Stability of the Grabert master equation

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We study the dynamics of a quantum system having Hilbert space of finite dimension d_{H}. Instabilities are possible provided that the master equation governing the system's dynamics contain nonlinear terms. Here we consider the nonlinear master equation derived by Grabert. The dynamics near a fixed point is analyzed by using the method of linearization, and by evaluating the eigenvalues of the Jacobian matrix. We find that all these eigenvalues are non-negative, and conclude that the fixed point is stable. This finding raises the question: under what conditions is instability possible in a quantum system having finite d_{H}?

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I. INTRODUCTION

Consider a given closed quantum system having Hilbert space of finite dimension d_{H}, whose master equation, which governs the time evolution of the reduced density matrix ρ, can be expressed as $d\rho/dt = \Theta(\rho) = \Theta_{\mathrm{u}}(\rho) - \Theta_{\mathrm{d}}(\rho)$. The first term, which is given by $\Theta_{\mathrm{u}}(\rho) = (i/\hbar)[\rho, \mathcal{H}]$, where $\mathcal{H} = \mathcal{H}^{\dagger}$ is the Hamiltonian of the closed system, represents unitary evolution, and the second one $\Theta_{\mathrm{d}}(\rho)$ represents the effect of coupling between the closed system and its environment. While it is commonly assumed that both the unitary term $\Theta_{\mathrm{u}}(\rho)$ and the damping term $\Theta_{\mathrm{d}}(\rho)$ are linear in ρ [1,2], in some cases the master equation can become nonlinear. Two types of nonlinearity are considered below [3,4]. For the first one, which is henceforth referred to as unitary nonlinearity, the unitary term $\Theta_{\mathrm{u}}(\rho)$ is replaced by a nonlinear term. In most cases, unitary nonlinearity originates from either the mean field approximation [5–8], or from a transformation mapping of the Hilbert space of finite dimension d_{H} into a space having infinite dimensionality (e.g., the Holstein-Primakoff transformation [9], which can yield a parametric instability in ferromagnetic resonators [10]). Here, we consider the second type, which is henceforth referred to as damping nonlinearity, and focus on the master equation that was proposed by Grabert [11], which has a damping term $\Theta_{\mathrm{d}}(\rho)$ nonlinear in ρ.

Grabert has shown that the invalidity of the quantum regression hypothesis gives rise to damping nonlinearity [11]. The nonlinear term added to the master equation ensures that the purity $\mathrm{Tr}\,\rho^2$ does not exceed unity [12,13], and that entropy is generated at a non-negative rate, as is expected from the second law of thermodynamics [14].

For some cases, nonlinear dynamics can be ruled out based on some specified assumptions. This was studied in Ref. [15] for the assumption that faster than light signaling is prohibited by the causality principle, and in Ref. [16] for the case of an isolated system. However, the nonlinear *damping* of the type the Grabert master equation (GME) has cannot be ruled out by these arguments.

The GME has a fixed point given by

$$\rho_0 = \frac{e^{-\beta\mathcal{H}}}{\mathrm{Tr}\,(e^{-\beta\mathcal{H}})}, \tag{1}$$

where $\beta = 1/(k_{\mathrm{B}}T)$ is the inverse of the thermal energy [11]. At the fixed point ρ_0 the system is in thermal equilibrium having Boltzmann distribution.

Here we explore the stability of this fixed point ρ_0 for the case where the Hamiltonian $\mathcal{H}$ of the closed system is time independent. In a basis of energy eigenstates of a time-independent Hamiltonian both matrices $\mathcal{H} = \mathrm{diag}(E_1, E_2, \cdots, E_{d_{\mathrm{H}}})$ and $\rho_0 = \mathrm{diag}(\rho_1, \rho_2, \cdots, \rho_{d_{\mathrm{H}}})$ are diagonal, where $\rho_n = e^{-\beta E_n}/\mathrm{Tr}(e^{-\beta\mathcal{H}})$ [see Eq. (1)].

For the case of thermal equilibrium, one may argue that the stability of ρ_0 is obvious. However, the stability of a driven system is anything but obvious. Note that in many cases the rotating wave approximation (RWA) is employed in order to model the dynamics of a given system under external driving, using a transformation into a rotating frame, in which the Hamiltonian becomes time-independent in the RWA. Thus, our conclusion, that the fixed point ρ_0 is stable for any time independent Hermitian $\mathcal{H}$, can be extended beyond the limits of thermal equilibrium.

The GME for the reduced density matrix ρ can be expressed as [12]

$$\frac{d\rho}{dt} = \Theta(\rho) = \Theta_{\mathrm{u}}(\rho) - \Theta_{\mathrm{d}}(\rho), \tag{2}$$

where the damping term is given by $\Theta_{\mathrm{d}}(\rho) = \Theta_{\mathrm{A}}(\rho) + \Theta_{\mathrm{B}}(\rho)$, where $\Theta_{\mathrm{A}}(\rho)$, which is given by $\Theta_{\mathrm{A}}(\rho) = \gamma_{\mathrm{E}}[Q, [Q, \rho]]$, is linear in ρ, and $\Theta_{\mathrm{B}}(\rho)$, which is given by $\Theta_{\mathrm{B}}(\rho) = \beta\gamma_{\mathrm{E}}[Q, [Q, \mathcal{H}]_{\rho}]$ is nonlinear. The constant $\gamma_{\mathrm{E}} > 0$ is a damping rate, the Hermitian operator $Q^{\dagger} = Q$ describes the interaction between the quantum system and its environment, and

$$A_{\rho} = \int_0^1 d\eta\ \rho^{\eta} A \rho^{1-\eta}. \tag{3}$$

Alternatively, the damping term $\Theta_{\rm d}(\rho)$ can be expressed as $\Theta_{\rm d}(\rho)=\beta\gamma_{\rm E}[Q,[Q,\mathcal{U}_{\rm H}]_\rho]$, where $\mathcal{U}_{\rm H}=\mathcal{H}+\beta^{-1}\log\rho$ is the Helmholtz free energy operator [11]. According to the master equation (2), the time evolution of the Helmholtz free energy $\langle\mathcal{U}_{\rm H}\rangle=\mathrm{Tr}(\mathcal{U}_{\rm H}\rho)$ is governed by

$$\frac{d\langle\mathcal{U}_{\rm H}\rangle}{dt}=-\beta\gamma_{\rm E}\,\mathrm{Tr}\,(\mathcal{C}_\rho\mathcal{C}), \tag{4}$$

where $\mathcal{C}=i[Q,\mathcal{U}_{\rm H}]$, and thus $d\langle\mathcal{U}_{\rm H}\rangle/dt\leqslant 0$ (since $\mathcal{C}^\dagger=\mathcal{C}$) [12,17], i.e., the Helmholtz free energy $\langle\mathcal{U}_{\rm H}\rangle$ is a monotonically decreasing function of time.

Note that the operator $\mathcal{C}$ vanishes at the fixed point ρ_0 given by Eq. (1). Alternatively, the Kubo identity [given by Eq. (4.2.17) of Ref. [17]] can be used to show that ρ_0 is a fixed point [11]. For some cases the existence of a limit cycle (i.e., periodic) solution for the GME (2) can be ruled out using Eq. (4). Along such a solution the condition $\mathcal{C}=0$ must be satisfied [since $\mathrm{Tr}(\mathcal{C}_\rho\mathcal{C})=0$ implies that $\mathcal{C}=0$ when $\mathrm{Tr}\,\rho^2<1$]. Hence, when $\rho=\rho_0$ is a unique solution of $\mathcal{C}=0$, a limit cycle solution can be ruled out.

A linear master equation can be derived by replacing the nonlinear term $\Theta_{\rm B}(\rho)$ by the term $(\beta'/\hbar)\gamma_{\rm E}[Q,[Q,\mathcal{H}]]$, where $\beta'>0$. It was shown in Ref. [18] (see also Appendix B of Ref. [8]) that such a linear master equation is stable provided that $\gamma_{\rm E}>0$. Below we analyze the stability of the nonlinear GME (2).

II. LINEARIZATION

The stability of the fixed point ρ_0 of the master equation (2) is explored by the method of linearization applied to the nonlinear term $\Theta_{\rm B}(\rho)$. In the vicinity of $\rho_0=\mathrm{diag}(\rho_1,\rho_2,\cdots,\rho_{d_{\rm H}})$ the density matrix ρ is expressed as $\rho=\rho_0+\epsilon\mathcal{V}$, where ϵ is a real small parameter. Let $u\rho u^\dagger=\rho_{\rm d}=\mathrm{diag}(\rho_1',\rho_2',\cdots,\rho_{d_{\rm H}}')$ be diagonal, where u is unitary, i.e., $u^\dagger u=1$. With the help of time-independent perturbation theory one finds that the eigenvalues ρ_n' of ρ are given by

$$\rho_n'=\rho_n+\epsilon(n|\mathcal{V}|n)+O(\epsilon^2), \tag{5}$$

and the unitary transformation u that diagonalizes ρ is given by

$$u=\sum_n\left(|n)+\sum_{k\neq n}\frac{\epsilon(k|\mathcal{V}|n)}{\rho_n-\rho_k}|k)\right)(n|+O(\epsilon^2), \tag{6}$$

or

$$u=1-i\epsilon F+O(\epsilon^2), \tag{7}$$

where the Hermitian matrix F is given by

$$F=\sum_{k\neq l}\frac{i(k|\mathcal{V}|l)}{\rho_l-\rho_k}|k)(l|, \tag{8}$$

$(k|\mathcal{V}|l)=\mathcal{V}_{kl}$ is the (k'th raw - l'th column) matrix element of $\mathcal{V}$, and $|k)(l|$ denotes a $d_{\rm H}\times d_{\rm H}$ matrix having entry 1 in the (k'th raw - l'th column), and entry 0 elsewhere.

Using the identity [12]

$$\int_0^1 x^\eta y^{1-\eta}d\eta=\mathcal{F}(x,y), \tag{9}$$

where

$$\mathcal{F}(x,y)=\frac{x-y}{\log x-\log y}, \tag{10}$$

one finds that (recall that $\rho_{\rm d}$ is diagonal)

$$\int_0^1 d\eta\;\rho_{\rm d}^\eta A\rho_{\rm d}^{1-\eta}=\mathcal{F}'\circ A, \tag{11}$$

where $\circ$ denotes the Hadamard matrix multiplication (element by element matrix multiplication), and where the matrix elements of $\mathcal{F}'$ are given by $\mathcal{F}'_{nm}=\mathcal{F}(\rho_n',\rho_m')$. Note that $\mathcal{F}'_{nm}=\mathcal{F}_{nm}+O(\epsilon)$, where $\mathcal{F}_{nm}=\mathcal{F}(\rho_n,\rho_m)$ [see Eq. (5)], hence, the following holds [see Eqs. (3) and (7) and note that $uAu^\dagger=A+i\epsilon[A,F]+O(\epsilon^2)$],

$$A_\rho=\mathcal{F}'\circ A+i\epsilon[F,\mathcal{F}\circ A]+i\epsilon\mathcal{F}\circ[A,F]+O(\epsilon^2), \tag{12}$$

where $\mathcal{F}'=\mathcal{F}+\epsilon(d\mathcal{F}/d\epsilon)+O(\epsilon^2)$.

The following holds [see Eq. (10)]

$$\mathcal{F}(x,y)=\frac{x+y}{2}f_{\rm d}\left(\frac{x-y}{x+y}\right), \tag{13}$$

where the function $f_{\rm d}(\eta)$ is given by

$$f_{\rm d}(\eta)=\frac{2\eta}{\log\frac{1+\eta}{1-\eta}}=\frac{\eta}{\tanh^{-1}\eta}. \tag{14}$$

The function $f_{\rm d}$ is symmetric, i.e., $f_{\rm d}(-\eta)=f_{\rm d}(\eta)$, and the following holds $f_{\rm d}(0)=1$ and $f_{\rm d}(\pm1)=0$. With the help of Eqs. (5) and (13) one finds that the matrix $d\mathcal{F}/d\epsilon$ is real, symmetric, and the following holds (no summation due to repeated indices n and m)

$$\left(\frac{d\mathcal{F}}{d\epsilon}\right)_{nm}=\frac{d\alpha_{nm}}{d\epsilon}F_{nm}+\alpha_{nm}\frac{d\eta_{nm}}{d\epsilon}F'_{nm}, \tag{15}$$

where $\alpha_{nm}=(\rho_n+\rho_m)/2$, $\eta_{nm}=(\rho_n-\rho_m)/(\rho_n+\rho_m)$, $F_{nm}=f_{\rm d}(\eta_{nm})$, and where $F'_{nm}=f'_{\rm d}(\eta_{nm})$. Moreover, $\mathrm{Tr}(d\mathcal{F}/d\epsilon)=0$ (note that $F_{nn}=1$ and $F'_{nn}=0$).

The $N_{\rm d}=d_{\rm H}^2-1$ Hermitian and traceless $d_{\rm H}\times d_{\rm H}$ generalized Gell-Mann matrices λ_n, which span the SU($d_{\rm H}$) Lie algebra, satisfy the orthogonality relation

$$\frac{\mathrm{Tr}\,(\lambda_a\lambda_b)}{2}=\delta_{ab}. \tag{16}$$

For the case $d_{\rm H}=2$ ($d_{\rm H}=3$) the matrices are called Pauli (Gell-Mann) matrices. The set $\{\lambda_a\}$ of $N_{\rm d}$ matrices can be divided into three subsets. The subset $\{\lambda_{{\rm X},(n,m)}\}$ contains $d_{\rm H}(d_{\rm H}-1)/2$ matrices given by $\lambda_{{\rm X},(n,m)}=|n)(m|+|m)(n|$, and the subset $\{\lambda_{{\rm Y},(n,m)}\}$ contains $d_{\rm H}(d_{\rm H}-1)/2$ matrices given by $\lambda_{{\rm Y},(n,m)}=-i|n)(m|+i|m)(n|$, where $1\leqslant m<n\leqslant d_{\rm H}$. The subset $\{\lambda_{{\rm Z},l}\}$ contains $d_{\rm H}-1$ diagonal matrices given by

$$\lambda_{{\rm Z},l}=\sqrt{\frac{2}{l(l+1)}}\left(-l|l+1)(l+1|+\sum_{j=1}^{l}|j)(j|\right), \tag{17}$$

where $1\leqslant l\leqslant d_{\rm H}-1$.

Any given density matrix ρ can be expanded as $\rho=d_{\rm H}^{-1}+\bar{K}\cdot\bar{\lambda}$, where $\bar{K}=(K_1,K_2,\cdots,K_{N_{\rm d}})$, K_n are all real, and $\bar{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_{N_{\rm d}})$. The purity is related to the generalized Bloch vector $\bar{K}$ by $\mathrm{Tr}\,\rho^2=d_{\rm H}^{-1}+2\bar{K}^2$ [see Eq. (16)].

Note that the approximation $f_{\mathrm{d}}(\eta) \simeq 1$ turns the GME into an equation linear in ρ [12] [see Eq. (14)]. This approximation, which is applicable when $|\bar{K}| \ll K_{\mathrm{B}}$, where $K_{\mathrm{B}} = \sqrt{(1-d_{\mathrm{H}}^{-1})/2}$ is the radius of the generalized Bloch sphere (GBS), allows the linearization of the GME around the center point of the GBS. On the other hand, to explore the stability of the steady state solution, we linearize below the GME around the point ρ_0. Note that close to the surface of the GBS, i.e., when the purity is close to unity, the nonlinearity of the GME becomes significant and non-negligible.

III. STABILITY

To explore the stability of the steady state solution ρ_0, it is convenient to express the perturbation $\epsilon\mathcal{V} = \rho - \rho_0$ as $\epsilon\mathcal{V} = \bar{\kappa}\cdot\bar{\lambda}$, where $\bar{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_{N_{\mathrm{d}}})$. In this notation the GME (2) becomes (repeated index implies summation)

$$\frac{d\kappa_b}{dt}\lambda_b = \Theta(\rho_0 + \kappa_b\lambda_b), \tag{18}$$

or [see Eq. (16)]

$$\frac{d\kappa_a}{dt} = \frac{1}{2}\operatorname{Tr}\left(\Theta(\rho_0 + \kappa_b\lambda_b)\lambda_a\right). \tag{19}$$

To first order in $\bar{\kappa}$

$$\frac{d\kappa_a}{dt} = \frac{1}{2}\operatorname{Tr}\left(\frac{\partial\Theta}{\partial\kappa_b}\lambda_a\kappa_b\right), \tag{20}$$

or in a vector form

$$\frac{d\bar{\kappa}}{dt} = J\bar{\kappa}, \tag{21}$$

where the Jacobian matrix J is given by $J = J_{\mathrm{u}} - J_{\mathrm{A}} - J_{\mathrm{B}}$, and where

$$J_\Sigma = \frac{1}{2}\operatorname{Tr}\left(\frac{\partial\Theta_\Sigma}{\partial\kappa_b}\lambda_a\right), \tag{22}$$

with $\Sigma \in \{\mathrm{u}, \mathrm{A}, \mathrm{B}\}$.

The system's stability depends on the set of eigenvalues of the Jacobian matrix J, which is denoted by $\mathcal{S}$. The system is stable provided that real$(\xi) < 0$ for any $\xi \in \mathcal{S}$. It was shown in Appendix B of Ref. [8] that such a system is stable provided that J_{u}, J_{A}, and J_{B} are all real, J_{u} is antisymmetric, all diagonal elements of $J_{\mathrm{A}} + J_{\mathrm{B}}$ are positive, and d_{H} is finite. Properties of the matrices J_{u}, J_{A}, and J_{B} are analyzed below.

The matrix J_{u}, which governs the unitary evolution, is given by [recall the trace identity $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$]

$$J_{\mathrm{u}} = \frac{i}{2\hbar}\operatorname{Tr}\left([\lambda_b, \mathcal{H}]\lambda_a\right) = \frac{i}{2\hbar}\operatorname{Tr}\left(\mathcal{H}[\lambda_a, \lambda_b]\right), \tag{23}$$

hence J_{u} is *real* and *antisymmetric* provided that $\mathcal{H}$ is Hermitian (note that $i[\lambda_b, \lambda_a]$ is Hermitian).

The matrix J_{A} is given by

$$J_{\mathrm{A}} = \frac{\gamma_{\mathrm{E}}}{2}\operatorname{Tr}\left([Q, [Q, \lambda_b]]\lambda_a\right) = \frac{\gamma_{\mathrm{E}}}{2}\operatorname{Tr}\left(-[Q, \lambda_b][Q, \lambda_a]\right). \tag{24}$$

Both matrices $i[Q, \lambda_a]$ and $i[Q, \lambda_b]$ are Hermitian, provided that Q is Hermitian, hence J_{A} is *real* (recall that γ_{E} is positive). The *diagonal* elements of J_{A} are *positive* since $-[Q, \lambda_b][Q, \lambda_a]$ is positive-definite for the case $a = b$.

The diagonal elements of the matrix J_{B} can be evaluated using the linearization of the term A_ρ given by Eq. (12). For the case where the perturbation $\mathcal{V} = (\rho - \rho_0)/\epsilon$ is a generalized Gell-Mann matrix, i.e., $\mathcal{V} \in \{\lambda_a\}$, the following holds [see Eq. (8)]:

$$F = \begin{cases} \frac{\lambda_{\mathrm{Y},(n,m)}}{\rho_n - \rho_m} & \text{if } \mathcal{V} = \lambda_{\mathrm{X},(n,m)} \\ -\frac{\lambda_{\mathrm{X},(n,m)}}{\rho_n - \rho_m} & \text{if } \mathcal{V} = \lambda_{\mathrm{Y},(n,m)} \end{cases}, \tag{25}$$

and [see Eq. (12), and note that, according to Eq. (5), $\mathcal{F}' = \mathcal{F} + O(\epsilon^2)$ when all diagonal elements of the perturbation vanish, e.g., when $\mathcal{V} \in \{\lambda_{\mathrm{X},(n,m)}\} \cup \{\lambda_{\mathrm{Y},(n,m)}\}$, and, according to Eqs. (7) and (8), $u = 1 + O(\epsilon^2)$ when the perturbation is diagonal, e.g., when $\mathcal{V} \in \{\lambda_{\mathrm{Z},l}\}$]

$$\frac{dA_\rho}{d\epsilon} = \begin{cases} \frac{[\mathcal{F}\circ A, \lambda_{\mathrm{Y},(n,m)}] - \mathcal{F}\circ[A, \lambda_{\mathrm{Y},(n,m)}]}{i(\rho_n - \rho_m)} & \text{if } \mathcal{V} = \lambda_{\mathrm{X},(n,m)} \\ \frac{[\mathcal{F}\circ A, \lambda_{\mathrm{X},(n,m)}] - \mathcal{F}\circ[A, \lambda_{\mathrm{X},(n,m)}]}{(-i)(\rho_n - \rho_m)} & \text{if } \mathcal{V} = \lambda_{\mathrm{Y},(n,m)} \\ \frac{d\mathcal{F}'}{d\epsilon}\circ A & \text{if } \mathcal{V} = \lambda_{\mathrm{Z},(n,m)} \end{cases}. \tag{26}$$

The diagonal elements of $J_{\mathrm{A}} + J_{\mathrm{B}}$ are evaluated by using Eq. (26) with different values of the perturbation $\mathcal{V}$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{\mathrm{Z},l}$, which is labeled by j_l, is given by [see Eqs. (22), (24), and (26)]

$$j_l = \frac{\gamma_{\mathrm{E}}}{2}\operatorname{Tr}\left(-[Q, \lambda_{\mathrm{Z},l}]^2\right) + \frac{\beta\gamma_{\mathrm{E}}}{2}\operatorname{Tr}\left(\left[Q, \frac{d\mathcal{F}}{d\epsilon}\circ[Q, \mathcal{H}]\right]\lambda_{\mathrm{Z},l}\right), \tag{27}$$

where the term $d\mathcal{F}/d\epsilon$ is evaluated according to Eq. (15) for the case where the perturbation is given by $\mathcal{V} = \lambda_{\mathrm{Z},l}$. In terms of the elements of the diagonal matrix $\lambda_{\mathrm{Z},l} = \operatorname{diag}(\nu_1, \nu_2, \cdots, \nu_{d_{\mathrm{H}}})$ one finds using Eq. (5) that $\rho_n' = \rho_n + \epsilon\nu_n + O(\epsilon^2)$, hence $(d\mathcal{F}/d\epsilon)_{nm} = d_{nm}$, where

$$d_{nm} = \frac{\nu_{nm}F_{nm}}{2\varkappa_{nm}}\left(1 + \frac{(\varkappa_{nm} - \eta_{nm})F'_{nm}}{F_{nm}}\right), \tag{28}$$

$\nu_{nm} = \nu_n - \nu_m$ and $\varkappa_{nm} = (\nu_n - \nu_m)/(\nu_n + \nu_m)$. The following holds $d_{nm} = d_{mn}$, hence Eq. (27) yields

$$j_l = \gamma_{\mathrm{E}}\sum_{n<m}\zeta_{nm}\nu_{nm}^2|q_{nm}|^2, \tag{29}$$

where $\zeta_{nm} = 1 + d_{nm}e_{nm}/\nu_{nm}$, $e_{nm} = \beta(E_n - E_m)$, and where q_{nm} are the matrix elements of the operator Q (recall that it is assumed that $Q^\dagger = Q$, i.e., $q_{mn} = q_{nm}^*$). With the help of the relation $\eta_{nm} = -\tanh(e_{nm}/2)$ [see Eq. (1)] one finds that $\zeta_{nm} = \zeta(\eta_{nm}, \varkappa_{nm})$, where the function $\zeta(\eta, \varkappa)$ is given by [see Eq. (14) and note that $1 - (1/(1-\eta^2))(\eta/\tanh^{-1}\eta) = \eta F'(\eta)/F(\eta)$]

$$\zeta(\eta, \varkappa) = \frac{f_{\mathrm{d}}(\eta)}{1-\eta^2}\left(1 - \frac{\eta}{\varkappa}\right). \tag{30}$$

The following holds [see Eq. (17), and note that only the cases for which $\upsilon_{nm} \neq 0$, i.e., the cases that can contribute to j_l, are listed]

$$-\frac{1}{\varkappa_{nm}}=\begin{cases}\frac{l-1}{l+1} & n\leqslant l \text{ and } m=l+1\\ 1 & n\leqslant l \text{ and } m>l+1\\ 1 & n=l+1 \text{ and } m>l+1\end{cases}, \tag{31}$$

hence $0\leqslant(-1/\varkappa)\leqslant 1$ for all terms contributing to j_l, hence $\zeta_{nm}v_{nm}^2\geqslant 0$ for these terms, and consequently $j_l\geqslant 0$.

The diagonal matrix element corresponding to the generalized Gell-Mann matrix $\lambda_{\mathrm{X},(2,1)}$ ($\lambda_{\mathrm{Y},(2,1)}$) is labeled by j_{X} (j_{Y}). We show below that both j_{X} and j_{Y} are non-negative. The proof is applicable for all other diagonal elements, corresponding to all generalized Gell-Mann matrices $\lambda\in\{\lambda_{\mathrm{X},(n,m)}\}\cup\{\lambda_{\mathrm{Y},(n,m)}\}$ with $(n,m)\neq(2,1)$, since the ordering of the energy eigenvectors is arbitrary.

With the help of Eqs. (22), (24), and (26) one finds that [the subscript (2, 1) is omitted for brevity]

$$\frac{j_{\mathrm{X}}}{\frac{\gamma_{\mathrm{E}}}{2}}=\mathrm{Tr}\,(-[Q,\lambda_{\mathrm{X}}][Q,\lambda_{\mathrm{X}}]) +\mathrm{Tr}\left(\beta\left[Q,\frac{[\mathcal{F}\circ[Q,\mathcal{H}],\lambda_{\mathrm{Y}}]-\mathcal{F}\circ[[Q,\mathcal{H}],\lambda_{\mathrm{Y}}]}{i(\rho_2-\rho_1)}\right]\lambda_{\mathrm{X}}\right), \tag{32}$$

and

$$\frac{j_{\mathrm{Y}}}{\frac{\gamma_{\mathrm{E}}}{2}}=\mathrm{Tr}\,(-[Q,\lambda_{\mathrm{Y}}][Q,\lambda_{\mathrm{Y}}]) +\mathrm{Tr}\left(\beta\left[Q,\frac{[\mathcal{F}\circ[Q,\mathcal{H}],\lambda_{\mathrm{X}}]-\mathcal{F}\circ[[Q,\mathcal{H}],\lambda_{\mathrm{X}}]}{(-i)(\rho_2-\rho_1)}\right]\lambda_{\mathrm{Y}}\right), \tag{33}$$

hence

$$\frac{j_{\mathrm{X}}}{\gamma_{\mathrm{E}}}=q_{\mathrm{d}}^2+4\upsilon q_{12}''^2+\sum_{n=1}^{2}\sum_{m\geqslant 3}G_{nm}|q_{nm}|^2, \tag{34}$$

and

$$\frac{j_{\mathrm{Y}}}{\gamma_{\mathrm{E}}}=q_{\mathrm{d}}^2+4\upsilon q_{12}'^2+\sum_{n=1}^{2}\sum_{m\geqslant 3}G_{nm}|q_{nm}|^2, \tag{35}$$

where $q_{\mathrm{d}}=q_{11}-q_{22}$,

$$\upsilon=1-\frac{(\mathcal{F}_{11}+\mathcal{F}_{22}-2\mathcal{F}_{12})e_{12}}{2(\rho_1-\rho_2)}, \tag{36}$$

$q_{12}'=\mathrm{Re}\,q_{12}$, $q_{12}''=\mathrm{Im}\,q_{12}$, and where

$$G_{nm}=1+\frac{(\mathcal{F}_{1m}-\mathcal{F}_{2m})e_{nm}}{\rho_1-\rho_2}. \tag{37}$$

With the help of Eqs. (1), (13), and (14) one finds that [note that $e_{nm}=-\log(\rho_n/\rho_m)=\log((1-\eta_{nm})/(1+\eta_{nm}))=-2\eta_{nm}/f_{\mathrm{d}}(\eta_{nm})$]

$$\upsilon=\frac{1}{f_{\mathrm{d}}(\eta_{12})}, \tag{38}$$

and that $G_{1m}=G(\rho_1/\rho_m,\rho_2/\rho_m)$ and $G_{2m}=G(\rho_2/\rho_m,\rho_1/\rho_m)$, where the function G is given by

$$G(r_1,r_2)=1-\frac{\frac{r_1-1}{\log r_1}-\frac{r_2-1}{\log r_2}}{r_1-r_2}\log r_1, \tag{39}$$

or

$$G(r_1,r_2)=\frac{r_2-1}{r_2\log r_2}\frac{\log\frac{r_1}{r_2}}{\frac{r_1}{r_2}-1}, \tag{40}$$

hence $\upsilon\geqslant 1$ [since $0\leqslant f_{\mathrm{d}}(\eta_{12})\leqslant 1$] and $G_{nm}\geqslant 0$ [see Eq. (40), and note that for non-negative r_1 and r_2, both the first factor, which depends on r_2 only, and the second one, which depends on r_1/r_2 only, are non-negative], and thus both j_{X} and j_{Y} are non-negative. This concludes the proof that the GME steady state ρ_0 is stable.

Some of the inequalities that have been used above for exploring the stability of the GME can be used to derive other general properties. The contributions to the rate of energy (entropy) change due to the terms Θ_{u} and Θ_{A} in the GME (2) are denoted by $\mathcal{R}_{\mathrm{Hu}}$ and $\mathcal{R}_{\mathrm{HA}}$ ($\mathcal{R}_{\mathrm{Su}}$ and $\mathcal{R}_{\mathrm{SA}}$), respectively. For a time independent Hamiltonian $\mathcal{H}$, both energy and entropy do not change due to the unitary term Θ_{u}, i.e., $\mathcal{R}_{\mathrm{Hu}}=0$ and $\mathcal{R}_{\mathrm{Su}}=0$. With the help of the inequality $(\rho_n-\rho_m)(E_n-E_m)\leqslant 0$ one finds that $\mathcal{R}_{\mathrm{HA}}\geqslant 0$, and with the help of the inequality $\mathcal{F}(\rho_1,\rho_2)\geqslant 0$ one finds that $\mathcal{R}_{\mathrm{SA}}\geqslant 0$, i.e., the linear damping term Θ_{A} gives rise to increase in both energy and entropy (provided that $\gamma_{\mathrm{E}}>0$).

IV. SUMMARY

In summary, the dynamics governed by the GME (2) in the vicinity of the steady state ρ_0 depends on the d_{H}^2-1 diagonal element of the Jacobian matrix $J_{\mathrm{A}}+J_{\mathrm{B}}$. Our derived expressions for the eigenvalues, given by Eqs. (29), (34) and (35), can be used to evaluate statistical properties of the system near its steady state ρ_0. We find that all these eigenvalues are non-negative, and conclude that the steady state ρ_0 is stable.

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