Analytical theory of the propagation of a dissipative soliton in a nonequilibrium resonant medium

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A nonlinear integrodifferential equation of the "reaction-diffusion" type is derived for an optical pulse propagating in a gain resonant two-level medium with the inhomogeneous broadening of the quantum transitions. The stable exact analytical solution of this equation in the form of a dissipative optical soliton with an asymmetric temporal profile is found and analyzed. The temporal duration of this soliton is much longer than the characteristic phase relaxation time but much shorter than the energy relaxation time. It is shown that the formation of such a soliton requires the presence of linear losses, created by the equilibrium part of the medium. It is noted that the found soliton solution qualitatively coincides with the dissipative soliton recently discovered experimentally in a laser microcavity.

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I. INTRODUCTION

Recently, there has been a significant increase in interest in the study of dissipative optical solitons, which are stable localized bunches of light energy [1–4]. Such energy bunches are capable of propagating over long distances in nonlinear active media. Atoms or molecules of the active medium are in excited (nonequilibrium) states. The energy stored in atoms can be transferred to an optical pulse, contributing to its amplification. For the formation of a dissipative soliton, the amplification of the optical pulse must be compensated for by dissipative energy losses. Such losses can be caused by the processes of intra-atomic relaxation, electrical conductivity, diffusion, thermal conductivity, etc. Propagating in an active medium, a dissipative optical soliton transfers some of the atoms from excited states to the ground state. As a result, the state of the medium after the propagation of the dissipative soliton differs from its initial state.

Many types of the dissipative solitons have been studied to date. Both resonant and nonresonant solitons were studied. Dissipative solitons interact inelastically with each other. For example, mutual annihilation of solitons or the formation of bound soliton states can occur [1–8].

Dissipative optical solitons can be used in systems for the transmission and processing of information [1,2]. In addition, it is proposed to use the dissipative solitons for the control by a motion of various micro- and nano-objects [9]. Nonresonant dissipative solitons in the laser and fiber systems are usually studied on the basis of generalized versions of the complex Ginzburg–Landau equation [3,4,10–15]. In this case, the cubic nonlinearity as well as the cubic-quintic and saturating nonlinearities are considered [3,4,16].

For describing resonant dissipative solitons, the systems of the nonlinear Maxwell-Bloch equations are used [10]. In this connection, mention should be made of works in which various versions of the dissipative nonlinear Maxwell-Bloch system were used without using the slowly varying envelope approximation [15–22]. In particular, solutions in the form of unipolar dissipative solitons were found.

In many theoretical works, solutions in the form of dissipative solitons were found by means of numerical simulations. Such methods are universal. Therefore they have certain advantages over analytical approaches. On the other hand, it is very difficult to find important dependences of the obtained solutions on various parameters using the numerical simulations. In this case, the restrictions imposed on the parameters of the medium are very important. It is very difficult to find quantitative dependences of the amplitude, temporal duration, and velocity of a dissipative soliton on the parameters of the medium using only numerical simulation. This circumstance seems to be very important for the attempts to experimental confirmation of the theoretical predictions. To find answers on these and other important questions, the analytical approaches are more preferable.

The present work is devoted to an analytical study of the possibility of formation of a dissipative optical soliton in an absorbing medium containing the impurity resonant atoms with an inverted population of quantum levels. The article is organized as follows. In Sec. II an integrodifferential equation of the reaction-diffusion type is derived from the dissipative system of the Maxwell-Bloch in the approximation of fast phase relaxation. In Sec. III the exact solution of this equation in the form of a dissipative soliton is found and its physical analysis is carried out. It is shown that this soliton is stable with respect to small perturbations of its profile. It is also noted that the obtained solution is in qualitative agreement with the dissipative soliton recently observed experimentally in a laser microcavity [23]. In addition, the parameters of the soliton were estimated for its possible experimental observations in a ruby crystal and in rubidium vapor. Section IV

summarizes the results of the work and formulates the main conclusions.

II. DERIVATION OF A NONLINEAR EQUATION

Let the optical pulse propagate in a two-level resonant medium along the z axis. Then this propagation is described by a system of equations of the following form:

$$\frac{\partial u}{\partial t} = -\Delta v - \frac{u}{T_2}, \tag{1}$$

$$\frac{\partial v}{\partial t} = \Delta u - \frac{v}{T_2} + \Omega w, \tag{2}$$

$$\frac{\partial w}{\partial t} = -\Omega v - \frac{w + 1/2}{T_1}, \tag{3}$$

$$\frac{\partial \Omega}{\partial z} + \frac{n_m}{c}\frac{\partial \Omega}{\partial t} = \beta \int_{-\infty}^{+\infty} v g(\Delta) d\Delta - \gamma_m \Omega. \tag{4}$$

Here u and v are the in-phase and quadrature components of the envelope of the resonant transition dipole moment, respectively, w is the population inversion of this transition (for the ground state of the atom we have $w = -1/2$, and for the excited state we have $w = +1/2$), T_1 and T_2 are the times of irreversible relaxations of energy and phase, respectively, $\Delta = \omega_0 - \omega$ is the detuning the carrier frequency ω of the laser pulse from the resonant transition frequency ω_0 of the selected atom, $\Omega = 2dE/\hbar$ is the local Rabi frequency of the laser pulse, d is the matrix element of the dipole moment of the resonant transition, $\hbar$ is the Planck constant, E is the envelope of the pulse electric field E associated with this field by means of the relation $E = 2E\cos[\omega(t - n_m z/c)]$, t is the time, n_m is the refractive index of the medium, c is the speed of light in vacuum, $\beta = 4\pi d^2 n\omega/\hbar c$, n is the concentration of two-level atoms, $g(\Delta)$ is the contour of inhomogeneous broadening of resonant quantum transitions in the form of a Lorentzian,

$$g(\Delta) = \frac{1}{\pi}\frac{T_2^*}{1 + (T_2^*\Delta)^2}, \tag{5}$$

where T_2^* is the characteristic time of reversible phase relaxation due to inhomogeneous broadening, and γ_m is the coefficient of linear nonresonant losses of the medium. In the case when $\gamma_m = 0$, the system (1)–(4) takes the form of the well-known Maxwell-Bloch equations [2,24–28].

For two-level atoms in the media, the relaxation times T_1 and T_2 are very differ from each other. In these cases the situations are possible when $T_2/T_1 \sim 10^{-5} - 10^{-9}$ [28], i.e., an inequality $T_2 \ll T_1$ is valid. Below we will assume that the temporal duration τ_p of the laser pulse satisfies the condition $\min(T_2, T_2^*) \ll \tau_p \ll T_1$.

Introducing the reduced relaxation time $T_r = \frac{T_2 T_2^*}{T_2 + T_2^*}$, where T_2^* is the time of reversible phase relaxation caused by the inhomogeneous broadening, we rewrite this condition in the form

$$T_r \ll \tau_p \ll T_1. \tag{6}$$

Due to the condition (6) we can put in Eq. (3), $T_1 = \infty$.

Now the problem is to exclude the material variables from (1)–(4). As a result, we must derive a nonlinear equation for the envelope of a pulse electric field. Introducing a complex variable $R = u + iv$, we rewrite (1) and (2) as follows:

$$\frac{\partial R}{\partial t} = \left(i\Delta - \frac{1}{T_2}\right)R + i\Omega w. \tag{7}$$

The solution of this equation, taking into account the fact that $R = 0$ at $\Omega = 0$, has the form

$$R(t) = i\int_0^\infty \Omega(t - t')w(t - t')e^{-(1/T_2 - i\Delta)t'}dt'. \tag{8}$$

Due to the inequality (6), the functions Ω and w in (8) are very weakly dependent on time in comparison with the exponential. In this case the Taylor series expansion with respect to parameter t' is valid:

$$\Omega(t - t')w(t - t') = \Omega(t)w(t) - t'\frac{\partial}{\partial t}(\Omega(t)w(t)) + \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t'^k\frac{\partial^k}{\partial t^k}(\Omega w). \tag{9}$$

Substituting (9) into (8), after integration we arrive at the Crisp expansion [29]:

$$R = i\sum_{k=0}^{\infty}(-1)^k\left(\frac{T_2}{1 - iT_2\Delta}\right)^{k+1}\frac{\partial^k}{\partial t^k}(\Omega w). \tag{10}$$

The left-hand side of inequality (6) means that the spectral width $\delta\omega_p \sim 1/\tau_p$ of the optical pulse is small in comparison with the width $\delta\omega_{tr} = 1/T_2 + 1/T_2^* = 1/T_r$ of the resonant quantum transition. Therefore the interaction of two-level atoms with an optical pulse under condition (6) is selective. Consequently, the change of the population inversion averaged over the contour of the inhomogeneous broadening $\langle w\rangle \equiv \int_{-\infty}^{+\infty} g(\Delta)w d\Delta$ is small. Taking into account this circumstance and keeping the first three terms in the expansion (10), we write

$$v = \mathrm{Im}R = \frac{T_2\Omega}{1 + (T_2\Delta)^2}w - w_{-\infty}T_2^2\frac{1 - (T_2\Delta)^2}{(1 + (T_2\Delta)^2)^2}\frac{\partial\Omega}{\partial t} + w_{-\infty}T_2^3\frac{1 - 3(T_2\Delta)^2}{(1 + (T_2\Delta)^2)^3}\frac{\partial^2\Omega}{\partial t^2}. \tag{11}$$

Here, in the second and third terms, we put approximately $w \approx w_{-\infty}$, where $w_{-\infty}$ is the initial population inversion of the resonant transition, when $t = -\infty$.

Now we substitute (11) into (3) under condition $T_1 = \infty$. Since the change of the population inversion w is small, we will take into account only the first term in the right-hand side of Eq. (11). Then

$$\frac{\partial w}{\partial t} = -w\frac{T_2\Omega^2}{1 + (T_2\Delta)^2} \approx -w_{-\infty}\frac{T_2\Omega^2}{1 + (T_2\Delta)^2}.$$

Integrating, we will have

$$w = w_{-\infty}\left[1 - \frac{\theta}{1 + (T_2\Delta)^2}\right], \tag{12}$$

where

$$\theta = T_2\int_{-\infty}^{\tau}\Omega^2 d\tau'. \tag{13}$$

After substituting Eqs. (12) and (13) into (11), we arrive at the expression

$$v = w_{-\infty}\left\{\frac{T_2\Omega}{1+(T_2\Delta)^2}\left[1-\frac{T_2}{1+(T_2\Delta)^2}\int_{-\infty}^{t}\Omega^2 dt'\right] - T_2^2\frac{1-(T_2\Delta)^2}{(1+(T_2\Delta)^2)^2}\frac{\partial\Omega}{\partial t} + T_2^3\frac{1-3(T_2\Delta)^2}{(1+(T_2\Delta)^2)^3}\frac{\partial^2\Omega}{\partial t^2}\right\}.$$

From here and from (5) we find

$$\langle v\rangle \equiv \int_{-\infty}^{+\infty} g(\Delta)v d\Delta = w_{-\infty}\left[T_r\Omega - T_r^2\left(1+\frac{T_2}{2T_2^*}\right)\Omega\int_{-\infty}^{t}\Omega^2 dt' - T_r^2\frac{\partial\Omega}{\partial t} + T_r^3\frac{\partial^2\Omega}{\partial t^2}\right]. \quad (14)$$

Substituting (14) into (4), we obtain

$$\frac{\partial\Omega}{\partial z} = \gamma\Omega - \varepsilon\Omega\int_{-\infty}^{\tau}\Omega^2 d\tau' + \sigma\frac{\partial^2\Omega}{\partial\tau^2}, \quad (15)$$

where

$$\gamma = \Gamma - \gamma_m, \quad \varepsilon = \Gamma T_r\left(1+\frac{T_2}{2T_2^*}\right), \quad \sigma = \Gamma T_r^2, \quad (16)$$

$\tau = t - z/v_0$, and the gain Γ and the linear group velocity v_0 are determined by the expressions

$$\Gamma = w_{-\infty}\beta T_r, \quad (17)$$

$$\frac{1}{v_0} = \frac{n_m}{c} + \Gamma T_r. \quad (18)$$

Equation (15) is a nonlinear integrodifferential equation of the "reaction-diffusion" type. Here the temporal diffusion occurs due to the reversible and irreversible phase relaxations [see the last term on the right-hand side of (15)].

The nonlinearity in Eq. (15) has the temporal nonlocality. This is the fundamental difference between Eq. (15) and well-known equations of the "reaction-diffusion" type with the local nonlinear sources [30–32].

After multiplying Eq. (15) by $2T_2\Omega$ and integrating with respect to τ, taking into account (13), we have

$$\frac{\partial\theta}{\partial z} = 2\gamma\theta - \frac{\varepsilon}{T_2}\theta^2 + \sigma\left[\frac{\partial^2\theta}{\partial\tau^2} - 2T_2\int_{-\infty}^{\tau}\left(\frac{\partial\Omega}{\partial\tau'}\right)^2 d\tau'\right]. \quad (19)$$

If we neglect the second term in the square brackets of Eq. (19), then this equation becomes the Fisher equation [31]. However, in our case we cannot neglect the second term in square brackets of Eq. (19), since its value is of the same order as the first term in these brackets.

After averaging Eq. (12) over the Lorentz contour (5) we will have

$$\langle w\rangle \equiv \int_{-\infty}^{+\infty} w g(\Delta) d\Delta = w_{-\infty}\left(1-\frac{T_r}{T_2}\theta\right). \quad (20)$$

Thus we can say that the dynamics of the averaged population inversion of the resonant transition is describing by Eq. (19).

III. EXACT SOLUTION IN THE FORM OF A DISSIPATIVE SOLITON

Using the solution of the Fischer equation in the form of a dissipative soliton, one can obtain a similar solution of Eq. (19). To do this, let us note that using (13), the second term in square brackets of Eq. (19) can be written in the following form:

$$2T_2\int_{-\infty}^{\tau}\left(\frac{\partial\Omega}{\partial\tau'}\right)^2 d\tau' = 2\int_{-\infty}^{\tau}\left(\frac{\partial}{\partial\tau'}\sqrt{\frac{\partial\theta}{\partial\tau'}}\right)^2 d\tau'. \quad (21)$$

Let us substitute the soliton solution of the Fisher equation into (19). This solution looks like [33]

$$\theta = A(1+\tanh\xi)^2, \quad (22)$$

where

$$\xi = \frac{\tau - z/V'}{\tau_p} = \frac{t - z/V}{\tau_p}, \quad (23)$$

and A, τ_p, and $1/V = 1/v_0 + 1/V'$ are the unknown constants.

After substituting Eqs. (22) and (23) into (19) and taking into account Eq. (21), we obtain

$$A = \frac{15}{32}\frac{\gamma}{\varepsilon}T_2, \quad \tau_p = 2\sqrt{\frac{2\sigma}{\gamma}}, \quad \frac{1}{V} = \frac{1}{v_0} - 3\sqrt{\frac{\gamma\sigma}{2}}. \quad (24)$$

Now, using Eqs. (13), (22), (23), and (24), we have a solution of Eq. (15) in the form of a dissipative soliton:

$$\Omega = \Omega_0\,\mathrm{sech}\xi\sqrt{1+\tanh\xi}, \quad (25)$$

where

$$\Omega_0 = \left(\frac{15}{32\sqrt{2}}\right)^{1/2}\left(\frac{\gamma^3}{\varepsilon^2\sigma}\right)^{1/4}. \quad (26)$$

Thus the expression (25) is an exact solution of Eq. (15). In this solution the variable ξ has the form (23). In turn, the constants τ_p, $1/V$, and Ω_0 are determined by the equalities (24) and (26).

It is obvious that the parameters τ_p and V have the meaning of the soliton temporal duration and the soliton velocity, respectively. In turn, the parameter Ω_0 is proportional to the amplitude of the soliton (see below). Let us study the stability of the found dissipative soliton. For this we introduce a dimensionless parameter Q which is proportional to the soliton energy:

$$Q = \theta_{|\tau\to+\infty} = T_2\int_{-\infty}^{+\infty}\Omega^2 d\tau. \quad (27)$$

Passing to the limit $\tau\to+\infty$ in Eq. (19), we write

$$\frac{dQ}{dz} = 2\gamma Q - \frac{\varepsilon}{T_2}Q^2 - 2\sigma T_2\int_{-\infty}^{+\infty}\left(\frac{\partial\Omega}{\partial\tau}\right)^2 d\tau. \quad (28)$$

Due to the last term on the right-hand side of (28), it is not possible to obtain a closed equation for the dynamic parameter Q. Note that for soliton (25) this parameter is determined by the expression $Q_\infty \equiv 15\gamma T_2/8\varepsilon$.

We will consider the small deviations of parameter Q from Q_∞: $|Q - Q_\infty| \ll Q_\infty$. In this case, in the last term of Eq. (28) we can approximate the Rabi frequency Ω to be replaced by expression (25). As a result, we obtain $\int_{-\infty}^{+\infty} (\frac{\partial\Omega}{\partial\tau})^2 d\tau = \frac{\Omega_0^2}{\tau_p}$. Now, using Eqs. (26) and (24), after the simple mathematical transformations we rewrite (28) in the form

$$\frac{dY}{dz} = \frac{15}{8}\gamma(1-Y)(Y-1/15), \tag{29}$$

where $Y = Q/Q_\infty$.

Equation (29) cannot be considered as an analog of the McCall–Hahn area theorem in the theory of self-induced transparency (SIT), since after deriving this equation the exact solution (25) was used. Therefore this equation is valid under the condition that the value of the parameter Y is very close to unity. Consequently, we can use Eq. (29) after considering the small deviations from the exact solution (25). In this case we can write $Y = 1 + y$, where $|y| \ll 1$. Then, linearizing Eq. (29) with respect to y, we obtain $\frac{dy}{dz} = -\frac{7}{4}\gamma y$. Thus the small deviations of the parameter Q from the value Q_∞ decrease exponentially. This is an essential argument in favor of the stability of dissipative soliton (25).

Let us now turn to the physical analysis of the obtained solution. Using Eqs. (16)–(18), we can express the constants τ_p, V, and Ω_0 through the physical parameters of the medium. Here we take into account that usually inequality $T_2^* \ll T_2$ is valid for many media. The situations are possible when $T_2^*/T_2 \sim 10^{-2} - 10^{-5}$ [34]. Under the condition $T_2^* \ll T_2$, we have $T_r \approx T_2^*$. Then from Eqs. (16)–(18), (24), (26), and (20) we find

$$\tau_p = 2.83\frac{T_2^*}{\sqrt{1-\gamma_m/\Gamma}}, \tag{30}$$

$$\frac{1}{V} = \frac{n_m}{c} + \left(1 - 2.12\sqrt{1-\frac{\gamma_m}{\Gamma}}\right)\Gamma T_2^*, \tag{31}$$

$$\Omega_0 = \frac{0.81}{\sqrt{T_2 T_2^*}}\left(1-\frac{\gamma_m}{\Gamma}\right)^{3/4}, \tag{32}$$

$$\frac{\langle w\rangle}{w_{-\infty}} = 1 - 0.94\frac{T_2^*}{T_2}\left(1-\frac{\gamma_m}{\Gamma}\right)(1+\tanh\xi)^2. \tag{33}$$

The temporal envelope profile (25) of a dissipative soliton is asymmetric: the front of the soliton is steeper than its tail (Fig. 1, top). The peak value $\Omega_m = \sqrt{32/27}\Omega_0 \approx 1.09\Omega_0$ of the Rabi frequency is achieved at $\xi = \xi_m = \ln 2/2 \approx 0.35$. The propagation of this soliton is accompanied by a running front (33) of the averaged population inversion (Fig. 1, bottom).

Similar solitons were discovered earlier by the numerical simulation methods [3,35,36]. It is important to note that such an asymmetric dissipative soliton was observed recently in a laser microcavity [23].

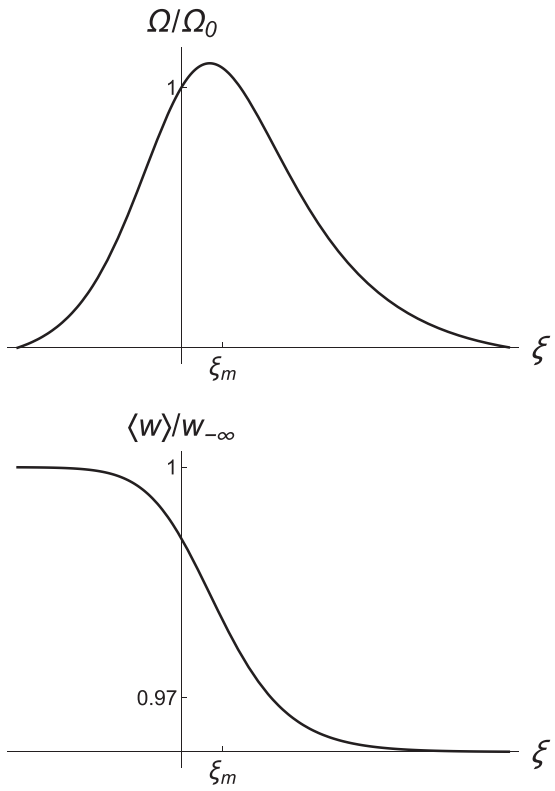

FIG. 1. Profiles of the normalized Rabi frequency of a dissipative soliton (top) and the accompanying averaged normalized population inversion of the quantum levels of the resonant transition (bottom): $T_2^*/T_2 = 0.2$, $\gamma_m/\Gamma = 0.9$.

The left-hand side of inequality (6) under the condition $T_2^* \ll T_2$ takes the form $T_2^* \ll \tau_p$. From this, as well as from (30), it follows that

$$0 < 1 - \frac{\gamma_m}{\Gamma} \ll 1. \tag{34}$$

In accordance with the left-hand side of double inequality (34), the resonant amplification should prevail over the nonresonant losses of the medium: $\Gamma > \gamma_m$. On the other hand, in accordance with the right-hand side of the double inequality (34), this dominance should be insignificant: $(\Gamma - \gamma_m)/\Gamma \ll 1$.

The averaged population inversion decreases very insignificantly [see Eqs. (33), (34), and condition $T_2^* \ll T_2$]. It was noted above that this occurs due to the selective nature of the interaction of two-level atoms with a laser pulse [see condition (6)]. However, this turns out to be sufficient to compensate for energy losses. This compensation leads to the formation of a dissipative soliton.

For the possibility of observing the dissipative soliton discussed here, numerical estimates of its parameters are very important. For a quantum transition $\bar{E}(2E) \to 4A_2(\pm 1/2)$ in a ruby crystal at liquid helium temperature, we have [37] $\omega \approx \omega_0 = 2.4 \times 10^{15}\mathrm{s}^{-1}$, $T_1 \sim 10^{-3}$s, $T_2 \approx 5 \times 10^{-8}$s, $T_2^* \approx 3 \times 10^{-10}$s, $d = 4.8 \times 10^{-21}$SGSE. For the concentration of resonance transitions we have $n = 1.6 \times 10^{19}\mathrm{cm}^{-3}$ [37]. Then $\Gamma \sim \beta T_r \approx \beta T_2^* \sim 50\,\mathrm{cm}^{-1}$. Assuming also $1 - \gamma_m/\Gamma \sim 0.1$,

for the temporal duration of the soliton we have from Eq. (30) that $\tau_p \sim 10T_2^* \sim 3 \times 10^{-9}$s.

For the resonant line in the rubidium vapor, which was used in [38], we have $\omega \approx \omega_0 = 2.4 \times 10^{15}\text{s}^{-1}$, $T_2 \sim 5.6 \times 10^{-8}$s, $T_2^* \sim 8 \times 10^{-10}$s, $d = 4.4 \times 10^{-18}$SGSE, $n \sim 10^{13}\text{cm}^{-3}$. Under these conditions the temperature is 80–100 °C. Then $\Gamma \sim 10^2\text{cm}^{-1}$. Under the condition $1 - \gamma_m/\Gamma \sim 0.1$ from (30), for this case we find $\tau_p \sim 10^{-8}$s.

Let us estimate the intensity of the soliton. Since the soliton intensity $I \sim \Omega^2$, then from (25) we obtain

$$I = I_0 \text{sech}^2\xi(1 + \tanh\xi), \tag{35}$$

where $I_0 = \frac{c\hbar^2\Omega_0^2}{16\pi d^2 n_m}$. Using also (32), we write $I_0 \sim \frac{c\hbar^2}{16\pi d^2 T_2 T_2^*}(1-\frac{\gamma_m}{\Gamma})^{3/2}$. Substituting here the values of the above parameters, we find for the intensity of a soliton in a crystal ruby $I_0 \sim 10^4\text{W/cm}^2$. For a dissipative soliton in rubidium vapor, we will have $I_0 \sim 10^{-2}\text{W/cm}^2$.

Thus the intensity of the considered dissipative soliton in rubidium vapor is six orders of magnitude lower than the intensity of a similar soliton in a ruby crystal. Note that a similar situation occurs for the effect of self-induced transparency [37,38]. This circumstance is due to the large value of the dipole moment of the resonance transition in a gas and a small value of this parameter in solids. Let us note that the temporal soliton duration in both cases lies inside of the range 1–10 ns.

Let us discuss the question related to the soliton velocity using expressions (24) and (31). From (24) it follows that the velocity of the dissipative soliton is greater than the linear group velocity: $V > v_0$. At the same time, as can be seen from (31) and (34), we have $V < c/n_m$. As a result, taking into account (18) for the range of possible values of the dissipative soliton velocity we write

$$\frac{c}{n_m + c\Gamma T_2^*} < V < \frac{c}{n_m}. \tag{36}$$

Substituting into (31) the above parameters for a ruby crystal, we find $V \sim 10^{-2}c$. In the case of rubidium vapor, we will have $V \sim 10^{-3}c$.

Thus under the conditions of a possible experiment, the velocity of a dissipative soliton can be much less than the speed of light. Let us now briefly consider the question of the formation of a dissipative soliton. Using Eq. (19), three characteristic scale lengths can be determined. Obviously, the scale length l of the linear amplification will be determined by the relation $l \sim 1/\gamma$.

Assuming in (19) $\partial\theta/\partial z \sim \theta/l_{\text{non}} \sim \varepsilon\theta^2$, we find the characteristic nonlinear scale length: $l_{\text{non}} \sim 1/(\varepsilon\theta) \sim (\varepsilon T_2 \tau_p \Omega_0^2)$. We define the diffusion scale length l_d in a similar way: $\partial\theta/\partial z \sim \theta/l_d \sim \sigma\partial^2\theta/\partial t^2 \sim \sigma\theta/\tau_p^2$. From here we find $l_d \sim \tau_p^2/\sigma$. Using the expressions (22),(24), (16), and (17), for all three scale lengths we have

$$l \sim l_{\text{non}} \sim l_d \sim \frac{1}{\Gamma - \gamma_m}.$$

Substituting here the above values of the parameters Γ and γ_m/Γ for the crystal ruby and for the rubidium vapor we find $l \sim l_{\text{non}} \sim l_d \sim 0.1$ cm. This is the characteristic distance at which a dissipative soliton can be formed.

The above estimates allow us to state that it is preferable to observe a dissipative soliton in gases than in solids. First, the gases do not need to be cooled to low temperatures. Second, the intensities of dissipative solitons in gases are much lower than the intensities of similar solitons in solids. Third, in gaseous media it is much easier to change the concentrations of both resonant and nonresonant atoms. Therefore it is easier to satisfy condition (34) in gases.

IV. CONCLUSION

Thus in this work we obtained a nonlinear equation, Eq. (15), of the reaction-diffusion type, which describes the propagation of an optical pulse in a medium containing the gain resonant transitions (two-level atoms). In this case, the pulse duration satisfies condition (6), which is equivalent to the inequality (34). Equation (15) contains a source with the nonlocal nonlinearity. It is shown here that Eq. (15) is equivalent to the Eq. (19), which is a generalization of the well-known Fisher equation. This equation describes dynamic processes of a physical, chemical, and biological nature [32]. The generalization of the Fisher equation here consists in the presence of an integral term in the square brackets (19).

An exact analytical solution of Eq. (15) is found and analyzed in the form of a localized dissipative soliton with the asymmetric temporal profile. There is reason to believe that this solution describes an experimentally observed dissipative soliton in a semiconductor laser microcavity [23].

It is important that the formation of such dissipative soliton requires the presence of a linear absorption of the electromagnetic field by the nonresonant atoms of the medium [see the last term on the right-hand side of Eq. (4)].

The analytical study carried out in this work made it possible to determine the propagation velocity interval (36) of a dissipative soliton in the resonant gain medium. The expressions for the soliton temporal duration (30) and its amplitude (32) have been obtained here. These results are not obvious and cannot be obtained on the basis of numerical simulations. In addition, using the analytical approach, it was possible to show that the considered dissipative soliton is stable with respect to small perturbations of its profile. On the other hand, the study of the formation process of the soliton considered here is possible, most likely, only with the using of numerical simulations.

For very powerful solitons, the assumption about a small change in the population difference of the quantum states may turn out to be incorrect. This can lead to equations describing the formation and propagation of the resonant dissipative solitons.

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