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Parametrized entanglement monotone

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Entanglement concurrence has been widely used for featuring entanglement in quantum experiments. As an entanglement monotone it is related to specific quantum Tsallis entropy. Our goal in this paper is to propose a parametrized bipartite entanglement monotone which is named as q-concurrence inspired by general Tsallis entropy. We derive an analytical lower bound for the q-concurrence of any bipartite quantum entanglement state by employing positive partial transposition criterion and realignment criterion, which shows an interesting relationship to the strong separability criteria. The parametrized entanglement monotone is used to characterize bipartite isotropic states. Finally, we provide a computational method to estimate the q-concurrence for any entanglement by superposing two bipartite pure states. It shows that the superposition operations can at most increase one ebit for the q-concurrence in the case that the two states being superposed are biorthogonal or one-sided orthogonal. These results reveal a series of phenomena about the entanglement, which may be interesting in quantum communication and quantum information processing.

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I. INTRODUCTION

Quantum entanglement as one of the most remarkable phenomena of quantum mechanics reveals the fundamental insights into the nature of quantum correlations. It is the key of many interesting quantum tasks such as quantum teleportation [1], quantum dense coding [2], quantum secret sharing [3], and quantum cryptography [4]. A fundamental problem is how to justify whether a given quantum composite system state is entangled or separable. So far, there are two important entanglement criteria for the bipartite entanglement. One is positive partial transpose (PPT) criterion [5], which implies the partial transposition satisfying $\rho^{T_A} \geqslant 0$ for any separable state ρ_{AB}. The PPT criterion is a necessary and sufficient condition of its separability for pure states and $2 \otimes 2$ and $2 \otimes 3$ mixed states, but in general not sufficient for higher dimensions [5,6]. The other is complementary operational criterion which is called the realignment criterion [7–9]. For a separable ρ_{AB}, the realignment operation $\mathcal{R}(\rho)$ satisfies $\|\mathcal{R}(\rho)\|_1 \leqslant 1$. Both entanglement criteria are widely used in quantum experiments and quantum applications [10].

Entanglement measure as another approach is also used to quantify entanglement [11,12]. There are some interesting entanglement measures for bipartite entangled systems, such as the concurrence [13–15], entanglement of formation [16,17], negativity [18,19], Tsallis-q entropy of entanglement [20], and Rényi-α entropy of entanglement [21,22]. Howbeit, the explicit computation of these measures for arbitrary states is a formidable task because of the extremization for mixed states. So far, analytical results are only available for special measures and two-qubit states or special higher-dimensional mixed states [15,23–27]. Moreover, these entanglement measures are also related to the PPT criterion and the realignment criterion. Recently, some efforts have been made towards the analytical lower bounds of concurrence [28–31]. Specially, a completely analytical and powerful lower bound for the concurrence was found in Ref. [29] by relating this quantity to the PPT and realignment criteria. Further, in Ref. [30] the authors sharpened this bound by relating concurrence to the local uncertainty relations and the correlation matrix criterion. Based on PPT and realignment, a lower bound for genuine tripartite entanglement concurrence was obtained in Ref. [31]. Thereby, the development of analytical lower bounds for various entanglement measures is of great interest. One of the goals in this paper is to construct an analytical lower bound of an entanglement monotone. This bound includes the result in Ref. [29] as a special case.

In fact, as one reason of the difficult quantification of mixed states, any entanglement generated by superposing two pure states cannot be simply featured by two individual states being superposed. Entanglement of superpositions is first introduced by Linden *et al.* [32], who found an upper bound on the entanglement of formation for the superposition in terms of the entanglement of two individual states being superposed. Their bound is then improved by Gour [33]. So far, the entanglement of superpositions has been addressed in terms of different entanglement measures [34–40]. Although it is difficult to exactly estimate the entanglement measure of the superposed entanglement from individual states being superposed, however, it may be helpful for exploring the nature of quantum entanglement by investigating the superposed entanglement. Additionally, it provides direct results for the approximate quantification of mixed entanglement [41–43].

The concurrence for a bipartite pure state $|\psi\rangle_{AB}$ is defined by $C(|\psi\rangle_{AB}) = \sqrt{2(1 - \mathrm{Tr}\rho_A^2)}$ [13]. It plays a major role in

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entanglement distributions such as entanglement swapping and remote preparation of bipartite entangled states [44]. In fact, the concurrence for pure states is related to specific Tsallis entropy [45,46] as $C(|\psi\rangle_{AB}) = \sqrt{2T_2(\rho_A)}$ for $q = 2$. Noteworthily, Tsallis entropy provides a generalization of traditional Boltzmann-Gibbs statistical mechanics and enables us to find a consistent treatment of dynamics in many nonextensive physical systems such as long-range interactions, long-time memories, and multifractal structures [47]. Tsallis entropy also provides many intriguing applications in the realms of quantum information theory [48–51]. Hence a natural problem is how to construct an entanglement measure from general Tsallis entropy with $q \geqslant 2$. Our goal in this paper is to solve this problem.

The outline of the rest is as follows. In Sec. II, we propose a parametrized bipartite entanglement monotone which is related to general Tsallis entropy for any $q \geqslant 2$. The so-called q-concurrence is actually an entanglement monotone. We prove an analytical lower bound for the q-concurrence by using the PPT and realignment criteria. Moreover, we valuate the q-concurrence for isotropic states. In Sec. III, we investigate the entanglement of the superposition of two pure states by using the q-concurrence in terms of two states being superposed. The entanglement of the superposed state can be expressed explicitly when two input states are biorthogonal or one-sided orthogonal. As a result, the superposing operation can only increase at most one ebit in terms of the q-concurrence in both cases. The last section concludes the paper.

II. ENTANGLEMENT MONOTONE

Before giving our definition, we recall a well-known bipartite entanglement monotone. For any arbitrary bipartite pure state $|\psi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence [13,14] is given by

$$C(|\psi\rangle_{AB}) = \sqrt{2\left(1 - \mathrm{Tr}\rho_A^2\right)}, \tag{1}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ is the reduced density matrix of the subsystem A by tracing out the subsystem B.

The concurrence defined in Eq. (1) can be regarded as a function of specific Tsallis entropy of $q = 2$ [45,46], i.e., $C(|\psi\rangle_{AB}) = \sqrt{2T_2(\rho_A)}$ for $q = 2$. In this section, we define another parametrized entanglement monotone named the q-concurrence which is related to general Tsallis entropy for any $q \geqslant 2$.

A. *q*-concurrence

Definition 1. For an arbitrary bipartite pure state $|\psi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the q-concurrence is defined as

$$C_q(|\psi\rangle_{AB}) = 1 - \mathrm{Tr}\rho_A^q \tag{2}$$

for any $q \geqslant 2$, where ρ_A is the reduced density operator of the subsystem A.

It is clear that $C_q(|\psi\rangle_{AB}) = 0$ if and only if $|\psi\rangle_{AB}$ is a separable state, i.e., $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$. The q-concurrence may be concerned with Schatten q-norm for the positive semidefinite matrices, where the Schatten q-norm [52] is defined as

$$\|A\|_q = (\mathrm{Tr}A^q)^{1/q}. \tag{3}$$

It will be a useful tool to prove the subadditivity inequality in the following Lemma 1.

Suppose a pure state $|\psi\rangle_{AB}$ defined on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ has the Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{m} \sqrt{\lambda_i}|a_i\rangle_A|b_i\rangle_B. \tag{4}$$

It is apparent that the reduced density matrices ρ_A and ρ_B have the same spectra of $\{\lambda_i\}$. Hence we have

$$C_q(|\psi\rangle_{AB}) = 1 - \mathrm{Tr}\rho_A^q = 1 - \mathrm{Tr}\rho_B^q. \tag{5}$$

This implies that

$$C_q(|\psi\rangle) = 1 - \sum_{i=1}^{m} \lambda_i^q, \tag{6}$$

where $C_q(|\psi\rangle)$ satisfies $0 \leqslant C_q(|\psi\rangle) \leqslant 1 - m^{1-q}$. The lower bound is obtained for product states, while the upper bound is achieved for the maximally entangled pure states $\frac{1}{\sqrt{m}}\sum_{i=1}^{m}|ii\rangle$.

For a mixed state ρ_{AB} on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we define its q-concurrence via the convex-roof extension as follows:

$$C_q(\rho_{AB}) = \inf_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C_q(|\psi_i\rangle_{AB}), \tag{7}$$

where the infimum is taken over all the pure-state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, with $\sum_i p_i = 1$ and $p_i \geqslant 0$.

So far, several results have been made for the requirements that a reasonable measure of entanglement should fulfill [11,53,54]. Specially, it has been proposed in Ref. [55] that the monotonicity under local operations and classical communication (LOCC) has to satisfy as the only requirement of any entanglement measure. This kind of entanglement measure is then defined as entanglement monotone. In fact, Vidal [55] states that it is an entanglement monotone E if the following conditions hold.

(i) $E(\rho) \geqslant 0$ for any state ρ and $E(\rho) = 0$ if ρ is fully separable.

(ii) For a pure state $|\Psi\rangle$, the measure is a function of the reduced density operator $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$, i.e., $E(|\Psi\rangle) = f(\rho_A)$, where the function f has the following properties: (a) f is invariant under any unitary transformation U, i.e., $f(U\rho_A U^\dagger) = f(\rho_A)$; (b) f is concave, i.e., $f[\lambda\rho_1 + (1-\lambda)\rho_2] \geqslant \lambda f(\rho_1) + (1-\lambda)f(\rho_2)$ for $\lambda \in (0, 1)$.

(iii) For a mixed state ρ, the measure $E(\rho)$ is defined as the convex-roof extension, i.e.,

$$E(\rho) = \inf_{\{p_i,|\psi_i\rangle\}} \left\{\sum_i p_i E(|\psi_i\rangle) | \sum_i p_i|\psi_i\rangle\langle\psi_i| = \rho\right\}, \tag{8}$$

where the minimum is taken over all possible pure-state decompositions of ρ.

These conditions (i)–(iii) formalize intuitive properties of an entanglement monotone. From this point of view any entanglement monotone could be regarded as a measure of entanglement. We present the following Lemma 1 for verifying that the q-concurrence defined in Eq. (7) is a proper entanglement monotone.

Lemma 1. Define the function

$$F_q(\rho) = 1 - \mathrm{Tr}\rho^q \tag{9}$$

for any density matrix ρ and $q \geqslant 2$. $F_q(\rho)$ satisfies the following properties.

(i) *Non-negativity*. $F_q(\rho) \geqslant 0$ for any density operator ρ, where the equality holds for pure states.

(ii) *Symmetry*. $F_q(\rho_A) = F_q(\rho_B)$ for a pure state ρ_{AB} of the composite system AB.

(iii) *Subadditivity*. For a general bipartite state ρ_{AB}, $F_q(\rho_{AB})$ satisfies the inequalities:

$$|F_q(\rho_A) - F_q(\rho_B)| \leqslant F_q(\rho_{AB}) \leqslant F_q(\rho_A) + F_q(\rho_B). \tag{10}$$

(iv) *Concavity and quasiconvexity*. F_q is concave, i.e.,

$$\sum_i p_i F_q(\rho_i) \leqslant F_q\left(\sum_i p_i\rho_i\right), \tag{11}$$

where $\{p_i\}$ is a probability distribution and ρ_i's are density matrices. The equality holds if and only if ρ_i's are identical for all $p_i > 0$. Moreover, F_q is quasiconvex, i.e.,

$$F_q\left(\sum_i p_i\rho_i\right) \leqslant \sum_i p_i^q F_q(\rho_i) + 1 - \sum_i p_i^q, \tag{12}$$

where the equality holds if and only if ρ_i's have supports on orthogonal subspaces, i.e., $\rho_i = |\psi_i\rangle\langle\psi_i|$, and $\{|\psi_i\rangle\}$ are orthogonal.

The proof of Lemma 1 is provided in Appendix A. Next we prove $C_q(\rho)$ is a proper entanglement monotone.

Proposition 1. The q-concurrence $C_q(\rho)$ in Eq. (7) is an entanglement monotone.

Proof. From the non-negativity in Lemma 1 and Eq. (7), it follows that $C_q(\rho) \geqslant 0$ for any density matrix ρ, where the equality holds iff ρ is separable. Furthermore, from the concavity in Lemma 1, we know that $F_q[\lambda\rho_1 + (1-\lambda)\rho_2] \geqslant \lambda F_q(\rho_1) + (1-\lambda)F_q(\rho_2)$ for any density matrices ρ_1 and ρ_2 and $\lambda \in (0, 1)$. Thus $C_q(|\phi\rangle_{AB})$ is a concave function of ρ_A. Finally, $C_q(|\phi\rangle_{AB})$ is invariant under local unitary transformations from the invariance of $\mathrm{Tr}\rho^q$. Then, the convex-roof extension of the q-concurrence $C_q(\rho)$ for mixed states is a proper entanglement monotone [55]. ■

Note that, for a given bipartite entanglement $|\psi\rangle$, the entanglement monotone $C_q(|\psi\rangle)$ in Eq. (2) is invariant under the local unitary operations. From the Cayley-Hamilton Theorem, the reduced density matrix ρ_A of rank d satisfies a characteristic equation as $\sum_{j=0}^d a_j\rho_A^j = 0$. From the spectra decomposition of $\rho_A = \sum_{i=1}^d \lambda_i|\phi_i\rangle\langle\phi_i|$, the q-concurrence in Eq. (6) satisfies a linear equation as $\sum_{q=0}^d a_q C_q(|\psi\rangle) - \sum_i a_i C_0(|\psi\rangle) = 0$. Hence all the q-concurrences with $q \leqslant d$ will be evaluated for any d-dimensional pure states. However, for a mixed state ρ_{AB} the q-concurrences $C_q(\rho_{AB})$ in Eq. (7) do not satisfy the characteristic equation of the density matrix ρ_A or ρ_{AB}. This implies that each q-concurrence of $C_q(\rho_{AB})$ may provide different meanings for featuring the entanglement.

The proposed q-concurrence is equivalent to the previous Tsallis-q [20] for specific q. However, they are different for large enough q. In fact, it is easy to show that the q-concurrence can be used to detect the bipartite entanglement, while the Tsallis-q entanglement cannot when $q \to \infty$, namely, the Tsallis-q cannot be applied for large q. In fact, as a parametrized generalization of the Boltzmann-Gibbs entropy, the Tsallis entropy is specially interesting in long-range systems such as the motion of cold atoms in dissipative optical lattices [56,57], spin glass relaxation [58], or trapped ion [59]. The present q-concurrences for large q may be applicable for featuring these long-range entangled systems beyond the standard concurrence [13] and the Tsallis-q entanglement [20].

B. Lower bound on the *q*-concurrence

In contrast to the simple case of pure entangled states in Eq. (2), the quantification of mixed states is still challenging due to the optimization procedures [10]. Fortunately, we present an effective operational way to detect the q-concurrence for any bipartite quantum state, which manifests an essential quantitative relation among the q-concurrence, PPT criterion, and realignment criterion. Before presenting the lower bound, we recall two separability criteria.

PPT criterion [5,6]. Given a bipartite state $\rho_{AB} = \sum_{ijkl}\rho_{ij,kl}|ij\rangle\langle kl|$. If ρ_{AB} is separable, then the partial transposition ρ^{T_A} with respect to the subsystem A has the non-negative spectrum, i.e., $\rho^{T_A} \geqslant 0$. The partial transpose ρ^{T_A} is given by $\rho^{T_A} = [\sum_{ijkl}\rho_{ij,kl}|ij\rangle\langle kl|]^{T_A} = \sum_{ijkl}\rho_{ij,kl}|kj\rangle\langle il|$, where the subscripts i and k are the row and column indices for the subsystem A, respectively, while j and l are such indices for the subsystem B.

Realignment criterion [7–9]. Let $\mathcal{R}$ be the realignment operation on the joint system $\rho_{AB} = \sum_{ijkl}\rho_{ij,kl}|ij\rangle\langle kl|$. The output is given by $\mathcal{R}(\rho) = \sum_{ijkl}\rho_{ij,kl}|ik\rangle\langle jl|$. If ρ_{AB} is separable, then $\|\mathcal{R}(\rho)\|_1 \leqslant 1$, where $\|X\|_1$ denotes the trace norm defined by $\|X\|_1 = \mathrm{Tr}\sqrt{XX^\dagger}$ [60].

According to these criteria, a given state ρ is entangled if the trace norms $\|\rho^{T_A}\|_1$ or $\|\mathcal{R}(\rho)\|_1$ are strictly larger than 1.

For the pure state defined in Eq. (4), it is straightforward to prove [29]

$$\|\rho^{T_A}\|_1 = \|\mathcal{R}(\rho)\|_1 = \left(\sum_{i=1}^m\sqrt{\lambda_i}\right)^2. \tag{13}$$

Besides, we know that

$$\left(\sum_{i=1}^m\sqrt{\lambda_i}\right)^2 \leqslant m\sum_{i=1}^m\lambda_i = m \tag{14}$$

from the Cauchy-Schwarz inequality, where $\{\lambda_i\}$ is a probability distribution.

Theorem 1. For any mixed entanglement state ρ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with the dimension of m and n ($m \leqslant n$), respectively, the q-concurrence $C_q(\rho)$ satisfies the following inequality:

$$C_q(\rho) \geqslant \frac{\left[\max\left\{\|\rho^{T_A}\|_1^{q-1}, \|\mathcal{R}(\rho)\|_1^{q-1}\right\} - 1\right]^2}{m^{2q-2} - m^{q-1}}. \quad (15)$$

Proof. Consider the optimal decomposition of ρ as $\rho = \sum_i p_i\rho_i$ in order to achieve the infimum of $C_q(\rho)$ in Eq. (7), where ρ_i's are pure states with $\rho_i = |\psi_i\rangle\langle\psi_i|$. First, we will prove that

$$C_q(\rho_i) \geqslant \frac{\left(\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}, \quad (16)$$

$$C_q(\rho_i) \geqslant \frac{\left[\|\mathcal{R}(\rho_i)\|_1^{q-1} - 1\right]^2}{m^{2q-2} - m^{q-1}}. \quad (17)$$

In fact, note that the function $g(\lambda) = \lambda^q$ is convex for $q \geqslant 2$ and $\lambda \in (0, 1)$. This means that $h(\lambda_1, \ldots, \lambda_m) = \sum_{k=1}^m \lambda_k^q$ is Schur convex [52]. Since the uniform distribution of $\{\frac{1}{m}, \ldots, \frac{1}{m}\}$ is majorized by any other distribution $\{\lambda_1, \ldots, \lambda_m\}$, i.e., $\{\frac{1}{m}, \ldots, \frac{1}{m}\} \prec \{\lambda_1, \ldots, \lambda_m\}$. For the Schur-convex function $h(\lambda_1, \ldots, \lambda_m)$ it follows that [52]

$$\sum_{k=1}^m \lambda_k^q \geqslant \frac{1}{m^{q-1}}. \quad (18)$$

From Eq. (6), we have

$$\begin{aligned} C_q(\rho_i) &= 1 - \sum_{k=1}^m \lambda_{ik}^q \\ &= \frac{\left(\sum_{j=1}^m \sqrt{\lambda_{ij}}\right)^{2q-2} - \sum_{k=1}^m \lambda_{ik}^q \left(\sum_{j=1}^m \sqrt{\lambda_{ij}}\right)^{2q-2}}{\left(\sum_{j=1}^m \sqrt{\lambda_{ij}}\right)^{2q-2}} \\ &\geqslant \frac{\left(\sum_{k=1}^m \sqrt{\lambda_{ik}}\right)^{2q-2} - 1}{\left(\sum_{k=1}^m \sqrt{\lambda_{ik}}\right)^{2q-2}} \quad (19) \\ &= \frac{\left[\left(\sum_{j=1}^m \sqrt{\lambda_{ij}}\right)^{2q-2} - 1\right]^2}{\left(\sum_{k=1}^m \sqrt{\lambda_{ik}}\right)^{2q-2}\left[\left(\sum_{k=1}^m \sqrt{\lambda_{ik}}\right)^{2q-2} - 1\right]} \\ &\geqslant \frac{\left(\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}, \quad (20) \end{aligned}$$

where the inequality (19) is due to the inequality $-\sum_{k=1}^m \lambda_{ik}^q (\sum_{j=1}^m \sqrt{\lambda_{ij}})^{2q-2} \geqslant -1$, which can be proved by using the inequalities (14) and (18). Moreover, for a pure state $\rho_i = |\psi_i\rangle\langle\psi_i|$, we have $\|\rho_i^{T_A}\|_1 = (\sum_{k=1}^m \sqrt{\lambda_{ik}})^2 \leqslant m$ as shown in Eq. (13). This implies the inequality (20).

From Eq. (20), we have

$$\sum_i p_i C(\rho_i) \geqslant \frac{\sum_i p_i\left(\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}. \quad (21)$$

In what follows, we prove that

$$\left(\|\rho^{T_A}\|_1^{q-1} - 1\right)^2 \leqslant \sum_i p_i\left(\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2. \quad (22)$$

In fact, for $\rho^{T_A} = \sum_i p_i \rho_i^{T_A}$, we obtain

$$\begin{aligned} \left(\|\rho^{T_A}\|_1^{q-1} - 1\right)^2 &= \left(\left\|\sum_i p_i\rho_i^{T_A}\right\|_1^{q-1} - 1\right)^2 \\ &\leqslant \left(\sum_i p_i\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2 \quad (23) \\ &\leqslant \sum_i p_i\|\rho_i^{T_A}\|_1^{2q-2} \\ &\quad -2\sum_i p_i\|\rho_i^{T_A}\|_1^{q-1} + 1 \quad (24) \\ &= \sum_i p_i\left(\|\rho_i^{T_A}\|_1^{q-1} - 1\right)^2. \quad (25) \end{aligned}$$

Note that we have $\|\sum_i p_i\rho_i^{T_A}\|_1^{q-1} \leqslant \sum_i p_i\|\rho_i^{T_A}\|_1^{q-1}$ from the convexity of function $f(x) = \|x\|_1^{q-1}$ with $q \geqslant 2$. Moreover, $\|\sum_i p_i\rho_i^{T_A}\|_1^{q-1} \geqslant 1$ and $\sum_i p_i\|\rho_i^{T_A}\|_1^{q-1} \geqslant 1$ from $\|\rho_i^{T_A}\|_1^{q-1} \geqslant 1$ for any density matrix ρ_i and $q \geqslant 2$. This follows the inequality (23). The inequality (24) is obtained from the convexity of the function $f(x) = x^2$, i.e.,

$$\left(\sum_i p_i\|\rho_i^{T_A}\|_1^{q-1}\right)^2 \leqslant \sum_i p_i\|\rho_i^{T_A}\|_1^{2q-2}. \quad (26)$$

Thereby, by substituting Eq. (25) into Eq. (21) we obtain

$$C_q(\rho) \geqslant \frac{\left(\|\rho^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}. \quad (27)$$

From Eq. (13), similar to Eq. (20), we can prove that

$$C_q(\rho) \geqslant \frac{\left[\|\mathcal{R}(\rho)\|_1^{q-1} - 1\right]^2}{m^{2q-2} - m^{q-1}}. \quad (28)$$

Combining (27) and (28), we get that

$$C_q(\rho) \geqslant \max\left\{\frac{\left(\|\rho^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}, \frac{\left[\|\mathcal{R}(\rho)\|_1^{q-1} - 1\right]^2}{m^{2q-2} - m^{q-1}}\right\}. \quad (29)$$

Note that $\|\rho^{T_A}\|_1^{q-1} \geqslant 1$ and $\|\mathcal{R}(\rho)\|_1^{q-1} \geqslant 1$ for any density matrix ρ and $q \geqslant 2$. The inequality (15) follows from the inequality (29). This completes the proof. ■

For $q = 2$, the inequality (15) in Theorem 1 is reduced to $C_2(\rho) \geqslant \frac{[\max\{\|\rho^{T_A}\|_1, \|\mathcal{R}(\rho)\|_1\} - 1]^2}{m(m-1)}$, i.e., $\sqrt{2C_2(\rho)} \geqslant \sqrt{\frac{2}{m(m-1)}} \max\{\|\rho^{T_A}\|_1, \|\mathcal{R}(\rho)\|_1\} - 1$, which is just the result (6) in Ref. [29]. The interesting feature is that the lower bound in the right side of (15) depends on the entropy parameter related to the detailed states. This allows us to choose proper q depending on m and n such that $C_q(\rho) \gg 0$ for experimental entanglement detection of specific state ρ.

Example 1. Consider a 5-qubit Heisenberg model with a random magnetic field in the z direction. Its Hamiltonian is given by

$$\begin{aligned} H_5 = &\, 5\vec{\sigma}_1 \cdot \vec{\sigma}_5 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_3 + 4\vec{\sigma}_2 \cdot \vec{\sigma}_4 \\ &+ 6\vec{\sigma}_3 \cdot \vec{\sigma}_5 + \sum_{i=1}^5 h_j\sigma_j^z, \quad (30) \end{aligned}$$

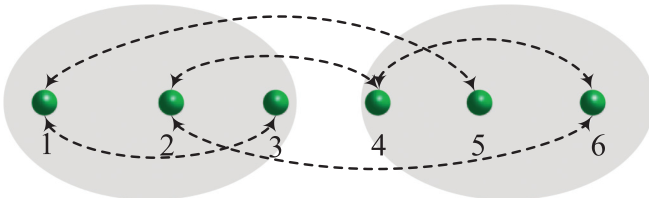


FIG. 1. Schematic six-body system with long-range correlations. The dotted line denotes a long-range correlation between two particles. Two gray regions represent a bipartite cut of $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

where we suppose that the pairs of $(1, 5)$, $(1, 3)$, $(2, 4)$, $(3, 5)$ are correlated with each other; herein, $\vec{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ represents a vector of Pauli matrices on qubit j and $h_j \in [-10, 10]$ denotes the strength of the disorder. For a 6-qubit system, the Hamiltonian is given by

$$H_6 = 5\vec{\sigma}_1\cdot\vec{\sigma}_5 + 2\vec{\sigma}_1\cdot\vec{\sigma}_3 + 4\vec{\sigma}_2\cdot\vec{\sigma}_4 + \vec{\sigma}_4\cdot\vec{\sigma}_6 + 6\vec{\sigma}_2\cdot\vec{\sigma}_6 + \sum_{i=1}^{6} h_j\sigma_j^z, \qquad (31)$$

where the pairs of (1,5),(1,3),(2,4),(2,6),(4,6) have correlation with each other (as shown in Fig. 1). Moreover, the Hamiltonian for a 7-qubit system is given by

$$H_7 = 5\vec{\sigma}_1\cdot\vec{\sigma}_5 + 2\vec{\sigma}_1\cdot\vec{\sigma}_3 + 4\vec{\sigma}_2\cdot\vec{\sigma}_4 + \vec{\sigma}_4\cdot\vec{\sigma}_6 + 6\vec{\sigma}_2\cdot\vec{\sigma}_6 + 3\vec{\sigma}_5\cdot\vec{\sigma}_7 + \sum_{i=1}^{7} h_j\sigma_j^z, \qquad (32)$$

where the pairs of (1,5),(1,3),(2,4),(2,6),(4,6),(5,7) are correlated with each other. Consider the initial state $|+\rangle^{\otimes 5}$, $|+\rangle^{\otimes 6}$, $|+\rangle^{\otimes 7}$, respectively, and the evolution time 500, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. We can obtain the q-concurrence defined in Eq. (2) and Tsallis-q entanglement as shown in Fig. 2. From Fig. 2, the Tsallis entanglement decreases in terms of entropy parameter q. This implies that the Tsallis entanglement may be unapplicable for large q because it cannot be distinguished from separable states when the experimental noise is involved. However, the present q-concurrence tends to 1 for large q. This is greatly different from separable states. This implies that the q-concurrence can be applied in this case. This holds for any bipartite entangled pure states. Another example is for characterizing the entangled ground state of many-body systems.

Example 2. Isotropic states are a class of mixed states on $\mathbb{C}^d\otimes\mathbb{C}^d$ which are invariant under the operation $U\otimes U^*$ with any unitary transformation U. Such mixed states are generally expressed as [7]

$$\rho_F = \frac{1-F}{d^2-1}(\mathbb{1} - |\Psi^+\rangle\langle\Psi^+|) + F|\Psi^+\rangle\langle\Psi^+|, \qquad (33)$$

where $\mathbb{1}$ denotes the identity operator, $|\Psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |ii\rangle$, and F is the fidelity of ρ_F with respect to $\rho_\Psi = |\Psi^+\rangle\langle\Psi^+|$, i.e., $F = f_{\Psi^+}(\rho_F) = \langle\Psi^+|\rho_F|\Psi^+\rangle$, which satisfies $0 \leqslant F \leqslant 1$.

For isotropic states ρ_F, there are lots of analytic results in the entanglement of formation [26], the tangle and concurrence [24], and Rényi α-entropy entanglement [61]. Inspired by the techniques [24,26,61], the q-concurrence $C_q(\rho_F)$ for these states will be derived by an extremization as follows.

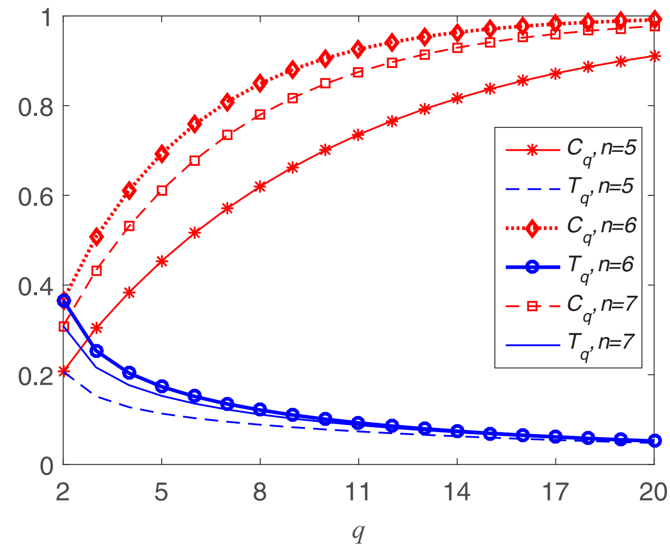


FIG. 2. Comparison of the q-concurrence and Tsallis-q entanglement in terms of q. Here, we consider three evolution systems with 5, 6, and 7 qubits.

Lemma 2. The q-concurrence for isotropic states ρ_F on Hilbert space $\mathbb{C}^d\otimes\mathbb{C}^d$ $(d\geqslant 2)$ is given by

$$C_q(\rho_F) = co(\xi(F, q, d)), \qquad (34)$$

where $F\in(1/d, 1]$ and $co(\,)$ denotes the largest convex function that is upper bounded by a given function $\xi(F, q, d)$ defined as

$$\xi(F, q, d) = 1 - \gamma^{2q} - (d-1)\delta^{2q}, \qquad (35)$$

with $\gamma = \sqrt{F}/\sqrt{d} + \sqrt{(d-1)(1-F)}/\sqrt{d}$ and $\delta = \sqrt{F}/\sqrt{d} - \sqrt{1-F}/\sqrt{d(d-1)}$.

The evaluation is essentially algebraic and quite tedious, as shown in Appendix B. For convenience, we take $q = 2$ as an example. The 2-concurrence of an two-qubit isotropic state ρ_F is given by [24]

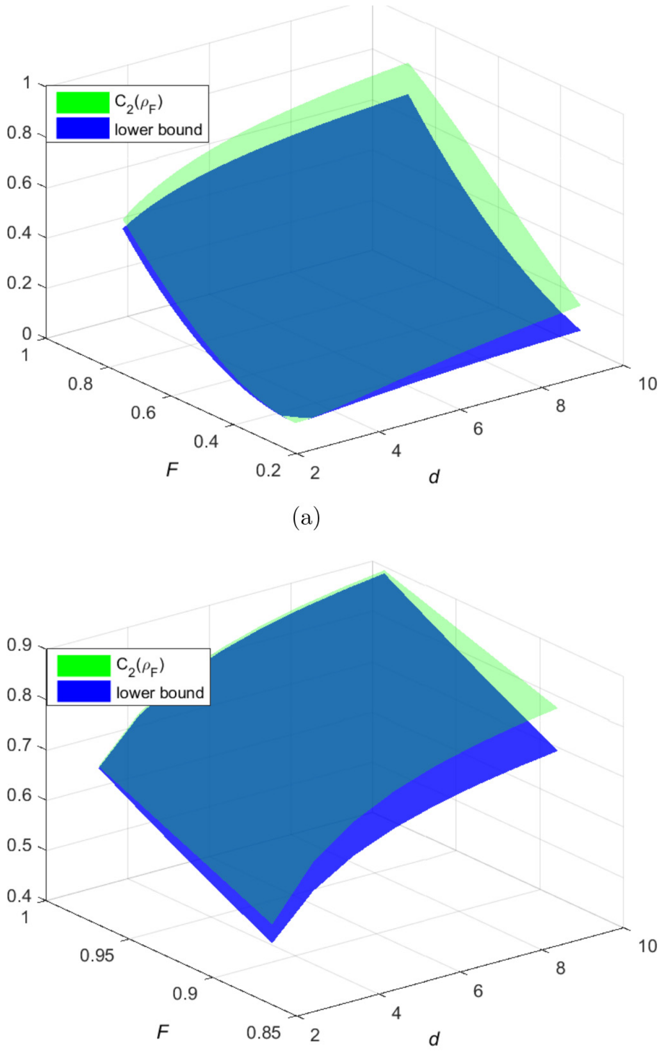
$$C_2(\rho_F) = \begin{cases} 0, & 0\leqslant F\leqslant \frac{1}{2}, \\ \frac{(1-2F)^2}{2}, & \frac{1}{2}\leqslant F\leqslant 1. \end{cases} \qquad (36)$$

Moreover, from Theorem 1, it implies $C_2(\rho_F)\geqslant (1-2F)^2/2 = C_2(\rho_F)$. This implies that the upper bound gives the exact value of the 2-concurrence for this qubit system. For arbitrary $d\geqslant 3$, the 2-concurrence $C_2(\rho_F)$ is given by

$$C_2(\rho_F) = \begin{cases} 0, & F\leqslant \frac{1}{d}, \\ \xi(F, 2, d), & \frac{1}{d}\leqslant F\leqslant \frac{4(d-1)}{d^2}, \\ \frac{dF-d}{d-1} + \frac{d-1}{d}, & \frac{4(d-1)}{d^2}\leqslant F\leqslant 1. \end{cases} \qquad (37)$$

Moreover, it is shown that $\|\rho_F^{T_A}\|_1 = \|\mathcal{R}(\rho_F)\|_1 = dF$ for $F > 1/d$ [8,19]. From Theorem 1 we get that

$$C_2(\rho_F) \geqslant \frac{(dF-1)^2}{d^2-d}. \qquad (38)$$



(a)

(b)

FIG. 3. 2-concurrence (green) of isotropic states ρ_F and lower bounds (blue) in Example 1. (a) $3 \leqslant d \leqslant 10$ and $1/d \leqslant F \leqslant 4(d-1)/d^2$. (b) $3 \leqslant d \leqslant 10$ and $4(d-1)/d^2 \leqslant F \leqslant 1$.

For simplicity, $C_2(\rho_F)$ in Eq. (37) and its lower bounds in Eq. (38) are shown in Fig. 3 with $3 \leqslant d \leqslant 10$. Especially, it illustrates that the present lower bound is very close to the exact value of the 2-concurrence, which shows the tightness of our bounds.

The lower bound in Theorem 1 can be used to detect the q-concurrence for all the entangled states of the two-qubit or qubit-qutrit system because the PPT criterion is necessary and sufficient for the separability in both cases [5,6]. Unfortunately, it cannot detect all the other entangled states due to the limitation of the PPT criterion [5,6] and the realignment criterion [7,8]. Thus it is intriguing to explore other bounds for the q-concurrence of general mixed states.

III. q-CONCURRENCE OF SUPERPOSITION STATES

Assume that a state $|\Gamma\rangle$ is generated by superposing two pure states $|\Phi\rangle$ and $|\Psi\rangle$, i.e., $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$. Our goal in this section is to explore the q-concurrence for these superposition states. We discuss how the entanglement of superpositions of some given pure states is related to the entanglement contained in input states. In detail, we consider four cases: two component states in the superposition are biorthogonal states, one-sided orthogonal states, orthogonal states, and arbitrary states.

A. Biorthogonal states

Definition 2. Two bipartite states $|\Phi\rangle_{AB}$ and $|\Psi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ are biorthogonal if they satisfy

$$\mathrm{Tr}_B[\mathrm{Tr}_A(|\Phi\rangle\langle\Phi|)\mathrm{Tr}_A(|\Psi\rangle\langle\Psi|)] = 0, \tag{39}$$

$$\mathrm{Tr}_A[\mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)\mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)] = 0. \tag{40}$$

For two biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$ we get up to local unitary transformations [33] that

$$|\Phi\rangle = \sum_{i=1}^{d_1} a_i|i\rangle_A|i\rangle_B,$$
$$|\Psi\rangle = \sum_{i=1}^{d} b_i|i+d_1\rangle_A|i+d_1\rangle_B, \tag{41}$$

where a_i, b_i are positive constants. In this case, the q-concurrence of the superposition state $|\Gamma\rangle$ will be evaluated as follows.

Theorem 2. Given two biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$, then the q-concurrence of the superposition $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ satisfies

$$C_q(|\Gamma\rangle) = |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2), \tag{42}$$

where $h_q(t) = 1 - t^q - (1-t)^q$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Proof. From Eq. (41) the reduced states of the system A for $|\Phi\rangle$ and $|\Psi\rangle$ are diagonal in the same basis. More specifically, from Eq. (2), we have $C_q(|\Gamma\rangle) = F_q(\rho_A)$ with $\rho_A = \mathrm{Tr}_B(|\Gamma\rangle\langle\Gamma|)$. It follows that the first d_1 eigenvalues of ρ_A are given by $\{|\alpha|^2 a_i^2, i = 1, 2, \dots, d_1\}$, and all the remaining eigenvalues are given by $\{|\beta|^2 b_j^2, j = 1, 2, \dots, d\}$. Thus, from Definition 1, we get

$$\begin{aligned} C_q(|\Gamma\rangle) &= 1 - \sum_i \left(|\alpha|^2 a_i^2\right)^q - \sum_j \left(|\beta|^2 b_j^2\right)^q \\ &= |\alpha|^{2q}\left(1 - \sum_i a_i^{2q}\right) + |\beta|^{2q}\left(1 - \sum_j b_i^{2q}\right) \\ &\quad + 1 - |\alpha|^{2q} - |\beta|^{2q} \\ &= |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2). \end{aligned} \tag{43}$$

From Lemma 1 for any density matrices ρ and σ we get the following inequalities:

$$F_q(|\alpha|^2\rho + |\beta|^2\sigma) \geqslant |\alpha|^2F_q(\rho) + |\beta|^2F_q(\sigma), \tag{44}$$

$$F_q(|\alpha|^2\rho + |\beta|^2\sigma) \leqslant |\alpha|^{2q}F_q(\rho) + |\beta|^{2q}F_q(\sigma) + h_q(|\alpha|^2). \tag{45}$$

Moreover, from Lemma 1, Eq. (45) holds iff ρ and σ are orthogonal. Since $|\Phi\rangle$ and $|\Psi\rangle$ are biorthogonal, their reduced density matrices ρ_A and σ_A are orthogonal. Thus, from Eq. (45), we get Eq. (42). ■

Note that the entanglement of the superposition is related to the average of the entanglement of two states being superposed. For convenience, we define the increase of q-concurrence entanglement for the superposition state $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ as follows:

$$\Delta C_q(|\Gamma\rangle) = C_q(|\Gamma\rangle) - [|\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle)]. \quad (46)$$

For the biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$, we obtain the following Corollary 1.

Corollary 1. Given two biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$, the increase of the q-concurrence of the superposition state $|\Gamma\rangle$ is upper bounded by one ebit, i.e., $\Delta C_q(|\Gamma\rangle) \leqslant 1$.

Proof. From Theorem 2, we obtain

$$C_q(|\Gamma\rangle) - [|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle)] = h_q(|\alpha|^2). \quad (47)$$

Moreover, since $|\alpha|, |\beta| \leqslant 1$, it is obvious that

$$\begin{aligned}\Delta C_q(|\Gamma\rangle) &= C_q(|\Gamma\rangle) - [|\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle)] \\ &\leqslant C_q(|\Gamma\rangle) - [|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle)].\end{aligned} \quad (48)$$

Thereby, combining Eqs. (47) and (48), we get

$$\Delta C_q(|\Gamma\rangle) \leqslant h_q(|\alpha|^2) \leqslant 1, \quad (49)$$

which implies that the increase of the q-concurrence for the superposition state $|\Gamma\rangle$ cannot be greater than one ebit. ■

Example 3. Consider the superposition state

$$|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle, \quad (50)$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, where $|\Phi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ and $|\Psi\rangle = \cos\phi|22\rangle + \sin\phi|33\rangle$ are generalized bipartite entangled states for $\theta, \phi \in (0, \pi/2)$. Note that $|\Phi\rangle$ and $|\Psi\rangle$ are biorthogonal states. From Eq. (2) it is easy to check that $C_q(|\Phi\rangle) = 1 - \cos^{2q}\theta - \sin^{2q}\theta$, $C_q(|\Psi\rangle) = 1 - \cos^{2q}\phi - \sin^{2q}\phi$, and

$$\begin{aligned}C_q(|\Gamma\rangle) = 1 &- |\alpha|^{2q}(\cos^{2q}\theta + \sin^{2q}\theta) \\ &- |\beta|^{2q}(\cos^{2q}\phi + \sin^{2q}\phi).\end{aligned} \quad (51)$$

Moreover, we have $h_q(|\alpha|^2) = 1 - |\alpha|^{2q} - |\beta|^{2q}$. Thus Eq. (42) holds for the superposition state $|\Gamma\rangle$ in Eq. (50), which is consistent with Theorem 2. From Eq. (46) we have

$$\begin{aligned}\Delta C_q(|\Gamma\rangle) &= C_q(|\Gamma\rangle) - [|\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle)] \\ &= (|\alpha|^2 - |\alpha|^{2q})(\cos^{2q}\theta + \sin^{2q}\theta) \\ &\quad + (|\beta|^2 - |\beta|^{2q})(\cos^{2q}\phi + \sin^{2q}\phi).\end{aligned} \quad (52)$$

To show the result in Corollary 1, we take the special case of $\alpha = \beta = 1/\sqrt{2}$ and $q = 4$. As shown in Fig. 4, it is apparent that the function $\Delta C_q(|\Gamma\rangle)$ of θ and ϕ satisfies $\Delta C_q(|\Gamma\rangle) \leqslant 1$.

B. One-sided orthogonal states

Definition 3. Two bipartite states $|\Phi\rangle_{AB}$ and $|\Psi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ are one-sided orthogonal if they satisfy only one of Eqs. (39) and (40).

Without loss of generality, we assume that two one-sided orthogonal states satisfy Eq. (39). Up to local unitary transformations [33], we have

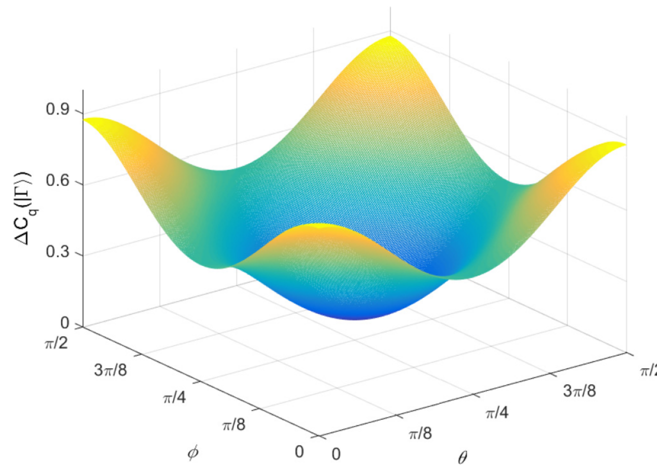


FIG. 4. Increase $\Delta C_q(|\Gamma\rangle)$ of the q-concurrence for the superposition state $|\Gamma\rangle$ in Example 2. Here, $\alpha = \beta = 1/\sqrt{2}$ and $q = 4$.

$$\begin{aligned}|\Phi\rangle &= \sum_{i=1}^{d_1} a_i |i\rangle_A |i\rangle_B, \\ |\Psi\rangle &= \sum_{i=1}^{d} b_i |i\rangle_A |i + d_1\rangle_B,\end{aligned} \quad (53)$$

where a_i and b_i are positive constants.

Now, consider the case of $|\Phi\rangle$ and $|\Psi\rangle$ being one-sided orthogonal but not necessarily biorthogonal, i.e., they satisfy Eq. (53).

Theorem 3. Given two one-sided orthogonal states $|\Phi\rangle$ and $|\Psi\rangle$, the q-concurrence of the superposition state $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ is given by

$$\begin{aligned}C_q(|\Gamma\rangle) &= |\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) \\ &\quad + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|,\end{aligned} \quad (54)$$

where ρ_A and ρ_B are reduced density matrices of ρ_{AB}, $\rho_{AB} := |\alpha|^2|\Phi\rangle\langle\Phi| + |\beta|^2|\Psi\rangle\langle\Psi|$, with $|\alpha|^2 + |\beta|^2 = 1$, and h_q is defined in Theorem 2.

Proof. Note that $\||\Gamma\rangle\| = \sqrt{\langle\Gamma|\Gamma\rangle} = \sqrt{|\alpha|^2 + |\beta|^2} = 1$. It means that $|\Gamma\rangle$ is normalized. From Eq. (53) we know that

$$\mathrm{Tr}_B(|\Gamma\rangle\langle\Gamma|) = |\alpha|^2 \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) + |\beta|^2 \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|), \quad (55)$$

which is also defined as the reduced density matrix ρ_A by tracing ρ_{AB} over the subsystem B. Hence we get that $\mathrm{Tr}_B(|\Gamma\rangle\langle\Gamma|) = \rho_A$. From Eq. (2) it follows that

$$C_q(|\Gamma\rangle) = F_q(\rho_A). \quad (56)$$

Note that $\mathrm{Tr}_A(|\Gamma\rangle\langle\Gamma|) \neq \rho_B$ because ρ_B is defined as the reduced density matrix by tracing ρ_{AB} over the subsystem A, i.e.,

$$\rho_B = |\alpha|^2 \mathrm{Tr}_A(|\Phi\rangle\langle\Phi|) + |\beta|^2 \mathrm{Tr}_A(|\Psi\rangle\langle\Psi|), \quad (57)$$

while

$$\begin{aligned}\mathrm{Tr}_A(|\Gamma\rangle\langle\Gamma|) &= |\alpha|^2 \mathrm{Tr}_A(|\Phi\rangle\langle\Phi|) + \alpha\beta^* \mathrm{Tr}_A(|\Phi\rangle\langle\Psi|) \\ &\quad + \alpha^*\beta\, \mathrm{Tr}_A(|\Psi\rangle\langle\Phi|) + |\beta|^2 \mathrm{Tr}_A(|\Psi\rangle\langle\Psi|).\end{aligned} \quad (58)$$

From Eqs. (53) and (57) the first d_1 eigenvalues of ρ_B are given by $\{|\alpha|^2 a_i^2 | i = 1, 2, \ldots, d_1\}$, and all the rest are shown as $\{|\beta|^2 b_j^2 | j = 1, 2, \ldots, d\}$. Thus, according to the definition of $F_q(\rho)$ in Eq. (9), we get

$$\begin{aligned} F_q(\rho_B) &= 1 - \sum_i \left(|\alpha|^2 a_i^2\right)^q - \sum_j \left(|\beta|^2 b_j^2\right)^q \\ &= |\alpha|^{2q}\left(1 - \sum_i a_i^{2q}\right) + |\beta|^{2q}\left(1 - \sum_j b_i^{2q}\right) \\ &\quad + 1 - |\alpha|^{2q} - |\beta|^{2q} \\ &= |\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) + h_q(|\alpha|^2). \end{aligned} \tag{59}$$

By utilizing Eqs. (45) and (55) we have

$$\begin{aligned} F_q(\rho_A) &\leqslant |\alpha|^{2q} F_q[\mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)] \\ &\quad + |\beta|^{2q} F_q[\mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)] + h_q(|\alpha|^2) \\ &= |\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) + h_q(|\alpha|^2). \end{aligned} \tag{60}$$

From Eqs. (59) and (60) $F_q(\rho_B) \geqslant F_q(\rho_A)$ for the one-sided orthogonal states defined in Eq. (39). Due to Eqs. (56) and (59) we obtain

$$\begin{aligned} C_q(|\Gamma\rangle) &= |\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) \\ &\quad + h_q(|\alpha|^2) - [F_q(\rho_B) - F_q(\rho_A)]. \end{aligned} \tag{61}$$

In a similar manner, for one-sided orthogonal states in Eq. (40) we get that

$$\begin{aligned} C_q(|\Gamma\rangle) &= |\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) \\ &\quad + h_q(|\alpha|^2) - [F_q(\rho_A) - F_q(\rho_B)]. \end{aligned} \tag{62}$$

Combining Eqs. (61) and (62), we have completed the proof. ■

Note that $F_q(\rho_A) = F_q(\rho_B)$ holds for biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$. Thus we can obtain Eq. (42) for Theorem 2. Moreover, since $|\Phi\rangle$ and $|\Psi\rangle$ are orthogonal pure states, we have $F_q(\rho_{AB}) = h_q(|\alpha|^2)$. This implies $h_q(|\alpha|^2) \geqslant |F_q(\rho_A) - F_q(\rho_B)|$ from the triangle inequality of $F_q(\rho_{AB})$ in Lemma 1.

Similar to Corollary 1, we get the following result for one-sided orthogonal states.

Corollary 2. Given two one-sided orthogonal states $|\Phi\rangle$ and $|\Psi\rangle$, the increase of the q-concurrence for the superposition state $|\Gamma\rangle$ is no more than one ebit, i.e., $\Delta C_q(|\Gamma\rangle) \leqslant 1$.

Proof. From Theorem 3 we have

$$\begin{aligned} &C_q(|\Gamma\rangle) - [|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle)] \\ &\quad = h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|. \end{aligned} \tag{63}$$

Note that

$$h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)| \leqslant h_q(|\alpha|^2), \tag{64}$$

where the equality holds for the biorthogonal states $|\Phi\rangle$ and $|\Psi\rangle$.

Clearly, Eq. (48) holds for one-sided orthogonal states $|\Phi\rangle$ and $|\Psi\rangle$. Thus, combining Eqs. (63), (64), and (48), we obtain $\Delta C_q(|\Gamma\rangle) \leqslant h_q(|\alpha|^2) \leqslant 1$. This completes the proof. ■

Example 4. Consider the superposition state

$$|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle \tag{65}$$

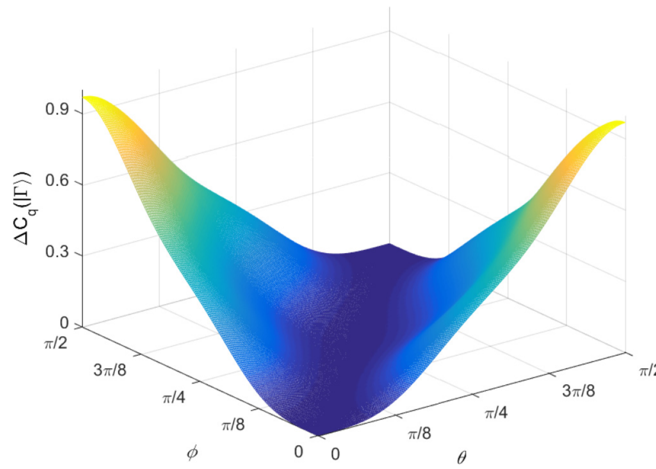


FIG. 5. Increase of the q-concurrence $\Delta C_q(|\Gamma\rangle)$ for the superposition state $|\Gamma\rangle$ in Example 3. Here, $\alpha = \beta = 1/\sqrt{2}$ and $q = 6$.

with $|\alpha|^2 + |\beta|^2 = 1$, where $|\Phi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ and $|\Psi\rangle = \cos\phi|02\rangle + \sin\phi|13\rangle$ are one-sided orthogonal entangled states for $\theta, \phi \in (0, \pi/2)$. From Eq. (2), it is easy to calculate that $C_q(|\Phi\rangle) = 1 - \cos^{2q}\theta - \sin^{2q}\theta$, $C_q(|\Psi\rangle) = 1 - \cos^{2q}\phi - \sin^{2q}\phi$, and

$$\begin{aligned} C_q(|\Gamma\rangle) &= 1 - (|\alpha|^2\cos^2\theta + |\beta|^2\cos^2\phi)^q \\ &\quad - (|\alpha|^2\sin^2\theta + |\beta|^2\sin^2\phi)^q. \end{aligned} \tag{66}$$

According to Eq. (9), one can check that

$$\begin{aligned} F_q(\rho_A) &= 1 - (|\alpha|^2\cos^2\theta + |\beta|^2\cos^2\phi)^q \\ &\quad - (|\alpha|^2\sin^2\theta + |\beta|^2\sin^2\phi)^q \end{aligned} \tag{67}$$

and

$$\begin{aligned} F_q(\rho_B) &= 1 - |\alpha|^{2q}\cos^{2q}\theta - |\alpha|^{2q}\sin^{2q}\theta \\ &\quad - |\beta|^{2q}\cos^{2q}\phi - |\beta|^{2q}\sin^{2q}\phi. \end{aligned} \tag{68}$$

Moreover, note that $h_q(|\alpha|^2) = 1 - |\alpha|^{2q} - |\beta|^{2q}$. It is easy to verify Eq. (54) for the superposition state $|\Gamma\rangle$ defined in Eq. (65). This is consistent with Theorem 3.

Similar to Eq. (52), we can calculate $\Delta C_q(|\Gamma\rangle)$ of the superposition state $|\Gamma\rangle$ defined in Eq. (65). For convenience, we take $\alpha = \beta = 1/\sqrt{2}$ and $q = 6$ for numerical evaluations. From Fig. 5, it indicates that $\Delta C_q(|\Gamma\rangle) \leqslant 1$, which is consistent with Corollary 2.

C. Arbitrary states

For general cases, two pure states that are superposed are not orthogonal. This means that the superposition state is not normalized, if we define $|\Gamma'\rangle = \frac{1}{c_+}|\Gamma\rangle$ with $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ as the normalized state of $|\Gamma\rangle$, where c_+ is the normalization constant. By using Theorem 1, we prove the following inequality for its q-concurrence.

Theorem 4. Given any two different states $|\Phi\rangle$ and $|\Psi\rangle$, then the q-concurrence of the superposition

$|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ satisfies

$$C_q(|\Gamma'\rangle) \leqslant \frac{2}{c_+^2}[|\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] \quad (69)$$

or

$$C_q(|\Gamma'\rangle) \leqslant \frac{2}{c_+^2}[|\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] - \frac{c_-^2}{c_+^2}\frac{\left(\|\sigma^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}, \quad (70)$$

where $|\alpha|^2 + |\beta|^2 = 1$ and σ is the density operator of $|\Gamma'_-\rangle$, i.e., $\sigma = |\Gamma'_-\rangle\langle\Gamma'_-|$. σ^{T_A} stands for a partial transpose with respect to the subsystem A. $\|X\|_1$ denotes the trace norm.

Proof. Consider that Alice, in addition to Hilbert space $\mathcal{H}_A$, introduces an auxiliary state $|0\rangle_a$ and $|1\rangle_a$ on Hilbert space $\mathcal{H}_a$. And consider the state

$$|\Delta\rangle = \alpha|\Phi\rangle|0\rangle_a + \beta|\Psi\rangle|1\rangle_a. \quad (71)$$

Bob's reduced state of $|\Delta\rangle$ is given by

$$\rho_B = |\alpha|^2\text{Tr}_A(|\Phi\rangle\langle\Phi|) + |\beta|^2\text{Tr}_A(|\Psi\rangle\langle\Psi|). \quad (72)$$

According to Eq. (45), we get

$$F_q(\rho_B) \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2). \quad (73)$$

Since Bob can repeat the same operations by introducing an auxiliary state, in this case, for the reduced density matrix ρ_A, we have

$$F_q(\rho_A) \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2). \quad (74)$$

Thus, from Eqs. (73) and (74), we have

$$\max\{F_q(\rho_A), F_q(\rho_B)\} \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2). \quad (75)$$

Additionally, ρ_B may be written into

$$\rho_B = \frac{c_+^2}{2}\text{Tr}_A\left[\left(\frac{\alpha|\Phi\rangle + \beta|\Psi\rangle}{c_+}\right)\left(\frac{\alpha^*\langle\Phi| + \beta^*\langle\Psi|}{c_+}\right)\right] + \frac{c_-^2}{2}\text{Tr}_A\left[\left(\frac{\alpha|\Phi\rangle - \beta|\Psi\rangle}{c_-}\right)\left(\frac{\alpha^*\langle\Phi| - \beta^*\langle\Psi|}{c_-}\right)\right], \quad (76)$$

where $|\Gamma_\pm\rangle = \alpha|\Phi\rangle \pm \beta|\Psi\rangle$, $|\Gamma'_\pm\rangle = \frac{1}{c_\pm}|\Gamma_\pm\rangle$ is the normalized state of $|\Gamma_\pm\rangle$, and $c_\pm$ are the normalization constants of $|\Gamma_\pm\rangle$.

Now, using Eqs. (44) and (76), we get the following inequalities:

$$F_q(\rho_B) \geqslant \frac{c_+^2}{2}C_q(|\Gamma'_+\rangle) + \frac{c_-^2}{2}C_q(|\Gamma'_-\rangle). \quad (77)$$

In similar way, we obtain

$$F_q(\rho_A) \geqslant \frac{c_+^2}{2}C_q(|\Gamma'_+\rangle) + \frac{c_-^2}{2}C_q(|\Gamma'_-\rangle). \quad (78)$$

Similar to Eq. (75), Eqs. (77) and (78) can be written into

$$\min\{F_q(\rho_A), F_q(\rho_B)\} \geqslant \frac{c_+^2}{2}C_q(|\Gamma'_+\rangle) + \frac{c_-^2}{2}C_q(|\Gamma'_-\rangle). \quad (79)$$

If $\max\{F_q(\rho_A), F_q(\rho_B)\} = F_q(\rho_A)$, according to Eq. (75) we get

$$F_q(\rho_B) + F_q(\rho_A) \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) + F_q(\rho_B). \quad (80)$$

It means that

$$\min\{F_q(\rho_A), F_q(\rho_B)\} = F_q(\rho_B) \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - F_q(\rho_A) + F_q(\rho_B). \quad (81)$$

If $\max\{F_q(\rho_A), F_q(\rho_B)\} = F_q(\rho_B)$, we can also get

$$\min\{F_q(\rho_A), F_q(\rho_B)\} \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - F_q(\rho_B) + F_q(\rho_A). \quad (82)$$

Thus, combining Eqs. (81) and (82), we obtain

$$\min\{F_q(\rho_A), F_q(\rho_B)\} \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|. \quad (83)$$

Moreover, from Eqs. (79) and (83), we have

$$\frac{c_+^2}{2}C_q(|\Gamma'_+\rangle) + \frac{c_-^2}{2}C_q(|\Gamma'_-\rangle) \leqslant |\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|. \quad (84)$$

If $C_q(|\Gamma'_-\rangle) = 0$, i.e., the superposition state $|\Gamma'_-\rangle$ is separable, from Eq. (84) it is obvious that

$$C_q(|\Gamma'_+\rangle) \leqslant \frac{2}{c_+^2}[|\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|]. \quad (85)$$

If $C_q(|\Gamma'_-\rangle) > 0$, i.e., a superposition state $|\Gamma'_-\rangle$ defined on $m \otimes n (m \leqslant n)$ systems is entangled, from Eq. (16) we get

$$C_q(|\Gamma'_+\rangle) \leqslant \frac{2}{c_+^2}[|\alpha|^{2q}C_q(|\Phi\rangle) + |\beta|^{2q}C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] - \frac{c_-^2}{c_+^2}\frac{\left(\|\sigma^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}, \quad (86)$$

where σ is the density operator of $|\Gamma'_-\rangle$, i.e., $\sigma = |\Gamma'_-\rangle\langle\Gamma'_-|$. This completes the proof. ■

Corollaries 1 and 2 feature the maximal changes of the entanglement by the superposing two special states. For more general states, we get the following result.

Corollary 3. Given two arbitrary states $|\Phi\rangle$ and $|\Psi\rangle$, the increase of the q-concurrence for the superposition state $|\Gamma\rangle$

satisfies

$$C_q(|\Gamma'\rangle) - a[|\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle)] \leqslant a[h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] \quad (87)$$

or

$$C_q(|\Gamma'\rangle) - a[|\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle)] \leqslant a[h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] - \frac{ac_-^2\left(\|\sigma^{T_A}\|_1^{q-1} - 1\right)^2}{2m^{2q-2} - 2m^{q-1}}, \quad (88)$$

where $a = \frac{2}{c_+^2}$, and $\|\sigma^{T_A}\|_1$ is defined in Theorem 4.

Proof. Since $|\alpha|, |\beta| \leqslant 1$, we get that

$$|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) \leqslant |\alpha|^2 C_q(|\Phi\rangle) + |\beta|^2 C_q(|\Psi\rangle). \quad (89)$$

From Eqs. (69) and (70), Corollary 3 is obtained by a straightforward evaluation. ■

Example 5. Consider the superposition state

$$|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle, \quad (90)$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, where

$$|\Phi\rangle = \cos\theta|00\rangle + \frac{1}{\sqrt{2}}\sin\theta|11\rangle + \frac{1}{\sqrt{2}}\sin\theta|22\rangle,$$
$$|\Psi\rangle = \cos\phi|03\rangle + \frac{1}{\sqrt{2}}\sin\phi|11\rangle + \frac{1}{\sqrt{2}}\sin\phi|22\rangle, \quad (91)$$

which are entangled for $\theta, \phi \in (0, \pi/2)$. From Eq. (2), we obtain

$$C_q(|\Phi\rangle) = 1 - \cos^{2q}\theta - 2^{1-q}\sin^{2q}\theta, \quad (92)$$

$$C_q(|\Psi\rangle) = 1 - \cos^{2q}\phi - 2^{1-q}\sin^{2q}\phi. \quad (93)$$

Here, we consider $\alpha = \beta = 1/\sqrt{2}$ for the superposition state $|\Gamma\rangle$ in Eq. (90). It should be clear that

$$C_q(|\Gamma'_+\rangle) = 1 - \frac{2^q(\cos^2\theta + \cos^2\phi)^q + 2(\sin\theta + \sin\phi)^{2q}}{4^q c_+^{2q}}, \quad (94)$$

$$C_q(|\Gamma'_-\rangle) = 1 - \frac{2^q(\cos^2\theta + \cos^2\phi)^q + 2(\sin\theta - \sin\phi)^{2q}}{4^q c_-^{2q}}, \quad (95)$$

where $|\Gamma'_\pm\rangle$ is the normalized state of $|\Gamma_\pm\rangle$ and $c_\pm = \sqrt{1 \pm \sin\theta\,\sin\phi}$ is the normalization constant. According to Eq. (9), it is easy to check that

$$F_q(\rho_A) = 1 - \frac{(\cos^2\theta + \cos^2\phi)^q}{2^q} - \frac{(\sin^2\theta + \sin^2\phi)^q}{2^{2q-1}} \quad (96)$$

and

$$F_q(\rho_B) = 1 - \frac{\cos^{2q}\theta + \cos^{2q}\phi}{2^q} - \frac{(\sin^2\theta + \sin^2\phi)^q}{2^{2q-1}}. \quad (97)$$

Moreover, $h_q(|\alpha|^2) = 1 - 2^{1-q}$. Note that $C_q(|\Gamma'_-\rangle) = 0$ iff $\theta = \phi$. We present the upper bound in Eq. (69) from Theorem 4 and the entanglement of superposition state $|\Gamma\rangle$ in Eq. (90) in Fig. 6. It indicates that the present bound is close to the exact value of the entanglement for superposition state $|\Gamma\rangle$. Howbeit, there may be some entanglement values of superposition states that cannot be effectively evaluated, i.e., the present bound is larger than 1. Thus the present bound in Eq. (69) may be further improved.

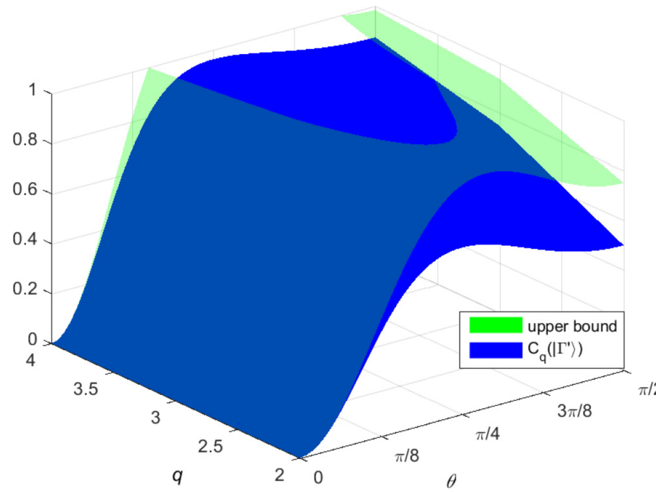


FIG. 6. q-concurrence (blue) and upper bounds (green) of the superposition state $|\Gamma\rangle$ in Example 4. Here, $2 \leqslant q \leqslant 4$ and $\theta \in (0, \pi/2)$. The upper bound of the q-concurrence is restricted to be no larger than 1.

When $C_q(|\Gamma'_-\rangle) > 0$, i.e., $\theta \neq \phi$, for convenience, we take $\theta = \pi/3$, $\phi = \pi/6$, and $q = 2$ as an example. We get $\|\sigma^{T_A}\|_1 = 2.2571$, since the superposition state is a $3 \otimes 4$ system, which implies that $m = 3$ in the right side of the inequality (70). From Eqs. (92)–(97), a straightforward calculation shows the upper bound in the inequality (70) being 0.8335, while $C_q(|\Gamma'_+\rangle) = 0.6663$ according to Eq. (70) in Theorem 4. This indicates that the upper bound in Eq. (70) may be further improved. In a similar manner, it is easy to verify the validity of Corollary 3.

In Theorem 4, if two states $|\Phi\rangle$ and $|\Psi\rangle$ are orthogonal, i.e., $\langle\Phi|\Psi\rangle = 0$, we have $c_\pm = 1$. We can obtain the following Corollary from Theorem 4.

Corollary 4. Given two orthogonal states $|\Phi\rangle$ and $|\Psi\rangle$ (not necessarily biorthogonal), the q-concurrence of the superposition state $|\Gamma\rangle = \alpha|\Phi\rangle + \beta|\Psi\rangle$ satisfies

$$C_q(|\Gamma'\rangle) \leqslant 2[|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] \quad (98)$$

or

$$C_q(|\Gamma'_+\rangle) \leqslant 2[|\alpha|^{2q} C_q(|\Phi\rangle) + |\beta|^{2q} C_q(|\Psi\rangle) + h_q(|\alpha|^2) - |F_q(\rho_A) - F_q(\rho_B)|] - \frac{\left(\|\sigma^{T_A}\|_1^{q-1} - 1\right)^2}{m^{2q-2} - m^{q-1}}. \quad (99)$$

In fact, the entanglement of the quantum channel can be reckoned as the communication capacity of this quantum channel. As an application, our method in Sec. III can provide

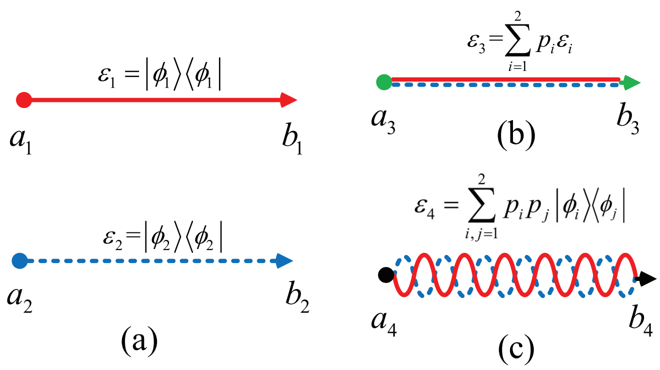


FIG. 7. Schematic three types of channels. (a) Quantum channels ε_1 and ε_2. (b) The classical mixing channel ε_3. (c) Quantum superposition channel ε_4. Here, the input a_i is sent through the channel ε_i corresponding to the output b_i.

a powerful tool for investigating the quantum superposition channels of quantum communication.

Example 6. Consider three types of channels. The first one is from two entangled channels associated with $\varepsilon_1 = |\phi_1\rangle\langle\phi_1|$ and $\varepsilon_2 = |\phi_2\rangle\langle\phi_2|$. The second one is a classically mixing channel of ε_1 and ε_2 with the probability distribution $\{p_i\}$, i.e., $\varepsilon_3 = \sum_{i=1}^{2} p_i\varepsilon_i$. The third one is a quantum superposition channel defined by $\varepsilon_4 = |\Gamma\rangle\langle\Gamma|$ with $|\Gamma\rangle = \sum_{i=1}^{2} p_i|\phi_i\rangle$, as shown in Fig. 7. Note that $C_q(|\phi_i\rangle)$ provides a quantum communication capacity associated with the channel ε_i. Hence it is reasonable to regard $p_1C_q(|\phi_1\rangle) + p_2C_q(|\phi_2\rangle)$ as the communication capacity of a classically mixing channel of ε_3 as shown in Fig. 7(b). On the other hand, $C_q(|\Gamma\rangle)$ may be considered as the communication capacity of the quantum superposition channel of ε_4 as shown in Fig. 7(c). From Corollaries 1 and 2, it follows that the quantum superposition channel of ε_4 can provide at most one qubit capacity larger than the classically mixing channel ε_3. This implies a quantitative relationship between two different channels independent of entropy parameter q. Thus the present method here should be self-interesting in quantum communication and quantum information processing.

IV. CONCLUSION

Given an entanglement, how much is it entangled? The entanglement monotone has been introduced to solve this problem by quantifying the degree of entanglement. In this paper, inspired by the general Tsallis entropy, we define a parametrized entanglement monotone as q-concurrence for any $q \geqslant 2$. We prove a lower bound of the q-concurrence for general states. The present bound is exact for two-qubit isotropic states. In addition, the parametrized entanglement monotone is finally applied for characterizing the superposition state in terms of two states being superposed, especially for biorthogonal and one-sided orthogonal states. It shows that the increase of the q-concurrence for the superposition state is upper bounded by one ebit in both cases. These results are interesting in the entanglement theory, quantum information processing, quantum communication, and quantum many-body theory.

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APPENDIX A: PROOF OF THE LEMMA 1

(I) Since the discrete spectra of ρ_i of ρ are in [0,1], we conclude the operator inequality $\rho^q \leqslant \rho$, where the equality holds iff ρ is pure state. It follows that $\mathrm{Tr}\rho^q \leqslant 1$, where the equality holds iff ρ is a pure state. This implies $F_q(\rho) \geqslant 0$, where the equality holds iff ρ is a pure state.

(II) From the Schmidt decomposition in Eq. (4), we know that the reduced density matrices ρ_A and ρ_B have the same spectra. From Eq. (9), it is easy to show that $F_q(\rho_A) = F_q(\rho_B)$.

(III) For $F_q(\rho_{AB}) \leqslant F_q(\rho_A) + F_q(\rho_B)$, the key is an inequality (see Theorem 2 in Ref. [62]) with the Schatten q-norm as

$$1 + \|\rho_{AB}\|_q^q \geqslant \|\rho_A\|_q^q + \|\rho_B\|_q^q. \tag{A1}$$

This can be rewritten into

$$\mathrm{Tr}\rho_A^q + \mathrm{Tr}\rho_B^q \leqslant 1 + \mathrm{Tr}\rho_{AB}^q, \tag{A2}$$

which is equivalent to the inequality:

$$1 - \mathrm{Tr}\rho_{AB}^q \leqslant 1 - \mathrm{Tr}\rho_A^q + 1 - \mathrm{Tr}\rho_B^q. \tag{A3}$$

It means that

$$F_q(\rho_{AB}) \leqslant F_q(\rho_A) + F_q(\rho_B), \tag{A4}$$

which completes the proof.

Similar to the von Neumann entropy, the subadditivity inequality leads to the triangle (or "Araki-Lieb") inequality [63]. For $|F_q(\rho_A) - F_q(\rho_B)| \leqslant F_q(\rho_{AB})$, the proof is inspired by Ref. [64]. Given a bipartite pure state $|\psi\rangle_{ABC}$, from the Schmidt decomposition of $|\psi\rangle_{ABC}$, the density matrices ρ_{AB} and ρ_C have the same nonzero eigenvalues. Hence $F_q(\rho_{AB}) = F_q(\rho_C)$. Similarly, we have $F_q(\rho_A) = F_q(\rho_{BC})$. Combining these with the inequality (A4), we get

$$F_q(\rho_A) - F_q(\rho_B) \leqslant F_q(\rho_{AB}). \tag{A5}$$

By symmetry, we also have

$$F_q(\rho_B) - F_q(\rho_A) \leqslant F_q(\rho_{AB}). \tag{A6}$$

Combining Eqs. (A5) and (A6), we have the claim that

$$|F_q(\rho_A) - F_q(\rho_B)| \leqslant F_q(\rho_{AB}). \tag{A7}$$

(IV) For $\sum_i p_iF_q(\rho_i) \leqslant F_q(\sum_i p_i\rho_i)$, we first prove $\lambda F_q(\rho) + \mu F_q(\sigma) \leqslant F_q(\lambda\rho + \mu\sigma)$ with $\lambda, \mu \geqslant 0$ and $\lambda + \mu = 1$. Here, from Minkowski's inequality [65] with positive semidefinite matrices ρ and σ, we get

$$[\mathrm{Tr}(\rho + \sigma)^r]^{1/r} \leqslant (\mathrm{Tr}\rho^r)^{1/r} + (\mathrm{Tr}\sigma^r)^{1/r} \tag{A8}$$

for $r \geqslant 2$. From Eq. (A8), we have

$$[\mathrm{Tr}(\lambda\rho+\mu\sigma)^r]^{1/r} \leqslant \lambda(\mathrm{Tr}\rho^r)^{1/r}+\mu(\mathrm{Tr}\sigma^r)^{1/r}, \quad (A9)$$

where $\lambda, \mu \geqslant 0$ and $\lambda+\mu=1$. Due to $r \geqslant 2$, from Eq. (A9), we get

$$\begin{aligned}\mathrm{Tr}(\lambda\rho+\mu\sigma)^r &\leqslant [\lambda(\mathrm{Tr}\rho^r)^{1/r}+\mu(\mathrm{Tr}\sigma^r)^{1/r}]^r\\ &\leqslant \lambda\,\mathrm{Tr}\rho^r+\mu\,\mathrm{Tr}\sigma^r,\end{aligned} \quad (A10)$$

where the inequality (A10) is obtained from the convexity of the function $y=x^r$ for $r \geqslant 2$. The inequality (A10) implies that

$$\lambda(1-\mathrm{Tr}\rho^r)+\mu(1-\mathrm{Tr}\sigma^r) \leqslant 1-\mathrm{Tr}(\lambda\rho+\mu\sigma)^r. \quad (A11)$$

By induction on i, we obtain the following inequality:

$$\sum_i p_i F_q(\rho_i) \leqslant F_q\left(\sum_i p_i\rho_i\right), \quad (A12)$$

where $\{p_i\}$ is the probability distribution corresponding to density operators ρ_i of ρ. The equality holds iff all the states ρ_i are identical.

For $F_q(\sum_i p_i\rho_i) \leqslant \sum_i p_i^q F_q(\rho_i)+(1-\sum_i p_i^q)$, similar with Lemma 1 in Ref. [66]. Suppose a joint state $\rho=\sum_i p_i\rho_i$; we get

$$\begin{aligned}F_q\left(\sum_i p_i\rho_i\right) &= 1-\mathrm{Tr}\left(\sum_i p_i\rho_i\right)^q\\ &\leqslant 1-\sum_i p_i^q \mathrm{Tr}\left(\rho_i^q\right)\\ &= \sum_i p_i^q\left[1-\mathrm{Tr}\left(\rho_i^q\right)\right]+1-\sum_i p_i^q\\ &= \sum_i p_i^q F_q\left(\rho_i^q\right)+1-\sum_i p_i^q.\end{aligned} \quad (A13)$$

The equality holds iff the states ρ_i have support on orthogonal subspaces. The proof is as follows: let λ_{ij} and e_{ij} be the eigenvalues and corresponding eigenvectors of ρ_i. Note that $p_i\lambda_{ij}$ and e_{ij} are the eigenvalues and eigenvectors of $\sum_i p_i\rho_i$. Thereby, we have

$$\begin{aligned}F_q\left(\sum_i p_i\rho_i\right) &= 1-\sum_i (p_i\lambda_{ij})^q\\ &= \sum_i p_i^q\left(1-\lambda_{ij}^q\right)+1-\sum_i p_i^q\\ &= \sum_i p_i^q F_q(\rho_i)+1-\sum_i p_i^q,\end{aligned} \quad (A14)$$

which completes the proof.

APPENDIX B: PROOF OF THE LEMMA 2

The proof is inspired by recent techniques [25,61,67] with local symmetry. The q-concurrence under the symmetry state ρ_F is given by

$$C_q(\rho_F) = co(\xi(F,q,d)), \quad (B1)$$

where the function $\xi(F,q,d)$ is defined as

$$\xi(F,q,d) = \inf\{C_q(|\psi\rangle) | f_{\Psi^+}(|\psi\rangle)=F,\ \mathrm{rank}(\rho_\psi)\leqslant d\}, \quad (B2)$$

where $\mathrm{rank}(\rho_\psi)$ denotes the rank of the density operator $\rho_\psi = |\psi\rangle\langle\psi|$.

The q-concurrence of the pure state $|\psi\rangle=\sum_{i=1}^d\sqrt{\lambda_i}|a_ib_i\rangle$ is given in terms of the Schmidt coefficients by

$$C_q(|\psi\rangle) = 1-\mathrm{Tr}\left(\rho_A^q\right) = 1-\sum_{i=1}^d \lambda_i^q. \quad (B3)$$

In order to evaluate $f_{\Psi^+}(|\psi\rangle)$, we decompose $|\psi\rangle$ into its Schmidt decomposition as $|\psi\rangle=\sum_{i=1}^d\sqrt{\lambda_i}|a_ib_i\rangle=(U_A\otimes U_B)\sum_{i=1}^d\sqrt{\lambda_i}|ii\rangle$. From a straightforward calculation, we get $f_{\Psi^+}(|\psi\rangle)=\frac{1}{d}|\sum_{i=1}^d\sqrt{\lambda_i}v_{ii}|^2$ [26], where $V=U_A^{\mathrm{T}}U_B$ and $v_{ij}=\langle i|V|j\rangle$.

Obviously, the value of $\xi(F,q,d)$ for $F\in(0,\frac{1}{d}]$ is easily obtained by setting $\lambda_1=1, v_{11}=\sqrt{F}$, which yields $\xi(F,q,d)=0$. For $F\in(\frac{1}{d},1]$, by using the Lagrange multipliers [24], one can minimize Eq. (B3) subject to the constraints

$$\sum_i \lambda_i = 1, \quad (B4)$$

$$\sum_i \sqrt{\lambda_i} = \sqrt{Fd}, \quad (B5)$$

with $Fd \geqslant 1$. And then, the condition for an extremum is given by

$$(\sqrt{\lambda_i})^{2q-1}+\mu_1\sqrt{\lambda_i}+\mu_2=0, \quad (B6)$$

where μ_1 and μ_2 denote the Lagrange multipliers. It is evident that $f(\sqrt{\lambda_i})=(\sqrt{\lambda_i})^{2q-1}$ is a convex function of $\sqrt{\lambda_i}$ for $q\geqslant 2$. Since a convex and a linear function cross each other in at most two points, this equation has maximally two possible nonzero solutions for $\sqrt{\lambda_i}$. Let γ and δ denote these two positive solutions. The Schmidt vectors $\vec{\lambda}=\{\lambda_1,\lambda_2,\ldots,\lambda_d\}$ have coefficients

$$\lambda_j = \begin{cases}\gamma^2, & j=1,\ldots,n,\\ \delta^2, & j=n+1,\ldots,n+m,\\ 0, & j=n+m+1,\ldots,d,\end{cases} \quad (B7)$$

where $n+m\leqslant d$ and $n\geqslant 1$. The minimization problem has been reduced into the following problem:

$$\begin{aligned}&\text{Given integers } n, m, n+m\leqslant d,\\ &\min C_q(|\psi\rangle)\end{aligned} \quad (B8)$$

$$\begin{aligned}&\text{such that } n\gamma^2+m\delta^2=1,\\ &n\gamma+m\delta=\sqrt{Fd},\end{aligned} \quad (B9)$$

where $C_q(|\psi\rangle)=1-n\gamma^{2q}-m\delta^{2q}$.

By solving Eq. (B9), we obtain

$$\gamma_{nm}^{\pm}(F) = \frac{n\sqrt{Fd}\pm\sqrt{nm(n+m-Fd)}}{n(n+m)} \quad (B10)$$

and

$$\begin{aligned}\delta_{nm}^{\pm}(F) &= \frac{\sqrt{Fd}-n\gamma_{nm}^{\pm}}{m}\\ &= \frac{m\sqrt{Fd}\mp\sqrt{nm(n+m-Fd)}}{m(n+m)}.\end{aligned} \quad (B11)$$

Since $\gamma_{mn}^- = \delta_{nm}^+$, the function in Eq. (B8) has the same value for γ_{nm}^+ and γ_{mn}^-. Therefore, we only need to consider the solutions of $\gamma_{nm} := \gamma_{nm}^+$. Since γ_{nm} is a proper solution of Eq. (B10), the quantity inside the square root has to be non-negative, which implies that $Fd \leqslant n + m$. On the other hand, δ_{nm} should be non-negative in Eq. (B11), which implies that $Fd \geqslant n$. In this regime, one can verify that $\delta_{nm}(F) \leqslant \sqrt{Fd}/(n+m) \leqslant \gamma_{nm}(F)$. Note that $n = 0$ is not defined. Hence we have $n \geqslant 1$.

To find the minimum of $C_q(|\psi\rangle)$ over all choices of n and m, we can perform the minimization explicitly by regarding n and m as continuous variables. It is completed by minimizing $C_q(|\psi\rangle)$ over the parallelogram defined by $1 \leqslant n \leqslant Fd$ and $Fd \leqslant n + m \leqslant d$. Note that the parallelogram collapses to a line when $Fd = 1$, i.e., the separability boundary. Within the parallelogram, we have $\gamma_{nm} \geqslant \delta_{nm} \geqslant 0$. $\gamma_{nm} = \delta_{nm}$ iff $n + m = Fd$, while $\delta_{nm} = 0$ iff $n = Fd$. We first calculate the derivatives of γ_{nm} and δ_{nm} with respect to n and m by differentiating the constraints (B9) as

$$\begin{aligned}\frac{\partial\gamma}{\partial n} &= \frac{1}{2n}\frac{2\gamma\delta-\gamma^2}{\gamma-\delta},\\ \frac{\partial\delta}{\partial n} &= -\frac{1}{2m}\frac{\gamma^2}{\gamma-\delta},\\ \frac{\partial\delta}{\partial m} &= -\frac{1}{2m}\frac{2\gamma\delta-\gamma^2}{\gamma-\delta},\\ \frac{\partial\gamma}{\partial m} &= \frac{1}{2n}\frac{\delta^2}{\gamma-\delta}.\end{aligned} \tag{B12}$$

These can be used in Eq. (B8) to calculate the partial derivatives of $C_q(|\psi\rangle)$ with respect to n and m as

$$\frac{\partial C_q}{\partial n} = (q-1)\gamma^{2q} - \frac{q\gamma^2\delta(\gamma^{2q-2}-\delta^{2q-2})}{\gamma-\delta} \tag{B13}$$

and

$$\begin{aligned}\frac{\partial C_q}{\partial m} &= (q-1)\delta^{2q} - \frac{q\delta^2\gamma(\gamma^{2q-2}-\delta^{2q-2})}{\gamma-\delta}\\ &\leqslant (q-1)\delta^{2q} - q\delta^2\gamma(\gamma+\delta) && \text{(B14)}\\ &\leqslant (q-1)\delta^{2q} - 2q\delta^4 && \text{(B15)}\\ &\leqslant (q-1-2q)\delta^4 && \text{(B16)}\\ &\leqslant 0, && \text{(B17)}\end{aligned}$$

where the inequality (B14) is confirmed because $f(q) = (\gamma^{2q-2} - \delta^{2q-2})/(\gamma-\delta)$ is an increasing function of q, i.e.,

$$\frac{\partial f}{\partial q} = \frac{(2q-2)(\gamma^{2q-3}-\delta^{2q-3})}{\gamma-\delta} \geqslant 0 \tag{B18}$$

for $q \geqslant 2$ and $\gamma \geqslant \delta$. The inequality (B15) holds for $\gamma \geqslant \delta$. The inequality (B16) is from $\nu(\delta) = \delta^{2q}$ being a decreasing function of $q \geqslant 2$. The inequality (B17) is obtained for $2q \geqslant q - 1$ with $q \geqslant 2$.

Now we introduce two parameters $u = m - n$ and $v = m + n$, which correspond to motions parallel and perpendicular to the $m + n = c$ (c is a constant) boundaries of the parallelogram. The derivative of $C_q(|\psi\rangle)$ with respect to u is given by

$$\begin{aligned}\frac{\partial C_q}{\partial u} &= \frac{\partial C_q}{\partial n}\frac{\partial n}{\partial u} + \frac{\partial C_q}{\partial m}\frac{\partial m}{\partial u}\\ &= \frac{1}{2}(q-1)(\delta^{2q}-\gamma^{2q})\\ &\quad -\frac{q(\gamma^{2q-2}-\delta^{2q-2})(\delta^2\gamma-\gamma^2\delta)}{2(\gamma-\delta)}\\ &\leqslant \frac{1}{2}(q-1)(\delta^{2q}-\gamma^{2q}) - \frac{q}{2}(\gamma+\delta)\gamma\delta(\delta-\gamma) && \text{(B19)}\\ &\leqslant \frac{1}{2}(q-1)(\delta^4-\gamma^4) + \frac{q}{2}(\gamma^2-\delta^2)\gamma\delta && \text{(B20)}\\ &\leqslant \frac{1}{2}[(\delta^4-\gamma^4)+(\gamma^2-\delta^2)\gamma\delta] && \text{(B21)}\\ &\leqslant -\frac{1}{2}[(\gamma^2-\delta^2)(\gamma^2+\delta^2-2\gamma\delta)] && \text{(B22)}\\ &= -\frac{1}{2}(\gamma+\delta)(\gamma-\delta)^3 \leqslant 0, && \text{(B23)}\end{aligned}$$

where the inequality (B19) holds for Eq. (B18). The inequality (B20) is from $g = (\delta^{2q} - \gamma^{2q})$ being a decreasing function of $q \geqslant 2$, i.e.,

$$\frac{\partial g}{\partial q} = 2q(\delta^{2q-1} - \gamma^{2q-1}) \leqslant 0. \tag{B24}$$

Let $h = \frac{1}{2}(q-1)(\delta^4-\gamma^4) + \frac{1}{2}q(\gamma^2-\delta^2)\gamma\delta$. We get

$$\frac{\partial h}{\partial q} = \frac{-(\gamma^2-\delta^2)(\gamma^2+\delta^2-\gamma\delta)}{2} \leqslant 0. \tag{B25}$$

Thus h is a decreasing function of $q \geqslant 2$. The inequality (B21) is achieved. The inequality (B22) holds for $\gamma\delta \leqslant 2\gamma\delta$.

From Eqs. (B17) and (B23), it is obvious that $\frac{\partial C_q}{\partial m} \leqslant 0$ within the parallelogram and $\frac{\partial C_q}{\partial u} \leqslant 0$ except on the boundary $m + n = Fd$, where it is zero. These results imply that the minimum of $C_q(|\psi\rangle)$ occurs at the vertex of $n = 1$ and $m = d - 1$. Thus we get the minimum of $C_q(|\psi\rangle)$ as

$$C_q(|\psi\rangle) = 1 - \gamma_{1,d-1}^{2q} - (d-1)\delta_{1,d-1}^{2q}. \tag{B26}$$

In this way, we derive an analytical expression of the function $\xi(F, q, d)$ as

$$\xi(F, q, d) = 1 - \gamma^{2q} - (d-1)\delta^{2q}, \tag{B27}$$

where γ and δ are defined as

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{d}}[\sqrt{F} + \sqrt{(d-1)(1-F)}],\\ \delta &= \frac{1}{\sqrt{d}}\left(\sqrt{F} - \frac{\sqrt{1-F}}{\sqrt{d-1}}\right).\end{aligned} \tag{B28}$$

Thus the q-concurrence for isotropic states $C_q(\rho_F) = co(\xi(F, q, d))$, and $\xi(F, q, d)$ has the form in Eq. (B27). This completes the proof of Lemma 2.

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