Class of Bell-Clauser-Horne inequalities for testing quantum nonlocality

Chen Qian,[*] Yang-Guang Yang [ORCID],[†] and Qun Wang[‡]

Interdisciplinary Center for Theoretical Study and Department of Modern Physics,
University of Science and Technology of China, Hefei, Anhui 230026, China
and Peng Huanwu Center for Fundamental Theory, Hefei, Anhui 230026, China

Cong-Feng Qiao[§]

School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
and Key Laboratory of Vacuum Physics, Chinese Academy of Sciences, Beijing 100049, China

Quantum nonlocality, one of the most important features of quantum mechanics, is normally connected in experiments with the violation of Bell-Clauser-Horne (Bell-CH) inequalities. We propose effective methods for the rearrangement and linear inequality to prove a large variety of Bell-CH inequalities. We also derive a set of Bell-CH inequalities using these methods which can be violated in some quantum entangled states.

I. INTRODUCTION

Quantum nonlocality is one of the most striking aspects of quantum mechanics without a classical analog in reality. It can be characterized as correlated outcomes when we measure two or more entangled quantum systems, even if these systems are spatially separated. Quantum nonlocality originates from a contradiction between local realism and the completeness of quantum mechanics pointed out by Einstein, Podolsky, and Rosen in 1935 [1], which was called the "EPR paradox" and led to a great challenge of the concept of "locality" taken for granted by most physicists. To test such a contradiction in a real physical system, Bell, Clauser, and other researchers formulated observable inequalities with correlation functions of measurement outcomes for entangled systems [2–4]. These inequalities were later called Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequalities. They proved that the quantum correlation cannot be fully described by any local hidden-variable theory. Later on, Clauser and Horne proposed the CH inequality in terms of probabilities [5], followed by many CH-like inequalities (we call them all Bell-CH inequalities in this paper). These inequalities have weaker auxiliary assumptions than Bell-CHSH inequalities in experimental considerations.

Violation of Bell-CH inequalities has been widely verified by many experiments in favor of the nonlocal feature of quantum mechanics [6–9]. However, observing the violation of these inequalities is not feasible in every entangled system [10]. One needs to generalize these inequalities in order to test them in experiments. The first attempt to find a systematic way of generalization was made by Froissart [11], and later attempt was made by Pitowsky [12–14], who introduced correlation polytopes to find Bell-CH inequalities inspired by Boole's method on probabilistic inequalities [15]. After a series of further works, such algorithms were built to find a set of Bell-CH inequalities based on the solution of convex problems [11,16–21]. A large variety of Bell-CH inequalities have been found [11,16,19,20,22–29]. In Refs. [27–29], the authors obtained a complete list of Bell-CH inequalities in the case of binary settings up to four measurements on each setting. The main goal of this paper is to prove these inequalities with analytical techniques and construct a class of Bell-CH inequalities. Our inequalities can be used as a substitute for the original ones for testing quantum nonlocality under some circumstances.

We consider a system consisting of two classical or quantum subsystems x and y. Alice chooses m measurement settings for a quantity that has k values on x, and Bob chooses n measurement settings for a quantity that has l values on y. We denote this case as $mnkl$. The system can be measured many times with different selected settings. We define P_{x_i} as the probability that Alice obtains a certain value a for measurement setting i on x, P_{y_j} as the probability that Bob obtains a certain value b for measurement setting j on y, and $P_{x_i y_j}$ as the probability that Alice obtains a for i and Bob obtains b for j at the same time. For a classical system labeled $mn22$, we have the so-called Bell-CH inequalities of binary settings which hold for certain integer coefficients C_{x_i}, C_{y_j}, and $C_{x_i y_j}$,

$$\sum_{i=1}^{m} C_{x_i} P_{x_i} + \sum_{j=1}^{n} C_{y_j} P_{y_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} C_{x_i y_j} P_{x_i y_j} \leqslant 0. \quad (1)$$

In our formulation, we define the corresponding Bell-CH-like inequalities in algebraic forms (we call them algebraic Bell-CH inequalities for short) as

$$B \sum_{i=1}^{m} C_{x_i} x_i + A \sum_{j=1}^{n} C_{y_j} y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} C_{x_i y_j} x_i y_j \leqslant 0, \quad (2)$$

[*]qianch18@mail.ustc.edu.cn
[†]Corresponding author: mathmuse@ustc.edu.cn
[‡]qunwang@ustc.edu.cn
[§]qiaocf@ucas.ac.cn

with A and B being the upper bounds of two sets of positive variables x_i and y_i, respectively. The Bell-CH inequalities can be derived from the algebraic ones in local hidden-variable theory. Among these inequalities, there are independent ones which we label $Imn22$ for Bell-CH inequalities and I_{mn22} for algebraic ones.

This paper is organized as follows. In this section, we have provided an introduction and the motivation of this work with definitions of Bell-CH inequalities and their algebraic forms. In Sec. II, we derive algebraic Bell-CH inequalities with two methods; one is from the rearrangement inequality, and the other is from the linear inequality. In Sec. III, we derive Bell-CH inequalities and obtain a different class of inequalities. In Sec. IV, we test our inequalities with special quantum entangled states. In Sec. V we summarize this work and draw conclusions.

II. ALGEBRAIC BELL-CH INEQUALITIES FOR BINARY SETTINGS

A. Method of rearrangement inequality

The rearrangement inequality is a well-known inequality in classical mathematics, from which one can prove many famous inequalities, such as the arithmetic-mean–geometric-mean inequality, Cauchy inequality, etc. In Appendix A we give a very brief introduction of the rearrangement inequality; readers can see Ref. [30] for a review of this topic.

In this section, we derive two low-order algebraic Bell-CH inequalities by applying the rearrangement inequality: algebraic Bell-CH inequalities I_{2222} and I_{3322}. In order to prove these inequalities, we use the maximum and minimum values of variables to enlarge the upper bound of polynomials in intermediate steps. Then we can rearrange the orders of variables in these polynomials and apply the rearrangement inequality to determine their signs. In this way, we find a profound connection between these inequalities and the classical rearrangement inequality.

1. I_{2222}

We define two sets of m and n positive real numbers

$$a = \{x_i, i = 1, \ldots, m\},$$
$$b = \{y_j, j = 1, \ldots, n\}, \tag{3}$$

with upper bounds A and B, respectively,

$$0 \leqslant x_i \leqslant A, \ 0 \leqslant y_j \leqslant B. \tag{4}$$

The maximum and minimum values of the numbers in set a are denoted $x_+(1, \ldots, m)$ and $x_-(1, \ldots, m)$, respectively, and the maximum and minimum values of the numbers in set b are denoted $y_+(1, \ldots, n)$ and $y_-(1, \ldots, n)$, respectively.

We define I_2 as a polynomial of the following form:

$$I_2 = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2 - x_1 B - A y_1, \tag{5}$$

which involves elements of set a and b in (3) with $m=n=2$. We now prove the I_{2222} inequality, $I_2 \leqslant 0$. Using boundary conditions

$$0 \leqslant x_-(1, 2) \leqslant x_1, x_2 \leqslant x_+(1, 2) \leqslant A,$$
$$0 \leqslant y_-(1, 2) \leqslant y_1, y_2 \leqslant y_+(1, 2) \leqslant B \tag{6}$$

in (4), we obtain

$$I_2 \leqslant x_1(y_1+y_2)+x_2(y_1 - y_2) - x_1 y_+(1, 2) - x_+(1, 2)y_1$$
$$\leqslant x_1(y_1+y_2) + x_2(y_1-y_2) - x_1 y_+(1, 2) - x_+(1, 2)y_1$$
$$-[x_1-x_+(1, 2)] y_-(1, 2)-x_-(1, 2) [y_1-y_+(1, 2)], \tag{7}$$

where we have replaced A and B with $x_+(1, 2)$ and $y_+(1, 2)$ in Eq. (5), respectively, to obtain the first line and added two positive quantities in the second inequality.

Since there are only two variables in each set, we always have

$$x_+(1, 2) + x_-(1, 2) = x_1 + x_2,$$
$$y_+(1, 2) + y_-(1, 2) = y_1 + y_2. \tag{8}$$

Using the above identity, inequality (7) becomes

$$I_2 \leqslant x_1(y_1 + y_2) + x_2(y_1 - y_2) - x_1(y_1 + y_2) - (x_1 + x_2)y_1$$
$$+x_+(1, 2)y_-(1, 2) + x_-(1, 2)y_+(1, 2)$$
$$= -(x_1 y_1 + x_2 y_2) + x_+(1, 2)y_-(1, 2) + x_-(1, 2)y_+(1, 2)$$
$$\equiv I_2^{(0)}. \tag{9}$$

We see in (9) that $I_2^{(0)}$ is the difference between the reversed sum and the unordered sum for sets a and b in (3) with $m = n = 2$. Using the rearrangement inequality, we obtain

$$I_2 \leqslant I_2^{(0)} \leqslant 0. \tag{10}$$

This concludes the proof of the I_{2222} inequality by applying the rearrangement inequality.

We can rewrite $I_2^{(0)}$ with the help of the Heaviside step function,

$$\theta(x) = \begin{cases} 1 & x > 0, \\ \frac{1}{2} & x = 0, \\ 0 & x < 0. \end{cases} \tag{11}$$

The maximum and minimum values can be put into the following form:

$$x_+(1, 2) = x_1\theta(x_1 - x_2) + x_2\theta(-x_1 + x_2),$$
$$x_-(1, 2) = x_1\theta(-x_1 + x_2) + x_2\theta(x_1 - x_2),$$
$$y_+(1, 2) = y_1\theta(y_1 - y_2) + y_2\theta(-y_1 + y_2),$$
$$y_-(1, 2) = y_1\theta(-y_1 + y_2) + y_2\theta(y_1 - y_2). \tag{12}$$

Inserting the above formula into $I_2^{(0)}$ and using the equation

$$x\theta(x) = \frac{x + |x|}{2}, \tag{13}$$

we obtain

$$I_2^{(0)} = -(x_1 - x_2)(y_1 - y_2)[\theta(x_1 - x_2)\theta(y_1 - y_2)$$
$$+ \theta(-x_1 + x_2)\theta(-y_1 + y_2)]$$
$$= -\tfrac{1}{2}[(x_1 - x_2)(y_1 - y_2) + |(x_1 - x_2)(y_1 - y_2)|]. \tag{14}$$

This form will be used in the next section.

2. I_{3322}

Like I_2 in (5), we can define I_3 as

$$I_3 = x_1 y_2 + x_1 y_3 + x_2 y_1 - x_2 y_2 + x_2 y_3 + x_3 y_1 + x_3 y_2 - x_3 y_3$$
$$-(x_1 + x_2)B - A(y_1 + y_2), \tag{15}$$

which involves elements of sets a and b in (3) with $m = n = 3$. We now prove the I_{3322} inequality, $I_3 \leqslant 0$. We use boundary conditions (4), or, explicitly,

$$0 \leqslant x_-(1,2,3) \leqslant x_1, x_2, x_3 \leqslant x_+(1,2,3) \leqslant A,$$
$$0 \leqslant y_-(1,2,3) \leqslant y_1, y_2, y_3 \leqslant y_+(1,2,3) \leqslant B, \qquad (16)$$

$$-x_2 y_2 \leqslant -x_2 y_-(1,2,3) - x_-(1,2,3)y_2 + x_-(1,2,3)y_-(1,2,3),$$
$$-x_3 y_3 \leqslant -x_3 y_+(1,2,3) - x_+(1,2,3)y_3 + x_+(1,2,3)y_+(1,2,3), \qquad (18)$$

we can enlarge the right-hand side of (17) to obtain

$$I_3 \leqslant x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2) - x_2 y_-(1,2,3) - x_-(1,2,3)y_2 + x_-(1,2,3)y_-(1,2,3)$$
$$-x_3 y_+(1,2,3) - x_+(1,2,3)y_3 + x_+(1,2,3)y_+(1,2,3) - (x_1 + x_2)y_+(1,2,3) - x_+(1,2,3)(y_1 + y_2). \qquad (19)$$

The inequalities in (18) are true because they can be put into the following form:

$$-[x_2 - x_-(1,2,3)][y_2 - y_-(1,2,3)] \leqslant 0,$$
$$-[x_3 - x_+(1,2,3)][y_3 - y_+(1,2,3)] \leqslant 0. \qquad (20)$$

After adding additional positive terms to enlarge the right-hand side of (19), we can derive

$$I_3 \leqslant x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2) - (x_1 + x_2 + x_3)y_+(1,2,3) - x_+(1,2,3)(y_1 + y_2 + y_3)$$
$$-[x_1 - x_+(1,2,3)]y_-(1,2,3) - x_2 y_-(1,2,3) - [x_3 - x_+(1,2,3)]y_-(1,2,3)$$
$$-x_-(1,2,3)[y_1 - y_+(1,2,3)] - x_-(1,2,3)y_2 - x_-(1,2,3)[y_3 - y_+(1,2,3)]$$
$$+x_-(1,2,3)y_-(1,2,3) + x_+(1,2,3)y_+(1,2,3)$$
$$= -(x_1 y_1 + x_2 y_2 + x_3 y_3) + x_+(1,2,3)y_-(1,2,3) + x_-(1,2,3)y_+(1,2,3)$$
$$+[x_1 + x_2 + x_3 - x_+(1,2,3) - x_-(1,2,3)][y_1 + y_2 + y_3 - y_+(1,2,3) - y_-(1,2,3)]. \qquad (21)$$

Finally, we define $I_3^{(0)}$ as the last polynomial of (21) for later use,

$$I_3^{(0)} = -(x_1 y_1 + x_2 y_2 + x_3 y_3) + x_+(1,2,3)y_-(1,2,3) + x_-(1,2,3)y_+(1,2,3) + x_r y_r, \qquad (22)$$

where x_r and y_r denote the middle terms in sets a and b as

$$x_r = x_1 + x_2 + x_3 - x_+(1,2,3) - x_-(1,2,3), \quad y_r = y_1 + y_2 + y_3 - y_+(1,2,3) - y_-(1,2,3). \qquad (23)$$

It is obvious that $I_3^{(0)}$ in Eq. (22) is the difference between the reversed sum and the unordered sum for sets a and b in (3) with $m = n = 3$. From the rearrangement inequality, we have

$$I_3 \leqslant I_3^{(0)} \leqslant 0. \qquad (24)$$

This concludes the proof of the I_{3322} inequality using the method of the rearrangement inequality.

B. Method of linear inequality

In Sec. II A, we derived two algebraic inequalities in connection with low-order Bell-CH and rearrangement inequalities. But it is not easy to generalize this method to higher-order cases. In Ref. [16], the authors obtained a specific type of Bell-CH inequalities and found the relationship between lower-order and higher-order inequalities of this type. Following Ref. [16], there has been a lot of discussion on the perspective of mathematics along this line [17,18,31,32]. Inspired by these works, we find a general method which we call the method of linear inequality to prove higher-order inequalities I_{mn22} from the lowest-order

and replace A and B in (15) with $x_+(1,2,3)$ and $y_+(1,2,3)$, respectively, to obtain

$$I_3 \leqslant x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2) - x_2 y_2 - x_3 y_3$$
$$-(x_1 + x_2)y_+(1,2,3) - x_+(1,2,3)(y_1 + y_2). \qquad (17)$$

Using the inequalities

one systematically. As a by-product, this method can be used to construct higher-order inequalities more effectively than the traditional method. We will prove a lemma using the property of monotonicity. Then we will apply the lemma to linear functions and obtain a theorem. Finally, we will use this method to prove a large variety of inequalities that have been obtained.

We consider two sets a and b defined by Eq. (3) with conditions (4). Starting from positivity inequalities

$$-x_i y_j \leqslant 0,$$
$$x_i(y_j - B) \leqslant 0, \qquad (25)$$
$$(x_i - A)y_j \leqslant 0,$$

we present a theorem for a specific type of algebraic Bell-CH inequalities.

Theorem 1. $I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k) \leqslant 0$ is an algebraic Bell-CH inequality for $k \geqslant 2$ with variables x_i and y_j in sets a and b, respectively, where $I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k)$ is defined

as

$$I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k) = \sum_{j=1}^{k} \sum_{i=1}^{k+1-j} x_i y_j - \sum_{i=2}^{k} x_i y_{k+2-i} - \sum_{i=1}^{k-1} (k-i) x_i B - A y_1. \tag{26}$$

Proof. Let $f(x_i, \{c_j^y\}) = x_i(\sum_j c_j^y y_j)$ and $g(y_i, \{c_j^x\}) = (\sum_j c_j^x x_j) y_i$, where x_i and y_j are variables in sets a and b and c_j^x and c_j^y are coefficients. Then we have the following linear inequalities:

$$f(x_i, \{c_j^y\}) \leqslant \theta\left(\sum_j c_j^y y_j\right) f(A, \{c_j^y\}) + \theta\left(-\sum_j c_j^y y_j\right) f(0, \{c_j^y\}),$$

$$g(y_i, \{c_j^x\}) \leqslant \theta\left(\sum_j c_j^x x_j\right) g(B, \{c_j^x\}) + \theta\left(-\sum_j c_j^x x_j\right) g(0, \{c_j^x\}). \tag{27}$$

We can prove $I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k) \leqslant 0$ with the help of (27) by the induction method. First, we should prove the case for $k = 2$, which is just the algebraic version of the original Bell-CH inequality. We have

$$I_{22}(x_1, x_2 | y_1, y_2) = (x_1 + x_2) y_1 + (x_1 - x_2) y_2 - x_1 B - A y_1$$

$$\leqslant \theta(x_1 - x_2) I_{22}(x_1, x_2 | y_1, B) + \theta(-x_1 + x_2) I_{22}(x_1, x_2 | y_1, 0)$$

$$= \theta(x_1 - x_2)[(x_1 + x_2) y_1 - x_2 B - A y_1] + \theta(-x_1 + x_2)[(x_1 + x_2) y_1 - x_1 B - A y_1], \tag{28}$$

where we have used (27) for $g(y_2, \{c_j^x\}) = (x_1 - x_2) y_2$ with $\{c_j^x\} = \{1, -1\}$. Then we can rewrite the right-hand side of the above inequality and obtain

$$I_{22}(x_1, x_2 | y_1, y_2) \leqslant \theta(x_1 - x_2)[(x_1 - A) y_1 + x_2(y_1 - B)] + \theta(-x_1 + x_2)[(x_2 - A) y_1 + x_1(y_1 - B)]$$

$$\leqslant 0. \tag{29}$$

Thus, the inequality holds for $k = 2$.

Then we assume the inequality holds for $k - 1$ with

$$I_{k-1,k-1}(x_1, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) \leqslant 0, \quad I_{k-1,k-1}(x_2, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) \leqslant 0. \tag{30}$$

According to the induction method, the inequality should hold for $I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k)$ in (26). We now apply (27) to I_{kk} for $g(y_k, \{c_j^x\}) = (x_1 - x_2) y_k$, with $\{c_j^x, j = 1, 2, \ldots, k\} = \{1, -1, 0, \ldots, 0\}$, as

$$I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k) \leqslant \theta(x_1 - x_2) I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_{k-1}, B) + \theta(-x_1 + x_2) I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_{k-1}, 0)$$

$$= \theta(x_1 - x_2)\left[\left(\sum_{i=1}^{k} x_i\right) y_1 + \left(\sum_{i=1}^{k-1} x_i - x_k\right) y_2 + \cdots + (x_1 + x_2 - x_3) y_{k-1} + (x_1 - x_2) B\right.$$

$$\left. - \sum_{i=1}^{k-1} (k-i) x_i B - A y_1\right] + \theta(-x_1 + x_2)\left[\left(\sum_{i=1}^{k} x_i\right) y_1 + \left(\sum_{i=1}^{k-1} x_i - x_k\right) y_2 + \cdots\right.$$

$$\left. + (x_1 + x_2 - x_3) y_{k-1} - \sum_{i=1}^{k-1} (k-i) x_i B - A y_1\right]. \tag{31}$$

We can then rewrite the right-hand side of the above inequality and obtain

$$I_{kk}(x_1, \ldots, x_k | y_1, \ldots, y_k) \leqslant \theta(x_1 - x_2)\left\{I_{k-1,k-1}(x_1, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) + \sum_{i=1}^{k-1}[x_2(y_i - B)]\right\}$$

$$+ \theta(-x_1 + x_2)\left\{I_{k-1,k-1}(x_2, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) + \sum_{i=1}^{k-1}[x_1(y_i - B)]\right\}$$

$$\leqslant I_{kk}^{(0)}(x_1, \ldots, x_k | y_1, \ldots, y_k)$$

$$\leqslant 0. \tag{32}$$

Here, $I_{kk}^{(0)}$ is given by

$$I_{kk}^{(0)}(x_1, \ldots, x_k | y_1, \ldots, y_k) = \max\left\{I_{k-1,k-1}(x_1, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) + \sum_{i=1}^{k-1}[x_2(y_i - B)],\right.$$

$$\left. I_{k-1,k-1}(x_2, x_3, \ldots, x_k | y_1, \ldots, y_{k-1}) + \sum_{i=1}^{k-1}[x_1(y_i - B)]\right\}. \tag{33}$$

So we prove in (32) that the inequality holds for k. This concludes the proof of the theorem.

The same method can also be applied to prove general inequalities I_{mn22} recursively. For example, we can prove I_3 in (15) as the symmetric version of $I_{33}(x_1, x_2, x_3|y_1, y_2, y_3)$: I_3 can be obtained from $I_{33}(x_1, x_2, x_3|y_1, y_2, y_3)$ by the transformation $x_1 \to A - x_1$, $y_2 \to B - y_2, y_3 \to B - y_3$ and then relabeling indices of $\{x_i\}$ and $\{y_j\}$ by interchanging $1 \leftrightarrow 3$. Hence, $I_3 \leqslant 0$ can be proved by using lower-order inequalities with the help of (27).

As another example, we can reconstruct and prove I_{5322} by applying linear inequalities. We use the polynomial $I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3)$ to represent I_{5322} as

$$
\begin{aligned}
I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3) \equiv & \, x_1y_1 - x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_1 + x_3y_2 + x_4y_1 - x_4y_3 \\
& -x_5y_1 + x_5y_2 - x_5y_3 - (x_1 + x_2 + x_3)B - A(y_1 + y_2) \\
\leqslant & \, 0.
\end{aligned}
\tag{34}
$$

To prove inequality (34), we can apply linear inequalities (27) for $f(x_4, \{c_j^y\}) = x_4(y_1 - y_3)$, with $\{c_j^y, j = 1, 2, 3\} = \{1, 0, -1\}$, and obtain

$$
\begin{aligned}
I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3) \leqslant & \, \theta(y_1 - y_3)I_{53}(x_1, x_2, x_3, A, x_5|y_1, y_2, y_3) + \theta(-y_1 + y_3)I_{53}(x_1, x_2, x_3, 0, x_5|y_1, y_2, y_3) \\
= & \, \theta(y_1 - y_3)[I_{22}(x_2, x_1|y_3, y_2) + I_{22}(x_3, x_5|y_2, y_1) + x_1(y_1 - B) - x_5y_3] \\
& +\theta(-y_1 + y_3)[I_{22}(x_2, x_5|y_2, y_3) + I_{22}(x_3, x_1|y_1, y_2) + x_1(y_3 - B) - x_5y_1] \\
\leqslant & \, 0,
\end{aligned}
\tag{35}
$$

where we have used inequality (29) for $I_{22}(x_2, x_1|y_3, y_2)$, $I_{22}(x_3, x_5|y_2, y_1)$, $I_{22}(x_2, x_5|y_2, y_3)$, and $I_{22}(x_3, x_1|y_1, y_2)$. It was shown in Refs. [33,34] that there is only one I_{5322}, and the explicit form was first found by [26]. Here, we give the proof of the corresponding algebraic one using the linear inequality method.

In the Supplemental Material [35], we summarize proofs of 257 algebraic Bell-CH inequalities using the linear inequality method. In these proofs, we can show that all inequalities can be reduced to second- and third-order ones.

III. BELL-CH INEQUALITIES FOR BINARY SETTINGS

According to "objective local theories" introduced by Clauser and Horne [5], any physical system, either a classical or quantum-mechanical one, can be considered as a state. The state is labeled by some local hidden variables λ without any other assumptions. For example, in a bipartite correlated system, $P(x, \lambda)$ describes the probability density of some certain measurement outcome x in one subsystem, and $P(x, y, \lambda)$ is the correlation probability density of measurement outcomes x and y in two subsystems.

From the definition of the state and local hidden variables, we can derive Bell-CH inequalities from algebraic ones. In a theory of local hidden variables, the physically detectable probability density $P(x)$ is related to a hidden variable λ, which is assumed to be drawn from the probability distribution $\rho(\lambda) \in [0, 1]$ as

$$
P(x) = \int P(x, \lambda)\rho(\lambda)d\lambda.
\tag{36}
$$

Likewise, the joint probability density $P(x, y)$ is obtained with

$$
P(x, y) = \int P(x, \lambda)P(y, \lambda)\rho(\lambda)d\lambda.
\tag{37}
$$

Let us take the derivation of the CH inequality as an example. From the I_{2222} inequality, replacing x_i/A and y_i/B with $P(x_i, \lambda)$ and $P(y_i, \lambda)$ for $i = 1, 2$, respectively, we obtain

$$
\begin{aligned}
& P(x_1, \lambda)[P(y_1, \lambda) + P(y_2, \lambda)] + P(x_2, \lambda)[P(y_1, \lambda) - P(y_2, \lambda)] \\
& - P(x_1, \lambda) - P(y_1, \lambda) \leqslant 0.
\end{aligned}
\tag{38}
$$

Multiplying the above inequality by $\rho(\lambda)$ and taking an integration over λ, we obtain the CH inequality

$$
I_{2,\text{CH}} \leqslant 0,
\tag{39}
$$

where $I_{2,\text{CH}}$ is defined as

$$
\begin{aligned}
I_{2,\text{CH}} = & \, P(x_1, y_1) + P(x_1, y_2) + P(x_2, y_1) - P(x_2, y_2) \\
& - P(x_1) - P(y_1).
\end{aligned}
\tag{40}
$$

In correspondence to (10), we obtain the inequality

$$
I_{2,\text{CH}} \leqslant I_{2,\text{CH}}^{(0)} \leqslant 0,
\tag{41}
$$

where $I_{2,\text{CH}}^{(0)}$ is defined as

$$
\begin{aligned}
I_{2,\text{CH}}^{(0)} = & -\frac{1}{2} \int \{[P(x_1, \lambda) - P(x_2, \lambda)][P(y_1, \lambda) - P(y_2, \lambda)] \\
& +|[P(x_1, \lambda) - P(x_2, \lambda)][P(y_1, \lambda) - P(y_2, \lambda)]|\}\rho(\lambda)d\lambda.
\end{aligned}
\tag{42}
$$

Here, we use Eq. (14) and make the replacements $x_i/A \to P(x_i, \lambda)$ and $y_i/B \to P(y_i, \lambda)$ for $i = 1, 2$, respectively, multiplied by $\rho(\lambda)$ and integrated over λ. We apply Jensen's inequality in Appendix B assuming the convex function $\varphi(x) = |x|$, which leads to an upper bound for $I_{2,\text{CH}}^{(0)}$ and then for $I_{2,\text{CH}}$,

$$
I_{2,\text{CH}} \leqslant I_{2,\text{CH}}^{(0)} \leqslant -\tfrac{1}{2}\{[P(x_1, y_1) - P(x_1, y_2) - P(x_2, y_1) + P(x_2, y_2)] + |P(x_1, y_1) - P(x_1, y_2) - P(x_2, y_1) + P(x_2, y_2)|\},
\tag{43}
$$

where we have employed inequality (41). We can easily see that the upper bound of $I_{2,\text{CH}}^{(0)}$ is less than or equal to zero.

Another approach to our class of inequalities is through the method of linear inequalities in deriving algebraic inequalities I_{mn22}. We take a type of Bell-CH inequality as an example. In correspondence to inequality (32), we obtain the inequality

$$I_{kk;Q} \leqslant I_{kk;Q}^{(0)} \leqslant 0, \tag{44}$$

in which we have defined

$$I_{kk;Q} \equiv (AB)^{-1} \int d\lambda \rho(\lambda) I_{kk}[AP(x_1, \lambda), \ldots, AP(x_k, \lambda)|BP(y_1, \lambda), \ldots, BP(y_k, \lambda)],$$

$$I_{kk;Q}^{(0)} \equiv (AB)^{-1} \int d\lambda \rho(\lambda) I_{kk}^{(0)}[AP(x_1, \lambda), \ldots, AP(x_k, \lambda)|BP(y_1, \lambda), \ldots, BP(y_k, \lambda)]. \tag{45}$$

Here, we have made the replacements in (32) $x_i/A \to P(x_i, \lambda)$ and $y_i/B \to P(y_i, \lambda)$ for $i = 1, \ldots, k$, respectively, multiplied by $\rho(\lambda)$ and integrated over λ. The inequality $I_{kk,Q}^{(0)} \leqslant 0$ leads to the following alternative inequality:

$$\max\left\{ I_{k-1,k-1;Q}^{(1)} + \sum_{i=1}^{k-1} [P(x_2, y_i) - P(x_2)], I_{k-1,k-1;Q}^{(2)} + \sum_{i=1}^{k-1} [P(x_1, y_i) - P(x_1)] \right\} \leqslant 0, \tag{46}$$

where $I_{k-1,k-1;Q}^{(1)}$ and $I_{k-1,k-1;Q}^{(2)}$ are defined as

$$I_{k-1,k-1;Q}^{(1)} \equiv \sum_{j=1}^{k-1} \sum_{i=1,i\neq2}^{k-j} P(x_i, y_j) - \sum_{i=3}^{k-1} P(x_i, y_{k+1-i}) - \sum_{i=1,i\neq2}^{k-2} (k-1-i)P(x_i) - P(y_1),$$

$$I_{k-1,k-1;Q}^{(2)} \equiv \sum_{j=1}^{k-1} \sum_{i=2}^{k-j} P(x_i, y_j) - \sum_{i=3}^{k-1} P(x_i, y_{k+1-i}) - \sum_{i=2}^{k-2} (k-1-i)P(x_i) - P(y_1). \tag{47}$$

Note that this inequality can be tested in physical systems.

Applying this method to $Imn22$, we can obtain a set of Bell-CH inequalities which is summarized in the Supplemental Material [35]. In the next section, we will test our inequalities with special quantum entangled states.

IV. TESTING OUR BELL-CH INEQUALITIES WITH QUANTUM ENTANGLED STATES

To test our class of Bell-CH inequalities we consider a simple quantum system with two qubits. The entangled states of two qubits we will use in the test are parameterized as

$$|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle. \tag{48}$$

For such quantum states, the correlation probabilities can be obtained from expectation values of operators acting on the Hilbert space,

$$P(x_i) = \langle\psi(\theta)|x_i \otimes I_y|\psi(\theta)\rangle,$$
$$P(y_j) = \langle\psi(\theta)|I_x \otimes y_j|\psi(\theta)\rangle, \tag{49}$$
$$P(x_i, y_j) = \langle\psi(\theta)|x_i \otimes y_j|\psi(\theta)\rangle,$$

with x_i (y_j) being the projectors measured by Alice (Bob) on x (y) and I_x (I_y) being the unit operators acting on x (y).

For entangled states (48), we define Q, Q_a, and Q_b as the maximum violations of $Imn22$ and the corresponding Bell-CH inequalities $Imn22^a$ and $Imn22^b$ presented in the Supplemental Material [35], respectively. For the test, we have calculated the maximum violations of our inequalities for all $Imn22$ listed in [35]. For states with the maximum violation characterized by θ_{max}, we have also calculated the resistance

to noise λ_{max} defined through the mixture state

$$\rho = \lambda|\psi(\theta_{max})\rangle\langle\psi(\theta_{max})| + (1-\lambda)\frac{I}{4}, \tag{50}$$

so that it does not violate the inequality marginally, where $1 - \lambda$ is the parameter for white noise. In Table I, we give some selected results of 32 $Imn22$ and our corresponding Bell-CH inequalities from the complete table in the Supplemental Material [35].

For all $Imn22$, we find that our Bell-CH inequalities have positive violation by entangled states (48). According to the preceding section, our inequalities can be decomposed into groups of inequalities formed by combinations of some original low-order $Imn22$; the nonlocality implied by the violation of high-order $Imn22$ can be replaced by that of our inequalities. Moreover, the resistance to noise for some of these inequalities is lower than that of the original ones; hence, our inequalities could be better candidates for testing nonlocality in some physical systems.

V. SUMMARY AND CONCLUSIONS

In this work effective methods for the rearrangement and linear inequality were employed to prove the Bell-CH inequalities. Alternative types of Bell-CH inequalities were found to be violated by entangled states. The main results are summarized as follows. First, a large variety of $Imn22$ ($m, n \leqslant 5$) can be easily derived through the rearrangement inequality and the linear inequality. Second, along with $I2222$ or the CH inequality, we gave an inequality in (43) using the rearrangement inequality method. Third, all original $Imn22$ can be replaced by the maximum of lower-order $Imn22$ combinations using

TABLE I. Maximum violations Q, Q_a, and Q_b; related parameters θ_{max}/π, θ_{max}^a/π, and θ_{max}^b/π; and resistances to noise λ_{max}, λ_{max}^a, and λ_{max}^b of 32 $Imn22$ (here we use the original names in the literature to represent them) and corresponding Bell-CH inequalities $Imn22^a$ and $Imn22^b$.

Name	Q	θ_{max}/π	λ_{max}	Q_a	θ_{max}^a/π	λ_{max}^a	Q_b	θ_{max}^b/π	λ_{max}^b
I_{3322}	0.25	0.25	0.8	0.2071	0.25	0.7836			
I_{4422}^4	0.056	0.1316	0.9728	0.2361	0.2332	0.864	0.4142	0.25	0.7071
I_{4422}^{14}	0.4103	0.238	0.8298	0.4282	0.2377	0.8034	0.3793	0.2304	0.7981
I_{4422}^{16}	0.2407	0.219	0.8791	0.226	0.2362	0.8691	0.2071	0.25	0.8579
I_{4422}^{18}	0.1812	0.168	0.9508	0.2983	0.2195	0.9096	0.5436	0.2278	0.7863
J_{4422}^3	0.7249	0.2448	0.838	0.7337	0.2419	0.8267			
J_{4422}^4	0.6862	0.2452	0.8138	0.6579	0.2426	0.7917			
J_{4422}^5	0.5007	0.2404	0.8749	0.4434	0.2303	0.8494			
J_{4422}^7	0.3642	0.2304	0.8917	0.6057	0.2376	0.7675			
J_{4422}^8	0.2657	0.19	0.9186	0.2772	0.2057	0.9002			
J_{4422}^{12}	0.4198	0.25	0.9147	0.5797	0.2407	0.8381			
J_{4422}^{13}	0.5629	0.2422	0.8766	0.5541	0.2213	0.8633	0.636	0.2421	0.8251
J_{4422}^{15}	0.6133	0.2433	0.8412	0.5668	0.2416	0.8411			
J_{4422}^{21}	0.5441	0.2397	0.8655	0.5717	0.2255	0.8279			
J_{4422}^{30}	0.2459	0.1832	0.9242	0.4201	0.2398	0.8264			
J_{4422}^{31}	0.2133	0.1847	0.9336	0.3796	0.2432	0.8556			
J_{4422}^{34}	0.4075	0.2377	0.8804	0.3925	0.2318	0.8515			
J_{4422}^{42}	0.6012	0.2362	0.8331	0.6722	0.2462	0.7881			
J_{4422}^{51}	0.6678	0.2419	0.8046	0.675	0.2466	0.7874			
J_{4422}^{88}	0.616	0.2468	0.7851	0.5956	0.2446	0.7705			
J_{4422}^{113}	0.8196	0.2399	0.83	0.8484	0.2448	0.793			
N_{4422}^6	0.5972	0.2496	0.8543	0.5695	0.2426	0.8404			
N_{4422}^9	0.7399	0.2445	0.8352	0.8259	0.2404	0.7841			
N_{4422}^{10}	0.5	0.25	0.8333	0.4331	0.2449	0.822			
A_{10}	0.4154	0.229	0.8082	0.3944	0.2388	0.7918			
A_{11}	0.4561	0.2379	0.7933	0.3944	0.2388	0.7918			
A_{13}	0.4031	0.2375	0.8128	0.4142	0.25	0.7836	0.4142	0.25	0.7836
A_{16}	0.416	0.2402	0.8278	0.4353	0.2447	0.7751	0.3944	0.2388	0.7601
A_{34}	0.514	0.2461	0.7956	0.535	0.2476	0.7659			
A_{69}	0.3304	0.2245	0.8833	0.4459	0.2393	0.8177	0.5148	0.25	0.7727
A_{83}	0.6962	0.2438	0.798	0.62	0.2424	0.784			
A_{88}	0.0768	0.1575	0.9702	0.197	0.2356	0.9103	0.25	0.25	0.8571

the linear inequality method, which can be violated by some entangled states in Q_a and Q_b.

This work can help us understand the mathematical structures of Bell-CH inequalities. A number of interesting topics are open for future studies. One could investigate a set of Bell-CH inequalities for multipartite systems using the method of the linear inequality. Appropriate ways might be found to test our Bell-CH inequalities, especially inequality (43), in systems of optics, high-energy physics, or condensed matter.

APPENDIX A: REARRANGEMENT INEQUALITY

The rearrangement inequality (or the permutation inequality) is

$$x_n y_1 + \cdots + x_1 y_n \leqslant x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n$$
$$\leqslant x_1 y_1 + \cdots + x_n y_n \tag{A1}$$

for every permutation of $\sigma(i)$ ($i = 1, \ldots, n$), with n real numbers $x_1, \ldots, x_n$ which satisfy

$$x_1 \leqslant \cdots \leqslant x_n \tag{A2}$$

and n real numbers $y_1, \ldots, y_n$ which satisfy

$$y_1 \leqslant \cdots \leqslant y_n. \tag{A3}$$

If the numbers are different, that is to say, $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, then the lower bound is obtained only for the permutation which reverses the order, i.e., $\sigma(i) = n - i + 1$ for all $i = 1, \ldots, n$, and the upper bound is obtained only for the identity permutation, i.e., $\sigma(i) = i$ for all $i = 1, \ldots, n$.

APPENDIX B: JENSEN'S INEQUALITY

Suppose $f(x)$ is a non-negative measurable function satisfying

$$\int_{-\infty}^{\infty} f(x)dx = 1, \tag{B1}$$

which is a probability density function in the probabilistic view. Jensen's inequality about convex integrals is

$$\varphi\left(\int_{-\infty}^{\infty} g(x)f(x)dx\right) \leqslant \int_{-\infty}^{\infty} \varphi(g(x))f(x)dx \tag{B2}$$

for any real-valued measurable function $g(x)$ and φ is convex over $g(x)$. If $g(x) = x$, then Jensen's inequality reduces to

$$\varphi\left(\int_{-\infty}^{\infty} xf(x)dx\right) \leqslant \int_{-\infty}^{\infty} \varphi(x)f(x)dx. \tag{B3}$$

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[35] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.103.062203 for the list of each inequality and our corresponding inequalities and a table of complete numerical results.