Analytic approach to dynamics of the resonant and off-resonant Jaynes-Cummings systems with cavity losses

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(Received 15 March 2021; accepted 10 May 2021; published 28 May 2021)

An analytic approach to investigate the zero-temperature time evolution of the Jaynes-Cummings system with cavity losses is developed. With realistic coupling between the cavity and the environment assumed, a simple master equation is derived, leading to the explicit analytic solution for the resonant case. This solution is suitable for analyses not only on single-excitation states but also on many-excitation states, which enables us to investigate the photon coherent state and to observe sharp collapses and revivals under dissipation. For the off-resonant case, on the other hand, the present study presents an analytic, systematic method instead. We examine the small and large detuning limits and discuss the condition where the widely used phenomenological treatment is justified. Explicit evaluations of the time evolutions for various initial states with finite detuning are also presented.

DOI: 10.1103/PhysRevA.103.053713

I. INTRODUCTION

The Jaynes-Cummings (JC) model [1] is one of the simplest models for matters interacting with a quantized mode of the electromagnetic field and contains fertile physics such as spontaneous emission, Rabi oscillation, and collapses and revivals of the atomic-state probabilities [1–4]. These phenomena have been observed in experiments in optical cavities [5–8]. The JC model is now being applied to quantum informatics as a way of realizing the controlled-NOT gate, which plays an indispensable role in this field [9–12].

From the viewpoint of experiments in cavities, the noises due to the interaction between the system that we focus on and the environment, such as photon losses, are inevitable and sometimes suppress expected quantum phenomena [13–16]. To discuss the effects of the noises, many attempts have been made to formulate the JC system interacting with the environment, especially with the GKSL-type master equation technique [17,18].

In this context, the JC system with cavity losses has been investigated theoretically in analytical and numerical ways over the years [19–24], most of which have focused on collapses and revivals. These publications have typically treated the master equation of the form

$$\dot{\rho} = -i[H_{\mathrm{JC}}, \rho] + \gamma\left(a\rho a^{\dagger} - \tfrac{1}{2}a^{\dagger}a\rho - \tfrac{1}{2}\rho a^{\dagger}a\right), \qquad (1.1)$$

where ρ is the density matrix for the JC system that we focus on, H_{JC} is the JC Hamiltonian, $a^{\dagger}$ (a) is the field creation (annihilation) operator, and γ is the damping rate independent of the energy levels. Since the JC system describes a Rydberg atom interacting with a resonant cavity of a high-quality factor and the photons are lost due to the imperfection of the cavity, this master equation is apparently correct. However, this master equation is actually derived with an *ad hoc* approximation, and the microscopic process of the cavity losses is obscure. The condition where this phenomenological master equation is justified is also unclear. Against this background, Scala *et al.* [25,26] assumed a realistic microscopic interaction between the JC system and the environment, derived a master equation for the resonant case using the standard technique [27], and analyzed the derived microscopic master equation. While the microscopic master equation gives a precise description, the expression is complicated and not easy to handle. In particular, it is quite difficult to treat many-photon initial conditions analytically. Although the technique used in the derivation is also applicable to the off-resonant case, the resulting master equation will be much more complicated.

In this paper, we present an alternative analytic method to discuss the JC system with cavity losses on the basis of the microscopic treatment. We derive a simpler master equation by explicitly using the analytic solution to the JC model. In this formalism, at zero temperature, we find the analytic solutions for both the resonant and the off-resonant cases under the approximation usually used. The obtained solutions are suitable for analyses not only of single-excitation cases, which were examined by Scala *et al.* [25], but also of multiexcitation cases and the coherent state. In particular, for the resonant case, we can write down the time evolution for general initial states explicitly. Utilizing these solutions, we discuss some specific cases. Furthermore, using the formula for finite-detuning cases, we examine the large and small detuning limits and discuss the condition where the phenomenological treatment, Eq. (1.1), is justified.

This paper is organized as follows. In Sec. II, we review the properties of the pure JC model and rederive its analytic solution. In Sec. III, using the obtained expressions, we derive

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a master equation with the standard method. In Sec. IV, we derive an analytic solution to the derived master equation for the resonant case, i.e., when the detuning Δ is 0, and discuss the behavior of the initially multiexcitation states including the coherent state. In Sec. V, we derive an analytic solution for finite Δ, discuss the large- and small-Δ limits, and evaluate the time evolution for some specific cases. Section VI is devoted to discussion and a summary.

II. THE JAYNES-CUMMINGS MODEL AND ITS ANALYTIC SOLUTION

A. Energy eigenvalues and energy eigenstates

We consider a two-level atom with energy separation $\hbar\omega_0$ and a quantized single mode of the electromagnetic field with frequency ω. We denote the ground state and the excited state of the atom $|g\rangle$ and $|e\rangle$, respectively. We introduce an interaction between the atom and the quantized mode. In the rotating wave approximation, the system is described by the JC Hamiltonian

$$H_{\rm JC} = \tfrac{1}{2}\omega_0\sigma_z + \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \lambda(a\sigma_+ + a^\dagger\sigma_-) \quad (\hbar = 1), \tag{2.1}$$

where $a^\dagger$ (a) creates (annihilates) a quantized mode of energy ω and $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$ and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ are the operators for the atom. The magnitude of the interaction is represented by λ. Using the relation $\sigma_z = 2\sigma_+\sigma_- - 1$, we can rewrite the Hamiltonian as [28]

$$H_{\rm JC} = \omega N + C, \tag{2.2}$$

where $N = a^\dagger a + \sigma_+\sigma_-$ is the total excitation and $C = -\Delta\sigma_z + \lambda(\sigma_+ a + \sigma_- a^\dagger)$ is the remaining part of $H_{\rm JC}$. The detuning $\Delta = \frac{1}{2}(\omega - \omega_0)$ represents the deviation of the photon energy from the energy separation of the atom. Making use of the fact that the operators N and C commute, we obtain the eigenstates and eigenvalues

$$|E_0\rangle = |g, 0\rangle \quad \text{for} \quad E_0 = \Delta, \tag{2.3}$$

$$|E_{n+}\rangle = \cos\theta_n|g, n\rangle + \sin\theta_n|e, n-1\rangle \quad \text{for} \quad E_{n+} = n\omega + \varepsilon_n, \tag{2.4}$$

$$|E_{n-}\rangle = \sin\theta_n|g, n\rangle - \cos\theta_n|e, n-1\rangle \quad \text{for} \quad E_{n-} = n\omega - \varepsilon_n, \tag{2.5}$$

with

$$\varepsilon_n = \sqrt{\Delta^2 + n\lambda^2}, \tag{2.6}$$

$$\cos\theta_n = \frac{\lambda}{|\lambda|}\sqrt{\frac{\varepsilon_n + \Delta}{2\varepsilon_n}}, \quad \sin\theta_n = \sqrt{\frac{\varepsilon_n - \Delta}{2\varepsilon_n}}. \tag{2.7}$$

Note that C is also diagonal with respect to the energy eigenstates, and the corresponding eigenvalues are given by

$$C|E_0\rangle = \Delta|E_0\rangle, \tag{2.8}$$

$$C|E_{\pm n}\rangle = \pm\varepsilon_n|E_{\pm n}\rangle. \tag{2.9}$$

B. Time evolution in the Heisenberg picture

We introduce operators in the Heisenberg picture,

$$\begin{aligned} a(t) &= e^{iH_{\rm JC}t} a e^{-iH_{\rm JC}t} \\ &= e^{iCt} e^{i\omega Nt} a e^{-i\omega Nt} e^{-iCt} \\ &= e^{-i\omega t} e^{iCt} a e^{-iCt}, \end{aligned} \tag{2.10}$$

where we have used the fact that N and C commute in the second equality. We can put time dependence only on the left or right of the operator a, which makes its handling easy in the later procedure. Rewriting the operator e^{iCt} by Euler's formula, we obtain

$$a(t) = e^{-i\omega t}\cos(\sqrt{C^2}t) a e^{-iCt} + e^{-i\omega t}\frac{C}{\sqrt{C^2}}\sin(\sqrt{C^2}t) a e^{-iCt}, \tag{2.11}$$

where the sign of C is expressed as $C/\sqrt{C^2}$. On the other hand, we find the relation

$$f(C^2)a = a f(C^2 - \lambda^2 I), \tag{2.12}$$

where $f(x)$ and I are a function of x and the identity operator, respectively. In the following, the identity operator I will be suppressed for simplicity. Using this relation, we can put the time-dependent operators on the right of a as

$$\begin{aligned} a(t) = {} & e^{-i\omega t} a \cos(\sqrt{C^2 - \lambda^2}t) e^{-iCt} \\ & + i e^{-i\omega t}\frac{C}{\sqrt{C^2}} a \sin(\sqrt{C^2 - \lambda^2}t) e^{-iCt}. \end{aligned} \tag{2.13}$$

Finally, we use Euler's formula again, which yields

$$a(t) = P_+ a e^{i(-\omega + \sqrt{C^2 - \lambda^2} - C)t} + P_- a e^{i(-\omega - \sqrt{C^2 - \lambda^2} - C)t}, \tag{2.14}$$

where

$$P_\pm = \frac{1}{2}\left(1 \pm \frac{C}{\sqrt{C^2}}\right) \tag{2.15}$$

are projection operators which project states depending on the positive or negative eigenvalues of C, i.e.,

$$P_\pm|E_{n\pm}\rangle = |E_{n\pm}\rangle, \tag{2.16}$$

$$P_\mp|E_{n\pm}\rangle = 0 \tag{2.17}$$

for $n \geqslant 1$ and

$$P_+|E_0\rangle = \begin{cases} |E_0\rangle & (\Delta \geqslant 0), \\ 0 & (\Delta < 0), \end{cases} \tag{2.18}$$

$$P_-|E_0\rangle = \begin{cases} 0 & (\Delta \geqslant 0), \\ |E_0\rangle & (\Delta < 0) \end{cases} \tag{2.19}$$

for the ground state. Note that other definitions of $P_\pm$ on $|E_0\rangle$ are also possible when $\Delta = 0$. Similarly, we can apply Euler's formula to e^{-iCt} using Eq. (2.12). Then we obtain another expression:

$$a(t) = e^{i(-\omega + C - \sqrt{C^2 + \lambda^2})t} a P_+ + e^{i(-\omega + C + \sqrt{C^2 + \lambda^2})t} a P_-. \tag{2.20}$$

III. MICROSCOPIC DERIVATION OF THE MASTER EQUATION

In this section, we derive a master equation for the Jaynes-Cummings system in the usual manner. We assume the Hamiltonian for the environment and the interaction between the JC system and the environment given by

$$H_{\mathrm{B}} = \sum_k \omega_k b_k^\dagger b_k, \tag{3.1}$$

$$H_{\mathrm{int}} = (a + a^\dagger) \otimes B, \tag{3.2}$$

where $b_k^\dagger$ (b_k) is a creation (annihilation) operator for a boson in the environment with wave number k, ω_k is the energy of a boson, and B is given by

$$B = \sum_k g_k (b_k + b_k^\dagger), \tag{3.3}$$

with g_k characterizing the coupling between the JC system and the environment. We denote the density matrices for the JC system, the environment, and the total system $\rho_{\mathrm{JC}}(t)$, $\rho_{\mathrm{B}}(t)$, and $\rho_{\mathrm{tot}}(t)$, respectively. The first two density matrices are written in terms of the total density matrix as

$$\rho_{\mathrm{JC}}(t) = \mathrm{Tr}_{\mathrm{B}} \rho_{\mathrm{tot}}(t), \tag{3.4}$$

$$\rho_{\mathrm{B}}(t) = \mathrm{Tr}_{\mathrm{JC}} \rho_{\mathrm{tot}}(t), \tag{3.5}$$

where Tr_{B} and $\mathrm{Tr}_{\mathrm{JC}}$ stand for partial traces taken over the degrees of freedom for the environment and the JC system, respectively.

Here we assume that the environment is always at thermal equiblium, i.e.,

$$\rho_{\mathrm{B}}(t) = \rho_{\mathrm{B}} \equiv e^{-\beta H_{\mathrm{B}}} / \mathrm{Tr}_{\mathrm{B}} e^{-\beta H_{\mathrm{B}}}, \tag{3.6}$$

where $\beta = 1/k_B T$, with k_B and T being the Boltzmann constant and the temperature, respectively. The time development of the total system is described by the von Neumann equation

$$\dot{\rho}_{\mathrm{tot}}(t) = -i[H_{\mathrm{tot}}, \rho_{\mathrm{tot}}(t)], \tag{3.7}$$

where $H_{\mathrm{tot}} = H_{\mathrm{JC}} + H_{\mathrm{B}} + H_{\mathrm{int}}$ is the total Hamiltonian. After the Born and the Markov approximations, we obtain a time evolution equation in the interaction picture,

$$\dot{\rho}_{\mathrm{JC}}^{\mathrm{I}}(t) = -\int_0^\infty d\tau \mathrm{Tr}_{\mathrm{B}} \big[H_{\mathrm{int}}^{\mathrm{I}}(t), \big[H_{\mathrm{int}}^{\mathrm{I}}(t-\tau), \rho_{\mathrm{JC}}^{\mathrm{I}}(t) \otimes \rho_{\mathrm{B}} \big] \big], \tag{3.8}$$

where the superscript I denotes the interaction picture,

$$O^{\mathrm{I}}(t) = e^{i(H_{\mathrm{JC}}+H_{\mathrm{B}})t} O e^{-i(H_{\mathrm{JC}}+H_{\mathrm{B}})t}, \tag{3.9}$$

for an operator O in the Schrödinger picture.

Using the explicit form of H_{int} and tracing out the degrees of freedom for the environment, we obtain

$$\begin{aligned} \dot{\rho}_{\mathrm{JC}}^{\mathrm{I}}(t) = &-\int_0^\infty d\tau g(\tau)[a^{\mathrm{I}}(t) + a^{\mathrm{I}\dagger}(t)] \\ &\times [a^{\mathrm{I}}(t-\tau) + a^{\mathrm{I}\dagger}(t-\tau)] \rho_{\mathrm{JC}}^{\mathrm{I}}(t) \\ &+ \int_0^\infty d\tau g(\tau)[a^{\mathrm{I}}(t-\tau) + a^{\mathrm{I}\dagger}(t-\tau)] \\ &\times \rho_{\mathrm{JC}}^{\mathrm{I}}(t)[a^{\mathrm{I}}(t) + a^{\mathrm{I}\dagger}(t)] + \mathrm{H.c.}, \end{aligned} \tag{3.10}$$

where $g(\tau) = \mathrm{Tr}_{\mathrm{B}}[B^{\mathrm{I}}(\tau) B^{\mathrm{I}}(0) \rho_{\mathrm{B}}]$ is the two-time correlation function. In each term in Eq. (3.10), we can put together the time dependence of $a(t)$'s at one place using Eqs. (2.14) and (2.20) and the completeness condition,

$$I = |E_0\rangle\langle E_0| + \sum_{n=1}^\infty \sum_\pm |E_{n\pm}\rangle\langle E_{n\pm}|, \tag{3.11}$$

on both sides of $\rho_{\mathrm{JC}}^{\mathrm{I}}(t)$ in the second term. Carrying out the secular approximation, we obtain

$$\begin{aligned} \dot{\rho}_{\mathrm{JC}}^{\mathrm{I}} = &-\frac{1}{2} \sum_\pm \{ P_\pm a^\dagger \gamma(\omega - C \pm \sqrt{C^2 + \lambda^2}) a P_\pm, \rho_{\mathrm{JC}}^{\mathrm{I}}(t) \} \\ &- \frac{1}{2} \sum_\pm \{ P_\pm a \gamma(-\omega - C \pm \sqrt{C^2 - \lambda^2}) a^\dagger P_\pm, \rho_{\mathrm{JC}}^{\mathrm{I}}(t) \} \\ &+ \sum_\pm \gamma(\omega - C \pm \sqrt{C^2 + \lambda^2}) \mathcal{P}^{\mathrm{diag}} \big[a P_\pm \rho_{\mathrm{JC}}^{\mathrm{I}}(t) P_\pm a^\dagger \big] \\ &+ \sum_\pm \gamma(-\omega - C \pm \sqrt{C^2 - \lambda^2}) \mathcal{P}^{\mathrm{diag}} \big[a^\dagger P_\pm \rho_{\mathrm{JC}}^{\mathrm{I}}(t) P_\pm a \big], \end{aligned} \tag{3.12}$$

where $\mathcal{P}^{\mathrm{diag}}$ is a projection operator for a density matrix onto the diagonal eigenbasis with respect to the Hamiltonian, which is defined by

$$\begin{aligned} \mathcal{P}^{\mathrm{diag}}[\rho] = &|E_0\rangle\langle E_0|\rho|E_0\rangle\langle E_0| \\ &+ \sum_{n,\pm} |E_{n\pm}\rangle\langle E_{n\pm}|\rho|E_{n\pm}\rangle\langle E_{n\pm}|. \end{aligned} \tag{3.13}$$

The damping rate $\gamma(\Omega)$ is given by

$$\begin{aligned} \gamma(\Omega) &= \int_{-\infty}^\infty d\tau e^{i\Omega\tau} g(\tau) \\ &= 2\pi \sum_k g_k^2 [\delta(\Omega - \omega_k)(N(\omega_k) + 1) \\ &\qquad + \delta(\Omega + \omega_k) N(\omega_k)], \end{aligned} \tag{3.14}$$

where $N(\omega_k) = (e^{\beta\omega_k} - 1)^{-1}$ is the average boson number of the environment, and we have omitted the unitary part since it is usually negligibly small. The argument Ω is an operator, and thus $\gamma(\Omega)$ is also an operator. Note that the general formula, Eq. (3.12), is still valid for finite temperatures and finite Δ.

IV. DYNAMICS OF THE RESONANT JC SYSTEM WITH CAVITY LOSSES

A. General formalism

In the following, we consider zero-temperature cases and assume a cavity in the one-dimensional space with an electromagnetic field. This cavity is characterized by $g_k = \sqrt{c\gamma/2L}$ ($\gamma > 0$), where c is the speed of light, γ^{-1} is the lifetime of the cavity mode, and L is the size of the one-dimensional space. The operators b and $b^\dagger$ are for photons, and the dispersion relation $\omega_k = c|k|$ is assumed. Substituting this for Eq. (3.14) and carrying out the k summation, we obtain

$$\gamma(\Omega) = \begin{cases} \gamma & (\Omega > 0), \\ 0 & (\Omega < 0). \end{cases} \tag{4.1}$$

This condition after all results in the effective Lorentzian coupling between the atom and the external electromagnetic field [29]. We further assume $\omega \gg \lambda, \Delta$. Then the excitation number-raising terms in Eq. (3.12) (the second and fourth terms) vanish. These conditions simplify the master equation, Eq. (3.12), and we obtain

$$\dot{\rho}^{\mathrm{I}}_{\mathrm{JC}}(t) = -\frac{\gamma}{2}\sum_{\pm}\{P_\pm a^\dagger a P_\pm, \rho^{\mathrm{I}}_{\mathrm{JC}}(t)\} + \gamma\sum_{\pm}\mathcal{P}^{\mathrm{diag}}[aP_\pm\rho^{\mathrm{I}}_{\mathrm{JC}}(t)P_\pm a^\dagger]. \tag{4.2}$$

Let us examine the dynamics described by Eq. (4.2) in the case of $\Delta = 0$. To solve Eq. (4.2), we decompose the density matrix $\rho^{\mathrm{I}}_{\mathrm{JC}}$ into the H_{JC}-diagonal and H_{JC}-off-diagonal sectors as

$$\rho^{\mathrm{I}}_{\mathrm{JC}}(t) = \rho^{\mathrm{I,diag}}_{\mathrm{JC}}(t) + \rho^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t), \tag{4.3}$$

with

$$\rho^{\mathrm{I,diag}}_{\mathrm{JC}}(t) = \mathcal{P}^{\mathrm{diag}}[\rho^{\mathrm{I}}_{\mathrm{JC}}(t)], \tag{4.4}$$

$$\rho^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t) = \rho^{\mathrm{I}}_{\mathrm{JC}}(t) - \mathcal{P}^{\mathrm{diag}}[\rho^{I}_{\mathrm{JC}}(t)]. \tag{4.5}$$

Note that the equation is closed in each sector. For each sector, the master equations are given by

$$\dot{\rho}^{\mathrm{I,diag}}_{\mathrm{JC}}(t) = -\gamma A\rho^{\mathrm{I,diag}}_{\mathrm{JC}}(t) + \gamma\mathcal{P}^{\mathrm{diag}}[a\rho^{\mathrm{I,diag}}_{\mathrm{JC}}(t)a^\dagger], \tag{4.6}$$

$$\dot{\rho}^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t) = -\frac{\gamma}{2}(A\rho^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t) + \rho^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t)A), \tag{4.7}$$

with

$$A \equiv \sum_{\pm}P_\pm a^\dagger a P_\pm = N - \tfrac{1}{2}(1 - P_0) \tag{4.8}$$

and $P_0 = |E_0\rangle\langle E_0|$. For the H_{JC}-off-diagonal sector, Eq. (4.7) is easily solved and we obtain the solution

$$\rho^{\mathrm{I,off\text{-}diag}}_{\mathrm{JC}}(t) = e^{-\frac{\gamma}{2}At}\rho^{\mathrm{off\text{-}diag}}_{\mathrm{JC}}(0)e^{-\frac{\gamma}{2}At}. \tag{4.9}$$

In the following, we concentrate on the H_{JC}-diagonal sector. Applying the transformation [30]

$$\rho^{\mathrm{I,diag}}_{\mathrm{JC}}(t) = e^{-\gamma At}\tilde{\rho}^{\mathrm{I,diag}}_{\mathrm{JC}}(t), \tag{4.10}$$

we obtain

$$\dot{\tilde{\rho}}^{\mathrm{I,diag}}_{\mathrm{JC}}(t) = \gamma\mathcal{P}^{\mathrm{diag}}[e^{\gamma At}ae^{-\gamma At}\tilde{\rho}^{\mathrm{I,diag}}_{\mathrm{JC}}(t)a^\dagger] = \gamma\mathcal{K}_1(t)\tilde{\rho}^{\mathrm{I,diag}}_{\mathrm{JC}}(t), \tag{4.11}$$

with $\mathcal{K}_1(t)$ being the linear operator for a density matrix defined by

$$\mathcal{K}_1(t)[\rho] = e^{-\gamma t(1-\frac{1}{2}P_0)}\mathcal{P}^{\mathrm{diag}}[a\rho a^\dagger], \tag{4.12}$$

where we have used the relation $[a, A] = (1 - \frac{1}{2}P_0)a$. The solution to Eq. (4.12) is formally written as

$$\tilde{\rho}^{\mathrm{I,diag}}_{\mathrm{JC}}(t) = Te^{\gamma\int_0^t\mathcal{K}_1(t')dt'}\rho^{\mathrm{diag}}_{\mathrm{JC}}(0), \tag{4.13}$$

where T represents the time-ordered product. Since $aP_0 = 0$ and the product of $\mathcal{K}_1$'s at different times is written as

$$\mathcal{K}_1(t_1)\mathcal{K}_1(t_2)\ldots\mathcal{K}_1(t_n) = e^{-\gamma(1-\frac{1}{2}P_0)t_1}e^{-\gamma(t_2+\ldots+t_n)}(\mathcal{K}_2)^n, \tag{4.14}$$

with

$$\mathcal{K}_2[\rho] = \mathcal{P}^{\mathrm{diag}}[a\rho a^\dagger], \tag{4.15}$$

the "commutation relation,"

$$\mathcal{K}_1(t_1)\ldots\mathcal{K}_1(t_i)\ldots\mathcal{K}_1(t_j)\ldots\mathcal{K}_1(t_n) = \mathcal{K}_1(t_1)\ldots\mathcal{K}_1(t_j)\ldots\mathcal{K}_1(t_i)\ldots\mathcal{K}_1(t_n), \tag{4.16}$$

holds for $1 < i < j \leqslant n$. Then we can carry out the time-ordered product and we obtain

$$Te^{\gamma\int_0^t\mathcal{K}_1(t')dt'} = 1 + \gamma\int_0^t dt' e^{-\gamma(1-\frac{1}{2}P_0)t'}e^{(1-e^{-\gamma t'})\mathcal{K}_2}\mathcal{K}_2. \tag{4.17}$$

Performing the remaining integral, we obtain the time evolution of the H_{JC}-diagonal sector,

$$\rho^{\mathrm{diag}}_{\mathrm{JC}}(t) = e^{-\gamma At}\left[1 + \sum_{n=0}^{\infty}c_n(t)(\mathcal{K}_2)^{n+1}\right]\rho^{\mathrm{diag}}_{\mathrm{JC}}(0), \tag{4.18}$$

where

$$c_n(t) = \sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\frac{1 - e^{-(1+k-\frac{1}{2}P_0)\gamma t}}{1 + k - \frac{1}{2}P_0}. \tag{4.19}$$

B. Multiexcitation cases

Our formulations Eq. (4.17) and Eq. (4.18) are suitable for handling multiexcitation cases. To consider the time evolution of these cases, it is convenient to see the behavior of $\mathcal{K}_2$ acting on the H_{JC}-diagonal states. Let us introduce a basis for the H_{JC}-diagonal sector of the density matrix,

$$\Pi_{n\pm} = |E_{n+}\rangle\langle E_{n+}| \pm |E_{n-}\rangle\langle E_{n-}| \quad (n \geqslant 1), \qquad \Pi_0 = |E_0\rangle\langle E_0|. \tag{4.20}$$

Then we can show

$$\mathcal{K}_2\Pi_{n+} = (n - \tfrac{1}{2})\Pi_{n-1,+} \quad (n \geqslant 2), \tag{4.21}$$

$$\mathcal{K}_2\Pi_{n-} = \sqrt{n(n-1)}\Pi_{n-1} \quad (n \geqslant 2), \tag{4.22}$$

$$\mathcal{K}_2\Pi_{1+} = \Pi_0, \tag{4.23}$$

$$\mathcal{K}_2\Pi_{1-} = \mathcal{K}_2\Pi_0 = 0. \tag{4.24}$$

This recurrence equation is easily solved and we obtain

$$(\mathcal{K}_2)^l\Pi_{n+} = \left(n - l + \frac{1}{2}\right)_l\Pi_{n-l,+}, \tag{4.25}$$

$$(\mathcal{K}_2)^l\Pi_{n-} = \sqrt{\frac{n!(n-1)!}{(n-l)!(n-l-1)!}}\Pi_{n-l,-} \tag{4.26}$$

for $n \geqslant l+1$, where $(n)_l$ is the Pochhammer symbol defined by $(n)_l = n(n+1)\dots(n+l-1)$. Back in the original basis, we obtain

$$(\mathcal{K}_2)^l|E_{n\pm}\rangle\langle E_{n\pm}| = \frac{1}{2}\left[\left(n-l+\frac{1}{2}\right)_l \pm \sqrt{\frac{n!(n-1)!}{(n-l)!(n-l-1)!}}\right]|E_{n-l,+}\rangle\langle E_{n-l,+}| + \frac{1}{2}\left[\left(n-l+\frac{1}{2}\right)_l \mp \sqrt{\frac{n!(n-1)!}{(n-l)!(n-l-1)!}}\right]|E_{n-l,-}\rangle\langle E_{n-l,-}| \quad (4.27)$$

for $n \geqslant l+1$ and

$$(\mathcal{K}_2)^n|E_{n\pm}\rangle\langle E_{n\pm}| = \frac{(2n-1)!!}{2^n}|E_0\rangle\langle E_0| \quad (4.28)$$

for $n = l(\geqslant 1)$.

Using these relations, we evaluate the time evolution $\rho_{gn}(t)$ for the multiphoton initial state $\rho_{gn}(0) = |g,n\rangle\langle g,n|$ with Eqs. (3.9), (4.10), and (4.17), which is given by

$$\rho_{gn}(t) = \frac{1}{2}e^{-\gamma(n-\frac{1}{2})t}\Pi_{n+} + \frac{1}{2}\sum_{m=1}^{n-1} e^{-\gamma(m-\frac{1}{2})t}\frac{(1-e^{-\gamma t})^{n-m}}{(n-m)!}\left(m+\frac{1}{2}\right)_{n-m}\Pi_{m+} + \frac{(2n-1)!!}{(n-1)!2^n}B\left(n,\frac{1}{2};1-e^{\gamma t}\right)|E_0\rangle\langle E_0| + \frac{1}{2}e^{-\gamma(n-\frac{1}{2})t}(e^{-2i\sqrt{n}\lambda t}|E_{n+}\rangle\langle E_{n-}| + \text{H.c.}), \quad (4.29)$$

where

$$B(a,b;z) = \int_0^z x^{a-1}(1-x)^{b-1}dx \quad (4.30)$$

is the incomplete beta function. Let us discuss the probability that we observe the ground state of the atom, P_g, and the average photon number, $\langle n_{\text{photon}}\rangle$, which are defined by

$$P_g(t) = \sum_{k=0}^{\infty}\langle g,k|\rho(t)|g,k\rangle \quad (4.31)$$

and

$$\langle n_{\text{photon}}\rangle = \sum_{k=1}^{\infty} k\langle g,k|\rho(t)|g,k\rangle + \sum_{k=1}^{\infty} k\langle e,k|\rho(t)|e,k\rangle, \quad (4.32)$$

respectively. Substituting Eq. (4.29) for Eqs. (4.31) and (4.32), we obtain

$$P_g = \frac{1}{2} + \frac{(2n-1)!!}{(n-1)!2^{n+1}}B\left(n,\frac{1}{2};1-e^{-\gamma t}\right) + \frac{1}{2}e^{-\gamma(n-\frac{1}{2})t}\cos(2\sqrt{n}\lambda t) \quad (4.33)$$

and

$$\langle n_{\text{photon}}\rangle = \frac{(2n-1)!!}{2^n(n-1)!}e^{-\gamma t/2}(1-e^{-\gamma t})^{n-1} \times {}_2F_1\left(1,1-n,\frac{1}{2};-\frac{1}{e^{\gamma t}-1}\right) + \frac{1}{2}e^{-\gamma(n-\frac{1}{2})t}\cos(2\sqrt{n}\lambda t), \quad (4.34)$$

where

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (4.35)$$

is the Gaussian hypergeometric function. Figures 1(a), 1(b) and 1(c) show P_g for the initial states $|1,g\rangle\langle 1,g|$, $|3,g\rangle\langle 3,g|$, and $|5,g\rangle\langle 5,g|$, respectively. Several remarks on the above results follow.

1. The period of oscillation becomes shorter as the initial photon number increases. $|n,g\rangle\langle n,g|$ $(n=1,3,5)$ contains the H_{JC}-off-diagonal part, which oscillates with the period inversely proportional to the energy difference between the two energy eigenstates with the total excitation number n.

2. The decay of the oscillation becomes faster as the initial photon number increases. This is because the oscillatory part is typically suppresed by the factor $e^{-\gamma A t}$, and the eigenvalues of A are almost proportional to the total excitation number [see Eq. (4.8)].

3. The times that the initial states take to decay into the ground state $|E_0\rangle\langle E_0|$ are not much different with different initial photon numbers. However, as the initial photon number increases, the period for which P_g stays around 0.5 becomes longer.

C. Coherent state

In this subsection, we consider the time evolution of the product state of the photon coherent state and atomic ground state, which is known to show collapses and revivals without cavity losses. The photon coherent state is given by

$$|\alpha\rangle = e^{-\frac{\alpha^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad (4.36)$$

where α is the parameter that characterizes the coherent state and we assume α is real here and in the following. The average photon number is given by α^2. Using $|\alpha\rangle$, we set the initial state as

$$\rho_\alpha(0) = |g\alpha\rangle\langle g\alpha| \equiv (|g\rangle\otimes|\alpha\rangle)(\langle g|\otimes\langle\alpha|). \quad (4.37)$$

We again decompose the density matrix into the $H_{\rm JC}$-diagonal and $H_{\rm JC}$-off-diagonal parts with respect to the energy eigenstates. We first consider the $H_{\rm JC}$-diagonal part, given by

$$\rho_\alpha^{\rm diag}(0) = e^{-\alpha^2}|E_0\rangle\langle E_0| + \frac{e^{-\alpha^2}}{2}\sum_{k=1}^{\infty}\frac{\alpha^{2k}}{k!}\Pi_{k+}. \tag{4.38}$$

Using Eqs.(3.9), (4.10), and (4.17), after some algebra, we obtain

$$\rho_\alpha^{\rm diag}(t) = e^{-\alpha^2}\left[1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\alpha^{2n}}{(n-1)!n!}\frac{(2n-1)!!}{2^n}B\left(n,\frac{1}{2};1-e^{-\gamma t}\right)\right]\Pi_0 + \frac{e^{-\alpha^2}}{2}\sum_{k=1}^{\infty}e^{-(k-\frac{1}{2})\gamma t}\left[\frac{\alpha^{2k}}{k!}{}_1F_1\left(k+\frac{1}{2},k+1;\alpha^2(1-e^{-\gamma t})\right)\right]\Pi_{k+}, \tag{4.39}$$

where

$${}_1F_1(a,b;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{4.40}$$

is the confluent hypergeometric function. The remaining $H_{\rm JC}$-off-diagonal part is very easily treated and from Eqs. (3.9) and (4.9) we obtain

$$\rho_\alpha^{\text{off-diag}}(t) = \sum_{k=1}^{\infty}\frac{e^{-\alpha^2}\alpha^{2k}}{2\cdot k!}e^{-\frac{\gamma}{2}(2k-1)t}\left(e^{-2i|\lambda|\sqrt{k}t}|E_{k+}\rangle\langle E_{k-}| + e^{2i|\lambda|\sqrt{k}t}|E_{k-}\rangle\langle E_{k+}|\right), \tag{4.41}$$

where we have omitted the different-excitation-number off-diagonal states since these terms do not affect P_g. For this state, we obtain P_g [Eq. (4.33)] as

$$P_g(t) = \frac{1}{2}(1+e^{-\alpha^2}) + \frac{e^{-\alpha^2}}{4}\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n-1)!n!}\frac{\alpha^{2n}}{2^n}B\left(n,\frac{1}{2};1-e^{-\gamma t}\right) + \sum_{n=1}^{\infty}\frac{e^{-\alpha^2}\alpha^{2n}}{2\cdot n}e^{-\gamma(n-\frac{1}{2})t}\cos 2\lambda\sqrt{n}t, \tag{4.42}$$

which is shown in Fig. 2 for the $\gamma/\lambda = 10^{-4}$, 10^{-3}, and 10^{-2} cases. With small γ/λ, we can observe clear collapses and revivals. However, as the decay rate γ becomes large, the long-time oscillation and the revivals are supressed more. This result is consistent with a numerical calculation for the phenomenological master equation [20].

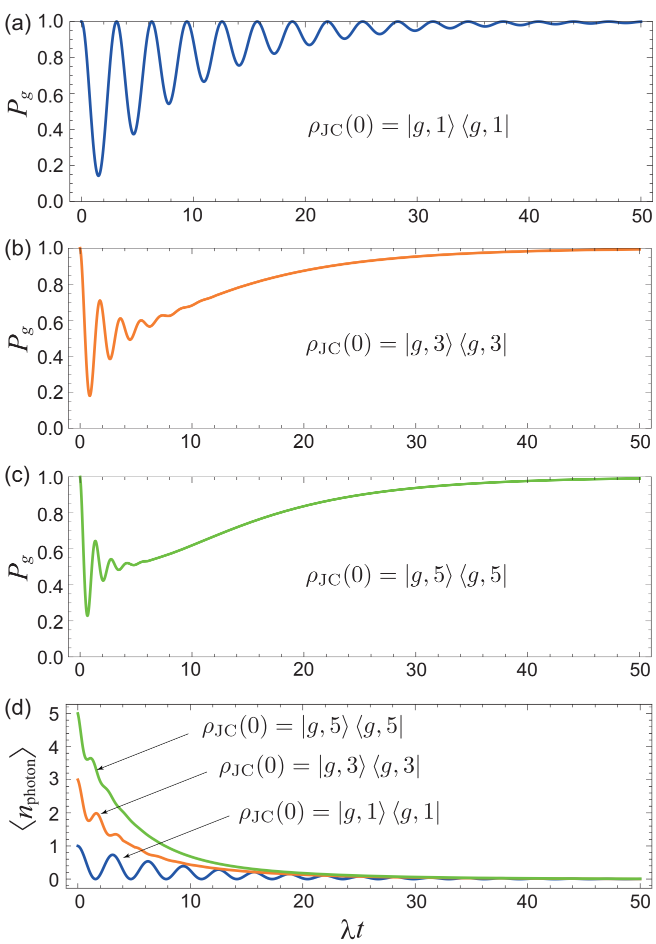

FIG. 1. (a–c) Probabilities of observing the atomic ground-state P_g as functions of the dimensionless time λt for the initial states $|n,g\rangle\langle n,g|$ with (a) $n=1$, (b) $n=3$, and (c) $n=5$. (d) Averaged photon number $\langle n_{\rm photon}\rangle$ as functions of dimensionless time λt for the same initial states.

V. DYNAMICS OF THE OFF-RESONANT JC SYSTEM WITH CAVITY LOSSES

In this section, we consider the off-resonant cases ($|\Delta| \neq 0$) and assume the same interaction with the cavity as in Sec. IV. After deriving an analytic solution, we discuss the behavior of the time evolution for large- and small-$|\Delta|$ limits. Also, we actually evaluate the time evolution for some initial conditions using the obtained analytic solution and compare the result with that derived from the phenomenological master equation.

A. General formalism

We start from Eq. (4.2). The $H_{\rm JC}$-off-diagonal dynamics is still described by Eq. (4.9) by substituting $\tilde{A}$ for A with

$$\tilde{A} = \sum_{\pm}P_\pm a^\dagger a P_\pm = N - \frac{1}{2} + \frac{\Delta}{2C}. \tag{5.1}$$

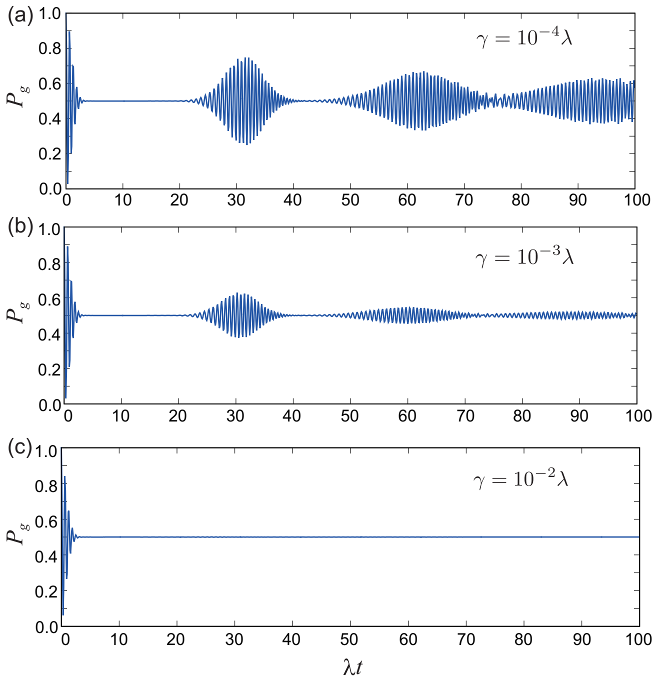

FIG. 2. The probability P_g for the coherent initial state $|g,\alpha\rangle\langle g,\alpha|$ is shown as a function of the dimensionless time λt for the cases (a) $\gamma/\lambda = 10^{-4}$, (b) $\gamma/\lambda = 10^{-3}$, and (c) $\gamma/\lambda = 10^{-2}$. We set $\alpha = 5$ (initial average photon number is 25).

For $H_{\rm JC}$-diagonal states, on the other hand, we again perform a transform, $\rho_{\rm JC}^{\rm I,diag}(t) = e^{-\gamma\tilde{A}t}\tilde{\rho}_{\rm JC}^{\rm I,diag}(t)$, and obtain the transformed master equation as follows:

$$\dot{\tilde{\rho}}_{\rm JC}^{\rm I\,diag}(t) = \gamma\mathcal{P}^{\rm diag}[e^{\gamma\tilde{A}t}ae^{-\gamma\tilde{A}t}\tilde{\rho}_{\rm JC}^{\rm I,diag}(t)a^\dagger]. \qquad (5.2)$$

Here we rewrite the time-dependent part as

$$\begin{aligned} e^{\gamma\tilde{A}t}ae^{-\gamma\tilde{A}t} &= e^{-\gamma t}e^{\gamma\frac{\Delta}{2C}t}ae^{-\gamma\frac{\Delta}{2C}t} \\ &= e^{-\gamma t}[e^{-\frac{1}{2}\gamma\Delta t(\frac{1}{\varepsilon_{N+1}}-\frac{1}{C})}aP_+ + e^{\frac{1}{2}\gamma\Delta t(\frac{1}{\varepsilon_{N+1}}+\frac{1}{C})}aP_-], \end{aligned} \qquad (5.3)$$

where we have used the relation obtained in a similar way to Eq. (2.20),

$$ae^{-\gamma\frac{\Delta}{2C}t} = e^{\frac{\gamma\Delta t}{2\varepsilon_{N+1}}}aP_- + e^{-\frac{\gamma\Delta t}{2\varepsilon_{N+1}}}aP_+. \qquad (5.4)$$

Then the master equation for the $H_{\rm JC}$-diagonal part reads

$$\dot{\tilde{\rho}}_{\rm JC}^{\rm I,diag}(t) = \sum_{i=1,2}\gamma e^{-\gamma t\kappa_i(C)}\mathcal{Q}_i\tilde{\rho}_{\rm JC}^{\rm I,diag}(t), \qquad (5.5)$$

where

$$\kappa_1(C) = 1 - \frac{\Delta}{2}\left(\frac{1}{C} - \frac{1}{\sqrt{C^2+\lambda^2}}\right), \qquad (5.6)$$

$$\kappa_2(C) = 1 - \frac{\Delta}{2}\left(\frac{1}{C} + \frac{1}{\sqrt{C^2+\lambda^2}}\right), \qquad (5.7)$$

$$\mathcal{Q}_1[\rho] = \mathcal{P}^{\rm diag}[aP_+\rho P_+a^\dagger], \qquad (5.8)$$

$$\mathcal{Q}_2[\rho] = \mathcal{P}^{\rm diag}[aP_-\rho P_-a^\dagger]. \qquad (5.9)$$

The formal solution to Eq. (5.5) is given by

$$\tilde{\rho}_{\rm JC}^{\rm I,diag}(t) = T\exp\left[\int_0^t\sum_i\gamma e^{-\gamma t'\kappa_i(C)}\mathcal{Q}_i dt'\right]\rho_{\rm JC}^{\rm I,diag}(0). \qquad (5.10)$$

After the series expansion and the integration, we obtain

$$\tilde{\rho}_{\rm JC}^{\rm I,diag}(t) = \rho_{\rm JC}^{\rm I,diag}(0) + \sum_{k=1}^{\infty}\mathcal{R}_k(t)[\rho_{\rm JC}^{\rm I,diag}(0)] \qquad (5.11)$$

with

$$\mathcal{R}_k(t) = \sum_{\{i_1,\dots,i_k\}=1,2} I_k(\gamma t;\kappa_{i_k}(\sigma_{i_{k-1}}\varepsilon_{N+k-1}),\dots,\kappa_{i_2}(\sigma_{i_1}\varepsilon_{N+1}),\kappa_{i_1}(C))\mathcal{Q}_{i_1}\dots\mathcal{Q}_{i_k}, \qquad (5.12)$$

where each variable for the summation $i_1,\dots,i_k$ takes 1 or 2; σ_{i_j} is the sign, $+1$ for $i_j = 1$ and -1 for $i_j = 2$; and $I_k(t;a_1,\dots,a_k)$ is the function determined by the recurrence formula,

$$I_1(\tau;a_1) = \frac{1}{a_1}(1-e^{-a_1\tau}), \qquad (5.13)$$

$$\begin{aligned} I_k(\tau;a_1,\dots,a_k) = \frac{1}{a_1}&[I_{k-1}(\tau;a_2,a_3,a_4,\dots,a_n) \\ &- I_{k-1}(\tau;a_1+a_2,a_3,a_4,\dots,a_n)]. \end{aligned} \qquad (5.14)$$

The details of the derivation are shown in the Appendix.

B. Small- and large-$|\Delta|$ limits

First, let us consider the case $|\Delta| \to 0$. In this condition, Eq. (5.1) reproduces Eq. (4.8), and thus the solution for $|\Delta| \to 0$ is smoothly connected to the $\Delta = 0$ case.

Next, we consider the large-$|\Delta|$ limit. In the following, we assume $|\sqrt{n_{\rm ave}}\lambda/\Delta| \ll 1$, where $n_{\rm ave}$ is the average photon number of the initial state. In this limit, $\cos\theta_n$ and $\sin\theta_n$ [defined in Eq. (2.7)] approach unity and 0, respectively, and $|g,n\rangle$ and $|e,n-1\rangle$ decouple. Therefore, we can divide the master equation into the $|E_+\rangle(=|g,n\rangle)$ part and the $|E_-\rangle(=-|e,n-1\rangle)$ part:

$$\dot{\tilde{\rho}}_{\rm JC}^{+}(t) = \gamma e^{-\gamma t\kappa_1(\varepsilon_N)}\mathcal{Q}_1[\tilde{\rho}_{\rm JC}^{+}(t)], \qquad (5.15)$$

$$\dot{\tilde{\rho}}_{\rm JC}^{-}(t) = \gamma e^{-\gamma t\kappa_2(-\varepsilon_N)}\mathcal{Q}_2[\tilde{\rho}_{\rm JC}^{-}(t)], \qquad (5.16)$$

where $\tilde{\rho}_{\rm JC}^{\pm}(t)$ represents $P_\pm\tilde{\rho}_{\rm JC}^{\rm I,diag}(t)P_\pm$. Furthermore, $\kappa_1(\varepsilon_N)$ and $\kappa_2(-\varepsilon_N)$ are evaluated as

$$\kappa_1(\varepsilon_N) = 1 + O\left(\left(\frac{\sqrt{N}\lambda}{\Delta}\right)^2\right), \qquad (5.17)$$

$$\kappa_2(-\varepsilon_N) = 1 + O\left(\left(\frac{\sqrt{N}\lambda}{\Delta}\right)^2\right). \qquad (5.18)$$

TABLE I. Regions where the phenomenological and present master equations are justified.

	$\lvert\sqrt{n_{\mathrm{ave}}}\lambda/\Delta\rvert \ll 1$	$\lvert\sqrt{n_{\mathrm{ave}}}\lambda/\Delta\rvert \gg 1$
Phenomenological	Valid	Not valid[1]
Present	Valid	Valid

[1]For the single-excitation states with $\Delta = 0$, the long-time evolution is well described by the phenomenological master equation [25].

Hereafter, we assume that the total excitation number is small enough and we can neglect the higher-order terms with respect to $|\sqrt{N}\lambda/\Delta|$, which is regarded as the order of $|\sqrt{n_{\mathrm{ave}}}\lambda/\Delta|$. We expect that this condition is still maintained for the coherent state, which contains infinitely many photons, since there is an effective cutoff in the photon number determined by α. By this condition, we do not need to write $P_\pm$ in $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and we obtain the master equations for $|\Delta| \to \infty$ limit as follows:

$$\dot{\tilde{\rho}}^{\pm}_{\mathrm{JC}}(t) = \gamma e^{-\gamma t} a \tilde{\rho}^{\pm}_{\mathrm{JC}}(t) a^\dagger. \tag{5.19}$$

In this limit, we find $\tilde{A} \to a^\dagger a$, and the master equation in the Schrödinger picture $\rho^{\pm}_{\mathrm{JC}}(t)\ (= e^{-\gamma\tilde{A}t}\tilde{\rho}^{\pm}_{\mathrm{JC}})$ is expressed by

$$\dot{\rho}^{\pm}_{\mathrm{JC}}(t) = -\frac{\gamma}{2} a^\dagger a \rho^{\pm}_{\mathrm{JC}}(t) - \frac{\gamma}{2}\rho^{\pm}_{\mathrm{JC}}(t) a^\dagger a + \gamma a \rho^{\pm}_{\mathrm{JC}}(t) a^\dagger. \tag{5.20}$$

In this form, we can readily confirm the trace-preserving property. Similarly, we obtain the H_{JC}-off-diagonal part,

$$\begin{aligned}\dot{\rho}^{\mathrm{off-diag}}_{\mathrm{JC}}(t) = &-i\big[H_{\mathrm{JC}}, \rho^{\mathrm{off-diag}}_{\mathrm{JC}}(t)\big] - \frac{\gamma}{2} a^\dagger a \rho^{\mathrm{off-diag}}_{\mathrm{JC}}(t) \\ &- \frac{\gamma}{2}\rho^{\mathrm{off-diag}}_{\mathrm{JC}}(t) a^\dagger a.\end{aligned} \tag{5.21}$$

The master equations, Eqs. (5.19) and (5.21), are immediately integrated, yielding

$$\begin{aligned}\rho^{\pm}_{\mathrm{JC}}(t) &= e^{-\gamma a^\dagger a t} e^{(1-e^{-\gamma t})\mathcal{K}_3} \rho^{\pm}_{\mathrm{JC}}(0), \\ \rho^{\mathrm{off-diag}}_{\mathrm{JC}}(t) &= e^{-iH_{\mathrm{JC}}t - \frac{\gamma}{2}a^\dagger a t}\rho^{\mathrm{off-diag}}_{\mathrm{JC}}(0) e^{iH_{\mathrm{JC}}t - \frac{\gamma}{2}a^\dagger a t},\end{aligned} \tag{5.22}$$

with $\mathcal{K}_3\rho = a\rho a^\dagger$.

Through these analyses, we find two important facts. First, Eqs. (5.20) and (5.21) are the same as the phenomenological master equation, Eq. (1.1), apart from the jump term in the H_{JC}-off-diagonal part. This fact suggests that the phenomenological master equation is justified for the case of $|\sqrt{n_{\mathrm{ave}}}\lambda/\Delta| \ll 1$. (The region where the phenomenological and present master equations are justified is summarized in Table I.) This suggestion is confirmed for some examples in the following section. Second, $|E_{1-}\rangle (= -|e, 0\rangle)$ does not decay to $|E_0\rangle$ because $\mathcal{K}_3|E_{1-}\rangle = 0$. This reflects the fact that $|g, n\rangle$ and $|e, n-1\rangle$ decouple in the large-$|\Delta|$ limit. This effect is observed in the numerical evaluations shown later and Ref. [31].

C. Examples

1. Single-excitation state

Using the obtained solution, we can immediately write the time evolution for single-excitation initial states as follows.

(i) $\rho_{\mathrm{JC}}(0) = |E_{1+}\rangle\langle E_{1+}|$:

$$\begin{aligned}\rho_{\mathrm{JC}}(t) &= e^{-\gamma\tilde{A}t}\Bigg[|E_{1+}\rangle\langle E_{1+}| + \sum_{i=1,2} I_1(\gamma t; \kappa_i(C))\mathcal{Q}_i |E_{1+}\rangle\langle E_{1+}|\Bigg] \\ &= e^{-\gamma\tilde{A}t}|E_{1+}\rangle\langle E_{1+}| \\ &\quad + \frac{1}{2}\left(1 + \frac{\Delta}{\sqrt{\Delta^2+\lambda^2}}\right) I_1(\gamma t; \kappa_1(\Delta))|E_0\rangle\langle E_0| \\ &= e^{-\gamma\kappa_1(\Delta)t}|E_{1+}\rangle\langle E_{1+}| + (1 - e^{-\gamma\kappa_1(\Delta)t})|E_0\rangle\langle E_0|.\end{aligned} \tag{5.23}$$

(ii) $\rho_{\mathrm{JC}}(0) = |E_{1-}\rangle\langle E_{1-}|$:

$$\begin{aligned}\rho_{\mathrm{JC}}(t) &= e^{-\gamma\tilde{A}t}\Bigg[|E_{1-}\rangle\langle E_{1-}| + \sum_{i=1,2} I_1(\gamma t; \kappa_i(C))\mathcal{Q}_i |E_{1-}\rangle\langle E_{1-}|\Bigg] \\ &= e^{-\gamma\tilde{A}t}|E_{1-}\rangle\langle E_{1-}| \\ &\quad + \frac{1}{2}\left(1 - \frac{\Delta}{\sqrt{\Delta^2+\lambda^2}}\right) I_1(\gamma t; \kappa_2(\Delta))|E_0\rangle\langle E_0| \\ &= e^{-\gamma\kappa_2(\Delta)t}|E_{1-}\rangle\langle E_{1-}| + (1 - e^{-\gamma\kappa_2(\Delta)t})|E_0\rangle\langle E_0|.\end{aligned} \tag{5.24}$$

Let us compare the time evolutions described by the microscopic master equation [Eq. (5.5)] and the phenomenological one [Eq. (1.1)]. We consider the Bell-type initial states $|g, 1\rangle\langle g, 1|$ and $|e, 0\rangle\langle e, 0|$ with finite $|\Delta/\lambda|$. For these initial states, the H_{JC}-diagonal part is described by the linear combination of the results above, while the H_{JC}-off-diagonal part is described by Eqs. (4.9) and (5.1). The time evolutions of the probability that we observe the ground state $|E_0\rangle\langle E_0|$, P_{0g}, are shown in Figs. 3(a)–3(d). In the large-$|\Delta|$ cases, we find two tendencies. First, the time evolution described by the phenomenological master equation approaches that by the microscopic one. These results support the fact that the microscopic master equation almost coincides with the phenomenological one in the large-$|\Delta|$ limit. Second, we find that $|e, 0\rangle\langle e, 0|$ decays much more slowly than $|g, 1\rangle\langle g, 1|$ as mentioned by Gonzalez *et al.* [31]. This is explained by the decoupling as follows. While $|g, 1\rangle\langle g, 1|$ can decay into $|E_0\rangle\langle E_0|$ by operator a directly, $|e, 0\rangle\langle e, 0|$ cannot decay or rarely turns into $|g, 1\rangle\langle g, 1|$ by the Rabi oscillation due to large $|\Delta|$.

2. Three-photon state

Let us consider the initial three-photon state as an example of multiphoton cases. We fix the initial condition at $\rho_{\mathrm{JC}}(0) = |g, 3\rangle\langle g, 3|$. The time evolution of P_g [Eq. (4.33)] is shown in Fig. 4(a) with $\Delta = 0.1\lambda$, Fig. 4(b) with $\Delta = \lambda$, and Fig. 4(c) with $\Delta = 5\lambda$. As $|\Delta|$ becomes larger, the oscillation becomes faster. This oscillation derives from the H_{JC}-off-diagonal part of $\rho_{\mathrm{JC}}(0)$, and its period is dominated by the energy difference $2\sqrt{\Delta^2 + 3\lambda^2}$. Furthermore, the oscillation is suppresed by the factor $e^{-\gamma\tilde{A}t}$ typically, and the eigenvalue of $\tilde{A}$ is almost proportional to the total excitation number [see Eq.(5.1)]. Therefore, the oscillation is more suppresed with larger $|\Delta|$.

The average photon number is shown in Fig. 4(d). The solid (blue), dotted (orange), and dashed (green) lines correspond to the conditions for Figs. 4(a), 4(b) and 4(c), respectively.

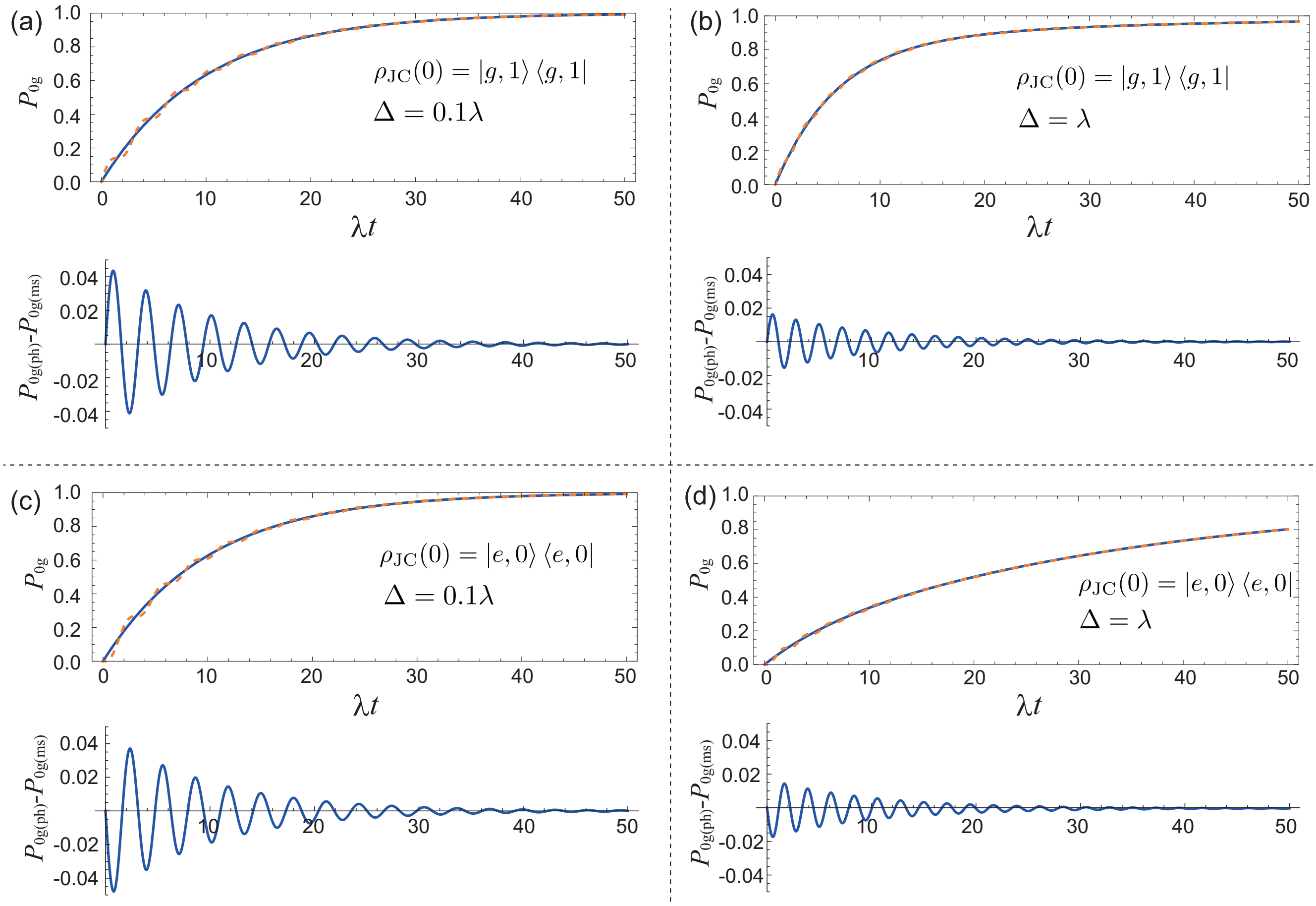

FIG. 3. Comparison of the probabilities of observing the ground state evaluated by the microscopic (present) vs the phenomenological master equations. The initial state and detuning are set to (a) $|g, 1\rangle\langle g, 1|$ and $\Delta = 0.1\lambda$, (b) $|g, 1\rangle\langle g, 1|$ and $\Delta = \lambda$, (c) $|e, 0\rangle\langle e, 0|$ and $\Delta = 0.1\lambda$, and (d) $|e, 0\rangle\langle e, 0|$ and $\Delta = \lambda$. The damping rate is set to $\gamma = 0.2\lambda$ in all cases. The time evolutions are shown as functions of the dimensionless time λt. Solid [blue; denoted $P_{0g(\mathrm{ms})}$] and dashed [orange; denoted $P_{0g(\mathrm{ph})}$] lines represent the probabilities calculated from the microscopic and phenomenological master equations, respectively. The differences for each set of time evolutions are also shown.

Apart from the fluctuation derived from the Rabi oscillation, the average photon number decays to 0 at a rate almost independent of $|\Delta|$.

3. Coherent state

Finally, let us consider the initial coherent state $\rho_{\mathrm{JC}}(0) = |g, \alpha\rangle\langle g, \alpha|$. Figure 5 shows the P_g values [Eq. (4.33)] as functions of the dimensionless time λt for $\Delta = \{\lambda, 3\lambda, 5\lambda\}$ and $\gamma = \{2 \times 10^{-3}, 10^{-2}\}$. We can see that the collapse-revival period becomes longer and the collapses and revivals become clearer as $|\Delta|$ becomes larger. These tendencies are consistent with that observed by Gonzalez *et al.* [31]. For larger γ, as in the case with $\Delta = 0$, revivals are supressed more rapidly but P_g takes a relatively long time to decay into 1 compared with the case with a few photons presented in the previous subsection.

VI. DISCUSSION AND SUMMARY

Let us compare the present master equation with the previous result for the resonant ($\Delta = 0$) case, where the present result should be equivalent to Scala's [25]. This is shown as follows. In the first two terms in Eq. (3.12), we replace the projection operator $P_\pm$ with the sum of the projection operators for the eigenstates, $|E_0\rangle\langle E_0|$, $|E_{n+}\rangle\langle E_{n+}|$, and $|E_{n-}\rangle\langle E_{n-}|$. Since the combinations $a^\dagger\gamma(\omega - C \pm \sqrt{C^2 + \lambda^2})a$ and $a\gamma(-\omega - C \pm \sqrt{C^2 - \lambda^2})a^\dagger$ do not change the total excitation number, the excitation numbers of the resultant projection operators on both sides should be the same. Similarly, putting the completeness condition, Eq. (3.11), for both sides of $\rho_{\mathrm{JC}}^{\mathrm{I}}$ in the last two terms, we obtain an eigenstate-based expression. After a lengthy but straightforward calculation, we obtain completely the same expression as Scala's. Although the present result and Scala's result are equivalent, the present method is suitable for the resonant cases with many photons and the off-resonant cases. This difference is summarized in Table II.

It is also to be noted that we have assumed throughout this paper that the coupling between the cavity and the environment is flat. This condition is no longer maintained for cavities in two- or three-dimensional space. However, this condition is not the essential assumption for obtaining a closed form of the solution. When we set a specific coupling, the same discussion for the resonant or off-resonant cases in the present paper will

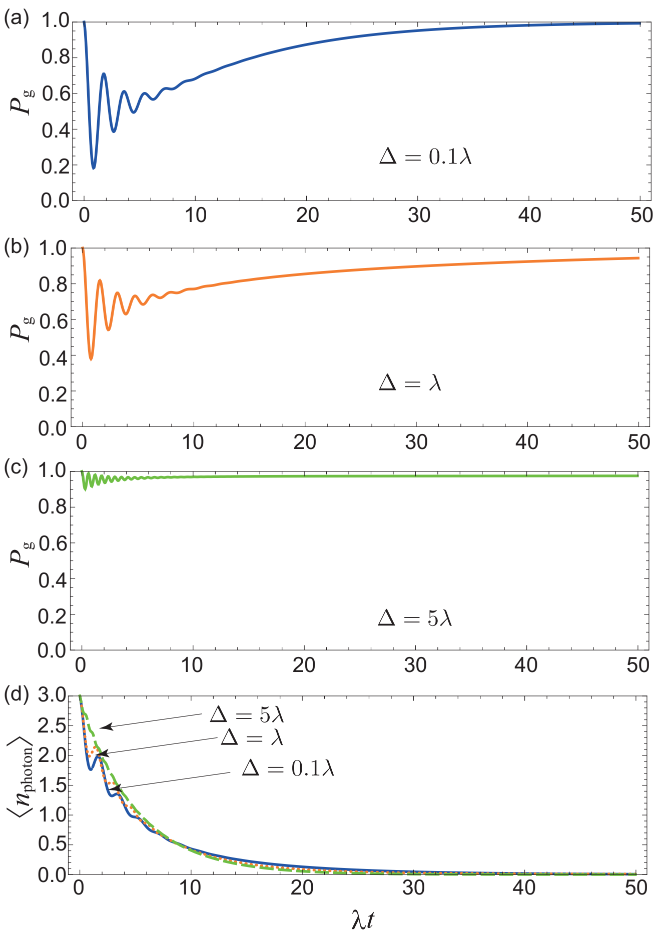

FIG. 4. (a–c) Probabilities P_g for the initial state of three photons $|g, 3\rangle\langle g, 3|$ as functions of the dimensionless time λt with (a) $\Delta = 0.1\lambda$, (b) $\Delta = \lambda$, and (c) $\Delta = 5\lambda$. (d) Time evolution of the average photon number $\langle n_{\text{photon}} \rangle$. Solid (blue), dotted (orange), and dashed (green) lines correspond to the parameters in (a), (b), and (c), respectively.

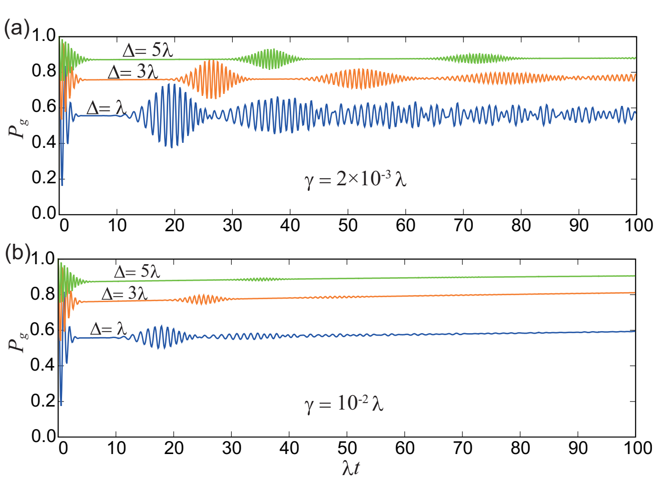

FIG. 5. Probability P_g for the coherent initial state $|\alpha, g\rangle\langle \alpha, g|$ is shown as a function of the dimensionless time λt for $\gamma/\lambda = 2 \times 10^{-3}$ and 2×10^{-2}. We set $\alpha = 3$ (initial average photon number is 9).

TABLE II. Differences between the present paper and Ref. [25].

	$\Delta = 0$	$\Delta \neq 0$
Scala *et al.* [25]	Solved for single-excitation cases	—
Present	Solved generally	Systematic method developed

be applied and we will be able to obtain expressions suitable for analytic analyses.

In summary, assuming a realistic coupling between the cavity and the environment, we have derived a master equation for the JC system with cavity losses. The derived equation is simple, and we can write the analytic solution explicitly for the resonant case ($\Delta = 0$) at zero temperature. This solution is suitable for the analysis of many-photon states. Using this solution, we clarified the many-photon effect on decay: The more photons exist, the slower the decay rate of the atom becomes. Also, we confirmed the clear collapses and revivals under dissipation. For the off-resonant case, on the other hand, we developed an analytic way to describe the time evolution. This method is systematic, although not explicit, and we can evaluate the time evolution similarly. As examples, we examined the single- and multiexcitation and coherent initial states and revealed their various behaviors. Also, we have discussed the limits of $\Delta \to 0$ and ∞ and suggested the condition that justifies the widely used master equation, Eq. (1.1): $|\Delta/\lambda|$ is sufficiently large and the initial excitation number is small enough. The present analytic methods are not only exact but also easy to handle, in particular for the resonant cases, and will be useful for analyses of experiments.

ACKNOWLEDGMENTS

We thank A. Watanabe for fruitful discussions. H.N. was partly supported by the Institute for Advanced Theoretical and Experimental Physics, Waseda University, and by a Waseda University Grant for Special Research Projects (Project No.2020C-272).

APPENDIX: RECURRENCE FORMULA FOR I_n

Let us first consider $\mathcal{R}_2(t)$. The integral is given by

$$\mathcal{R}_2(t)[\rho] = \sum_{\{i_1,i_2\}} \int_0^t dt_1 \int_0^{t_1} dt_2 \gamma^2 e^{-\gamma t_1 \kappa_{i_1}(C)} \times \mathcal{Q}_{i_1}[e^{-\gamma t_2 \kappa_{i_2}(C)} \mathcal{Q}_{i_2} \rho]. \tag{A1}$$

Using the relation $af(N) = f(N+1)a$ for a function of the total excitation f, we get the exponential terms out of $\mathcal{Q}$'s:

$$\mathcal{R}_2(t)[\rho] = \sum_{\{i_1,i_2\}} \int_0^t dt_1 \int_0^{t_1} dt_2 \gamma^2 e^{-\gamma t_1 \kappa_{i_1}(C)} \times e^{-\gamma t_2 \kappa_{i_2}(\sigma(i_2)\varepsilon_{N+1})} \mathcal{Q}_{i_1} \mathcal{Q}_{i_2} \rho. \tag{A2}$$

By the same procedure as above, we obtain

$$\mathcal{R}_k(t)[\rho] = \sum_{\{i_1,\dots,i_k\}} \int_0^t dt_1 \dots \int_0^{t_{k-1}} dt_k \gamma^k e^{-\gamma t_1 \kappa_{i_1}(C)} e^{-\gamma t_2 \kappa_{i_2}(\sigma(i_i)\varepsilon_{N+1})} \dots e^{-\gamma t_k \kappa_{i_k}(\sigma(i_k)\varepsilon_{N+k-1})} \mathcal{Q}_{i_1} \dots \mathcal{Q}_{i_k} \rho. \tag{A3}$$

Therefore, the integral we need to evaluate is of the form

$$I_n(t; a_1, \dots, a_n) \equiv \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1 e^{-(a_n t_n + \dots + a_1 t_1)}. \tag{A4}$$

For this integral, we find a recurrence relation,

$$I_1(t; a_1) = \frac{1}{a_1}(1 - e^{-a_1 t}), \tag{A5}$$

$$I_n(t; a_1, \dots, a_n) = \int_0^t e^{-a_n t'} I_{n-1}(t'; a_1, \dots, a_{n-1}) dt' \quad (n \geqslant 2). \tag{A6}$$

This relation is useful for numerical computation. Furthermore, to obtain an analytically handy expression, we explicitly calculate $I_2(t_2; a_1, a_2)$:

$$I_2(t; a_1, a_2) = \frac{1}{a_1}[I_1(t; a_2) - I_1(t; a_1 + a_2)]. \tag{A7}$$

Then we obtain, for example,

$$\begin{aligned} I_3(t; a_1, a_2, a_3) &= \frac{1}{a_1}\left[\int_0^t e^{-a_3 t'} I_1(t'; a_2) dt' - \int_0^t e^{-a_3 t'} I_1(t'; a_1 + a_2) dt'\right] \\ &= \frac{1}{a_1}[I_2(t; a_2, a_3) - I_2(t; a_1 + a_2, a_3)] \end{aligned} \tag{A8}$$

or, generally,

$$I_n(t; a_1, \dots, a_n) = \frac{1}{a_1}[I_{n-1}(t; a_2, a_3, a_4, \dots, a_n) - I_{n-1}(t; a_1 + a_2, a_3, a_4, \dots, a_n)]. \tag{A9}$$

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