathfrak{L}(B_0 \rightarrow B_1)$ . Similarly,  $\text{Herm}(A \rightarrow B) \subset \mathfrak{L}(A \rightarrow B)$  is a real vector space consisting of all the linear maps that take elements in  $\text{Herm}(A_0 \rightarrow A_1)$  to elements in  $\text{Herm}(B_0 \rightarrow B_1)$ . In other terms, maps in  $\text{Herm}(A \rightarrow B)$  take Hermitian-preserving maps to Hermitian-preserving maps. Linear maps in  $\mathfrak{L}(A \rightarrow B)$  and  $\text{Herm}(A \rightarrow B)$  will be called *supermaps*, and will be denoted by capital greek letters  $\Theta, \Upsilon, \Omega$ , etc. In the following, to avoid confusion with the notation for linear and Hermitian-preserving maps, whenever we mean linear or Hermitian-preserving maps, the systems involved will have a subscript, to make it clear that we are not considering pairs of systems. In this setting, the identity supermap in  $\mathfrak{L}(A \rightarrow A)$  will be denoted by  $\mathbb{1}_A$ .

We will use square brackets to denote the action of a supermap  $\Theta_{A \rightarrow B} \in \mathfrak{L}(A \rightarrow B)$  on a linear map  $\mathcal{N}_A \in \mathfrak{L}(A_0 \rightarrow A_1)$ . For example,  $\Theta_{A \rightarrow B}[\mathcal{N}_A]$  is a linear map in  $\mathfrak{L}(B_0 \rightarrow B_1)$  obtained from the action of the supermap  $\Theta$  on the map  $\mathcal{N}$ . Moreover, for a simpler notation, the identity supermap will not often appear explicitly in equations; e.g.,  $\Theta_{A \rightarrow B}[\mathcal{N}_{RA}]$  will mean  $(\mathbb{1}_R \otimes \Theta_{A \rightarrow B})[\mathcal{N}_{RA}]$ . Instead, the action of linear map (e.g., quantum channel)  $\mathcal{N}_A \in \mathfrak{L}(A_0 \rightarrow A_1)$  on a matrix  $\rho \in \mathfrak{B}(A_0)$  will be written with parentheses, i.e.,  $\mathcal{N}_A(\rho_{A_0}) \in \mathfrak{B}(A_1)$ .

Finally, we adopt the following convention concerning partial traces: when a system is missing, we take the partial trace over it. This applies to matrices as well as to maps. For example, if  $M_{AB}$  is a matrix on  $A_0A_1B_0B_1$ ,  $M_{AB_0}$  denotes the partial trace on the missing system  $B_1$ :  $M_{AB_0} := \text{Tr}_{B_1}[M_{AB}]$ .

**B. Supermaps**

In Refs. [50–52] it was shown that it is possible to construct the Choi matrix  $\mathbf{J}_{AB}^\Theta$  of a quantum supermap  $\Theta_{A \rightarrow B}$ . In particular, we can associate two linear maps with  $\Theta_{A \rightarrow B}$  [52]. The first is the map  $\mathcal{P}_{AB}^\Theta$ , defined as

$$\mathcal{P}_{AB}^\Theta := \Theta_{\tilde{A} \rightarrow B}[\Phi_{\tilde{A}\tilde{A}}^+],$$

where the map  $\Phi_{\tilde{A}\tilde{A}}^+$  acts on  $\rho \in \mathfrak{B}(A_0\tilde{A}_0)$  as

$$\Phi_{\tilde{A}\tilde{A}}^+(\rho_{A_0\tilde{A}_0}) = \text{Tr}[\rho_{A_0\tilde{A}_0} \phi_{A_0\tilde{A}_0}^+] \phi_{A_1\tilde{A}_1}^+, \tag{1}$$

with  $\phi_{A_0\tilde{A}_0}^+ := |\phi^+\rangle\langle\phi^+|_{A_0\tilde{A}_0}$  and  $|\phi^+\rangle_{A_0\tilde{A}_0} = \sum_j |jj\rangle_{A_0\tilde{A}_0}$  is the unnormalized maximally entangled state (expressed in the Choi basis). In other terms, the CP map  $\Phi_{\tilde{A}\tilde{A}}^+$  can be viewed as a generalization of  $\phi_{A_0\tilde{A}_0}^+$ . With this construction,  $\mathbf{J}_{AB}^\Theta$  can be defined as the Choi matrix of the map  $\mathcal{P}_{AB}^\Theta$ .

The second representation of a supermap is in terms of a linear map  $\mathcal{Q}^\Theta : \mathfrak{B}(A_1B_0) \rightarrow \mathfrak{B}(A_0B_1)$ , which is defined as the map satisfying

$$\mathbf{J}_{AB}^\Theta := \mathcal{Q}_{A_1\tilde{B}_0 \rightarrow A_0B_1}^\Theta (\phi_{A_1\tilde{A}_1}^+ \otimes \phi_{B_0\tilde{B}_0}^+),$$

or as  $\mathcal{Q}^\Theta := \mathbb{1}_A \otimes \Theta_{A \rightarrow B}[\mathcal{S}_A]$ , where  $\mathcal{S}_A$  is the swap from  $A_1$  to  $A_0$ . With this second construction, it is apparent that the Choi matrix  $\mathbf{J}_{AB}^\Theta$  of the supermap can be defined also as the Choi matrix of the the map  $\mathcal{Q}^\Theta$ . On top of being useful for the definition of the Choi matrix of a supermap, these two representations of a supermap,  $\mathcal{P}^\Theta$  and  $\mathcal{Q}^\Theta$ , will play a useful role in the study of the entanglement of bipartite channels.

A *superchannel* is a supermap  $\Theta_{A \rightarrow B} \in \mathfrak{L}(A \rightarrow B)$  that takes quantum channels to quantum channels even when tensored with the identity supermap [40,51–56]. More precisely,  $\Theta_{A \rightarrow B} \in \mathfrak{L}(A \rightarrow B)$  is called a superchannel if it satisfies the following two conditions:

- (1) For any trace-preserving map  $\mathcal{N}_A \in \mathfrak{L}(A_0 \rightarrow A_1)$ , the map  $\Theta_{A \rightarrow B}[\mathcal{N}_A]$  is a trace-preserving map in  $\mathfrak{L}(B_0 \rightarrow B_1)$ .
- (2) For any system  $R = (R_0, R_1)$  and any bipartite CP map  $\mathcal{N}_{RA} \in \text{CP}(R_0A_0 \rightarrow R_1A_1)$ , the map  $\Theta_{A \rightarrow B}[\mathcal{N}_{RA}]$  is also CP.

We will also say that a supermap  $\Theta_{A \rightarrow B} \in \mathfrak{L}(A \rightarrow B)$ , is *completely positive* if it satisfies the second condition above [40,52]. Therefore, a superchannel is a CP supermap that takes trace-preserving maps to trace-preserving maps [52,56]. We will denote the set of superchannels from

$A$  to  $B$  by  $\mathfrak{S}(A \rightarrow B)$ . Note that  $\mathfrak{S}(A \rightarrow B) \subset \mathfrak{L}(A \rightarrow B)$ . In particular, for the Choi matrix of a superchannel, we have  $\mathbf{J}_{A_1 B_0}^\ominus = I_{A_1 B_0}$  and  $\mathbf{J}_{A_0 B_1}^\ominus = \mathbf{J}_{A_0 B_0}^\ominus \otimes u_{A_1}$ .

The definitions seen so far are abstract; nevertheless, superchannels are *physical* objects that can be realized in terms of pre- and postprocessing that are both quantum channels [40,52]. Indeed, if  $\Theta \in \mathfrak{S}(A \rightarrow B)$ , then there exist a Hilbert space  $E$ , with  $|E| \leq |A_0 B_0|$ , and two CPTP maps  $\mathcal{F} \in \text{CPTP}(B_0 \rightarrow EA_0)$  and  $\mathcal{E} \in \text{CPTP}(EA_1 \rightarrow B_1)$  such that, for all  $\mathcal{N}_A \in \mathfrak{L}(A_0 \rightarrow A_1)$ ,

$$\Theta[\mathcal{N}_A] = \mathcal{E}_{EA_1 \rightarrow B_1} \circ \mathcal{N}_{A_0 \rightarrow A_1} \circ \mathcal{F}_{B_0 \rightarrow EA_0}.$$

### C. Quantum combs

Quantum combs are multipartite channels with a well-defined causal structure [50,51,57–60]. They generalize the notion of superchannels to objects that take several channels as input, and output a channel (see Refs. [50,51] for more details, and a further generalization where the input and the output of combs are combs themselves). We will denote a comb with  $n$  channel slots as input by  $\mathcal{C}_n$ , and its action on  $n$  channels by  $\mathcal{C}_n[\mathcal{N}_1, \dots, \mathcal{N}_n]$ . The causal relation between the different slots ensures that each such comb can be realized with  $n + 1$  channels  $\mathcal{E}_1, \dots, \mathcal{E}_{n+1}$ . We therefore associate a quantum channel

$$\mathcal{Q}^{\mathcal{C}_n} := \mathcal{E}_{n+1} \circ \mathcal{E}_n \circ \dots \circ \mathcal{E}_1$$

with every comb. Note that the quantum channel  $\mathcal{Q}^{\mathcal{C}_n}$  has a causal structure in the sense that the input to  $\mathcal{E}_k$  cannot affect the output of  $\mathcal{E}_{k-1}$  for any  $k = 2, \dots, n + 1$ .

The Choi matrix of the comb is defined as the Choi matrix of  $\mathcal{Q}^{\mathcal{C}_n}$ . Owing to the causal structure of  $\mathcal{Q}^{\mathcal{C}_n}$ , the marginals of the Choi matrix of  $\mathcal{C}_n$  satisfy similar relations to the marginals of the Choi matrix of a superchannel (see Refs. [50,51] for more details).

## III. DYNAMICAL ENTANGLEMENT THEORY

Recall that with one ebit, thanks to quantum teleportation [9], we can simulate a qubit channel from Alice to Bob using LOCC [23–25], and vice versa [38,39]. Therefore, one ebit (a *static* resource) is equivalent to a dynamical one: a qubit channel. Considering *bipartite channels* [46] in  $\text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  (see Fig. 1), we can understand the qubit identity channel from  $A_0$  to  $B_1$  as the maximal resource under LOCC as long as  $|A_1| = |B_0| = 1$ . It is maximal because, by using it, every other channel can be implemented between  $A_0$  and  $B_1$ .

Now let us generalize this situation by analyzing what the maximal resource is when all systems are nontrivial, and specifically  $|A_0| = |A_1| = |B_0| = |B_1| = d$ . In Fig. 2 we show that the SWAP operation is a maximal resource. Note that the SWAP operator can produce two e-dits, and can also be simulated by two e-dits. Therefore, the entanglement of the SWAP operator is two e-dits. Note also that the SWAP operator is the maximal resource even if the set of free operations allows only one-way classical communication. On the other hand, in the quantum resource theory in which free operations consists of only local operations and shared entanglement (LOSE) [61], but no classical communication, then two noiseless channels,

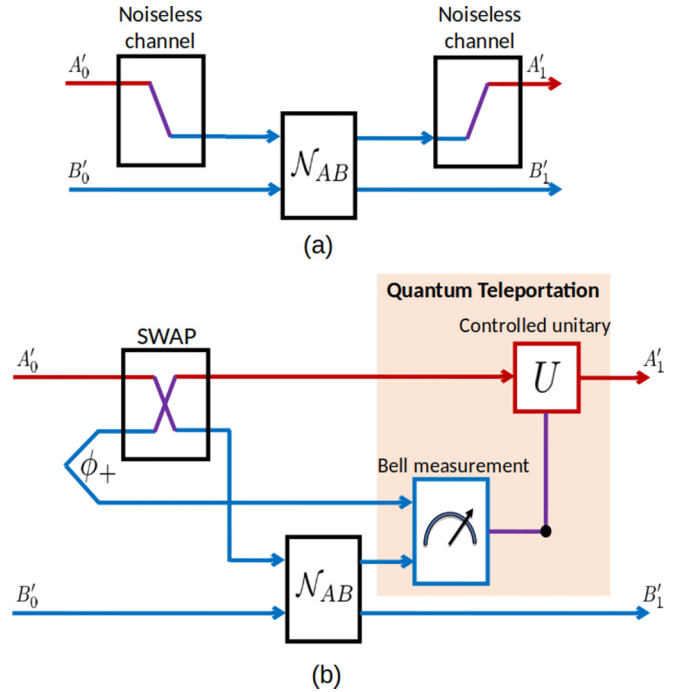


FIG. 2. (a) Simulation of an arbitrary channel  $\mathcal{N}_{A'B'}$  with two noiseless channels. (b) Simulation of an arbitrary channel  $\mathcal{N}_{A'B'}$  with the SWAP resource and one-way LOCC.

one from  $A_0 \rightarrow B_1$  and one from  $B_0 \rightarrow A_1$ , are more resourceful than the SWAP operator. This is because the SWAP operator is restricted to act *simultaneously* on both input systems. This example demonstrates that in general, two channels  $\mathcal{N}_{A_0 \rightarrow B_1}$  and  $\mathcal{M}_{B_0 \rightarrow A_1}$  can be more resourceful than their tensor product  $\mathcal{N}_{A_0 \rightarrow B_1} \otimes \mathcal{M}_{B_0 \rightarrow A_1}$  since they can be used *at different times*.

The fact that a tuple of  $n$  channels can be a greater resource than their tensor product was also discussed in Ref. [62] (cf. also Ref. [48]). In the following, however, we will focus mainly on a *single* resource at a time, in this case a single bipartite channel.

### A. Simulation of channels: Cost and distillation

Following Refs. [48,62–64], in Fig. 3 we illustrate the most general LOCC superchannel that can act on a bipartite

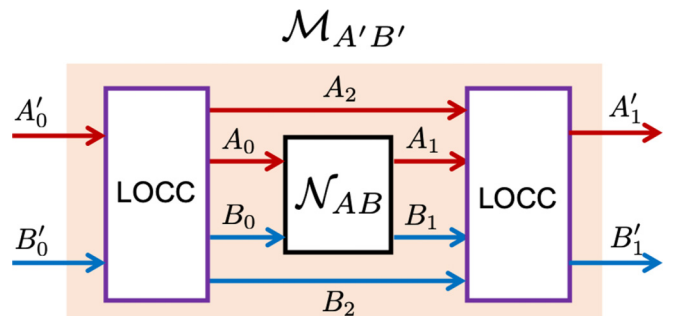
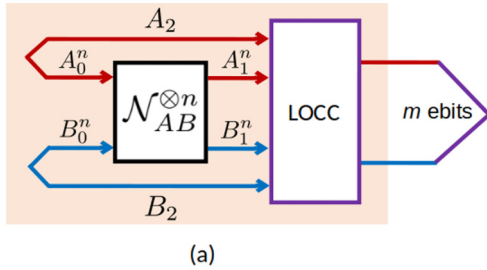
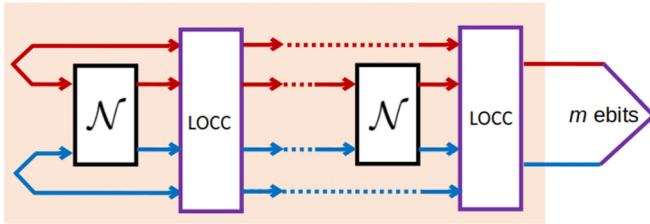


FIG. 3. The action of an LOCC superchannel on one copy of a bipartite channel  $\mathcal{N}_{AB}$ . The resulting channel is  $\mathcal{M}_{A'B'}$ . Note that this superchannel uses the dynamical resource  $\mathcal{N}_{AB}$  to simulate another channel  $\mathcal{M}_{A'B'}$ .



(a)



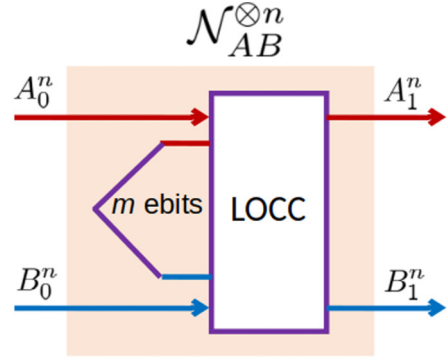
(b)

FIG. 4. The action of an LOCC superchannel on  $n$  copies of the bipartite channel  $\mathcal{N}_{AB}$ . (a) Parallel strategy for the distillation of  $m$  ebits out of  $n$  copies of  $\mathcal{N}_{AB}$ . (b) Adaptive strategy for the distillation of  $m$  ebits out of  $n$  subsequent uses of  $\mathcal{N}_{AB}$ .

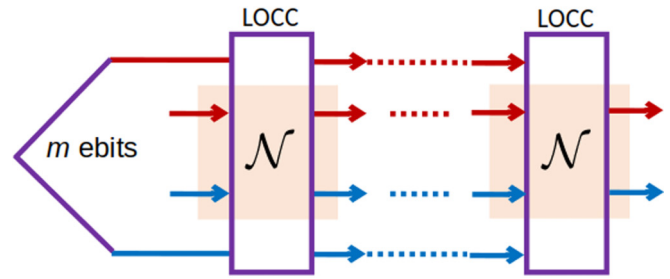
channel. The superchannel consists of a preprocessing and a postprocessing channel that are both LOCC. Moreover, the side channel, corresponding to the memory in the realization of a superchannel, consists of two parts:  $A_2$  on Alice’s side and  $B_2$  on Bob’s side. We denote the set of such superchannels by  $\text{LOCC}(AB \rightarrow A'B')$ .

The discussion at the beginning of Sec. III shows that ebits remain the units to quantify the entanglement of a bipartite channel. Indeed, two ebits can be used to simulate any bipartite channel in which the two input and two output systems are all qubits. Therefore, even in the resource theory of entanglement of bipartite channels one can define operational tasks in a very similar fashion to the state domain. For example, in Figs. 4(a) and 4(b) we illustrate parallel [65] and adaptive strategies [44,66–69] to distill static entanglement out of a dynamical resource. Since the parallel scheme is a special instance of the adaptive strategy, the distillable entanglement cannot be smaller when using the adaptive scheme. However, in Sec. VI we will see that there are bipartite entangled channels from which *no* distillation is possible, no matter what strategy is applied. This generalizes the notion of bound entanglement [28] to bipartite channels.

Similar to distillation, also the entanglement cost of a bipartite channel can be divided into two types: parallel and adaptive. In the parallel scheme, the goal is to simulate  $\mathcal{N}_{AB}^{\otimes n}$ , i.e.,  $n$  copies of  $\mathcal{N}_{AB}$  all acting simultaneously [see Fig. 5(a)]. On the other hand, the goal of the adaptive scheme is to simulate  $n$  copies of  $\mathcal{N}_{AB}$  in a time sequential order [see Fig. 5(b)]. Both schemes use ebits to simulate the channels. For the same reason as for the distillation case, note that the cost of simulating  $n$  sequentially ordered channels cannot be greater than the cost in the parallel scheme. Owing to the complexity of the adaptive scheme, in this paper we will focus mostly on the parallel one.



(a)



(b)

FIG. 5. The cost of simulating a bipartite channel. (a) Parallel strategy: consumption of  $m$  ebits to simulate  $\mathcal{N}_{AB}^{\otimes n}$ . (b) Adaptive strategy: consumption  $m$  ebits to simulate  $n$  subsequent uses of  $\mathcal{N}_{AB}$ .

Now we are ready to give the formal definitions of entanglement costs and distillable entanglement of bipartite channels. First of all, note that in entanglement theory, the conversion distance for any two channels  $\mathcal{N}_{AB}$  and  $\mathcal{M}_{A'B'}$ , introduced in Ref. [48] and inspired by Ref. [70], is given by

$$d_{\text{LOCC}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) = \frac{1}{2} \inf_{\Theta \in \text{LOCC}(AB \rightarrow A'B')} \|\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'}\|_{\diamond},$$

where the optimization is over the set of LOCC superchannels. Typically, the computation of this quantity is NP hard. To see why, consider the special case in which  $\mathcal{N}_{AB}$  is a bipartite separable state (i.e.,  $|A_0| = |B_0| = 1$ ), and  $\mathcal{M}_{AB}$  is some (possibly entangled) bipartite state as well. In this case, the computation of the conversion distance would determine if the bipartite state  $\mathcal{M}_{AB}$  is entangled or not, but this is known to be NP hard [71,72].

Furthermore, we know that if  $\Theta \in \text{LOCC}(AB \rightarrow A'B')$  then the bipartite channel  $\mathcal{Q}_{AB \rightarrow A'B'}^{\Theta}$  is also LOCC, while the condition that  $\mathcal{Q}_{AB \rightarrow A'B'}^{\Theta}$  is LOCC is most likely insufficient to ensure that  $\Theta \in \text{LOCC}(AB \rightarrow A'B')$ . This adds another layer of complexity to the problem of computing  $d_{\text{LOCC}}$ . In Sec. IV we will see that this additional complexity persists even when considering simpler sets of operations, like PPT channels [32,35].

Since in entanglement theory there exists a unique (up to local unitaries) maximal static resource, the single-shot entanglement cost and entanglement distillation with error  $\varepsilon \geq 0$  are given respectively by

$$\begin{aligned} \text{COST}_{\text{LOCC},\varepsilon}^{(1)}(\mathcal{N}_{AB}) &:= \log_2 \min_{m \in \mathbb{Z}_+} \{m : d_{\text{LOCC}}(\phi_m^+ \rightarrow \mathcal{N}_{AB}) \leq \varepsilon\} \end{aligned}$$

and

$$\begin{aligned} \text{DISTILL}_{\text{LOCC},\varepsilon}^{(1)}(\mathcal{N}_{AB}) &:= \log_2 \max_{m \in \mathbb{Z}_+} \{m : d_{\text{LOCC}}(\mathcal{N}_{AB} \rightarrow \phi_m^+) \leq \varepsilon\}, \end{aligned}$$

where  $\phi_m^+$  is a (normalized) maximally entangled state with Schmidt rank  $m$ . Here the optimizations are over this Schmidt rank  $m$ . Then the entanglement cost and the distillable entanglement of a dynamical resource in the asymptotic regime are defined respectively as

$$\text{COST}_{\text{LOCC}}(\mathcal{N}_{AB}) := \lim_{\varepsilon \rightarrow 0^+} \liminf_n \frac{1}{n} \text{COST}_{\text{LOCC},\varepsilon}^{(1)}(\mathcal{N}_{AB}^{\otimes n})$$

and

$$\begin{aligned} \text{DISTILL}_{\text{LOCC}}(\mathcal{N}_{AB}) &:= \lim_{\varepsilon \rightarrow 0^+} \limsup_n \frac{1}{n} \text{DISTILL}_{\text{LOCC},\varepsilon}^{(1)}(\mathcal{N}_{AB}^{\otimes n}). \end{aligned}$$

These definitions assume the parallel scheme. In the adaptive scheme, the entanglement cost and the distillable entanglement are defined accordingly, as per Ref. [48].

### B. Measures of dynamical entanglement

In this section we discuss a few measures that quantify the entanglement of a bipartite channel. We also examine the form that the complete family of resource measures introduced in Ref. [48] takes in entanglement theory.

A function  $E : \text{CPTP}(A_0B_0 \rightarrow A_1B_1) \rightarrow \mathbb{R}$  is called a *measure of dynamical entanglement* if it does not increase under LOCC superchannels. It is called *dynamical entanglement monotone* if it is convex, and does not increase on average under LOCC superinstruments [56]. Some measures of dynamical resources are discussed in Refs. [43,44,48,62,64,66,73–79]. Specifically, for bipartite entanglement the *relative entropy of dynamical entanglement* can be defined as

$$E_{\text{rel}}(\mathcal{N}_{AB}) = \inf_{\mathcal{M} \in \text{LOCC}} D(\mathcal{N}_{AB} \| \mathcal{M}_{AB}).$$

Note that we are using the infimum rather than the minimum because the set of LOCC channels is not topologically closed [29].

Moreover, any measure of static entanglement  $E$  that is monotonic under separable channels (in particular, under LOCC) can be extended to bipartite channels in two different ways [48,62,64,74]. In the first, we consider the amortized extension (cf. also Refs. [73,80])

$$\begin{aligned} E^{(1)}(\mathcal{N}_{AB}) &:= \sup_{\sigma \in \mathfrak{D}(A'_0B'_0A_0B_0)} \{E(\mathcal{N}_{A_0B_0 \rightarrow A_1B_1}(\sigma_{A'_0B'_0A_0B_0})) \\ &\quad - E(\sigma_{A'_0B'_0A_0B_0})\}, \end{aligned}$$

where  $A'_0$  and  $B'_0$  are additional reference systems in Alice’s and Bob’s sides, respectively, and the optimization is over all density matrices on the system  $A'_0B'_0A_0B_0$ . The other extension is given by

$$E^{(2)}(\mathcal{N}_{AB}) := \sup_{\sigma \in \text{SEP}(A'_0A_0 : B'_0B_0)} E(\mathcal{N}_{A_0B_0 \rightarrow A_1B_1}(\sigma_{A'_0B'_0A_0B_0})),$$

where  $\text{SEP}(A'_0A_0 : B'_0B_0)$  denotes the set of separable states between Alice and Bob. Both of the above extensions of  $E$  can be proved to be nonincreasing under separable superchannels [74].

Now we introduce the complete family of dynamical entanglement measures, following our construction in Ref. [48]. For any (fixed) bipartite channel  $\mathcal{P} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ , define (see Ref. [48])

$$E_{\mathcal{P}}(\mathcal{N}_{AB}) := \sup_{\Theta \in \text{LOCC}(AB \rightarrow A'B')} \text{Tr}[J_{A'B'}^{\mathcal{P}} J_{A'B'}^{\Theta[\mathcal{N}]}],$$

where  $\mathcal{N}_A \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$ , and  $J$  is the Choi matrix of the channel in its superscript. Note again that we are using the supremum instead of the maximum because the set of LOCC channels is not closed. This function may not vanish on LOCC channels; if we want so, we need to subtract  $\sup_{\mathcal{M} \in \text{LOCC}(A'B')} \text{Tr}[J_{A'B'}^{\mathcal{P}} J_{A'B'}^{\mathcal{M}}]$ . As explained in Ref. [48], this defines a new non-negative measure of dynamical entanglement, which vanishes on LOCC channels. Furthermore, the set of functions  $\{E_{\mathcal{P}}\}$  is complete, in the sense that a bipartite channel  $\mathcal{N}_{AB}$  can be converted within the topological closure of LOCC superchannels into another bipartite channel  $\mathcal{E}_{A'B'}$  if and only if

$$E_{\mathcal{P}}(\mathcal{N}_{AB}) \geq E_{\mathcal{P}}(\mathcal{E}_{A'B'}) \tag{2}$$

for every  $\mathcal{P} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ .

A natural question to ask is whether it is possible to find another family of measures of dynamical entanglement that is finite, but at the same time complete. However, in Ref. [81] it was proved that any such complete family of entanglement measures *must* be infinite. Nevertheless, our family  $\{E_{\mathcal{P}}\}$  can be made *countable* since we can remove from it all the channels  $\mathcal{P}$  whose Choi matrix includes coefficients that are irrational. This can be done because, by construction, each function  $E_{\mathcal{P}}$  is continuous in  $\mathcal{P}$ . Since the set of all channels  $\mathcal{P}$  whose Choi matrices involve only rational coefficients is dense in the set of all Choi matrices, by continuity it follows that, if Eq. (2) holds for all such rational  $\mathcal{P}$ s, it holds also for all  $\mathcal{P} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ . We conclude that our family  $\{E_{\mathcal{P}}\}$  is optimal, in the sense that there is no other complete family of measures of dynamical entanglement that characterizes the LOCC entanglement of a bipartite channel more efficiently.

Despite the various interesting properties of the measures of dynamical entanglement discussed in this section, they are all extremely hard to compute due to the complexity of LOCC channels and superchannels. We leave the discussion of more computationally manageable measures to Sec. IV C.

### C. Entanglement of bipartite POVMs

We end this section on the general properties of the resource theory of dynamical entanglement with a short dis-

cussion on entanglement of bipartite POVMs. A bipartite channel  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  for which the output system  $A_1B_1$  is classical can be viewed as a POVM. In this case, the channel can be expressed as

$$\mathcal{N}_{AB}(\rho_{A_0B_0}) = \sum_{x,y} \text{Tr}[\rho_{A_0B_0} E_{A_0B_0}^{xy}] |xy\rangle\langle xy|_{A_1B_1},$$

where the set of matrices  $\{E_{A_0B_0}^{xy}\}_{x,y}$  forms a POVM, and  $\{|xy\rangle\}_{x,y}$  is an orthonormal basis of  $A_1B_1$ . Such channels are fully characterized by the condition  $\mathcal{D}_{A_1B_1} \circ \mathcal{N}_{AB} = \mathcal{N}_{AB}$ , where  $\mathcal{D}_{A_1B_1}$  is the completely dephasing channel on system  $A_1B_1$  (with respect to the fixed classical basis). Note that  $\mathcal{D}_{A_1B_1} \in \text{LOCC}(A_1B_1 \rightarrow A_1B_1)$ .

*Lemma 1.* Let  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  be a bipartite POVM. Then

$$E_{\text{rel}}(\mathcal{N}_{AB}) = \inf_{\substack{\mathcal{M} \in \text{LOCC} \\ \mathcal{D}_{A_1B_1} \circ \mathcal{M}_{AB} = \mathcal{M}_{AB}}} D(\mathcal{N}_{AB} \| \mathcal{M}_{AB}). \quad (3)$$

*Proof.* Clearly, by definition  $E_{\text{rel}}(\mathcal{N}_{AB})$  is less than or equal to the right-hand side of Eq. (3). Let us prove the converse inequality. We have

$$\begin{aligned} E_{\text{rel}}(\mathcal{N}_{AB}) &= \inf_{\mathcal{M} \in \text{LOCC}} D(\mathcal{N}_{AB} \| \mathcal{M}_{AB}) \\ &\geq \inf_{\mathcal{M} \in \text{LOCC}} D(\mathcal{D}_{A_1B_1} \circ \mathcal{N}_{AB} \| \mathcal{D}_{A_1B_1} \circ \mathcal{M}_{AB}), \end{aligned}$$

where the inequality follows from the generalized data-processing inequality [74]. Now recall that, being a POVM,  $\mathcal{D}_{A_1B_1} \circ \mathcal{N}_{AB} = \mathcal{N}_{AB}$ . Therefore,  $E(\mathcal{N}_{AB}) \geq \inf_{\mathcal{M} \in \text{LOCC}} D(\mathcal{N}_{AB} \| \mathcal{D}_{A_1B_1} \circ \mathcal{M}_{AB})$ . Hence we conclude

that

$$E_{\text{rel}}(\mathcal{N}_{AB}) = \inf_{\substack{\mathcal{M} \in \text{LOCC} \\ \mathcal{D}_{A_1B_1} \circ \mathcal{M}_{AB} = \mathcal{M}_{AB}}} D(\mathcal{N}_{AB} \| \mathcal{M}_{AB}).$$

The above lemma demonstrates that the relative entropy of entanglement of a bipartite POVM can be viewed as its relative entropy distance to the set of LOCC POVMs (rather than arbitrary bipartite LOCC channels).

Now, note that if systems  $A_1$  and  $B_1$  are classical, we can view them as a single classical system (since classical communication is free), and instead of using two indices  $x, y$  to characterize the POVM, it makes more sense to use just a single index, say  $x$ . In this setting, the above lemma can be used to calculate the relative entropy of process entanglement for a POVM  $\{N_{A_0B_0}^x\}$ . Consider the associated quantum-to-classical channel  $\mathcal{N}_{A_0B_0 \rightarrow X}(\rho_{A_0B_0}) = \sum_{x=1}^{|X|} \text{Tr}[\rho_{A_0B_0} N_{A_0B_0}^x] |x\rangle\langle x|_X$ , and an LOCC POVM  $\{F_{A_0B_0}^y\}$ , with its associated quantum-to-classical channel  $\mathcal{F}_{A_0B_0 \rightarrow Y}(\rho_{A_0B_0}) = \sum_{y=1}^{|Y|} \text{Tr}[\rho_{A_0B_0} F_{A_0B_0}^y] |y\rangle\langle y|_Y$ . Now, possibly by completing one of the two POVMs with some zero elements, we can always take  $X = Y$ . To calculate the channel divergence we have to evaluate  $\mathcal{N}_{A_0B_0 \rightarrow X}$  and  $\mathcal{F}_{A_0B_0 \rightarrow X}$  on any pure state  $\psi_{RA_0B_0}$ , where  $R$  is isomorphic to  $A_0B_0$  [82,83]. Recall that  $\psi_{RA_0B_0} = (I_R \otimes \sqrt{\gamma_{A_0B_0}} U_{A_0B_0}) \phi_{RA_0B_0}^+ (I_R \otimes U_{A_0B_0}^\dagger \sqrt{\gamma_{A_0B_0}})$ , where  $\gamma_A \in \mathfrak{D}(A_0B_0)$  and  $U_{A_0B_0}$  is some unitary. After some calculations, we obtain

$$E(\{N^x\}) = \inf_{\{F_x\} \in \text{LOCC}} \max_{\gamma, U} D\left(\sum_x U \sqrt{\gamma} (N^x)^T \sqrt{\gamma} U^\dagger \otimes |x\rangle\langle x| \left\| \sum_x U \sqrt{\gamma} (F^x)^T \sqrt{\gamma} U^\dagger \otimes |x\rangle\langle x| \right\| \right).$$

By the properties of  $D$ , we have finally

$$\begin{aligned} E(\{N^x\}) &= \inf_{\{F_x\} \in \text{LOCC}} \max_{\gamma} \sum_x D(\sqrt{\gamma} (N^x)^T \sqrt{\gamma} \| \sqrt{\gamma} (F^x)^T \sqrt{\gamma}). \end{aligned}$$

Using the protocol of entanglement swapping [84], we can use the entanglement of POVMs to produce static entanglement. This is illustrated in Fig. 6.

#### IV. NPT ENTANGLEMENT OF A BIPARTITE CHANNEL

Entanglement theory is hard to study due to the complexity of LOCC channels [29,85–88] and the fact that even determining whether a given state is entangled or not is known to be NP hard [71,72]. For this reason, much of the work in recent years on entanglement theory involved the replacement of LOCC with a larger set of free operations that are more computationally friendly (see, e.g., Ref. [89] and references therein). One such set is the set of separable operations (or in short SEP; cf. Sec. V) [30–32]; another one is the set of PPT operations [32,35]. Both sets are larger than LOCC, but the set of PPT operations is much larger than both LOCC

and SEP operations, as it contains, for instance, PPT bound entangled states [27,28] (viewed as PPT channels with trivial input). Yet, among them, the set of PPT operations has the simplest characterization, and can be used to provide insights into LOCC entanglement, including various bounds on LOCC tasks.

Bipartite states with positive (semidefinite) partial transpose (called PPT states) were first discussed in Refs. [36,37]

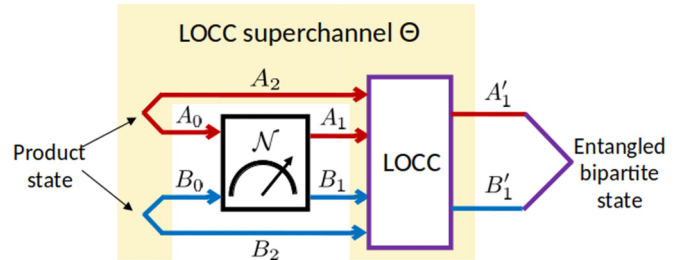


FIG. 6. Here  $A_1$  and  $B_1$  are classical systems. The action of an LOCC superchannel  $\Theta$  on a bipartite channel with classical output can produce an entangled state.

in the context of entanglement theory. A few years later Rains [32,35] defined PPT bipartite channels for the first time (of which LOCC or SEP channels are a special type), and used them to find an upper bound on the distillable entanglement. In this section we consider PPT superchannels [49], and use them for the study of entanglement of bipartite channels. We will see that several of the optimization problems introduced in the previous sections can be solved with SDPs in this theory of entanglement, called the theory of *NPT entanglement*. We start with a few notations that will be very useful in the following.

Denote the transpose supermap by  $\Upsilon_B \in \mathcal{L}(B \rightarrow B)$ :

$$\Upsilon_B[\mathcal{N}_B] := \mathcal{T}_{B_1} \circ \mathcal{N}_{B_0 \rightarrow B_1} \circ \mathcal{T}_{B_0},$$

for all  $\mathcal{N}_B \in \mathcal{L}(B_0 \rightarrow B_1)$ , where  $\mathcal{T}_{B_0}$  and  $\mathcal{T}_{B_1}$  are the transpose maps on the input and output systems, respectively. In Refs. [32,35] the symbol  $\Gamma$  was used to indicate the partial transpose of a bipartite channel; that is,

$$\mathcal{N}_{AB}^\Gamma := (\mathbb{1}_A \otimes \Upsilon_B)[\mathcal{N}_{AB}].$$

In the following we adopt the convention to always choose Bob's systems (i.e., those denoted by  $B$ ) to apply the (partial) transpose to. With these notations, the set of PPT maps in  $\text{CP}(A_0B_0 \rightarrow A_1B_1)$  is defined as

$$\begin{aligned} \text{PPT}(A_0B_0 \rightarrow A_1B_1) \\ = \{ \mathcal{N} \in \text{CP}(A_0B_0 \rightarrow A_1B_1) : \mathcal{N}_{AB}^\Gamma \geq 0 \}. \end{aligned}$$

Note that PPT maps are defined as general CP maps, not necessarily as channels. PPT maps have several useful properties. First,  $\mathcal{N}_{AB} \in \text{PPT}(A_0B_0 \rightarrow A_1B_1)$  if and only if its Choi matrix  $J_{AB}^\mathcal{N}$  satisfies  $J_{AB}^\mathcal{N} \geq 0$  and  $(J_{AB}^\mathcal{N})^{T_B} \geq 0$ . The former condition implies that  $\mathcal{N}_{AB}$  is a CP map, and the latter ensures that it is PPT. The latter follows from the identity

$$J_{AB}^{\mathcal{N}^\Gamma} = (J_{AB}^\mathcal{N})^{T_B}. \quad (4)$$

Furthermore, PPT maps have the property that they are completely PPT preserving [89], meaning that if  $\mathcal{N}_{AB} \in \text{PPT}(A_0B_0 \rightarrow A_1B_1)$ , then for every bipartite PPT quantum state  $\rho \in \mathfrak{D}(A'_0A_0B'_0B_0)$ , the matrix  $\mathcal{N}_{A_0B_0 \rightarrow A_1B_1}(\rho_{A'_0A_0B'_0B_0})$  has positive partial transpose. In other words,  $\mathcal{N}_{AB}$  takes PPT positive-semidefinite matrices to PPT positive-semidefinite matrices even when it is tensored with the identity.

Here we discuss two types of generalizations of PPT maps to supermaps. We call the first one *restricted PPT superchannels*, to distinguish it from the PPT supermaps we will study extensively in what follows. We will see that restricted PPT superchannels lead to a cumbersome entanglement theory on bipartite channels, similar to the one used in Refs. [47,67,90]. Further, here we consider bipartite channels, whereas in Ref. [47] the authors considered only one-way channels from Alice to Bob (i.e., the special case in which  $|B_0| = |A_1| = 1$ ).

A restricted PPT superchannel is depicted in Fig. 7. In the language of Ref. [48], it is a freely realizable superchannel: it consists of pre- and postprocessing channels  $\mathcal{E}$  and  $\mathcal{F}$  that are *both* PPT. Note that, at a first glance, this looks a very natural definition, and as discussed in Ref. [48], it is the most physical and natural one. Moreover, denoting this restricted PPT superchannel by  $\Theta$ , it is clear that if  $\mathcal{N}$  is a PPT channel then also the resulting map  $\Theta[\mathcal{N}]$  is PPT. Nonetheless,

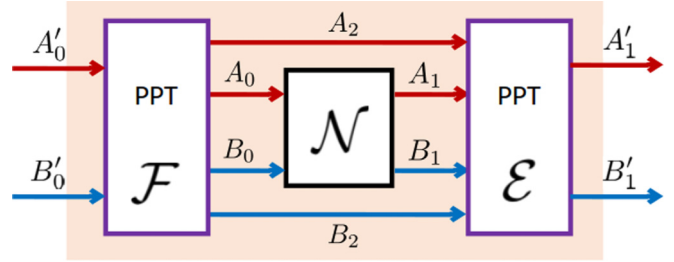


FIG. 7. The action of a restricted PPT superchannel on the bipartite channel  $\mathcal{N}_{AB}$ .

PPT channels are not physical. They do not arise from some *physical* constraint on a physical system. Therefore, the requirement that the superchannel  $\Theta$  be realized with PPT pre- and postprocessing channels does not make  $\Theta$  more physical. Moreover, as we will see, this definition does not lead to a simple resource theory, and as such, it loses its advantage of being a useful approximation to LOCC. For these reasons, we will adopt a more general definition of PPT superchannels that avoids the requirement that they be realized by PPT channels. However, before doing that, we first discuss some properties of restricted PPT superchannels.

*Proposition 2.* Let  $\Theta \in \mathfrak{S}(AB \rightarrow A'B')$  be a superchannel as in Fig. 7, where  $\mathcal{F} \in \text{PPT}(A'_0B'_0 \rightarrow A_2A_0B_0B_2)$  and  $\mathcal{E} \in \text{PPT}(A_2A_1B_1B_2 \rightarrow A'_1B'_1)$ . Then

$$(\mathbf{J}_{ABA'B'}^\Theta)^{T_{BB'}} \geq 0.$$

*Proof.* Since  $\mathbf{J}_{ABA'B'}^\Theta$  is the Choi matrix of the CPTP map

$$\mathcal{Q}_{A_1A'_0B_1B'_0 \rightarrow A_0A'_1B_0B'_1}^\Theta = \mathcal{E}_{A_2A_1B_1B_2 \rightarrow A'_1B'_1} \circ \mathcal{F}_{A'_0B'_0 \rightarrow A_2A_0B_0B_2}, \quad (5)$$

it is enough to show that the channel  $\mathcal{Q}^\Theta$  is PPT. Now,  $\mathcal{Q}^\Theta$  is PPT because it is defined as a composition of two PPT maps. Explicitly, we have

$$\begin{aligned} (\mathcal{Q}^\Theta)^\Gamma &= \mathcal{T}_{B_0B'_1} \circ \mathcal{E}_{A_2A_1B_1B_2 \rightarrow A'_1B'_1} \circ \mathcal{F}_{A'_0B'_0 \rightarrow A_2A_0B_0B_2} \circ \mathcal{T}_{B_1B'_0} \\ &= \mathcal{T}_{B'_1} \circ \mathcal{E}_{A_2A_1B_1B_2 \rightarrow A'_1B'_1} \circ \mathcal{T}_{B_1} \\ &\quad \circ \mathcal{T}_{B_0} \circ \mathcal{F}_{A'_0B'_0 \rightarrow A_2A_0B_0B_2} \circ \mathcal{T}_{B'_0} \\ &= \mathcal{T}_{B'_1} \circ \mathcal{E}_{A_2A_1B_1B_2 \rightarrow A'_1B'_1} \circ \mathcal{T}_{B_1B_2} \\ &\quad \circ \mathcal{T}_{B_0B_2} \circ \mathcal{F}_{A'_0B'_0 \rightarrow A_2A_0B_0B_2} \circ \mathcal{T}_{B'_0} \\ &= \mathcal{E}_{A_2A_1B_1B_2 \rightarrow A'_1B'_1}^\Gamma \circ \mathcal{F}_{A'_0B'_0 \rightarrow A_2A_0B_0B_2}^\Gamma. \end{aligned}$$

Since both  $\mathcal{E}$  and  $\mathcal{F}$  are PPT channels, the last line is a valid quantum channel. This completes the proof  $\blacksquare$

We believe that the converse of the proposition above does *not* hold. In other words, if the Choi matrix of  $\Theta$  has positive partial transpose, it does not necessarily mean that  $\Theta$  can be realized with pre- and postprocessing channels that are both PPT. However, to prove such a statement, one will need to provide an example, and then show that the proposed superchannel does not have any other realizations that involve only PPT pre- and postprocessing channels. Alternatively, the question can be rephrased as follows. Suppose we only know that the channel  $\mathcal{Q}^\Theta$  in the first line of Eq. (5) is a PPT channel; does it necessarily mean that there exist PPT channels  $\mathcal{E}$  and



$\mathcal{F}$  such that we can decompose  $\mathcal{Q}^\ominus$  as in the second line of Eq. (5)?

While there are no obvious reasons to believe that the answer is positive, we have not been able to prove it. If, instead, the answer were positive, it would mean that the set of restricted PPT superchannels is the same as the set of PPT superchannels we define below.

### A. PPT supermaps

In this section we define the set of PPT superchannels we are going to use in the following [49]. These superchannels have already featured in a number of works on quantum communication [91–93]. We believe that this set is strictly larger than the set of restricted PPT superchannels introduced above. However, as we discussed above, we have not been able to show this strict inclusion.

*Definition 3.* Let  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  be a CP supermap with systems  $A, B, A', B'$  all being composite systems with input and output dimensions.

(1)  $\Theta$  is *PPT preserving* if for any PPT map  $\mathcal{E} \in \text{PPT}(A_0B_0 \rightarrow A_1B_1)$ , the map  $\Theta[\mathcal{E}] \in \text{PPT}(A'_0B'_0 \rightarrow A'_1B'_1)$ .

(2)  $\Theta$  is *completely PPT preserving* if  $\mathbb{1}_{A''B''} \otimes \Theta$  is PPT preserving for any composite systems  $A'' = (A''_0, A''_1)$  and  $B'' = (B''_0, B''_1)$ .

(3)  $\Theta$  is a *PPT supermap* if, in addition to  $\Theta$ , also  $\Theta^\Gamma := \Upsilon_{B'} \circ \Theta \circ \Upsilon_B$  is a CP supermap.

*Remark 4.* Note that if  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  is a PPT CP supermap, and the dimensions  $|A_0| = |B_0| = |A_1| = |B_1| = 1$ , then  $\Theta$  can be viewed as a PPT map in  $\text{CP}(A'_0B'_0 \rightarrow A'_1B'_1)$ . Moreover, note that in the definition of a PPT supermap we require that both  $\Theta$  and  $\Theta^\Gamma$  are CP supermaps, in complete analogy with the definition of PPT CP maps.

We denote the set of all PPT CP supermaps by  $\text{PPT}(AB \rightarrow A'B')$ .

The landscape of PPT supermaps portrayed in Definition 3 is actually simpler. Indeed, completely PPT-preserving and PPT supermaps are the same notion (cf. also Ref. [49]).

*Theorem 5.* Let  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  be a CP supermap, and denote by  $\mathbf{J}_{ABA'B'}^\ominus$  its Choi matrix. Then, the following are equivalent:

- (1)  $\Theta$  is a PPT supermap.
- (2) The Choi matrix of  $\Theta$  satisfies

$$(\mathbf{J}_{ABA'B'}^\ominus)^{T_{BB'}} \geq 0.$$

- (3)  $\Theta$  is completely PPT preserving.

*Proof.* First we prove that statements (1) and (2) are equivalent. Consider the map  $\Phi_{AA'}^{+\sim} \in \text{CP}(A_0\tilde{A}_0 \rightarrow A_1\tilde{A}_1)$  defined in Eq. (1), which is completely positive, and it is the CP-map analog of the maximally entangled state. Recall also that one of the representations of a supermap  $\Theta \in \mathcal{L}(A \rightarrow A')$  is given by the map  $\mathcal{P}_{AA'}^\ominus = (\mathbb{1}_A \otimes \Theta)[\Phi_{AA'}^{+\sim}]$  whose Choi matrix is the Choi matrix of  $\Theta$ . Since here we consider a bipartite CP supermap  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$ , the map  $\mathcal{P}^\ominus$  is defined as

$$\mathcal{P}_{ABA'B'}^\ominus = (\mathbb{1}_{AB} \otimes \Theta_{\tilde{AB} \rightarrow A'B'})[\Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}],$$

where we have used the fact that  $\Phi_{ABA'B'}^{+\sim} = \Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}$ . Now, observe that

$$\mathcal{P}_{ABA'B'}^{\ominus\Gamma} = (\mathbb{1}_{AB} \otimes \Upsilon_{B'} \circ \Theta_{\tilde{AB} \rightarrow A'B'} \circ \Upsilon_{\tilde{B}})[\Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}] \quad (6)$$

and

$$\Upsilon_{\tilde{B}}[\Phi_{BB'}^{+\sim}] = \mathcal{T}_{\tilde{B}_1} \circ \Phi_{BB'}^{+\sim} \circ \mathcal{T}_{\tilde{B}_0} = \Upsilon_B[\Phi_{BB'}^{+\sim}], \quad (7)$$

where in the last equality we used the representation (1) of  $\Phi_{BB'}^{+\sim}$ , and the fact that  $(\phi_{B_0\tilde{B}_0}^{+\sim})^{T_{\tilde{B}_0}} = (\phi_{B_0\tilde{B}_0}^{+\sim})^{T_{B_0}}$  and  $(\phi_{B_1\tilde{B}_1}^{+\sim})^{T_{\tilde{B}_1}} = (\phi_{B_1\tilde{B}_1}^{+\sim})^{T_{B_1}}$ . Combining this with Eq. (6), we conclude that

$$\begin{aligned} \mathcal{P}_{ABA'B'}^{\ominus\Gamma} &= (\Upsilon_{B'} \otimes \Upsilon_B) \circ (\mathbb{1}_{AB} \otimes \Theta_{\tilde{AB} \rightarrow A'B'})[\Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}] \\ &= (\mathcal{P}_{ABA'B'}^\ominus)^\Gamma. \end{aligned} \quad (8)$$

Hence

$$\mathbf{J}_{ABA'B'}^{\ominus\Gamma} = (\mathbf{J}_{ABA'B'}^\ominus)^{T_{BB'}},$$

where we have used Eq. (4). This completes the proof of the equivalence between statements (1) and (2).

For the equivalence between statements (1) and (3), let  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  be a PPT supermap. Then, for any systems  $A''B''$  and any PPT bipartite CP map,  $\mathcal{N}_{A''B''AB}$ , we have

$$0 \leq \Theta_{AB \rightarrow A'B'}^\Gamma[\Upsilon_{B''B}[\mathcal{N}_{A''B''AB}]] = \Upsilon_{B''B}[\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{A''B''AB}]],$$

where the equality follows from the definition of  $\Theta_{AB \rightarrow A'B'}^\Gamma$ . In other words,  $(\mathbb{1}_{A''B''} \otimes \Theta)[\mathcal{N}_{A''B''AB}]$  is a PPT map, so  $\Theta$  is completely PPT preserving.

Conversely, let  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  be a CP supermap that is completely PPT preserving. Note that, by Eq. (7),  $\Phi_{ABA'B'}^{+\sim} = \Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}$  is a PPT map. Therefore, the CP map

$$\mathcal{P}_{ABA'B'}^\ominus = (\mathbb{1}_{AB} \otimes \Theta_{\tilde{AB} \rightarrow A'B'})[\Phi_{AA'}^{+\sim} \otimes \Phi_{BB'}^{+\sim}]$$

is PPT. From a similar relation to Eq. (8), it follows that  $\mathcal{P}_{ABA'B'}^{\ominus\Gamma} \geq 0$ , so  $\Theta$  is PPT. This completes the proof. ■

We end this section with a convenient property of the partial transpose operation. This will be very useful in the following.

*Proposition 6.* Let  $\Theta \in \mathcal{L}(AB \rightarrow A'B')$  be a bipartite supermap and let  $\mathcal{N} \in \mathcal{L}(A_0B_0 \rightarrow A_1B_1)$  be a bipartite map. Then,

$$(\Theta[\mathcal{N}])^\Gamma = \Theta^\Gamma[\mathcal{N}^\Gamma].$$

*Proof.* Note that

$$\begin{aligned} \Theta^\Gamma[\mathcal{N}^\Gamma] &= \Upsilon_{B'} \circ \Theta \circ \Upsilon_B[\Upsilon_B[\mathcal{N}_{AB}]] \\ &= \Upsilon_{B'} \circ \Theta[\mathcal{N}_{AB}] \\ &= (\Theta[\mathcal{N}_{AB}])^\Gamma. \end{aligned}$$

This completes the proof. ■

### B. Single-shot interconversions

In the resource theory of NPT static entanglement the conversion of one resource into another can be characterized by SDPs [94]. Here we show that for NPT dynamical entanglement, the conversion distance  $d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'})$ , defined in Ref. [48], can be computed by an SDP as long as we consider the PPT superchannels introduced in Definition 3, and *not* the restricted PPT operations illustrated in Fig. 7.

Now, recall that in NPT entanglement theory, the conversion distance is

$$\begin{aligned} d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) &= \frac{1}{2} \min_{\Theta \in \text{PPT}(AB \rightarrow A'B')} \|\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'}\|_\diamond. \end{aligned}$$

Now, the diamond norm can be expressed as the SDP [95]

$$\begin{aligned} & \frac{1}{2} \|\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'}\|_{\diamond} \\ &= \min_{\omega_{A'B'} \geq 0; \omega_{A'B'} \geq J_{A'B'}^{\Theta[\mathcal{N}]-\mathcal{M}}} \|\omega_{A'B'}\|_{\infty}. \end{aligned}$$

Now, in Ref. [74], it was shown that it can be written also as

$$\begin{aligned} & \frac{1}{2} \|\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'}\|_{\diamond} \\ &= \min \{ \lambda : \lambda \mathcal{Q}_{A'B'} \geq \Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'} \}, \end{aligned}$$

where  $\mathcal{Q}_B \in \text{CPTP}(B_0 \rightarrow B_1)$ . Therefore, calculating the conversion distance amounts to solving the following minimization problem:

$$\begin{aligned} \text{Find} \quad & d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) = \min \lambda \\ \text{Subject to} \quad & \lambda \mathcal{Q}_{A'B'} \geq \Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}] - \mathcal{M}_{A'B'}, \\ & \mathcal{Q} \text{ channel}, \\ & \Theta \text{ superchannel}. \end{aligned} \quad (9)$$

This can be rephrased in the following SDP form. Denote the Choi matrix of  $\lambda \mathcal{Q}_{A'B'}$  by  $\alpha_{A'B'}$ , and the Choi matrix of  $\Theta$  by  $\mathbf{J}_{ABA'B'}$ . Then, following Ref. [48], we can express Eq. (9) in terms of Choi matrices, finding the new optimization problem

$$d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) = \frac{1}{|A'_0 B'_0|} \min \text{Tr}[\alpha_{A'B'}]$$

subject to

$$\begin{aligned} \alpha_{A'B'} &\geq 0, \quad \alpha_{A'_0 B'_0} = \text{Tr}[\alpha_{A'_0 B'_0}] u_{A'_0 B'_0}, \\ \alpha &\geq \text{Tr}_{AB} [\mathbf{J}_{ABA'B'} ((J_{AB}^{\mathcal{N}})^T \otimes I_{A'B'})] - J_{A'B'}^{\mathcal{M}}, \\ \mathbf{J}_{ABA'B'} &\geq 0, \quad \mathbf{J}_{ABA'_0 B'_0} = \mathbf{J}_{A_0 B_0 A'_0 B'_0} \otimes u_{A_1 B_1}, \\ \mathbf{J}_{A_1 B_1 A'_0 B'_0} &= I_{A_1 B_1 A'_0 B'_0}, \quad \mathbf{J}_{ABA'B'}^{T_{BB'}} \geq 0. \end{aligned}$$

Clearly, the above optimization can be solved efficiently and algorithmically with an SDP. We can also express it in its dual form following Ref. [48]:

$$\begin{aligned} & d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) \\ &= \max \{ t | A_1 B_1 A'_0 B'_0 - \text{Tr}[\zeta_{A'B'} J_{A'B'}^{\mathcal{M}}] \}, \end{aligned} \quad (10)$$

subject to

$$\begin{aligned} & (J_{AB}^{\mathcal{N}})^T \otimes \zeta_{A'B'} - t I_{ABA'B'} \in \tilde{\mathfrak{J}}_{ABA'B'}^*, \\ & 0 \leq \zeta_{A'B'} \leq \eta_{A'_0 B'_0} \otimes I_{A'_1 B'_1}, \quad \text{Tr}[\eta_{A'_0 B'_0}] = 1, \end{aligned} \quad (11)$$

where  $\tilde{\mathfrak{J}}_{ABA'B'}^*$  is defined in Eq. (A1). We want to show that this dual problem is an SDP as well. To this end, from Eq. (A2), define

$$\beta_{ABA'_0 B'_0} := \frac{1}{|A_0 B_0|} (t I_{ABA'_0 B'_0} + Y_{ABA'_0 B'_0} + I_{A_0 B_0} \otimes Z_{A_1 B_1 A'_0 B'_0}),$$

where, like in Eq. (A2),  $Y$  is a Hermitian matrix such that  $Y_{AB} = 0$ , and  $Z$  is a Hermitian matrix such that  $\text{Tr}[Z_{A_1 B_1 A'_0 B'_0}] = 0$ . In this way, by the definition of  $\beta_{ABA'_0 B'_0}$ , and recalling Eq. (A2), we can rewrite Eqs. (10) and (11) as

$$d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'}) = \max \{ \text{Tr}[\beta_{ABA'_0 B'_0}] - \text{Tr}[\zeta_{A'B'} J_{A'B'}^{\mathcal{M}}] \},$$

subject to

$$\begin{aligned} & \beta \in \text{Herm}(ABA'_0 B'_0), \quad \beta_{AB} = u_{A_0 B_0} \otimes \beta_{A_1 B_1}, \\ & 0 \leq \zeta_{A'B'} \leq \eta_{A'_0 B'_0} \otimes I_{A'_1 B'_1}, \quad \text{Tr}[\eta_{A'_0 B'_0}] = 1, \\ & \left( (J_{AB}^{\mathcal{N}})^T \otimes \zeta_{A'B'} - |A_0 B_0| \beta \otimes I_{B'_1 A'_1} - P_{ABA'B'} \right)^{T_{BB'}} \geq 0, \\ & P_{ABA'B'} \geq 0. \end{aligned}$$

Hence, the computation of  $d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{M}_{A'B'})$  in the resource theory of NPT entanglement is an SDP optimization problem. We point out that if we considered restricted PPT superchannels, instead, the condition that  $\Theta$  is free would be expressed as the condition that the CPTP map  $\mathcal{Q}^{\Theta}$  has a decomposition into pre- and postprocessing that are both PPT channels, like in Eq. (5). This condition appears to be very cumbersome, and it is not clear if the determination of whether or not  $\Theta$  has the form (5) can be solved with an SDP.

### C. NPT entanglement measures

In the entanglement theory for static resources, functions that behave monotonically under PPT operations also behave monotonically under LOCC operations, as LOCC is a subset of PPT. Hence, any NPT entanglement measure is also an LOCC entanglement measure. The advantage of some of the NPT entanglement measures is that they can be computed with SDPs (see, e.g., the family of measures discussed in Ref. [96]). In this section we study a few of these measures.

#### 1. Negativity and logarithmic negativity of bipartite channels

A well-known NPT entanglement measure is the negativity [97]. It is defined on a bipartite quantum state  $\rho \in \mathfrak{D}(A_0 B_0)$  as

$$N(\rho_{A_0 B_0}) = \frac{\|\mathcal{T}_{B_0}(\rho_{A_0 B_0})\|_1 - 1}{2}.$$

The generalization of the negativity to bipartite channels can be done by replacing the input bipartite state  $\rho \in \mathfrak{D}(A_0 B_0)$  with input bipartite channel  $\mathcal{N} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$ , the trace norm with the diamond norm, and the transpose map  $\mathcal{T}_{B_0}$  with the transpose supermap  $\Upsilon_B$ . The negativity of the bipartite channel is therefore defined as

$$N(\mathcal{N}_{AB}) = \frac{\|\Upsilon_B[\mathcal{N}_{AB}]\|_{\diamond} - 1}{2}.$$

Furthermore, the logarithmic negativity is defined as

$$LN(\mathcal{N}_{AB}) = \log_2 \|\Upsilon_B[\mathcal{N}_{AB}]\|_{\diamond}.$$

To show that the above quantities are indeed good generalizations of the negativity and logarithmic negativity to bipartite channels, we show that they vanish on PPT bipartite channels, and that they behave monotonically under PPT superchannels. They vanish on PPT bipartite channels because if  $\mathcal{N}_{AB}$  is PPT then  $\Upsilon_B[\mathcal{N}_{AB}]$  is a quantum channel so its diamond norm is 1. To show the monotonicity property, let  $\Theta \in \text{PPT}(AB \rightarrow A'B')$  and observe that

$$\begin{aligned} \|\Upsilon_{B'} \circ \Theta[\mathcal{N}_{AB}]\|_{\diamond} &= \|(\Theta[\mathcal{N}_{AB}])^{\Gamma}\|_{\diamond} \\ &= \|\Theta^{\Gamma}[\mathcal{N}_{AB}^{\Gamma}]\|_{\diamond} \\ &\leq \|\Upsilon_B[\mathcal{N}_{AB}]\|_{\diamond}, \end{aligned}$$

where in the first equality we used the definition of the partial transpose of a channel, in the second Proposition 6, and finally the inequality follows from the fact that  $\Theta^\Gamma$  is a superchannel because  $\Theta$  is a PPT superchannel, and the fact that the diamond norm is contractive under superchannels [52]. Therefore, since both the negativity and the logarithmic negativity are increasing functions of  $\|\Upsilon_B[\mathcal{N}_{AB}]\|_\diamond$ , we conclude that they are nonincreasing under PPT superchannels.

**2. A complete set of computationally manageable measures of bipartite NPT dynamical entanglement**

We can use the same technique as above to generalize other measures of NPT static entanglement to NPT dynamical entanglement (see, e.g., Ref. [97]). Now we focus on the complete family of measures introduced in Ref. [48]. In the case of NPT entanglement, for any bipartite channel  $\mathcal{P}_{A'B'} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ , using the results in Ref. [48], we can write

$$f_{\mathcal{P}}(\mathcal{N}_{AB}) = \max_{\mathbf{J} \in \mathfrak{J}} \text{Tr} \left[ \mathbf{J}_{ABA'B'} \left( (J_{AB}^{\mathcal{N}})^T \otimes J_{A'B'}^{\mathcal{P}} \right) \right]$$

for every quantum channel  $\mathcal{N}_{AB}$ , where  $\mathfrak{J}$  is the set of Choi matrices of PPT superchannels (note that it is compact and convex). In other words,  $\mathbf{J}_{ABA'B'}$  is subject to the following constraints:

- (1)  $\mathbf{J}_{ABA'B'} \geq 0$ ;  $\mathbf{J}_{ABA'_0B'_0} = \mathbf{J}_{A_0B_0A'_0B'_0} \otimes \mathbf{I}_{A_1B_1}$ ;  $\mathbf{J}_{A_1B_1A'_0B'_0} = \mathbf{I}_{A_1B_1A'_0B'_0}$ .
- (2)  $\mathbf{J}_{ABA'B'}^{T_{BB'}} \geq 0$ .

The first group of conditions above ensures that  $\mathbf{J}_{ABA'B'}$  is the Choi matrix of a superchannel in  $\mathfrak{S}(AB \rightarrow A'B')$ ; the second condition guarantees that the superchannel is free, i.e., PPT. A key observation about the above optimization problem is that it is an SDP. As noted in Ref. [48], the family of convex functions  $\{f_{\mathcal{P}}\}$ , indexed by all  $\mathcal{P} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ , is complete, in the sense that there exists a PPT superchannel converting a bipartite channel  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  into another bipartite channel  $\mathcal{E} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$  if and only if

$$f_{\mathcal{P}}(\mathcal{N}_{AB}) \geq f_{\mathcal{P}}(\mathcal{E}_{A'B'}) \tag{12}$$

for every  $\mathcal{P} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$ .

One may argue that the above condition cannot be checked efficiently, as it involves an (uncountably) infinite number of measures of dynamical entanglement, labeled by all quantum channels  $\mathcal{P}$ . However, we have another way to determine whether two bipartite quantum channels  $\mathcal{N}_{AB}$  and  $\mathcal{E}_{A'B'}$  can be interconverted by PPT superchannels, which is to compute the conversion distance  $d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{E}_{A'B'})$ :  $d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{E}_{A'B'}) = 0$  if and only if  $\mathcal{N}_{AB}$  can be converted (exactly) into  $\mathcal{E}_{A'B'}$  by PPT superchannels. In Sec. IV B we showed that this can be done efficiently with an SDP.

Why do we consider this family of dynamical entanglement measures, then? Their significance is that they completely characterize the NPT entanglement of a single bipartite channel, whereas the computation of  $d_{\text{PPT}}(\mathcal{N}_{AB} \rightarrow \mathcal{E}_{A'B'})$  requires to know both of its inputs  $\mathcal{N}$  and  $\mathcal{E}$ , i.e., also the target channel. Hence, Eq. (12) demonstrates that the convertibility can be expressed in a monotonic form, similarly to Vidal’s monotones [26,98,99] in the theory of pure-state bipartite entanglement.

*Remark 7.* If we want measures of NPT dynamical entanglement that vanish on PPT channels, we can consider the measures  $G_{\mathcal{P}}(\mathcal{N}_{AB}) = f_{\mathcal{P}}(\mathcal{N}_{AB}) - \max_{\mathcal{M}} \text{Tr}[J_{A'B'}^{\mathcal{M}} J_{A'B'}^{\mathcal{P}}]$ . Here  $\mathcal{M}$  ranges over all PPT channels (again, a compact and convex set).

**3. The max-logarithmic negativity**

In Ref. [47] the authors considered a measure of NPT entanglement, which they called the  $\kappa$ -entanglement. For bipartite states, it is defined as

$$E_{\kappa}(\rho_{AB}) = \log_2 \inf \{ \text{Tr}[S_{AB}] : -S_{AB}^{T_B} \leq \rho_{AB}^{T_B} \leq S_{AB}^{T_B} ; S_{AB} \geq 0 \},$$

and for one-way channels  $\mathcal{E}_{A_0 \rightarrow B_1}$  as

$$E_{\kappa}(\mathcal{E}_{A_0 \rightarrow B_1}) = \log_2 \inf \{ \|J_{A_0}^{\mathcal{Q}}\|_{\infty} : -\mathcal{Q}^{\Gamma} \leq \mathcal{E}_{A_0 \rightarrow B_1}^{\Gamma} \leq \mathcal{Q}^{\Gamma} ; \mathcal{Q} \geq 0 \}.$$

The significance of this measure is that it has an operational interpretation as the exact asymptotic cost under PPT operations. Here we introduce the max-logarithmic negativity (MLN) (see also Ref. [96]), which has a similar operational interpretation, and is a generalization of the  $\kappa$ -entanglement to bipartite channels. However, as we will see, for bipartite channels, there are two possible generalizations of the quantity given in Ref. [47], and we define the MLN to be the maximum of the two. Explicitly, the MLN is defined as

$$LN_{\max}(\mathcal{N}_{AB}) = \max \{ LN_{\max}^{(0)}(\mathcal{N}_{AB}), LN_{\max}^{(1)}(\mathcal{N}_{AB}) \},$$

where

$$LN_{\max}^{(0)}(\mathcal{N}_{AB}) = \log_2 \inf \{ \|J_{A_0B_0}^{\mathcal{P}}\|_{\infty} : -\mathcal{P}_{AB}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \leq \mathcal{P}_{AB}^{\Gamma} ; \mathcal{P} \geq 0 \}$$

and

$$LN_{\max}^{(1)}(\mathcal{N}_{AB}) = \log_2 \inf \{ \|J_{A_0B_0}^{\mathcal{P}}\|_{\infty} : -\mathcal{P}_{AB}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \leq \mathcal{P}_{AB}^{\Gamma} ; \mathcal{P} \geq 0 \}.$$

The above quantities can be computed with SDP. In particular, they have a dual, giving an alternative expression for them:

$$LN_{\max}^{(0)}(\mathcal{N}_{AB}) = \log_2 \sup \{ \text{Tr}[J_{AB}^{\mathcal{N}}(V_{AB} - W_{AB})] \} \tag{13}$$

subject to

$$V_{AB} + W_{AB} \leq \rho_{A_0B_0} \otimes I_{A_1B_1}, \quad \rho \in \mathfrak{D}(A_0B_0),$$

$$V_{AB} \geq 0, \quad W_{AB} \geq 0,$$

and

$$LN_{\max}^{(1)}(\mathcal{N}_{AB}) = \log_2 \sup \{ \text{Tr}[J_{AB}^{\mathcal{N}}(V_{AB} - W_{AB})] \}$$

subject to

$$V_{AB} + W_{AB} \leq \rho_{A_0B_0}^{T_{B_0}} \otimes I_{A_1B_1}, \quad \rho \in \mathfrak{D}(A_0B_0),$$

$$V_{AB} \geq 0, \quad W_{AB} \geq 0.$$

These expressions can be obtained with the usual SDP techniques. By Sion’s minimax theorem, we can swap the

order of the infimum and the maximum in the definition of the MLN, so

$$LN_{\max}(\mathcal{N}_{AB}) = \log_2 \inf \left\{ \max \left\{ \|J_{A_0B_0}^{\mathcal{P}}\|_{\infty}, \|J_{A_0B_0}^{\mathcal{P}^{\Gamma}}\|_{\infty} \right\} \right\}$$

$$LN_{\max}(\mathcal{N}_{AB}) = \log_2 \inf \left\{ \max \left\{ \|P_{A_0B_0}\|_{\infty}, \|P_{A_0B_0}^{T_{B_0}}\|_{\infty} \right\} : -P_{AB}^{T_B} \leq (J_{AB}^{\mathcal{N}})^{T_B} \leq P_{AB}^{T_B}, P_{AB} \geq 0 \right\}. \quad (14)$$

Now we show here that many properties of the  $\kappa$ -entanglement discussed in Ref. [47] carry over to the max-logarithmic negativity, including the operational meaning of single-shot exact entanglement cost (cf. Sec. IV D). Moreover, we will see that the max-logarithmic negativity is monotonic under PPT superchannels, which we believe is a strictly larger set than the set discussed in Ref. [47], that is the set of restricted PPT superchannels, which can be implemented by PPT pre- and postprocessing, like in Fig. 7.

#### 4. Properties of the max-logarithmic negativity

Here we list a few key properties of the MLN. The first two show that it reduces to  $E_{\kappa}$  introduced in Ref. [47].

*Reduction to  $\kappa$ -entanglement for states.* A bipartite state can be viewed as a bipartite channel  $\mathcal{N}_{AB}$  with  $|A_0\rangle = |B_0\rangle = 1$ . In this case, in Eq. (14),  $P_{A_0B_0} = P_{A_0B_0}^{T_{B_0}} = \text{Tr}[P_{A_1B_1}]$ . Recalling that  $P_{A_1B_1} \geq 0$ , we have  $LN_{\max}(\mathcal{N}_{AB}) = \log_2 \inf \{\text{Tr}[P_{A_1B_1}]\}$ , subject to  $-P_{A_1B_1}^{T_{B_1}} \leq \rho_{A_1B_1}^{T_{B_1}} \leq P_{A_1B_1}^{T_{B_1}}$  and  $P_{A_1B_1} \geq 0$ . This expression coincides with  $E_{\kappa}(\rho_{A_1B_1})$ .

*Reduction to  $\kappa$ -entanglement for one-way channels.* For  $|B_0\rangle = |A_1\rangle = 1$ , the channel  $\mathcal{N}_{AB}$  can be viewed as a map  $\mathcal{E} \in \text{CPTP}(A_0 \rightarrow B_1)$  and

$$LN_{\max}(\mathcal{N}_{AB}) = E_{\kappa}(\mathcal{E}_{A_0 \rightarrow B_1}).$$

*Monotonicity.* Let  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  be a bipartite channel, and let  $\Theta \in \text{PPT}(AB \rightarrow A'B')$  be a PPT superchannel. Then,

$$LN_{\max}(\Theta_{AB \rightarrow A'B'}[\mathcal{N}_{AB}]) \leq LN_{\max}(\mathcal{N}_{AB}).$$

*Proof.* Recall that for any superchannel  $\Theta$  and bipartite channel  $\mathcal{N}_{AB}$  we have  $(\Theta[\mathcal{N}_{AB}])^{\Gamma} = \Theta^{\Gamma}[\mathcal{N}_{AB}^{\Gamma}]$  (see Proposition 6). Hence, from the expression

$$\begin{aligned} & LN_{\max}(\Theta[\mathcal{N}_{AB}]) \\ &= \log_2 \inf \left\{ \max \left\{ \|J_{A'_0B'_0}^{\mathcal{R}}\|_{\infty}, \|(J_{A'_0B'_0}^{\mathcal{R}})^{T_{B_0}}\|_{\infty} \right\} \right\}, \end{aligned}$$

subject to  $-\mathcal{R}_{A'B'}^{\Gamma} \leq \Theta^{\Gamma}[\mathcal{N}_{AB}^{\Gamma}] \leq \mathcal{R}_{A'B'}^{\Gamma}$  and  $\mathcal{R}_{A'B'} \geq 0$ , we can definitely write

$$\begin{aligned} & LN_{\max}(\Theta[\mathcal{N}_{AB}]) \\ &\leq \log_2 \inf \left\{ \max \left\{ \|J_{A'_0B'_0}^{\Theta[\mathcal{P}]}\|_{\infty}, \|J_{A'_0B'_0}^{\Theta^{\Gamma}[\mathcal{P}^{\Gamma}]}\|_{\infty} \right\} \right\}, \quad (15) \end{aligned}$$

where  $-(\Theta[\mathcal{P}_{AB}])^{\Gamma} \leq \Theta^{\Gamma}[\mathcal{N}_{AB}^{\Gamma}] \leq (\Theta[\mathcal{P}_{AB}])^{\Gamma}$  and  $\mathcal{P}_{AB} \geq 0$ . Indeed, this inequality follows because we have restricted  $\mathcal{R}_{A'B'}$  to CP maps of the form  $\Theta_{AB \rightarrow A'B'}[\mathcal{P}_{AB}]$ , where  $\mathcal{P} \in \text{CP}(A_0B_0 \rightarrow A_1B_1)$ , and  $\Theta$  is a PPT superchannel. Next, observe that, by the properties of the Choi matrices of super-

channels, subject to  $-\mathcal{P}_{AB}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \leq \mathcal{P}_{AB}^{\Gamma}$  and  $\mathcal{P}_{AB} \geq 0$ . The MLN is defined here in terms of the bipartite map  $\mathcal{P} \in \text{CP}(A_0B_0 \rightarrow A_1B_1)$ . Denoting its Choi matrix by  $P_{AB} \in \text{Herm}(AB)$ , we can express the MLN as

channels,

$$\begin{aligned} J_{A'_0B'_0}^{\Theta[\mathcal{P}]} &= \text{Tr}_{ABA_1B_1} \left[ \mathbf{J}_{ABA_1B_1}^{\Theta} \left( (J_{AB}^{\mathcal{P}})^T \otimes I_{A_1B_1} \right) \right] \\ &= \text{Tr}_{AB} \left[ \mathbf{J}_{ABA_0B'_0}^{\Theta} \left( (J_{AB}^{\mathcal{P}})^T \otimes I_{A_0B'_0} \right) \right] \\ &= \text{Tr}_{AB} \left[ \left( \mathbf{J}_{A_0B_0A'_0B'_0}^{\Theta} \otimes u_{A_1B_1} \right) \left( (J_{AB}^{\mathcal{P}})^T \otimes I_{A_0B'_0} \right) \right] \\ &= \frac{1}{|A_1B_1|} \text{Tr}_{A_0B_0} \left[ \mathbf{J}_{A_0B_0A'_0B'_0}^{\Theta} \left( (J_{A_0B_0}^{\mathcal{P}})^T \otimes I_{A'_0B'_0} \right) \right] \\ &=: \mathcal{D}_{A_0B_0 \rightarrow A'_0B'_0} \left( J_{A_0B_0}^{\mathcal{P}} \right), \end{aligned}$$

where  $\mathcal{D}$  is a CP map whose Choi matrix is given by  $J_{A_0B_0A'_0B'_0}^{\mathcal{D}} := \frac{1}{|A_1B_1|} \mathbf{J}_{A_0B_0A'_0B'_0}^{\Theta}$ . The fact that  $\Theta$  is a superchannel ensures that  $J_{A'_0B'_0}^{\mathcal{D}} = I_{A'_0B'_0}$ , so  $\mathcal{D}$  is *unital*. Now, the operator norm is contractive under CP unital maps; thus we conclude that  $\|J_{A'_0B'_0}^{\Theta[\mathcal{P}]}\|_{\infty} \leq \|J_{A_0B_0}^{\mathcal{P}}\|_{\infty}$ . Similarly, since  $\Theta^{\Gamma}$  is also a superchannel, we have  $\|J_{A'_0B'_0}^{\Theta^{\Gamma}[\mathcal{P}^{\Gamma}]}\|_{\infty} \leq \|J_{A_0B_0}^{\mathcal{P}^{\Gamma}}\|_{\infty}$ . Therefore, recalling Eq. (15),

$$LN_{\max}(\Theta[\mathcal{N}_{AB}]) \leq \log_2 \inf \left\{ \max \left\{ \|J_{A_0B_0}^{\mathcal{P}}\|_{\infty}, \|J_{A_0B_0}^{\mathcal{P}^{\Gamma}}\|_{\infty} \right\} \right\},$$

subject to  $-\mathcal{P}_{AB}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \leq \mathcal{P}_{AB}^{\Gamma}$  and  $\mathcal{P}_{AB} \geq 0$ , where we have used the fact that  $\Theta^{\Gamma}$  is a superchannel, so  $-\mathcal{P}_{AB}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \leq \mathcal{P}_{AB}^{\Gamma}$  implies  $-\Theta^{\Gamma}[\mathcal{P}_{AB}^{\Gamma}] \leq \Theta^{\Gamma}[\mathcal{N}_{AB}^{\Gamma}] \leq \Theta^{\Gamma}[\mathcal{P}_{AB}^{\Gamma}]$ . But the final expression we have obtained is precisely  $LN_{\max}(\mathcal{N}_{AB})$ . This completes the proof. ■

*Additivity.* For any two bipartite channels  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  and  $\mathcal{M} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$  we have

$$LN_{\max}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = LN_{\max}(\mathcal{N}_{AB}) + LN_{\max}(\mathcal{M}_{A'B'}).$$

In particular, note that this property implies that, for all positive integers  $n$ ,

$$LN_{\max}(\mathcal{N}_{AB}^{\otimes n}) = nLN_{\max}(\mathcal{N}_{AB}).$$

*Proof.* The proof follows from the facts

$$LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = LN_{\max}^{(0)}(\mathcal{N}_{AB}) + LN_{\max}^{(0)}(\mathcal{M}_{A'B'})$$

and

$$LN_{\max}^{(1)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = LN_{\max}^{(1)}(\mathcal{N}_{AB}) + LN_{\max}^{(1)}(\mathcal{M}_{A'B'}),$$

which can be proved with the same techniques as in Ref. [47], with the primal problem being used to show the  $\leq$  side, and the dual problem used to show the  $\geq$  side. For completeness, we include the proof in Appendix B. ■

#### D. Exact asymptotic NPT entanglement cost

In this section we generalize the operational interpretation given in Ref. [47] of  $E_\kappa$  to generic bipartite channels. This generalization will be fairly straightforward, and the ultimate reason for this is that we do not consider only restricted PPT superchannels, but rather generic PPT superchannels (see Sec. IV A). This makes the conditions involved closer to the case of bipartite states.

Following the same argument in Sec. III, in NPT entanglement theory, the maximally entangled state  $\phi_{A'_1 B'_1}^+$ , if suitably normalized, where  $|A'_1| = |B'_1| = m$ , can be regarded as the maximal resource: two maximally entangled states  $\phi^+$  are equivalent to the swap. This state can also be viewed as the channel  $\Phi_{A'B'}^+$  (cf. Sec. II B) with trivial inputs  $A'_0$  and  $B'_0$ . With this in mind, the single-shot exact resource cost to simulate a channel takes the form

$$E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) := \inf \{ \log_2 m : \mathcal{N}_{AB} = \Theta_{A'B' \rightarrow AB}[\Phi_{A'B'}^+] \}, \quad (16)$$

where the infimum is over all PPT superchannels  $\Theta$ ,  $|A'_0| = |B'_0| = 1$ , and  $|A'_1| = |B'_1| = m$ .

The following two lemmas will be used in the proof of the main theorem of this section (Theorem 10) that provides an operational meaning to the MLN. The first lemma provides an alternative expression for  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$ .

*Lemma 8.* Let  $\mathcal{N} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  be a bipartite channel. Then,

$$E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) = \inf \{ \log_2 m : -(m-1)\mathcal{R}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m+1)\mathcal{R}_{AB}^\Gamma \}, \quad (17)$$

where  $\mathcal{R} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  and  $m \in \mathbb{Z}_+$ .

*Proof.* The proof follows similar lines to the one in Ref. [47], but with states replaced by channels. We first prove that  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$  is less than or equal to the right-hand side of Eq. (17). Let  $m = |A'_1| = |B'_1|$  be a positive integer, and let  $\mathcal{R}_{AB}$  be a CPTP map satisfying

$$-(m-1)\mathcal{R}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m+1)\mathcal{R}_{AB}^\Gamma. \quad (18)$$

We need to show that there exists a PPT superchannel  $\Theta$  as in Eq. (16) with the same  $m$ . To this end, define the superchannel  $\Theta \in \mathfrak{S}(A'B' \rightarrow AB)$  (with  $|A'_1| = |B'_1| = m$  and  $|A'_0| = |B'_0| = 1$ ) on any CP map  $\mathcal{M}_{A'B'}$  as

$$\begin{aligned} \Theta_{A'B' \rightarrow AB}[\mathcal{M}_{A'B'}] &:= \mathcal{N}_{AB} \text{Tr}[\Phi_{A'B'}^+ \mathcal{M}_{A'B'}] \\ &\quad + \mathcal{R}_{AB} \text{Tr}[(I_{A'B'} - \Phi_{A'B'}^+) \mathcal{M}_{A'B'}], \end{aligned}$$

where we have used the fact that  $\mathcal{M}_{A'B'}$  and  $\Phi_{A'B'}^+$  can be viewed as matrices because their input dimensions are trivial, so the traces above are well defined. For a simpler notation, set  $A' \equiv A'_1$  and  $B' \equiv B'_1$ .

Note that  $\Theta$  above is indeed a superchannel, as it is CP, and sends channels to channels [56]. In addition, it satisfies  $\Theta[\Phi_{A'B'}^+] = \mathcal{N}_{AB}$ . We need to show that  $\Theta^\Gamma = \Upsilon_B \circ \Theta \circ \Upsilon_B$  is a superchannel too. For this purpose, let  $R = (R_0, R_1)$  be a reference system, and consider  $\mathcal{P}_{RA'B'} \in \text{CPTP}(R_0 \rightarrow R_1 A' B')$ , and observe that

$$\begin{aligned} \Theta^\Gamma[\mathcal{P}_{RA'B'}] &= \mathcal{N}_{AB}^\Gamma \otimes \text{Tr}_{A'B'}[(\phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}] \\ &\quad + \mathcal{R}_{AB}^\Gamma \otimes \text{Tr}_{A'B'}[(I_{A'B'} - \phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}], \end{aligned}$$

where the partial trace above is understood as follows: for any matrix  $X \in \mathfrak{B}(R_0)$ , the expression  $\text{Tr}_{A'B'}[(\phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}]$  is the map

$$\begin{aligned} &\text{Tr}_{A'B'}[(\phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}](X_{R_0}) \\ &:= \text{Tr}_{A'B'}[(\phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}(X_{R_0})]. \end{aligned}$$

Recall that  $(\phi_{A'B'}^+)^{T_{B'}} = \frac{1}{m} F_{A'B'}$ , where  $F_{A'B'}$  is the unitary SWAP (or flip) operator, and the factor  $\frac{1}{m}$  comes from the fact that here we are taking  $\phi_{A'B'}^+$  to be normalized. Therefore,

$$\text{Tr}_{A'B'}[(\phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}] = \frac{1}{m} \text{Tr}_{A'B'}[F_{A'B'} \mathcal{P}_{RA'B'}],$$

and

$$\begin{aligned} &\text{Tr}_{A'B'}[(I_{A'B'} - \phi_{A'B'}^+)^{T_{B'}} \mathcal{P}_{RA'B'}] \\ &= \text{Tr}_{A'B'} \left[ \left( I_{A'B'} - \frac{1}{m} F_{A'B'} \right) \mathcal{P}_{RA'B'} \right]. \end{aligned}$$

Following Ref. [47], we define  $\Pi_{A'B'}^\pm := \frac{1}{2}(I_{A'B'} \pm F_{A'B'})$  to be the orthogonal projections onto the symmetric and antisymmetric subspaces, respectively. Hence, substituting  $\Pi_{A'B'}^+ - \Pi_{A'B'}^-$  for  $F_{A'B'}$ , and  $\Pi_{A'B'}^+ + \Pi_{A'B'}^-$  for  $I_{A'B'}$ , yields (cf. Eqs. (68)–(73) in Ref. [47])

$$\begin{aligned} &\Theta^\Gamma[\mathcal{P}_{RA'B'}] \\ &= \frac{1}{m} \mathcal{N}_{AB}^\Gamma \otimes \text{Tr}_{A'B'}[F_{A'B'} \mathcal{P}_{RA'B'}] \\ &\quad + \mathcal{R}_{AB}^\Gamma \otimes \text{Tr}_{A'B'} \left[ \left( I_{A'B'} - \frac{1}{m} F_{A'B'} \right) \mathcal{P}_{RA'B'} \right] \\ &= \frac{1}{m} (\mathcal{N}_{AB}^\Gamma + (m-1)\mathcal{R}_{AB}^\Gamma) \otimes \text{Tr}_{A'B'}[\Pi_{A'B'}^+ \mathcal{P}_{RA'B'}] \\ &\quad + \frac{1}{m} ((m+1)\mathcal{R}_{AB}^\Gamma - \mathcal{N}_{AB}^\Gamma) \otimes \text{Tr}_{A'B'}[\Pi_{A'B'}^- \mathcal{P}_{RA'B'}]. \end{aligned} \quad (19)$$

By Eq. (18), the expression on the right-hand side of the equation above is a CPTP map. Hence,  $\mathbb{1}_R \otimes \Theta^\Gamma$  takes channels to channels; i.e.,  $\Theta^\Gamma$  is a superchannel, so  $\Theta$  is indeed a PPT superchannel. To summarize, we showed that, for any integer  $m$  for which there exists a channel  $\mathcal{R}_{AB}$  that satisfies Eq. (18), there exists a PPT superchannel  $\Theta$  achieving  $\Theta[\Phi_{A'B'}^+] = \mathcal{N}_{AB}$  with  $|A'_1| = |B'_1| = m$  (and  $|A'_0| = |B'_0| = 1$ ). Hence,  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$  cannot be greater than the right-hand side of Eq. (17). To complete the proof, we now prove the converse inequality; i.e., we show that  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$  is greater than or equal to the right-hand side of Eq. (17). Denote by  $\mathcal{G} \in \text{CPTP}(A'B' \rightarrow A'B')$  the twirling channel of the form

$$\mathcal{G}(\omega_{A'B'}) = \int (U_{A'} \otimes \bar{U}_{B'}) \omega_{A'B'} (U_{A'} \otimes \bar{U}_{B'})^\dagger dU_{A'B'} \quad (20)$$

with respect to the Haar probability measure,  $dU$ , over unitary matrices. It can be shown [100–102] that  $\mathcal{G}$  is actually the channel

$$\begin{aligned} \mathcal{G}(\omega_{A'B'}) &= \phi_{A'B'}^+ \text{Tr}[\phi_{A'B'}^+ \omega_{A'B'}] \\ &\quad + \frac{I_{A'B'} - \phi_{A'B'}^+}{m^2 - 1} \text{Tr}[(I_{A'B'} - \phi_{A'B'}^+) \omega_{A'B'}]. \end{aligned}$$

Note that, since  $|A'_0| = |B'_0| = 1$ , we can view the channel  $\mathcal{G}$  as a superchannel  $\mathcal{G}_{A'B' \rightarrow A'B'}$  taking channels (which are nothing but density matrices) in  $\text{CPTP}(A'_0 B'_0 \rightarrow A'_1 B'_1)$  to channels in the same set. In particular, this superchannel is self-adjoint, and satisfies  $\mathcal{G}[\Phi_{A'B'}^+] = \Phi_{A'B'}^+$ . The latter implies that if  $\Theta$  is a PPT superchannel such that  $\Theta[\Phi_{A'B'}^+] = \mathcal{N}_{AB}$ , then  $\Omega_{A'B' \rightarrow AB} := \Theta_{A'B' \rightarrow AB} \circ \mathcal{G}_{A'B'}$  is also a PPT superchannel that takes  $\Phi_{A'B'}^+$  to  $\mathcal{N}_{AB}$  (i.e., it achieves the same performance as  $\Theta$ ). Furthermore, by Eq. (20) the superchannel  $\Omega$  satisfies (cf. Eqs. (80)–(82) of Ref. [47])

$$\begin{aligned} \Omega_{A'B' \rightarrow AB}[\mathcal{M}_{A'B'}] &:= \mathcal{N}_{AB} \text{Tr}[\Phi_{A'B'}^+ \mathcal{M}_{A'B'}] \\ &\quad + \mathcal{R}_{AB} \text{Tr}[(I_{A'B'} - \Phi_{A'B'}^+) \mathcal{M}_{A'B'}], \end{aligned}$$

where

$$\mathcal{R}_{AB} := \frac{1}{1 - m^2} \Theta[I_{A'B'} - \Phi_{A'B'}^+].$$

Now, from the exact same lines leading to Eq. (19), it follows that, for  $\Omega_{A'B' \rightarrow AB}$  to be a PPT superchannel, it is necessary that for any  $\mathcal{P}_{RA'B'} \in \text{CPTP}(R_0 \rightarrow R_1 A'_1 B'_1)$ , the map on the right-hand side of Eq. (19) is a quantum channel. Since  $\Pi^+$  and  $\Pi^-$  are orthogonal projectors, each term must be a CP map, which yields Eq. (18). To summarize, if  $\Theta$  is a PPT superchannel that satisfies  $\Theta[\Phi_{A'B'}^+] = \mathcal{N}_{AB}$ , then  $\Omega$  is also a PPT superchannel that satisfies  $\Omega[\Phi_{A'B'}^+] = \mathcal{N}_{AB}$ ; the fact that  $\Omega$  is PPT forces each term of Eq. (19) to be a CP map which is equivalent to Eq. (18). Hence,  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$  cannot be smaller than the right-hand side of Eq. (17). This completes the proof. ■

The second lemma uses the previous one to link the single-shot exact PPT cost to the MLN.

*Lemma 9.* Let  $\mathcal{N} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  be a bipartite channel. Then,

$$\begin{aligned} \log_2 (2^{LN_{\max}(\mathcal{N}_{AB})} - 1) &\leq E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) \\ &\leq \log_2 (2^{LN_{\max}(\mathcal{N}_{AB})} + 2). \end{aligned}$$

*Proof.* First of all, we prove that the result of the previous lemma can be rewritten in a slightly modified version:

$$\begin{aligned} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) &= \inf \{ \log_2 m : -(m-1)\mathcal{R}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m+1)\mathcal{R}_{AB}^\Gamma \}, \end{aligned} \quad (21)$$

where  $\mathcal{R} \geq 0$ ,  $J_{A_0 B_0}^\mathcal{R} \leq I_{A_0 B_0}$ ,  $J_{A_0 B_0}^{\mathcal{R}^\Gamma} \leq I_{A_0 B_0}$ , and  $m \in \mathbb{N}$ . To see why, denote the second line of Eq. (21) by  $\tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$ . Then, by definition, we have  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) \leq \tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$  because if  $\mathcal{R}$  is a CPTP, then  $J_{A_0 B_0}^\mathcal{R} = J_{A_0 B_0}^{\mathcal{R}^\Gamma} = I_{A_0 B_0}$  [note that the condition  $-(m-1)\mathcal{R}_{AB}^\Gamma \leq (m+1)\mathcal{R}_{AB}^\Gamma$  implies in particular that  $\mathcal{R}^\Gamma \geq 0$ ]. Conversely, suppose  $\mathcal{R}$  satisfies  $J_{A_0 B_0}^\mathcal{R} \leq I_{A_0 B_0}$  and  $J_{A_0 B_0}^{\mathcal{R}^\Gamma} \leq I_{A_0 B_0}$ . Define  $\mathcal{P}$  as the map whose Choi matrix is given by

$$J_{AB}^\mathcal{P} := J_{AB}^\mathcal{R} + (I_{A_0 B_0} - J_{A_0 B_0}^\mathcal{R}) \otimes u_{A_1 B_1}.$$

Note that  $\mathcal{P}$  is a channel, and that both  $(I_{A_0 B_0} - J_{A_0 B_0}^\mathcal{R}) \otimes u_{A_1 B_1}$  and its partial transpose are positive semidefinite. Therefore,  $\mathcal{P}$  too satisfies the constraints

$$-(m-1)\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m+1)\mathcal{P}_{AB}^\Gamma,$$

so we can conclude that  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) \geq \tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$ . This proves that  $E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) = \tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})$ .

The rest of the proof employs similar techniques to Proposition 9 in Ref. [47], with a few exceptions. Continuing, we have

$$\begin{aligned} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) &= \tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}) \\ &\geq \log_2 \inf \{ m : -(m+1)\mathcal{R}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m+1)\mathcal{R}_{AB}^\Gamma, \mathcal{R} \geq 0, J_{A_0 B_0}^\mathcal{R} \leq I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{R}^\Gamma} \leq I_{A_0 B_0}, m \in \mathbb{N} \} \quad (22) \\ &= \log_2 \inf \{ m : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0, J_{A_0 B_0}^\mathcal{P} \leq (m+1)I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m+1)I_{A_0 B_0}, m \in \mathbb{N} \} \quad (23) \\ &\geq \log_2 \inf \{ m : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0, J_{A_0 B_0}^\mathcal{P} \leq (m+1)I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m+1)I_{A_0 B_0}, m \in \mathbb{R}_+ \} \quad (24) \\ &\geq \log_2 \inf \{ \max \{ \|J_{A_0 B_0}^\mathcal{P}\|_\infty, \|J_{A_0 B_0}^{\mathcal{P}^\Gamma}\|_\infty \} - 1 : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0 \} \\ &= \log_2 (2^{LN_{\max}(\mathcal{N}_{AB})} - 1), \end{aligned} \quad (25)$$

where in Eq. (22) we replaced  $m-1$  with  $m+1$ , so the infimum is on a less restricted set; in Eq. (23) we defined  $\mathcal{P}_{AB} := (m+1)\mathcal{R}_{AB}$ ; in Eq. (24) we removed the restriction that  $m$  is an integer; and the last inequality (25) follows from the fact that if  $J_{A_0 B_0}^\mathcal{P} \leq (m+1)I_{A_0 B_0}$  and  $J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m+1)I_{A_0 B_0}$ , then  $m \geq \max \{ \|J_{A_0 B_0}^\mathcal{P}\|_\infty, \|J_{A_0 B_0}^{\mathcal{P}^\Gamma}\|_\infty \} - 1$ .

For the other inequality, following similar lines, we get

$$\begin{aligned} 2^{E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})} &= 2^{\tilde{E}_{\text{PPT}}^{(1)}(\mathcal{N}_{AB})} \\ &\leq \inf \{ m : -(m-1)\mathcal{R}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq (m-1)\mathcal{R}_{AB}^\Gamma, \mathcal{R} \geq 0, J_{A_0 B_0}^\mathcal{R} \leq I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{R}^\Gamma} \leq I_{A_0 B_0}, m \in \mathbb{N} \} \quad (26) \\ &= \inf \{ m : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0, J_{A_0 B_0}^\mathcal{P} \leq (m-1)I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m-1)I_{A_0 B_0}, m \in \mathbb{N} \} \\ &= \inf \{ [m] : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0, J_{A_0 B_0}^\mathcal{P} \leq ([m]-1)I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq ([m]-1)I_{A_0 B_0}, m \in \mathbb{R}_+ \} \quad (27) \\ &\leq \inf \{ m : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0, J_{A_0 B_0}^\mathcal{P} \leq (m-2)I_{A_0 B_0}, J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m-2)I_{A_0 B_0}, m \in \mathbb{R}_+ \} \\ &= \inf \{ \max \{ \|J_{A_0 B_0}^\mathcal{P}\|_\infty, \|J_{A_0 B_0}^{\mathcal{P}^\Gamma}\|_\infty \} + 2 : -\mathcal{P}_{AB}^\Gamma \leq \mathcal{N}_{AB}^\Gamma \leq \mathcal{P}_{AB}^\Gamma, \mathcal{P} \geq 0 \} \\ &= 2^{LN_{\max}(\mathcal{N}_{AB})} + 2, \end{aligned} \quad (28) \quad (29)$$

where in Eq. (26) we replaced  $m + 1$  with  $m \in \mathbb{R}_+$ , obtaining a larger set; in Eq. (27) we set  $\mathcal{P} := (m - 1)\mathcal{R}$ ; in Eq. (29) we used the fact that  $m - 2 \leq \lfloor m \rfloor - 1$ , so the constraints  $J_{A_0 B_0}^{\mathcal{P}} \leq (m - 2)I_{A_0 B_0}$  and  $J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (m - 2)I_{A_0 B_0}$  imply the constraints  $J_{A_0 B_0}^{\mathcal{P}} \leq (\lfloor m \rfloor - 1)I_{A_0 B_0}$  and  $J_{A_0 B_0}^{\mathcal{P}^\Gamma} \leq (\lfloor m \rfloor - 1)I_{A_0 B_0}$  of Eq. (28), respectively. This completes the proof. ■

Recalling Ref. [48], the exact (parallel) NPT entanglement cost of the channel is defined as

$$E_{\text{PPT}}(\mathcal{N}_{AB}) = \limsup_n \frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n}).$$

The following result, which is the key theorem of this section, states that the exact PPT cost of a bipartite channel is given precisely by its max-logarithmic negativity.

*Theorem 10.* Let  $\mathcal{N} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  be a bipartite channel. Then,

$$E_{\text{PPT}}(\mathcal{N}_{AB}) = LN_{\max}(\mathcal{N}_{AB}).$$

*Proof.* The proof follows from the additivity property of  $LN_{\max}(\mathcal{N}_{AB})$  and Lemma 9. Specifically,

$$\begin{aligned} \frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n}) &\leq \frac{1}{n} \log_2(2^{LN_{\max}(\mathcal{N}_{AB}^{\otimes n})} + 2) \\ &= \frac{1}{n} \log_2(2^{nLN_{\max}(\mathcal{N}_{AB})} + 2). \end{aligned}$$

Conversely,

$$\begin{aligned} \frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n}) &\geq \frac{1}{n} \log_2(2^{LN_{\max}(\mathcal{N}_{AB}^{\otimes n})} - 1) \\ &= \frac{1}{n} \log_2(2^{nLN_{\max}(\mathcal{N}_{AB})} - 1). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{n} \log_2(2^{nLN_{\max}(\mathcal{N}_{AB})} - 1) &\leq \frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n}) \\ &\leq \frac{1}{n} \log_2(2^{nLN_{\max}(\mathcal{N}_{AB})} + 2). \end{aligned}$$

If we take the limit as  $n \rightarrow +\infty$ , the lower and upper bound of  $\frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n})$  have the same limit, equal to  $LN_{\max}(\mathcal{N}_{AB})$ . Therefore,  $\frac{1}{n} E_{\text{PPT}}^{(1)}(\mathcal{N}_{AB}^{\otimes n})$  has the limit (which will be equal to its limit superior). This allows us to conclude that  $E_{\text{PPT}}(\mathcal{N}_{AB}) = LN_{\max}(\mathcal{N}_{AB})$ . ■

In Ref. [43] we proved that the MLN is an upper bound for another entanglement measure, the NPT entanglement generation power  $E_g^{\text{PPT}}$  [62,64,74,103]:

$$E_g^{\text{PPT}}(\mathcal{N}_{AB}) \leq LN_{\max}(\mathcal{N}_{AB}).$$

## V. SEP entanglement of a bipartite channel

In the previous section we saw that extending the set of free operations beyond LOCC can be very fruitful. However, one may argue that the PPT operations allow for “too much” freedom, making NPT entanglement a rather crude approximation of LOCC entanglement. Here we consider a much smaller set: the set of separable superchannels (SEPS). Like before, SEPS do not necessarily have a realization similar to the one in Fig. 7, where the preprocessing and postprocessing are both SEP channels. Instead, we define SEPS using the Choi matrix formalism of superchannels. This simplifies the

set of operations, making them more useful for applications and calculations.

Recall that a channel  $\mathcal{N} \in \text{CPTP}(A_0 B_0 \rightarrow A_1 B_1)$  is called *separable* [30–32] if it has an operator-sum representation of the form

$$\mathcal{N}_{AB}(\rho_{A_0 B_0}) = \sum_j (X_{A_0}^j \otimes Y_{B_0}^j) \rho_{A_0 B_0} (X_{A_0}^j \otimes Y_{B_0}^j)^\dagger,$$

where  $X^j \in \mathfrak{B}(A_0)$ ,  $Y^j \in \mathfrak{B}(B_0)$ , and  $\sum_j (X_{A_0}^j)^\dagger X_{A_0}^j \otimes (Y_{B_0}^j)^\dagger Y_{B_0}^j = I_{A_0 B_0}$ . It is simple to check that the set SEP is precisely the set of completely resource nongenerating operations [13,48] in entanglement theory (see, e.g., Ref. [89] and references therein). Moreover, a bipartite channel is separable if and only if its Choi matrix is a separable matrix. This fact inspires us to define SEPS using the Choi formalism for superchannels.

*Definition 11.* Let  $\Theta \in \mathfrak{S}(A'B' \rightarrow A'B')$  be a bipartite superchannel. Then,  $\Theta$  is called a *separable superchannel* (SEPS) if its Choi matrix is separable; i.e., it can be expressed as

$$\mathbf{J}_{ABA'B'}^\Theta = \sum_j X_{AA'}^j \otimes Y_{BB'}^j,$$

where, for all  $j$ , the matrices  $X_{AA'}^j$  and  $Y_{BB'}^j$  are positive semidefinite. We denote by  $\text{SEPS}(AB \rightarrow A'B')$  the set of all bipartite SEPS from system  $AB$  to  $A'B'$ .

*Remark 12.* Note that clearly SEPS is a subset of PPT superchannels.

Definition 11 does *not* refer to the implementation of SEPS with pre- and postprocessing that are both SEP channels. On the other hand, however, if a bipartite superchannel  $\Theta$  consists of a SEP preprocessing channel  $\mathcal{E}$  and a SEP postprocessing channel  $\mathcal{F}$ , then the channel  $\mathcal{Q}^\Theta = \mathcal{F} \circ \mathcal{E}$  is also SEP (and also its Choi matrix  $\mathbf{J}^\Theta$ ), so we can conclude that  $\Theta$  is SEPS.

The next proposition shows that the set of SEPS is not “too large” in the sense that it cannot generate (dynamical) entanglement out of SEP channels. In this way, we establish that a superchannel  $\Theta$  is completely nonentangling (i.e., completely resource nongenerating) if and only if it is a SEPS.

*Proposition 13.*  $\Theta \in \text{SEPS}(AB \rightarrow A'B')$  if and only if, for every trace nonincreasing separable CP map  $\mathcal{N}_{A''A'B''B} \in \text{CP}(A''_0 A'_0 B'_0 B_0 \rightarrow A''_1 A'_1 B'_1 B_1)$ , the map

$$(\mathbb{1}_{A''B''} \otimes \Theta_{AB \rightarrow A'B'})[\mathcal{N}_{A''A'B''B}]$$

is a separable trace nonincreasing CP map in  $\text{CP}(A''_0 A'_0 B'_0 B'_0 \rightarrow A''_1 A'_1 B'_1 B'_1)$ .

*Proof.* Let  $\Theta$  be SEPS. Note that

$$\mathbf{J}_{A''B''ABA'B'}^{\mathbb{1} \otimes \Theta} = \mathbf{J}_{A''B''}^{\mathbb{1}} \otimes \mathbf{J}_{ABA'B'}^\Theta,$$

where

$$\begin{aligned} \mathbf{J}_{A''B''}^{\mathbb{1}} &= \phi_{A''_0 B''_0 \tilde{A}''_0 \tilde{B}''_0}^+ \otimes \phi_{A''_1 B''_1 \tilde{A}''_1 \tilde{B}''_1}^+ \\ &= \phi_{A''_0 \tilde{A}''_0}^+ \otimes \phi_{A''_1 \tilde{A}''_1}^+ \otimes \phi_{B''_0 \tilde{B}''_0}^+ \otimes \phi_{B''_1 \tilde{B}''_1}^+ \end{aligned}$$

is separable. Since  $\mathbf{J}_{ABA'B'}^\Theta$  is also separable, then  $\mathbb{1}_{A''B''} \otimes \Theta_{AB \rightarrow A'B'}$  is in SEPS too. Hence, it is enough to show that  $\Theta$  is resource non-generating [48]. Let  $\mathcal{M}_{AB}$  be a separable

bipartite CP map. Then,

$$J_{A'B'}^{\Theta[\mathcal{M}]} = \text{Tr}_{AB} \left[ \mathbf{J}_{ABA'B'}^{\Theta} \left( (J_{AB}^{\mathcal{M}})^T \otimes I_{A'B'} \right) \right]$$

is separable since both  $\mathbf{J}_{ABA'B'}^{\Theta}$  and  $J_{AB}^{\mathcal{M}}$  are separable.

Conversely, suppose  $\Theta \in \mathfrak{S}(AB \rightarrow A'B')$  is a completely nonentangling superchannel with respect to SEP channels. Recall the representation of  $\Theta$  given by  $\mathcal{P}^{\Theta}$  as in Sec. II B, where  $A$  and  $B$  are replaced by  $AB$  and  $A'B'$ , respectively. We have

$$\mathcal{P}_{ABA'B'}^{\Theta} = \Theta_{\widetilde{AB} \rightarrow A'B'}[\Phi_{ABAB}^+] = \Theta_{\widetilde{AB} \rightarrow A'B'}[\Phi_{AA}^+ \otimes \Phi_{BB}^+],$$

where we have used the fact that the CP map  $\Phi_{ABA'B'}^+$  splits in exactly the same way as its state counterpart  $\phi_{ABA'B'}^+$ . Since  $\Theta$  is completely nonentangling, it follows that the channel  $\mathcal{P}_{ABA'B'}^{\Theta}$  is separable, and therefore its Choi matrix  $\mathbf{J}_{ABA'B'}^{\Theta}$  is separable as well. Hence,  $\Theta$  is a SEPS. This completes the proof  $\blacksquare$ .

## VI. BOUND DYNAMICAL ENTANGLEMENT

We know that if the partial transpose of a bipartite entangled state yields a positive semidefinite matrix, then the state is not distillable under LOCC [27,28]. Such states are said to possess *bound entanglement* [28].

This condition can be elevated to bipartite channels. Let  $\mathcal{N}_{AB} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  be a bipartite channel whose partial transpose  $\mathcal{N}_{AB}^{\Gamma}$  is also a bipartite channel (i.e.,  $\mathcal{N}_{AB}$  is a PPT bipartite channel). We argue here that such channels cannot be used to distill entanglement. To see why, by contradiction, suppose that there exists  $n \in \mathbb{N}$  large enough and an LOCC superchannel  $\Theta$  converting  $\mathcal{N}_{AB}^{\otimes n}$  to a bipartite qubit state  $\rho_{A'B'} = \Theta[\mathcal{N}_{AB}^{\otimes n}]$ , where  $|A'_0| = |B'_0| = 1$  and  $|A'_1| = |B'_1| = 2$ . If  $\rho_{A'B'}$  is entangled, its partial transpose is not positive semidefinite [36,37]. On the other hand, on the right-hand side the partial transpose gives

$$(\Theta[\mathcal{N}_{AB}^{\otimes n}])^{\Gamma} = \Theta^{\Gamma}[(\mathcal{N}_{AB}^{\otimes n})^{\Gamma}] = \Theta^{\Gamma}[(\mathcal{N}_{AB}^{\Gamma})^{\otimes n}] \geq 0$$

for LOCC superchannels are in particular PPT, so  $\Theta^{\Gamma}$  is a superchannel. Recall also that we assume that  $\mathcal{N}_{AB}^{\Gamma}$  is a channel as well. Therefore, we get a contradiction.

Note that in the argument above we showed that PPT superchannels (which include in particular LOCC superchannels) cannot be used to distill entanglement from an arbitrarily large number of copies of a PPT channel. This further shows that our definition of the set of PPT superchannels, which in principle can be larger than the set of superchannels realizable with PPT pre- and postprocessing as in Fig. 7, is not so large such that PPT entanglement becomes distillable.

So far we have discussed the parallel scenario in which the superchannel  $\Theta$  acts on  $\mathcal{N}_{AB}^{\otimes n}$  in parallel, or at a single time. However, if one can use the channel repeatedly and sequentially, one can realize, e.g., a transformation of the form

$$\Theta_n[\mathcal{N}_{AB}] \circ \cdots \circ \Theta_2[\mathcal{N}_{AB}] \circ \Theta_1[\mathcal{N}_{AB}], \quad (30)$$

as illustrated in Fig. 8. More generally, in Fig. 9 we illustrate a PPT comb, which is not necessarily of the form given in Eq. (30). Instead, for a PPT comb we only require that the channel  $\mathcal{Q}_{A^{n+1}B^{n+1}} := \mathcal{E}_{n+1} \circ \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$  be PPT. The channel  $\mathcal{Q}_{A^{n+1}B^{n+1}}$  is illustrated in Fig. 10.

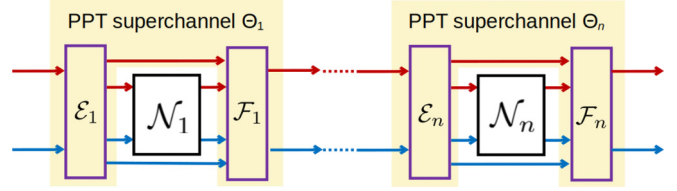


FIG. 8. Sequence of PPT superchannels applied to the channels  $\mathcal{N}_1, \dots, \mathcal{N}_n$ .

Now we argue that not even such a comb can convert  $n$  PPT bipartite channels  $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n$  to a single two-qubit entangled state. This in particular demonstrates that  $n$  adaptive uses of a PPT channel  $\mathcal{N}_{AB}$  in a PPT comb cannot produce a two-qubit entangled state. In other words, pure-state entanglement cannot be distilled by LOCC (not even by PPT combs) out of PPT bipartite channels. In other words, PPT entangled channels are bound entangled channels.

For this purpose, we note that a comb  $\mathcal{C}_n$  is uniquely characterized by the channel  $\mathcal{Q}_{A^{n+1}B^{n+1}}$ . Therefore, we define the partial transpose of  $\mathcal{C}_n$ , denoted  $\mathcal{C}_n^{\Gamma}$ , to be the supermap associated with  $\mathcal{Q}_{A^{n+1}B^{n+1}}^{\Gamma}$ . Consequently,  $\mathcal{C}$  is a PPT quantum comb if  $\mathcal{C}_n^{\Gamma}$  is a quantum comb. Moreover, note that

$$(\mathcal{C}_n[\mathcal{N}_1, \dots, \mathcal{N}_n])^{\Gamma} = \mathcal{C}_n^{\Gamma}[\mathcal{N}_1^{\Gamma}, \dots, \mathcal{N}_n^{\Gamma}],$$

as described in Fig. 11 for  $n = 2$ . This is the key reason why PPT quantum combs cannot produce entangled states from PPT channels.

*Proposition 14.* Let  $\mathcal{C}_n$  be a PPT quantum comb with  $n$  slots, as illustrated in Fig. 9, with  $|A_0^{(1)}| = |B_0^{(1)}| = 1$  and  $|A_1^{(n+1)}| = |B_1^{(n+1)}| = 2$ . Let  $\mathcal{N}_1, \dots, \mathcal{N}_n$  be  $n$  PPT bipartite channels with input and output dimensions compatible with the comb  $\mathcal{C}_n$ , i.e., such that  $\mathcal{C}_n[\mathcal{N}_1, \dots, \mathcal{N}_n] := \rho_{A_1^{(n+1)}B_1^{(n+1)}}$  is a well-defined two-qubit state. Then, the quantum state  $\rho_{A_1^{(n+1)}B_1^{(n+1)}}$  is separable.

*Proof.* The proof follows from the property that

$$\begin{aligned} (\rho_{A_1^{(n+1)}B_1^{(n+1)}})^{T_{B_1^{(n+1)}}} &= (\mathcal{C}_n[\mathcal{N}_1, \dots, \mathcal{N}_n])^{\Gamma} \\ &= \mathcal{C}_n^{\Gamma}[\mathcal{N}_1^{\Gamma}, \dots, \mathcal{N}_n^{\Gamma}] \\ &\geq 0 \end{aligned}$$

because  $\mathcal{C}_n^{\Gamma}$  is a quantum comb, and  $\mathcal{N}_1^{\Gamma}, \dots, \mathcal{N}_n^{\Gamma}$  are all CP maps, as  $\mathcal{N}_1, \dots, \mathcal{N}_n$  are PPT. So  $\rho_{A_1^{(n+1)}B_1^{(n+1)}}$  is a PPT two-qubit state, therefore separable [36,37].  $\blacksquare$

Note that the above proposition generalizes the notion of bound entanglement to multiple and possibly different dynamical resources. In the special case in which  $\mathcal{N}_1 = \dots =$

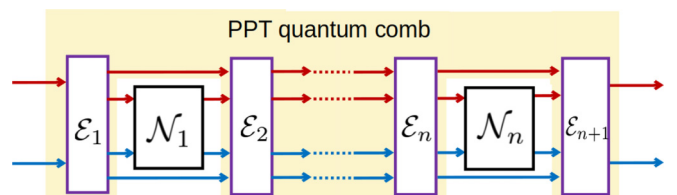


FIG. 9. PPT comb: any comb such that  $\mathcal{E}_{n+1} \circ \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1$  is a PPT channel.



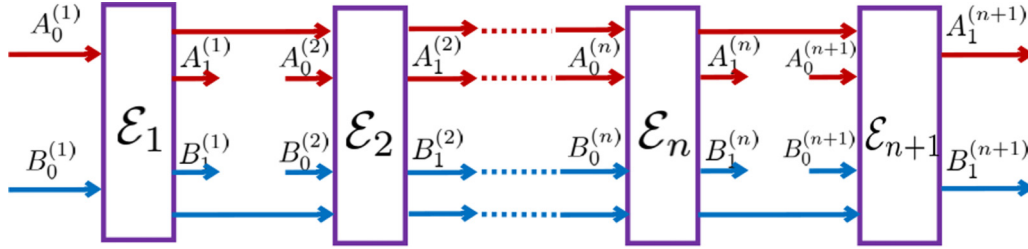


FIG. 10. A quantum comb is PPT if and only if the bipartite channel  $\mathcal{Q} \in \text{CPTP}(A_0^{(1)}B_0^{(1)} \dots A_0^{(n+1)}B_0^{(n+1)} \rightarrow A_1^{(1)}B_1^{(1)} \dots A_1^{(n+1)}B_1^{(n+1)})$  is PPT.

$\mathcal{N}_n \equiv \mathcal{N}$ , the above proposition implies that pure-state entanglement cannot be distilled from a PPT bipartite channel  $\mathcal{N}$ , not even with adaptive schemes. When  $\mathcal{N}$  has trivial input, we recover the PPT bound entanglement for states. When  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  has classical outputs  $A_1$  and  $B_1$ , we get bound entanglement for POVMs. Since the latter is a less studied one, we give here a simple example of a family of bipartite POVMs that are not local (i.e., cannot be implemented by LOCC), but at the same time they cannot produce distillable entanglement. To find other candidates for bound entangled channels, we must consider PPT channels that are not LOCC.

*Example 15.* Let  $\beta \in \mathfrak{D}(A_0B_0)$  be any PPT bound entangled state of a composite system  $A_0B_0$ , and define a binary POVM consisting of  $E_{A_0B_0} := \beta_{A_0B_0}$  and  $F_{A_0B_0} := I_{A_0B_0} - \beta_{A_0B_0}$ . We view this POVM as the bipartite channel  $\mathcal{E} \in \text{CPTP}(A_0B_0 \rightarrow X)$  (as already noted in Sec. III C, since the output is classical, there is no need to represent it with two classical systems, because classical communication is free) given by

$$\mathcal{E}_{A_0B_0 \rightarrow X}(\rho_{A_0B_0}) := \text{Tr}[E_{A_0B_0}\rho_{A_0B_0}]|0\rangle\langle 0|_X + \text{Tr}[F_{A_0B_0}\rho_{A_0B_0}]|1\rangle\langle 1|_X.$$

Since both  $E_{A_0B_0}$  and  $F_{A_0B_0}$  have positive partial transpose, it follows that  $\mathcal{E}$  above is a PPT channel, and, as such, it cannot produce distillable entanglement. This means that the POVM  $\{E_{A_0B_0}, F_{A_0B_0}\}$  is a bound entangled POVM.

VII. CONCLUSIONS AND OUTLOOK

In this article we studied quantum entanglement as a resource theory of processes, where the resources are bipartite channels (see Fig. 1). This paradigm encompasses several

interesting cases, including the already well-studied resource theory of entanglement of quantum states [1,2], but also the novel area of entanglement theory for POVMs.

The LOCC resource theory for dynamical entanglement is still very complicated to characterize from a mathematical point of view, so we also considered broader classes of free superchannels: separable superchannels (SEPS) [30–32] in Sec. V and PPT superchannels [32,35,49] in Sec. IV. The NPT resource theory is particularly simple to deal with, as all resource-theoretic protocols can be fully characterized by SDPs. This remarkable fact, which did not appear in a previous work on PPT superchannels [47], is a consequence of not restricting ourselves to freely realizable [48] PPT superchannels, i.e., to superchannels whose pre- and postprocessing are both PPT channels. This is not the only novelty with respect to Ref. [47]: since we considered the most general case of bipartite channels, we were able to generalize their notion of  $\kappa$ -entanglement in two possible ways, the maximum of which we call max-logarithmic negativity (see Sec. IV C). This has a nice operational characterization as the exact asymptotic entanglement cost of a bipartite channel under PPT superchannels.

Finally, we concluded with an analysis of bound entanglement for bipartite channels, showing that from a PPT channel we can distill *no* ebits under any PPT superchannels (therefore also under any LOCC or SEP superchannels), not even with an adaptive scheme. This generalizes the known result for PPT states [28]. We were also able to give an example of a bound entangled POVM (Sec. VI).

Clearly our work just looks at the surface of a whole unexplored world, but it opens the way to the study of the new area of entanglement of bipartite channels [104–111]. On a small level, one can generalize the analysis we did, and the results we obtained in this article. For example, one can try to characterize which PPT superchannels are freely realizable, i.e., restricted PPT channels (see Sec. IV A), and what the resulting resource theory looks like. One can also go a level up in complexity, and describe transitions under LOCC superchannels.

Possible easy directions for future work involve expanding our preliminary treatment of the entanglement of POVMs (Sec. III C to deal with concrete cases and examples, e.g., von Neumann measurements), studying the entanglement of bipartite unitary channels [103], or even achieving a complete characterization of the entanglement of the simplest instances of bipartite channels, i.e., those where every system is a qubit. Moreover, another interesting research direction is about witnesses. In Appendix A we introduce witnesses for PPT superchannels, but, as we note therein, the really

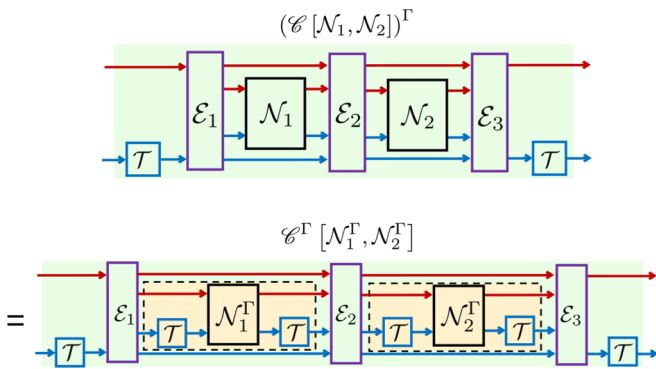


FIG. 11. The channel  $(\mathcal{C}_n[\mathcal{N}_1, \mathcal{N}_2])^\Gamma$  equals the channel  $\mathcal{C}_n^\Gamma[\mathcal{N}_1^\Gamma, \mathcal{N}_2^\Gamma]$ . Note that the yellow boxed areas correspond to the maps  $(\mathcal{N}_1^\Gamma)^\Gamma = \mathcal{N}_1$  and  $(\mathcal{N}_2^\Gamma)^\Gamma = \mathcal{N}_2$ .

interesting ones are for the LOCC theory, which have yet to be characterized.

On a grand scale, this work on entanglement theory leads to several areas that can be explored anew. Think, e.g., of multipartite entanglement [2], or of the whole zoo of entanglement measures [1,2]. One can also wonder if entangled bipartite channels can be used to draw a secret key from them [11]. Moreover, our results for LOCC superchannels can be translated to local operations and shared randomness (LOSR) superchannels [8,112–114], which are a strict subset of LOCC ones. LOSR superchannels were argued to be essential for the formulation of resource theories for nonlocality [8], as they define the relevant notion of dynamical entanglement in Bell and common-cause scenarios. This intriguing research direction deserves a comprehensive study in the future, in addition to theories of nonlocality that do not involve LOSR channels [115]. Finally, providing us with a more general angle, research developments in the resource theory of entanglement for bipartite channels can also help us get insights into one of the major open problems of quantum information theory: the existence of bound entangled states that are not PPT states [116–118].

To conclude, on an even more general and speculative level, one can introduce a resource theory of entanglement for higher-level generalizations of quantum channels [51,54,55], such as superchannels themselves, combs, or more exotic objects without a definite causal structure [53,119,120]. On such a general level, especially without causal definiteness, one can expect new and interesting behaviors, which need to be examined carefully.

#### ACKNOWLEDGMENTS

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#### APPENDIX A: NPT WITNESSES

Entanglement witnesses provide a simple “no-go” testing to determine whether a given resource (state, channel, or even superchannel) is free or not. Here we analyze the witnesses determining whether a bipartite superchannel is PPT or not,

for this is the most general case. Indeed, PPT states and PPT channels can be viewed as limiting cases of PPT superchannels when some of the input systems are trivial.

In NPT entanglement theory one can determine whether bipartite states, channels, or superchannels are PPT simply by checking the positivity of their partial transpose. Why do we study NPT witnesses then? The main reason is to distinguish them from LOCC entanglement witnesses, which are the more interesting ones. Indeed, LOCC entanglement witnesses play a central role in entanglement theory, as there is no simple or efficient way to determine if a resource is entangled or not. Therefore, distinguishing LOCC witnesses from NPT ones is necessary to understand which witnesses are truly physically meaningful.

*Definition 16.* A matrix  $W \in \text{Herm}(ABA'B')$  is an NPT witness if it is *not* positive semidefinite, and if it satisfies  $\text{Tr}[W_{ABA'B'} \mathbf{J}_{ABA'B'}^\Theta] \geq 0$  for all superchannels in  $\Theta \in \text{PPT}(AB \rightarrow A'B')$ .

Therefore, the set of all NPT witnesses can be viewed as the set of all matrices in  $\mathfrak{J}_{ABA'B'}^*$  that are not positive semidefinite, where  $\mathfrak{J}_{ABA'B'}^*$  is the dual of the cone generated by the Choi matrices of PPT bipartite superchannels,  $\mathfrak{J}_{ABA'B'}$ . In Ref. [48], we showed that the former can be expressed as

$$\mathfrak{J}_{ABA'B'}^* = \{W \in \text{Herm}(ABA'B') : \text{Tr}[W_{ABA'B'} \mathbf{J}_{ABA'B'}^\Theta] \geq 0\}, \quad (\text{A1})$$

for every PPT superchannel  $\Theta$ . Then  $\mathfrak{J}_{ABA'B'}^*$  is the set of all  $W \in \text{Herm}(ABA'B')$  such that  $\text{Tr}[W_{ABA'B'} \mathbf{J}_{ABA'B'}] \geq 0$  for all matrices  $\mathbf{J} \in \text{Herm}(ABA'B')$  with the following properties:

- (1)  $\mathbf{J}_{ABA'B'} \geq 0$ .
- (2)  $\mathbf{J}_{ABA_0B'_0} = \mathbf{J}_{A_0B_0A'_0B'_0} \otimes u_{A_1B_1}$ .
- (3)  $\mathbf{J}_{A_1B_1A'_0B'_0} = I_{A_1B_1A'_0B'_0}$ .
- (4)  $\mathbf{J}_{ABA'B'}^{T_{BB'}} \geq 0$ .

Note that the first three conditions ensure that  $\mathbf{J}_{ABA'B'}$  is the Choi matrix of a bipartite superchannel, and the last condition ensures that the superchannel is PPT.

The conditions above imply that all NPT witnesses  $W \in \text{Herm}(ABA'B')$  are of the form

$$W_{ABA'B'} = P_{ABA'B'} + X_{ABA'B'}^{T_{BB'}} + Y_{ABA_0B'_0} \otimes I_{A'_1B'_1} + I_{A_0B_0A'_1B'_1} \otimes Z_{A_1B_1A'_0B'_0}, \quad (\text{A2})$$

where  $P_{ABA'B'}, X_{ABA'B'} \geq 0$ ,  $Y_{ABA_0B'_0}$  is a Hermitian matrix such that  $Y_{AB} = 0$ , and  $Z$  is a Hermitian matrix such that  $\text{Tr}[Z_{A_1B_1A'_0B'_0}] = 0$ . Note that the Hilbert-Schmidt inner product between  $Y_{ABA_0B'_0} \otimes I_{A'_1B'_1}$  (or  $I_{A_0B_0A'_1B'_1} \otimes Z_{A_1B_1A'_0B'_0}$ ) and any Choi matrix of a superchannel is always zero, as shown in Ref. [52]. This is why they can be added to any NPT witness. Now we will use this form of NPT witnesses to express the PPT conversion distance as an SDP.

#### APPENDIX B: ADDITIVITY OF THE MAX-LOGARITHMIC NEGATIVITY

Here we prove only the additivity of  $LN_{\max}^{(0)}$ , as the proof of the additivity of  $LN_{\max}^{(1)}$  follows the exact same lines.

*Lemma 17.* For any two bipartite channels  $\mathcal{N} \in \text{CPTP}(A_0B_0 \rightarrow A_1B_1)$  and  $\mathcal{M} \in \text{CPTP}(A'_0B'_0 \rightarrow A'_1B'_1)$  we have

$$LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = LN_{\max}^{(0)}(\mathcal{N}_{AB}) + LN_{\max}^{(0)}(\mathcal{M}_{A'B'}).$$

*Proof.* For simplicity of the exposition, in some places we will omit the subscripts identifying the systems. By definition we have

$$LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = \log_2 \inf \left\{ \|J_{A_0B_0A'_0B'_0}^{\mathcal{P}}\|_{\infty} : -\mathcal{P}_{ABA'B'}^{\Gamma} \leq \mathcal{N}_{AB}^{\Gamma} \otimes \mathcal{M}_{A'B'}^{\Gamma} \leq \mathcal{P}_{ABA'B'}^{\Gamma}, \mathcal{P} \geq 0 \right\} \tag{B1}$$

$$\begin{aligned} &\leq \log_2 \inf \left\{ \|J_{A_0B_0A'_0B'_0}^{\mathcal{P}_1 \otimes \mathcal{P}_2}\|_{\infty} : -\mathcal{P}_1^{\Gamma} \leq \mathcal{N}^{\Gamma} \leq \mathcal{P}_1^{\Gamma}; -\mathcal{P}_2^{\Gamma} \leq \mathcal{M}^{\Gamma} \leq \mathcal{P}_2^{\Gamma}; \mathcal{P}_1, \mathcal{P}_2 \geq 0 \right\} \\ &= LN_{\max}^{(0)}(\mathcal{N}_{AB}) + LN_{\max}^{(0)}(\mathcal{M}_{A'B'}), \end{aligned} \tag{B2}$$

where the inequality follows from the fact that, if  $\mathcal{P}_1$  and  $\mathcal{P}_2$  satisfy the constraints in (B2), then  $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$  satisfies the constraints in (B1). The last equality follows from the multiplicativity of the operator norm under tensor product.

For the other direction, we use the dual expression in Eq. (13). Hence,

$$LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = \log_2 \sup \{ \text{Tr}[J^{\mathcal{N} \otimes \mathcal{M}}(V - W)] : V + W \leq \rho \otimes I; \rho \in \mathfrak{D}(A_0B_0A'_0B'_0); V, W \geq 0 \}.$$

Setting  $X := V + W$  and  $Y := V - W$ , we have

$$LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'}) = \log_2 \sup \{ \text{Tr}[J^{\mathcal{N} \otimes \mathcal{M}}Y] : X \leq \rho \otimes I; \rho \in \mathfrak{D}(A_0B_0A'_0B'_0); X \pm Y \geq 0 \} \tag{B3}$$

$$\geq \log_2 \sup \{ \text{Tr}[J^{\mathcal{N} \otimes \mathcal{M}}(Y_1 \otimes Y_2)] : X_1 \leq \rho_1 \otimes I; X_2 \leq \rho_2 \otimes I; X_1 \pm Y_1 \geq 0; X_2 \pm Y_2 \geq 0 \}, \tag{B4}$$

where  $\rho_1 \in \mathfrak{D}(A_0B_0)$  and  $\rho_2 \in \mathfrak{D}(A'_0B'_0)$  and the inequality follows from the fact that if  $X_1, X_2, \rho_1$ , and  $\rho_2$  satisfy the constraints in (B4), then  $X = X_1 \otimes X_2, Y = Y_1 \otimes Y_2$ , and  $\rho = \rho_1 \otimes \rho_2$  satisfy the constraints in (B3). In particular, let us show that if  $X_1 \pm Y_1 \geq 0$  and  $X_2 \pm Y_2 \geq 0$ , then we also have  $X_1 \otimes X_2 \pm Y_1 \otimes Y_2 \geq 0$ . First of all, observe that, from the assumptions  $X_1 \pm Y_1 \geq 0$  and  $X_2 \pm Y_2 \geq 0$ , we have

$$(X_1 \pm Y_1) \otimes (X_2 \pm Y_2) \geq 0,$$

from which

$$X_1 \otimes X_2 + Y_1 \otimes Y_2 \geq \mp(X_1 \otimes Y_2 + Y_1 \otimes X_2).$$

This means that

$$\langle \psi | X_1 \otimes X_2 + Y_1 \otimes Y_2 | \psi \rangle \geq \mp \langle \psi | X_1 \otimes Y_2 + Y_1 \otimes X_2 | \psi \rangle,$$

for all vectors  $\psi$ . This in turn means that

$$\langle \psi | X_1 \otimes X_2 + Y_1 \otimes Y_2 | \psi \rangle \geq 0,$$

for all vectors  $\psi$ , from which  $X_1 \otimes X_2 + Y_1 \otimes Y_2 \geq 0$ .

Similarly, from

$$(X_1 \pm Y_1) \otimes (X_2 \mp Y_2) \geq 0$$

we get that

$$X_1 \otimes X_2 - Y_1 \otimes Y_2 \geq \mp(Y_1 \otimes X_2 - X_1 \otimes Y_2),$$

which, by an argument similar to the one above, allows us to conclude that  $X_1 \otimes X_2 - Y_1 \otimes Y_2 \geq 0$ .

Combining both inequalities we obtained for  $LN_{\max}^{(0)}(\mathcal{N}_{AB} \otimes \mathcal{M}_{A'B'})$ , we prove the additivity. ■

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