Characterizing quantum networks: Insights from coherence theory

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Networks based on entangled quantum systems enable interesting applications in quantum information processing and the understanding of the resulting quantum correlations is essential for advancing the technology. We show that the theory of quantum coherence provides powerful tools for analyzing this problem. For that, we demonstrate that a recently proposed approach to network correlations based on covariance matrices can be improved and analytically evaluated for the most important cases.

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I. INTRODUCTION

Quantum networks [1–4] have recently attracted much interest as they have been identified as a promising platform for quantum information processing, such as long-distance quantum communication [5,6]. In an abstract sense, a quantum network consists of several sources, which distribute entangled quantum states to spatially separated nodes, then the quantum information is processed locally in these nodes. This may be seen as a generalization of a classical causal model [7,8], where the shared classical information between the nodes is replaced by quantum states. Clearly, it is important to understand the quantum correlations that arise in such a quantum network. Recent developments have shown that the network structure and topology leads to novel notions of nonlocality [9,10], as well as new concepts of entanglement and separability [11–13], which differ from the traditional concepts and definitions [14,15]. Dealing with these new concepts requires theoretical tools for their analysis. So far, examples of entanglement criteria for the network scenario have been derived using the mutual information [11,12], the fidelity with pure states [12,13], or covariance matrices build from measurement probabilities [16,17], but these ideas work either only for specific examples, or require numerical optimizations for their evaluation.

In this paper we demonstrate that the theory of quantum coherence provides powerful tools for analyzing correlations in quantum networks. In recent years, quantum coherence was under intense research, it was demonstrated that coherence is essential in quantum information applications and entanglement generation [18–21] and a resource theory of it has been developed [22–26]. We provide a direct link between the theory of multisubspace coherence [27,28] and the approach to quantum networks using covariance matrices established in Refs. [16,17]. This allows to solve analytically the criteria developed there for important cases; furthermore, some conjectures can be proved and, besides that, our methods can be applied to large networks for which tools based on numerical optimization are infeasible. We note that, since the covariance matrix approach is essentially a tool coming from classical causal models [16], our results demonstrate that results from the theory of *quantum* coherence are useful beyond the level of quantum states for the analysis of *classical* networks.

II. QUANTUM NETWORKS

The simplest nontrivial network is the triangle network, where three nodes are mutually connected by three sources that prepare bipartite quantum states that are subsequently shared with the nodes, see also Fig. 1(a). More generally, one has M sources, labeled by $m = 1, 2, \dots, M$ that independently produce quantum states ϱ_m, which are then distributed to N nodes, labeled by $n = 1, 2, \dots, N$. For every source m we denote by $\mathcal{C}_m$ the set of all connected nodes that have access to the state ϱ_m. The topology of the network captures the fact that not all vertices are connected to a single source, thus limiting the influence that each source can have on the different nodes.

At each node a measurement is performed that is described by a POVM $\mathbf{A}^{(n)} = \{A_x^{(n)}\}_x$. The observed probability distribution over the outcomes reads $p(x_1 \dots x_N) = \mathrm{tr}[(A_{x_1}^{(1)} \otimes \dots \otimes A_{x_N}^{(N)})\varrho_1 \otimes \dots \otimes \varrho_M]$. The central question is whether a given probability distribution may originate from a network with a given topology. We note that the set of probability distributions that are compatible with a given network topology is nonconvex and thus, in general, hard to characterize. One way to overcome this problem was put forward in Ref. [17]. The idea is to map the set of probability distributions compatible with the network to the space of covariance matrices, and then consider a convex relaxation of the problem.

For this purpose, a so-called feature map is defined that maps the outcomes x_n at each vertex n to a vector $\mathbf{v}_{x_n}^{(n)} \in \mathcal{V}_n$, where the $\mathcal{V}_n$ are some orthogonal vector spaces. Combining all the feature maps, one obtains a random vector $\mathbf{v}$ with components $\mathbf{v}_{x_1,\dots,x_N} = \mathbf{v}_{x_1}^{(1)} + \dots + \mathbf{v}_{x_N}^{(N)}$. The covariance matrix is then defined as

$$\Gamma(\mathbf{v}) = E(\mathbf{v}\mathbf{v}^\dagger) - E(\mathbf{v})E(\mathbf{v})^\dagger \tag{1}$$

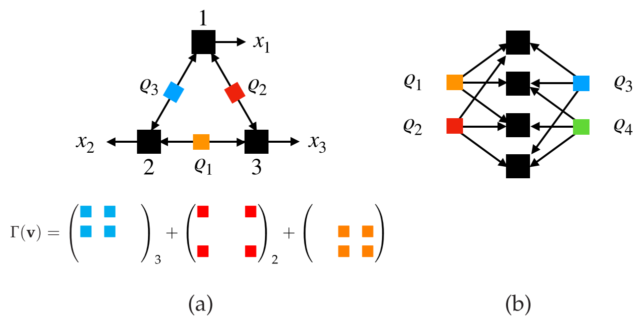


FIG. 1. (a) The triangle network consists of three nodes that produce measurement outcomes $x_1, \dots, x_3$ and three sources that distribute bipartite entanglement that is shared amongst the nodes. The covariance matrix of the triangle network has a 3×3 block structure and consists of three terms, where $(\square)_i$ denotes those blocks that are contributed by the source i. (b) A network consisting of four nodes that is 3-complete, i.e., it features four sources that distribute tripartite entanglement.

with $E(\mathbf{v}\mathbf{v}^\dagger) = \sum_{x_1,\dots,x_N} \mathbf{v}_{x_1,\dots,x_N} \mathbf{v}^\dagger_{x_1,\dots,x_N} P(x_1,\dots,x_N)$ and $E(\mathbf{v}) = \sum_{x_1,\dots,x_N} \mathbf{v}_{x_1,\dots,x_N} P(x_1,\dots,x_N)$. Due to the structure of $\mathbf{v}$, the covariance matrix has a natural block structure: Γ is an $N \times N$ block matrix with blocks $\Gamma_{\alpha\beta}$, and each block is a $r \times r$ matrix, with r being the dimension of $\mathcal{V}_n$. The standard covariance matrix formalism from mean values is a special instance of this notion, where one assigns to the outcomes x_n just real numbers and hence takes the $\mathcal{V}_n$ to be one dimensional. Here, however, we will assume that the feature map simply maps the outcomes x_n to orthogonal vectors $|x_n\rangle$, as for measurements with more than two outcomes the mean value contains less information in comparison with the probability distribution.

III. COVARIANCE MATRICES AND COHERENCE

The topology of the network imposes strong constraints on the structure of the covariance matrix. More precisely, the covariance matrix can be decomposed in a sum of positive matrices that have a certain block structure, corresponding to the sources [16,17]. The verification of this structure is then an instance of a semidefinite program (SDP) [29,30]. For simplicity, we will restrict our attention in the following to k-complete networks. This means that all sources distribute their states to $k < N$ parties and all possible k-partite sources are being used, so we have $M = \binom{N}{k}$ [see also Fig. 1(b)]. Our results can be extended to more complicated network topologies.

The criterion from Refs. [16,17] states that one has to find a decomposition of $\Gamma(\mathbf{v})$ into blocks Y_m according to

$$\text{find:} \quad Y_m \geqslant 0, \tag{2}$$

$$\text{subject to:} \quad Y_m = \Pi_m Y_m \Pi_m \quad \text{and} \quad \Gamma(\mathbf{v}) = \sum_{m=1}^{M} Y_m, \tag{3}$$

where $\Pi_m = \sum_{i \in \mathcal{C}_m} P_i$, with P_n being the projector onto $\mathcal{V}_n$; so Π_m is effectively a projector onto all spaces affected by the source m. To give an example, we depict this decomposition for the case of the triangle network in Fig. 1(a). Note that the formulation in Eqs. (2) and (3) is different from (but clearly equivalent to) the formulation in Ref. [17]. The advantage of our reformulation is that it allows to establish a link to the theory of quantum coherence.

When characterizing quantum coherence, one starts with a fixed basis $\{|\phi_i\rangle\}$ of the Hilbert space. The coherence of a quantum state is then given by the amount of off-diagonal elements of its density matrix, if expressed in this basis [19,22]. A pure state $|\psi\rangle$ is said to have coherence rank k, if it can be expressed using k elements of the basis $\{|\phi_i\rangle\}$, and a mixed state has coherence number k, if it can be written as a mixture of pure states with coherence rank k or less, but not as a mixture of pure states with coherence rank $k-1$ or less [20–22,27,31]. This can be extended to the notion of block coherence [28] by taking a set of orthogonal projectors $\{P_i\}$ such that any vector $|\psi\rangle$ can be decomposed as $|\psi\rangle = \sum_i |\psi_i\rangle$, where $|\psi_i\rangle = P_i|\psi\rangle$. The vector $|\psi\rangle$ is said to have block coherence rank k if exactly k terms in the decomposition do not vanish. The convex hull of rank one operators $|\psi\rangle\langle\psi|$ with block coherence rank k or less we denote as $\mathcal{BC}_k$. Then, an operator X has block coherence number $k+1$ if it is in $\mathcal{BC}_{k+1}$ but not in $\mathcal{BC}_k$. In general we have the inclusion $\mathcal{BC}_1 \subset \mathcal{BC}_2 \subset \cdots \subset \mathcal{BC}_N$. Note that the notions of coherence rank and coherence number are well studied and several criteria and properties are known [22].

Having this in mind, it is clear that Eqs. (2) and (3) are nothing but a reformulation of the notion of multisubspace coherence for the covariance matrix and we arrive at the first main result of this paper:

Observation 1. If a covariance matrix $\Gamma(\mathbf{v})$ has block coherence number $k+1$ with respect to the projectors Π_m, then it cannot have originated from a k-complete network.

IV. NETWORKS WITH DICHOTOMIC MEASUREMENTS

For dichotomic measurements, that is, measurements with two outcomes, one can expect from our discussion after Eq. (1) that the covariance matrix can be simplified. Indeed, with our feature map the blocks of the covariance matrix are always of the form $(\Gamma_{\alpha\beta})_{ij} = (p_{ij} - q_i r_j)$, where p_{ij} is a probability distribution, and $q_i = \sum_j p_{ij}$ and $r_j = \sum_i p_{ij}$ are its marginals. These blocks have vanishing row and column sums, so $(1,\dots,1)^T$ is a (left and right) eigenvector with eigenvalue zero. For the dichotomic case, the blocks are 2×2 matrices, so only one nonzero eigenvalue remains, and we must have $\Gamma_{\alpha\beta} \propto (\mathbb{1} - \sigma_x)$. So we have:

Observation 2. Consider a network of N vertices, where each node performs a dichotomic measurement. Then the covariance matrix $\Gamma(\mathbf{v})$ is of the form $\Gamma(\mathbf{v}) = C \otimes (\mathbb{1} - \sigma_x)$, where C is an $N \times N$ matrix.

So, for evaluating the criterion for k-completeness in the case of dichotomic measurements, one just has to check the k-level coherence of the matrix C. While this is, in general, still hard, the solution can directly be written down for the simplest nontrivial case of $k = 2$ [27]. Namely, it is known that a matrix X has coherence number less than or equal to two if and only if the so-called comparison matrix $M(X)$ defined by

$$(M[X])_{ij} = \begin{cases} |X_{ii}| & \text{if } i = j \\ -|X_{ij}| & \text{if } i \neq j \end{cases} \tag{4}$$

is positive semidefinite. Thus, we have:

Observation 3. If the comparison matrix $M(C)$ coming from the covariance matrix has a negative eigenvalue, then the observed probability distribution is incompatible with a network of bipartite sources.

V. EXAMPLE OF A GHZ-TYPE DISTRIBUTION

Consider the family of distributions that have previously been studied in Refs. [17,34]

$$P(x_1, \dots, x_N) = p\delta_0^{(N)} + q\delta_1^{(N)} + (1-p-q)\frac{1-\delta_0^{(N)}-\delta_1^{(N)}}{2^N-2}, \quad (5)$$

where $\delta_i^{(N)} = \prod_{j=1}^N \delta_{ix_j}$. For $p = q = \frac{1}{2}$ this corresponds to measuring locally σ_z on an N-particle Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = (|00\dots0\rangle + |11\dots1\rangle)/\sqrt{2}$. The covariance matrix for this distribution reads

$$C = \Delta\mathbb{1} + \chi|\mathbf{1}\rangle\langle\mathbf{1}|, \quad (6)$$

where $\Delta = 2^{N-2}(1-p-q)/(2^N-2)$, $\chi = \frac{1}{4}[1-(p-q)^2] - \Delta$ and $|\mathbf{1}\rangle = \sum_{n=1}^N |n\rangle$. From Eq. (4) we can conclude that C has coherence number less or equal two if and only if the matrix $M(C) = (\Delta + 2\chi)\mathbb{1} - \chi|\mathbf{1}\rangle\langle\mathbf{1}|$ is positive semidefinite. This matrix has eigenvalues $\lambda_1 = \Delta + 2\chi$ and $\lambda_2 = \Delta - (N-2)\chi$. It follows that C is incompatible with a 2-complete network if

$$q > p + \kappa - \sqrt{4\kappa p + (\kappa-1)^2}, \quad (7)$$

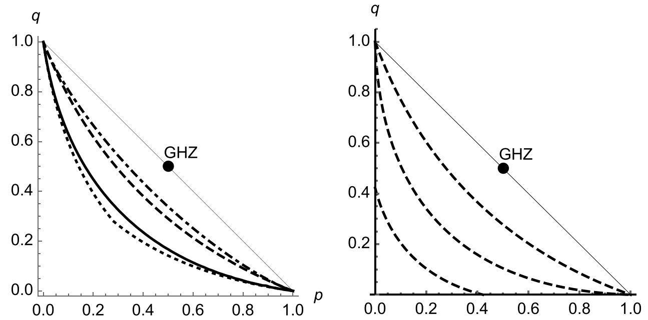
where $\kappa = [(N-1)2^{N-2}]/[(N-2)(2^{N-1}-1)]$. This criterion was already observed in Ref. [17], however only numerical evidence for its optimality was found. Applying our Observation 3, it also becomes evident that this criterion is indeed the optimal solution of the relaxation in Eqs. (2) and (3) for the GHZ-type distributions in Eq. (5). This relies on the fact that the criterion in Eq. (4) is necessary and sufficient to characterize two level coherence of an operator. We emphasize that if the criterion is not violated, this does not imply the compatibility with any 2-complete network.

VI. MULTILEVEL COHERENCE WITNESSES

Another possibility to detect the coherence number of a covariance matrix, and thus, ruling out large classes of states to be compatible with a network structure, are coherence witnesses. To illustrate the idea, let us consider again the GHZ-type distribution from Eq. (5). Due to the simple structure of the corresponding covariance matrix C in Eq. (6) we can completely characterize its multilevel coherence properties and so the underlying distributions according to their network topologies for arbitrary N. To that end, let us first recall the concept of coherence witnesses. Consider an arbitrary pure state $|\psi\rangle = \sum_{i=1}^M c_i|i\rangle$. A $(k+1)$-level coherence witness is given by $W_k = \mathbb{1} - \frac{1}{\sum_{i=1}^k |c_i^\downarrow|^2}|\psi\rangle\langle\psi|$, where $c_i^\downarrow$ denote the coefficients c_i reordered decreasingly according to their absolute values [27]. This means that $\mathrm{tr}[W_k\varrho] \geqslant 0$, if ϱ has coherence number k or less. For the maximally coherent state $|\psi^+\rangle = (\sum_{i=1}^N |i\rangle)/\sqrt{N}$ this witness is of the form $W_k = \mathbb{1} - |\mathbf{1}\rangle\langle\mathbf{1}|/k$. This witness can easily be proven to be optimal for the family of states $\varrho(\mu) = \mu|\psi^+\rangle\langle\psi^+| + (1-\mu)\frac{\mathbb{1}}{N}$. These states are, up to normalization and suitable choice of the parameter μ, equivalent to the covariance matrix C in Eq. (6). Thus we obtain $\mathrm{tr}[W_kC] = (1-1/k)\Delta + (1-N/k)\chi$. From this, it directly follows that C is incompatible with a k-complete network, if

$$q > p + \eta - \sqrt{4\eta p + (\eta-1)^2}, \quad (8)$$

with $\eta = (N-1)2^{N-2}/[(N-k)(2^{N-1}-1)]$. The results are shown for the case $N = 5$ and $k = 4, 3, 2$ in Fig. 2(b). Furthermore, we note that this technique can be applied to large networks where an approach based on SDPs would become infeasible, due to the rapidly growing number of terms in Eqs. (2) and (3), which grows as $\binom{N}{k}$.



(a) (b)

FIG. 2. (a) For the triangle network we compare the criterion in Eq. (7) (solid line) to the monogamy criterion in Eq. (10) (dashed line), the entropic constraint (dot-dashed line) from Refs. [32,33], and the inflation criterion (dotted line) from Ref. [10]. (b) Results of the GHZ-type distribution in Eq. (5) using Eq. (8) for $N = 5$ and $k = 4, 3, 2$. Everything above the lines is detected to be incompatible with the respective network structure.

VII. NETWORKS BEYOND DICHOTOMIC MEASUREMENTS

In the case of more than two outcomes per measurement, the block coherence number of the covariance matrix needs to be tested. For the case of networks involving only bipartite sources we have the following:

Observation 4. Let $\Gamma(\mathbf{v}) \in \mathcal{BC}_2$ be a covariance matrix with block coherence number two. Whenever the signs of some off-diagonal blocks are flipped such that the matrix remains symmetric, the resulting matrix will also remain positive semidefinite.

To see this, note that any matrix with block coherence number two can be written as a convex combination of pure states with coherence rank two, i.e., $|\psi\rangle = P_i|\psi\rangle + P_j|\psi\rangle$. For any such state, adding a minus sign in the density operator corresponds to the transformation $P_i|\psi\rangle + P_j|\psi\rangle \mapsto P_i|\psi\rangle - P_j|\psi\rangle$, under which the density operator remains positive semidefinite.

To demonstrate the power of this Observation, let us consider again the GHZ-type distribution, but now with three

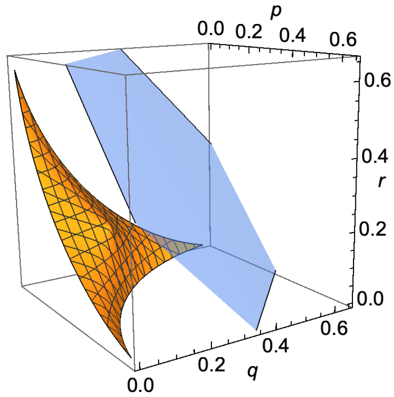


FIG. 3. Analysis of the GHZ-type distribution with three outcomes per measurement in Eq. (9) using Observation 4. The blue surface represents the normalization constraint.

outcomes per measurement,

$$P(x_1, x_2, x_3) = p\delta_0^{(3)} + q\delta_1^{(3)} + r\delta_2^{(3)} + (1 - p - q - r)\frac{1 - \delta_0^{(3)} - \delta_1^{(3)} - \delta_2^{(3)}}{3^3 - 3}. \quad (9)$$

A straightforward calculation provides a regime where this is incompatible with the triangle network, see Fig. 3.

VIII. CHARACTERIZING NETWORKS WITH MONOGAMY RELATIONS

Another possibility to characterize networks is to evaluate monogamy relations for the coherence between different subspaces [28]. The idea is that the amount of coherence that can be shared between one subspace and all other subspaces is limited if a certain block coherence number is imposed. To be more precise, for a trace one positive semidefinite block matrix $X = [X_{\alpha\beta}]_{\alpha,\beta=0}^N$ with block coherence number k it holds that $\sum_{\beta=1}^N \|X_{0\beta}\|_{\text{tr}} \leqslant \sqrt{k-1}\sqrt{\text{tr}[X_{00}](1 - \text{tr}[X_{00}])}$. If we consider the normalized matrix matrix $\tilde{C} = C/\text{tr}[C]$, evaluating such a monogamy relation provides a necessary criterion for C to have coherence number k. For the matrix in Eq. (6) this gives

$$\Delta - \left(\frac{\sqrt{N-1}}{\sqrt{k-1}} - 1\right)\chi \geqslant 0. \quad (10)$$

Hence, if this inequality is violated then the observed correlations are not compatible with a k-complete network. This is also shown in Fig. 2(a) for the triangle network. Although this test is in this case not as powerful as the analytical solution, it is easy to evaluate especially for large networks, since it requires only computing traces of smaller block matrices.

IX. FURTHER REMARKS

So far, we provided criteria to show that correlations are incompatible with a k-complete network. It would be interesting to derive also sufficient criteria for being compatible with a given network structure. In the framework of Ref. [17] this is not directly possible, as the criterion in Eqs. (2) and (3) is a convex relaxation of the original problem. Still, coherence theory allows to identify scenarios where the covariance matrix can be certified to have a small block coherence number k, so the covariance matrix approach must fail to prove incompatibility with a k-complete network.

Here we can make two small observations in this direction: (i) The following results from Ref. [27] can be directly applied to networks with dichotomic outcomes. Namely, if we have for the normalized matrix $\tilde{C} \geqslant \frac{N-k}{N-1}\Lambda(\tilde{C})$, where Λ is the fully decohering map, mapping any matrix to its diagonal part, then $\tilde{C}$ has coherence number k, implying that the test in Eqs. (2) and (3) for $(k+1)$-complete networks will fail. Furthermore we have that if $\text{tr}[\tilde{C}^2]/\text{tr}[\tilde{C}]^2 \leqslant 1/(N-1)$, then $\tilde{C}$ is two-level coherent. (ii) In the general case, if $M_b(\Gamma) \geqslant 0$, where $M_b(\Gamma)$ is the block comparison matrix defined by $(M_b[\Gamma])_{\alpha\beta} = (\|\Gamma_{\alpha\alpha}^{-1}\|)^{-1}$ for $\alpha = \beta$, and $(M_b[\Gamma])_{\alpha\beta} = -\|\Gamma_{\alpha\beta}\|$ for $\alpha \neq \beta$, with $\|X\|$ denoting the largest singular value of the block X, then $\Gamma \in \mathcal{BC}_2$. A detailed discussion is given in Appendix.

X. CONCLUSION

In this work we have established a connection between the theory of multilevel coherence and the characterization of quantum networks. To be precise, we showed that a recent approach based on covariance matrices leads to a well-studied problem in coherence theory; consequently, many results from the latter field can be transferred to the former. This provides a useful application of the resource theory of multilevel coherence outside of the usual realm of quantum states.

There are several interesting problems remaining for future work. First, it would be highly desirable to extend the covariance approach to the case where each node of the network can perform more than one measurement. This will probably lead to significantly refined tests for network topologies. Second, it seems to be promising to study the coherence in networks on the level of the resulting quantum state, and not the covariance matrix. This may shed light on the question which types of network correlations are useful for applications in quantum information processing. Third, it would be interesting to see if the covariance matrix approach can also be extended to generalized probabilistic theories.

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APPENDIX: SUFFICIENT CONDITIONS FOR BLOCK COHERENCE NUMBER TWO

Let the block matrix $X = [X_{\alpha\beta}] > 0$, with $X_{\alpha\beta} \in \mathbb{C}^{d\times d}$, be partitioned as follows:

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1K} \\ X_{21} & X_{22} & \cdots & X_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ X_{K1} & X_{K2} & \cdots & X_{KK} \end{bmatrix}. \quad (A1)$$

Definition 5 (from Ref. [35]). Let X be partitioned as in Eq. (A1). If the matrices $X_{\alpha\alpha}$ on the diagonal are nonsingular, and if

$$\left(\left\|X_{\alpha\alpha}^{-1}\right\|\right)^{-1} \geqslant \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{K} \|X_{\alpha\beta}\|, \tag{A2}$$

then X is called *block diagonally dominant*. Here $\|Y\|$ denotes the largest singular value, so for the positive $X_{\alpha\alpha}$ the expression $\|X_{\alpha\alpha}^{-1}\|^{-1}$ is the smallest eigenvalue of $X_{\alpha\alpha}$.

Observation 6. If X is positive and block diagonally dominant, then the block coherence number is smaller or equal to two, $bcn(X) \leqslant 2$.

Proof. Suppose X satisfies the hypothesis. Define 2×2 block matrices

$$G^{\alpha\beta} = \begin{bmatrix} |X_{\alpha\beta}| & X_{\alpha\beta} \\ X_{\alpha\beta}^{\dagger} & |X_{\alpha\beta}^{\dagger}| \end{bmatrix}, \tag{A3}$$

where $|X_{\alpha\beta}| = \sqrt{X_{\alpha\beta}^{\dagger}X_{\alpha\beta}}$ and the support of $G^{\alpha\beta}$ is the subspace α, β. Clearly, the $G^{\alpha\beta}$ are positive semidefinite and have block coherence number two. Next, consider the matrix $D = X - \sum_{\alpha=1}^{K}\sum_{\beta>\alpha} G^{\alpha\beta}$. Since $X > 0$ it is also Hermitian, and thus, $X_{\beta\alpha} = X_{\alpha\beta}^{\dagger}$, precisely as for $G^{\alpha\beta}$. From this we can conclude that the off-diagonal blocks of D vanish and the diagonal blocks are given by $D_{\alpha\alpha} = X_{\alpha\alpha} - \sum_{\beta\,=\,1,\,\beta\,\neq\,\alpha}^{K} |X_{\alpha\beta}|$. Furthermore, observe that $\lambda_{\min}(X_{\alpha\alpha}) \geqslant \sum_{\beta\,=\,1,\,\beta\,\neq\,\alpha}^{K} \lambda_{\max}(X_{\alpha\beta}) \geqslant \lambda_{\max}(\sum_{\beta\,=\,1,\,\beta\,\neq\,\alpha}^{K} X_{\alpha\beta})$, where the first inequality is due to Eq. (A2) and the second inequality is straightforward. This proves that, besides being block diagonal, D is also positive semidefinite. Thus X can be written as a positive sum of a block incoherent matrix D and matrices $G^{\alpha\beta}$ of block coherence number two, from which the statement follows. ■

The next concept that is needed is the so-called comparison matrix, which is defined as follows.

Definition 7 (from Ref. [36]). Let X be partitioned as in Eq. (A1) and $X_{\alpha\alpha}$ nonsingular. Then the block comparison matrix $M_b[X]$ is defined by

$$(M_b[X])_{\alpha\beta} = \begin{cases} \left(\left\|X_{\alpha\alpha}^{-1}\right\|\right)^{-1} & \text{for } \alpha=\beta \\ -\|X_{\alpha\beta}\| & \text{for } \alpha\neq\beta \end{cases}. \tag{A4}$$

From this definition it is evident that if the comparison matrix $M_b[X]$ exists and is (strictly) diagonally dominant, then X itself is (strictly) block diagonally dominant.

Definition 8 (M matrix). Let the matrix $A = (a_{ij})$ be a real matrix such that $a_{ij} \leqslant 0$ for $i \neq j$. Then A is called a nonsingular M matrix if and only if every real eigenvalue of A is positive.

Definition 9 (Def. 3.2. in Ref. [36]). If there exist nonsingular block diagonal matrices D and E such that $M_b[DXE]$ is a nonsingular M matrix, then X is said to be a nonsingular block H matrix.

Lemma 10 (Lemma 4. in Ref. [36]). If X is a nonsingular block H matrix, then there exist nonsingular block diagonal matrices D and E such that DXE is strictly block diagonally dominant.

Theorem 11. Let X be partitioned as in Eq. (A1) and positive semidefinite (but not necessarily strictly positive). If $M_b(X) \geqslant 0$, then X has $bcn(X) \leqslant 2$.

Proof. The proof follows the idea of Ref. [27]. First, define the operator $X_\epsilon = X + \epsilon\mathbb{1}$, for $\epsilon \geqslant 0$. Then, for $\epsilon > 0$ we have that $M_b[X_\epsilon] = M[X] + \epsilon\mathbb{1} > 0$. Evidently, since $M_b[X_\epsilon]$ is a real matrix with nonpositive off-diagonal entries and furthermore has only strictly positive eigenvalues it is a nonsingular M matrix, according to Def. 8. Then, according to Def. 9 X_ϵ is a nonsingular block H matrix. From the proof of Lemma 10 in Ref. [36] we can conclude that there exists a block diagonal matrix $D_\epsilon > 0$ such that $D_\epsilon X_\epsilon D_\epsilon$ is strictly block diagonally dominant. Then it follows from Observation 6 that strictly block diagonally dominant matrices can have at most block coherence number two. We find that $bcn(X_\epsilon) = bcn(D_\epsilon X_\epsilon D_\epsilon) \leqslant 2$, and since the block coherence number is lower semicontinuous we have $bcn(X) = bcn(\lim_{\epsilon\to 0^+} X_\epsilon) \leqslant \lim_{\epsilon\to 0^+} bcn(X_\epsilon) \leqslant 2$. ■

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