Wigner's friend and the quasi-ideal clock

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In 1962, Eugene P. Wigner [in *Philosophical Reflections and Syntheses* (Springer, Berlin, 1995), p. 247] introduced a thought experiment that highlighted the incompatibility in quantum theory between unitary evolution and wave function reduction in a measurement. This work resulted in a class of thought experiments often called Wigner's friend scenarios, which have been providing insights over many frameworks and interpretations of quantum theory. Recently, a no-go theorem obtained by D. Frauchiger and R. Renner [Nat. Commun. **9**, 3711 (2018)] brought attention back to the Wigner's friend and its potential of putting theories to the test. Many answers to this result pointed out how timing in the thought experiment could be yielding a paradox. In this work, we ask what would happen if the isolated friend in a Wigner's friend scenario did not share a time reference frame with the outer observer, and time were tracked by a quantum clock. For this purpose, we recollect concepts provided by the theory of quantum reference frames and the quantum resource theory of asymmetry, to learn how to internalize time in this scenario, and introduce a model for a feasible quantum clock proposed by M. P. Woods, R. Silva, and J. Oppenheim [Ann. Henri Poincaré **20**, 125 (2019)] called the quasi-ideal clock. Our results have shown that this approach produces no decoherent behavior, and the disagreement between the superobserver and its friend persists even for an imprecise clock on Wigner's side. However, the Gaussian spread of this clock model can control which observables do not raise a paradox, indicating the relevance of deepening this analysis.

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I. INTRODUCTION

Dynamics in quantum theory is given by two well-known postulates: one of them, the Schrödinger equation, describes the time evolution of isolated systems via unitary operators, while the second one, the measurement postulate, describes how a system is to be described after interacting with a measurement apparatus. To emphasize the incompatibility between these two descriptions, Eugene P. Wigner proposed a thought experiment later called the Wigner's friend [1], in which an observer in a laboratory would measure a physical binary property of a quantum system, this being followed by a second global measurement of the whole laboratory, made by an external superobserver. Given the right initial state of the quantum system and a set of measurements for the observers, they cannot agree on the probability distribution of the external observer's outcomes.

What is indeed happening in this protocol is the following: with respect to an ideal classical clock, the internal friend, who from now on is called Alice, performs her measurement over the quantum system. The superobserver, called Wigner in this work, measures the same ideal classical clock, or a synchronized and perfect copy of it, and when he is sure Alice is done with her measurement, he performs his own. It sounds unreasonable, however, to quantum mechanically describe a complex system such as a laboratory (which includes every degree of freedom of Alice and her measurement device) and not to do so with the clock.

In 2018, Daniela Frauchiger and Renato Renner [2] published an article pointing out how Wigner's friend scenarios (WFSs) could do more than highlight this fundamental incompatibility: they can be used as a test environment for the compatibility between assumptions about the world. Their extended WFS resulted in a no-go theorem stating that ($\mathcal{Q}$) the universal validity of the quantum theory, ($\mathcal{C}$) the consistency between predictions made by different agents, and ($\mathcal{S}$) single-world interpretations do not agree among themselves and shall lead to a paradox if simultaneously imposed in this scenario. Many responding articles argued, however, how other hidden assumptions could be producing the paradox instead of ($\mathcal{Q}$), ($\mathcal{C}$), and ($\mathcal{S}$), and the authors themselves claim that the definition of concepts such as *time* may be a source of the paradox instead.

In this work, we address the question of what would happen if we described the clock with respect to which Wigner performs his measurement as a quantum system. By imposing this condition, Wigner and Alice should no longer share a clock, since Alice's laboratory would cease to be isolated, and the Schrödinger equation would not apply. Wigner will not know when Alice's measurement is done but will know by a common established protocol when he is supposed to measure the laboratory state. Furthermore, to insert a source of uncertainty that could produce the desired decoherence, we equip Wigner with a specific quantum clock, proposed by Woods, Silva, and Oppenheim in 2019 [3].

This work is structured as follows: Section II revises the Wigner's friend scenarios of Wigner and Frauchiger-Renner and sketches our WFS. Section III briefly reviews the problem of marking time in quantum mechanics, calling upon the theory of quantum reference frames to provide us the necessary

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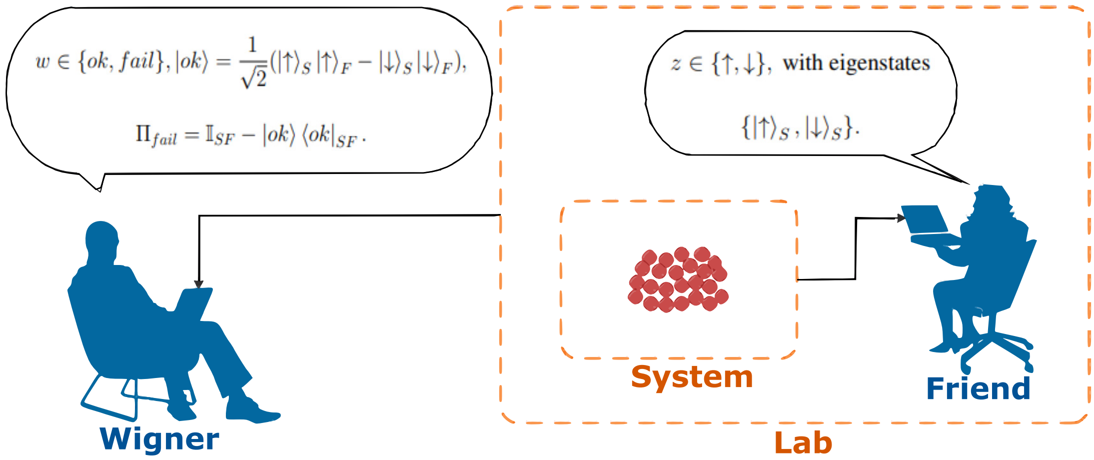

FIG. 1. Schematic of the simplest Wigner's friend scenario.

tools for our model and, also, introducing the Woods-Silva-Oppenheim clock states. Section IV finally proposes a model for the laboratory dynamics, and Sec. V derives our results, mainly stating that for the proposed models of laboratory and clock evolutions, the external observer still disagrees with the predictions of the internal agent for most of his possible measurement choices. We also analyze how the external agent's measurements are asymmetric with respect to time evolution. Finally, Sec. VI concludes with suggestions for further work.

II. WIGNER'S FRIEND SCENARIOS

The simplest WFS can be described by the scheme given in Fig. 1. Alice, in her laboratory, is going to measure σ_z over an ensemble of spin-$\frac{1}{2}$ particles. Let us assume that the ensemble is described by the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S). \tag{1}$$

Thus after performing a selective measurement, starting at time t_0 and finishing at time t_A with respect to the shared classical clock, she should describe the laboratory state as either

$$|\Psi_+\rangle = |\uparrow\rangle_S \otimes |\uparrow\rangle_A \quad \text{or} \quad |\Psi_-\rangle = |\downarrow\rangle_S \otimes |\downarrow\rangle_A, \tag{2}$$

where $\{|\uparrow\rangle_A, |\downarrow\rangle_A\}$ represent the state of Alice's measurement device, body and mind, and what else might exist in the laboratory and might change with the measurement.

Let us assume now that Wigner is going to perform a projection of the laboratory over the space associated with the state

$$|\text{ok}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S \otimes |\uparrow\rangle_A - |\downarrow\rangle_S \otimes |\downarrow\rangle_A), \tag{3}$$

registering the outcome "ok" if this projection is successful and "fail" otherwise. Thus from Alice's perspective Wigner is going to observe his outcomes at time t_W with probabilities

$$P_A(\text{ok}|\uparrow) = |\langle \text{ok}|\Psi_+\rangle|^2 = \frac{1}{2}, \tag{4}$$

$$P_A(\text{ok}|\downarrow) = |\langle \text{ok}|\Psi_-\rangle|^2 = \frac{1}{2}. \tag{5}$$

From Wigner's perspective, however, there is no state reduction in the laboratory, since it is isolated and can only evolve unitarily. Alice's measurement results for Wigner in a state at time t_A given by

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S \otimes |\uparrow\rangle_A + |\downarrow\rangle_S \otimes |\downarrow\rangle_A), \tag{6}$$

which is orthogonal to the projection Wigner aims to detect. Therefore, Wigner predicts his probability distribution at time t_W to be

$$P_W(\text{ok}) = |\langle \text{ok}|\Phi_+\rangle|^2 = 0; \tag{7}$$

$$P_W(\text{fail}) = 1 - P_W(\text{ok}) = 1. \tag{8}$$

The paradox will only vanish, i.e., Alice and Wigner will predict the same probability distribution only for

$$|\text{ok}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S \otimes |\uparrow\rangle_A \pm i\,|\downarrow\rangle_S \otimes |\downarrow\rangle_A). \tag{9}$$

Frauchiger and Renner propose a more complex version of this experiment, comprised of two laboratories. The protocol goes as follows, with n being the steps:

(a) $n = 00$: Alice, in her laboratory, measures the side of a quantum coin, given by the value $r \in \{h, t\}$. The coin is prepared in the state

$$|\psi\rangle_C = \frac{1}{\sqrt{3}}|h\rangle_C + \sqrt{\frac{2}{3}}|t\rangle_C. \tag{10}$$

If Alice gets $r = h$, she prepares a spin-$\frac{1}{2}$ particle in the state $|\downarrow\rangle_S$, and if she gets $r = t$, she prepares it in the state $\frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S)$. She then sends this spin-$\frac{1}{2}$ particle through a quantum channel to the neighbor laboratory.

(b) $n = 10$: Bob, in the neighbor laboratory, detects the spin $z \in \{+\frac{1}{2}, -\frac{1}{2}\}$ and nothing more.

(c) $n = 20$: After all measurements in the laboratories are carried out, the external observer Ursula performs a projection over Alice's laboratory with respect to the state

$$|\text{ok}\rangle_U = \frac{1}{\sqrt{2}}(|h\rangle_C \otimes |h\rangle_A - |t\rangle_C \otimes |t\rangle_A), \tag{11}$$

where $\{|h\rangle_A, |t\rangle_A\}$ represent Alice's device, body and mind, just as in the previous WFS. She registers $u = \text{ok}$ if the projection is successful and $u = \text{fail}$ otherwise.

(d) $n = 30$: Another superobserver, Wigner, does the same over Bob's laboratory, projecting it with respect to the state

$$|\text{ok}\rangle_W = \frac{1}{\sqrt{2}}(|\downarrow\rangle_S \otimes |\downarrow\rangle_B - |\uparrow\rangle_S \otimes |\uparrow\rangle_B), \tag{12}$$

where $\{|\uparrow\rangle_B, |\downarrow\rangle_B\}$ represent Bob's device, body and mind. He registers $w = \text{ok}$ if the projection is successful and $w = \text{fail}$ otherwise.

e. $n = 40$: If $u = \text{ok}$ and $w = \text{ok}$, the experiment is halted. Otherwise, it is reset.

Furthermore, every agent in this extended WFS shares three reasonable assumptions about the world:

(1) *Universal validity of quantum theory* ($\mathcal{Q}$): Any system can be correctly described by a state $|\psi\rangle$ in a Hilbert space, and its physical properties are given by projections of this state with respect to a family of Heisenberg projectors defined in a given time t_0, $\{\Pi_x(t_0)\}_{x\in\chi}$, being completed at time $t \geqslant t_0$. If $\langle\psi|\Pi_\xi(t_0)|\psi\rangle = 1$, then an agent can properly say, "I know that $x = \xi$ at time t."

(2) *Self-consistency* ($\mathcal{C}$): If agents A and B reason over the same theory, and agent A can state that "I know that agent B knows that $x = \xi$ at time t," then she can also say, "I know that $x = \xi$ at time t."

(3) *Single world* ($\mathcal{S}$): Physical quantities can have only one value at a given time t. In other words, if an agent can say, "I know that $x = \xi$ at time t," then he must *deny* that "I know that $x \neq \xi$ at time t."

It is important to emphasize that assumptions ($\mathcal{Q}$) and ($\mathcal{C}$) were explicitly assumed by Wigner in his original work, while assumption ($\mathcal{S}$) was implicitly assumed.

From the perspective of Ursula and Wigner, every measurement in the laboratory is described as a von Neumann measurement, and it is not hard to see that, at step $n = 21$, when every internal measurement is done, the global state yields to a probability distribution such that

$$P(w = \text{ok}, u = \text{ok}) = \frac{1}{12}. \tag{13}$$

However, even though Wigner, Ursula, and Bob, using assumptions ($\mathcal{Q}$) and ($\mathcal{C}$) all agree that Alice must detect $r = t$ at step $n = 01$ for the halting condition to be satisfied, when we assume that Alice indeed measured $r = t$ from her perspective, she will conclude that $w = fail$ at step $n = 31$, which must be false by assumption ($\mathcal{S}$).

We see by this argument how the time marking is relevant and can be confusing in this sort of thought experiment. Many works in the literature have pointed out how this timing might be generating the paradox instead of assumptions ($\mathcal{Q}$), ($\mathcal{C}$), and ($\mathcal{S}$). Sudbery [4] lists it among many other hidden assumptions in Frauchiger and Renner's work. Losada, Laura, and Lombardi [5] analyze the extended WFS under the consistent stories interpretation, concluding that this sequence of statements does not belong to the same consistent chain of events. Waaijer and Van Neerven [6] points out how agents' statements rely on registers from the past that are not in fact happening, which is forbidden for relational quantum mechanics. Baumann *et al.* [7] include a quantum ideal clock in a WFS, deriving some conditional probabilities that might rule out the paradox, and finally, Gambini, García-Pintos, and Pullin [8] propose that uncertainties in time and length measurements are fundamental to ensure the indistinguishability between a reduced state and a decohered one, claiming that this might solve the so-called Frauchiger-Renner paradox. These are a few examples of how time might play a crucial role in solving the Wigner's friend problem.

We here argue that one questionable feature of the Wigner's friend scenario is that Alice and Bob are actually sharing a time (classical) reference frame [Fig. 2(a)]. If we assume by (Q) that *any* physical system is going to be described by a quantum state, there is no reason why one should not describe the clock by a vector in a Hilbert space of its own. This assumption being made, we argue that the agents can no longer share a clock, because if they did so, then Alice's laboratory would be an open quantum system [Fig. 2(b)]. Henceforth, we will take Wigner's perspective [9], even though it should not make any difference.

III. QUANTUM TIME AND QUANTUM CLOCKS

To build a quantum operator capable of telling what time is it, we expect that it has some specific properties, such as [10]

$$U_t^\dagger T U_t = T + t, \tag{14}$$

where T is the time operator in the Schrödinger picture, U_t is a representation of an element of the uniparametric strongly continuous group generated by a Hamiltonianian H, and t is the parameter of the Schrödinger equation. Equation (14) is typically known as the *global covariance relation*. This immediately leads to the canonical commutation relation

$$[T, H] = i. \tag{15}$$

However, Wolfgang Pauli proposed an argument [11] that introduced pessimism into the construction of a time operator. This argument is hereby introduced as a theorem and goes as follows:

Theorem 1 (Pauli). Let $\mathcal{H}$ be a separable Hilbert space, and let $H, T \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators acting on this Hilbert space. Then, if T obeys a global covariance relation with each element of the uniparametric strongly continuous group of unitaries generated by H, i.e., if

$$U_t^\dagger T U_t = T + t, \quad \forall t \in \mathbb{R}, \tag{16}$$

then the spectra spec(H) and spec(T) are both equivalent to $\mathbb{R}$.

This was taken as a result that forbids the existence of a time operator, because it would take a Hamiltonian unbounded from below to recover the global covariance relation, which is not allowed by thermodynamics. However, it is important to highlight that the theorem does not rule out every possibility of building a time operator. It just states that (i) a time operator T with a spectrum equivalent to $\mathbb{R}$ and (ii) a Hamiltonian H bounded from below cannot be related to each other by Eq. (16).

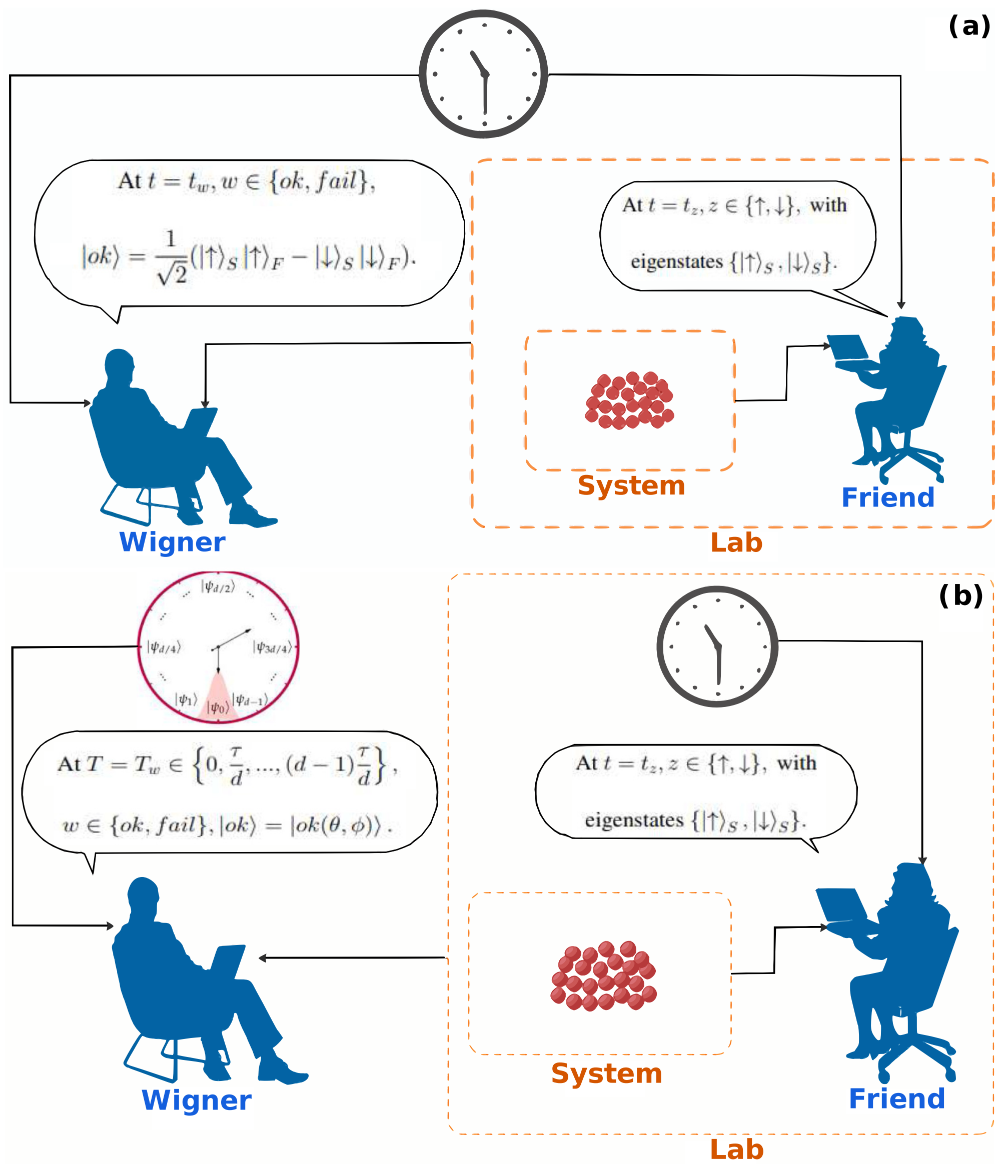

FIG. 2. Schematic of the simplest Wigner's friend scenario, including explicitly the time reference frame for two scenarios: (a) Alice and Wigner share a classical clock; and (b) Alice and Wigner have their own clocks, and Wigner's clock is a bounded quantum system with dimension d and period τ. In this scenario, Wigner is free to perform projective measurements parametrized by real variables (θ, ϕ).

A. Page-Wootters mechanism

The first step in solving this problem was taken by Paul Dirac in 1926 [12], in a procedure of extending the Hilbert space that would later be used by Bryce DeWitt in the construction of his constraint equation for quantum gravity [13]. What interests us is the solution proposed by Don Page and William Wootters in 1983 [14,15]. It consists of a universe described by a bipartite Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, whose dynamics is governed by the noninteracting Hamiltonian

$$H = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B. \tag{17}$$

In the Page-Wootters mechanism, the only states truly accessible to an observer are solutions of the constraint equation

$$H |\Psi\rangle\rangle = 0. \tag{18}$$

That is because, in this universe, the parametric time t in the Schrödinger equation is inaccessible. Instead, the dynamics of subsystem A is given *relationally* with respect to subsystem B. If there is a way of building a time operator T_B, with $\text{spec}(T_B) \equiv \mathbb{R}$, $[T_B, H_B] = i$ and nondegenerate eigenvectors $\{|\phi_B(t)\rangle\}_{t\in\text{spec}(T_B)}$, we are allowed to describe the local state of system A as

$$|\psi_A(t)\rangle = \frac{\langle\phi_B(t)|\Psi\rangle\rangle}{|\langle\phi_B(t)|\Psi\rangle\rangle|}, \tag{19}$$

and it is even possible to show that the Schrödinger equation is recovered in A, i.e.,

$$i\frac{d}{dt}|\psi_A(t)\rangle = H_A|\psi_A(t)\rangle\,, \tag{20}$$

except that now t is not a classical parameter but, rather, an eigenvalue of an operator. Once the global physical state $|\Psi\rangle\rangle$

is known, the dynamics in system A can thus be derived from it. This leads, however, to two important questions:

(1) Typically, an agent has no access to this physical state. Instead, what is known is a prepared state ρ, which in this work is always a product state, and whose evolution still depends on the Schrödinger equation parameter t. How does one start from ρ and obtain $|\Psi\rangle\rangle$, from which the relational description is to be derived?

(2) There is still a local problem in subsystem B, since our time operator might be fulfilling every condition to be ruled out by Pauli's argument. Is there a physical system capable of emulating every property of a quantum ideal clock but that is still described by a bounded Hamiltonian?

We aim to answer these questions in the following sections.

B. Internalizing time

To answer the first question, we call upon the theory of quantum reference frames [16]. It deals with problems where two parties, Alice and Bob, with their respective quantum systems, described with respect to their own quantum reference frames, communicate with each other in the absence of a shared classical reference frame. To illustrate this sort of problem, we can think of Alice and Bob scheduling a date for 2 p.m. at the park. However, Bob has just arrived from a distant country and has no idea what time zone they are in. He has his watch with him, but the lack of a classical reference frame between them makes it almost useless. In this situation, what should Bob do? To ensure that Alice will not be left alone waiting for him, he could just go to the park as soon as he can and sit on a bench until Alice shows up. What Bob is doing is essentially an average over every possible reference frame that might exist between him and Alice. In the theory of quantum reference frames, this operation is known as *G-twirling* and is given by

$$\mathcal{G}[\rho] = \int_{g\in G} U_g \rho U_g^\dagger dg, \tag{21}$$

where $g \in G$ is an element of the group of transformations between the reference frames in question, U_g is its representation, and dg is the Haar measure [17,18].

It is easy to show that, if G is the group of time translations described by the global Hamiltonian in Eq. (17), then $\mathcal{G}[\rho]$ is the static solution of the dynamical equation of motion in the density operator formalism. Indeed, for $\frac{\partial H_S}{\partial t} = \frac{\partial H_C}{\partial t} = 0$ and $\frac{\partial \rho}{\partial t} = 0$ in the Schrödinger picture, then

$$[\mathcal{G}[\rho], H] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} U_t[\rho, H]U_t^\dagger dt, \tag{22}$$

and since for these states $\frac{d\rho}{dt} = iU_t[\rho, H]U_t^\dagger$, then

$$[\mathcal{G}[\rho], H] = -i \lim_{T\to\infty} \frac{\rho(T) - \rho(-T)}{2T} = 0, \tag{23}$$

as $\rho(T)$ and $\rho(-T)$ have both finite eigenvalues for every T. The relative state of subsystem S is thus written as

$$\rho_S(t) = \frac{\mathrm{Tr}_C\{(\mathbb{I}_S \otimes \Pi_t^C)\mathcal{G}[\rho](\mathbb{I}_S \otimes \Pi_t^C)\}}{\mathrm{Tr}\{(\mathbb{I}_S \otimes \Pi_t^C)\mathcal{G}[\rho]\}}, \tag{24}$$

where Π_t^C are projectors over eigenspaces associated with the eigenvectors $|\phi_C(t)\rangle$ of T. A typical example is to think of a universe made of two qubits. Their initial states will both be described as $\rho = |+\rangle\langle+|$, and the noninteracting Hamiltonian is given by

$$H = \omega\big(\sigma_z^S \otimes \mathbb{I}_C + \mathbb{I}_S \otimes \sigma_z^C\big). \tag{25}$$

The G-twirling operation over the group generated by this Hamiltonian gives us the symmetrized state (in the computational basis for two qubits)

$$\mathcal{G}[\rho] = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

If we now define the time operator over the clock system C to be $T_C = \sigma_x^C$, in the sense that eigenvalues of this operator are shifted by the unitary U_t in discreet steps $t = \pi/\omega$, i.e.,

$$U_{\pi/\omega}|\pm\rangle = |\mp\rangle, \tag{27}$$

then we can obtain the relative state of system S as

$$\rho_S(\pm) = \frac{1}{4}\begin{pmatrix} 2 & \pm 1\\ \pm 1 & 2 \end{pmatrix}. \tag{28}$$

For this state, the probabilities of agreement between the system and the clock are $P(+|+) = P(-|-) = \frac{3}{4}$, while there will be a mistracking with probability $P(+|-) = P(-|+) = \frac{1}{4}$. This happens because the pair T_C, H_C does not constitute a canonical pair, and thus this qubit is not the best choice for a quantum clock.

C. Quasi-ideal clock states

To answer question 2 in Sec. III A, we turn to the quantum clock model proposed by A. Peres in 1980 [19]. Known as the Salecker-Wigner-Peres clock, due to the pioneering work of H. Salecker and Wigner himself on the formulation of a quantum clock in 1958 [20], it consists of a finite-dimensional system described by the Hamiltonian

$$H_C = \omega \sum_{n=0}^{d-1} n\,|n\rangle\langle n|\,. \tag{29}$$

To extract the canonical commutation $[T_C, H_C] = i$, we want to build a basis in which T_C is diagonal and that is related to the energy basis in the same way that the momentum basis is related to the position basis, since the momentum and position operators are canonically conjugated. This leads us to write the basis

$$|\theta_k\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi nk/d}\,|n\rangle\,, \tag{30}$$

with $k \in \{0, d-1\} \subset \mathbb{Z}$. This basis has interesting properties, such as the discreet shift in steps $\frac{\tau}{d} = \frac{2\pi}{\omega d}$, i.e.,

$$U_{\tau/d}\,|\theta_k\rangle = |\theta_{k+1}\rangle\,, \tag{31}$$

and each vector $|\theta_k\rangle$ is infinitely degenerated,

$$|\theta_k\rangle = |\theta_{k+md}\rangle\,, \quad m \in \mathbb{Z}. \tag{32}$$

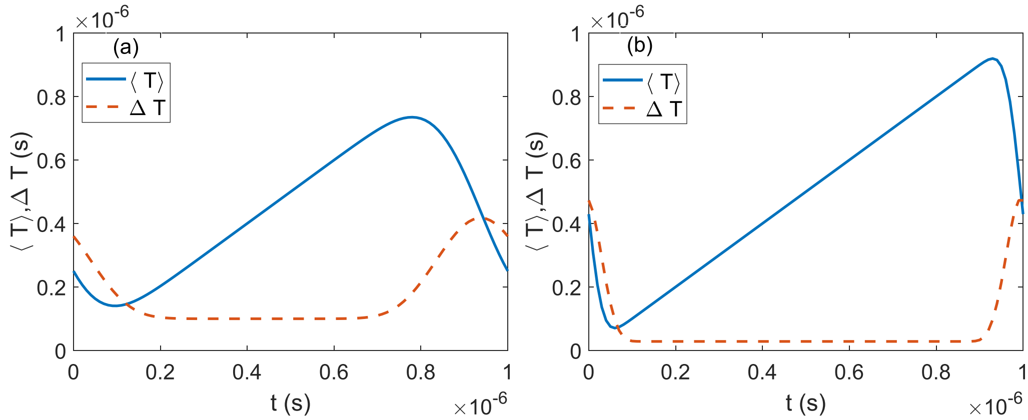

FIG. 3. Expectation value $\langle T_C \rangle$ (solid blue lines) and deviation ΔT_C (dashed red lines) for an SWP time operator over quasi-ideal clock states with $\tau = 1\ \mu$s, $\sigma = \sqrt{d}$, and (a) $d = 8$ and (b) $d = 100$.

Building a time operator diagonal on this basis,

$$T_C = \frac{\tau}{d} \sum_{k=0}^{d-1} k\,|\theta_k\rangle\langle\theta_k|\,, \tag{33}$$

it is expected that the canonical commuting relation should be obtained. However, it is easy to see that

$$\langle\theta_k|[T_C, H_C]|\theta_k\rangle = 0, \quad \forall\, k \in \{0, d-1\}. \tag{34}$$

However, Peres applied this model of a clock in three classic problems of quantum mechanics and showed that this system could indeed keep track of dynamics with an imprecision due to the evolution in discreet steps. He also argued that any attempt to increase the clock dimension to improve its precision would eventually lead to interaction between the system and the clock, affecting (and eliminating, in many cases) the observed phenomenon.

An improvement of this clock model was recently given by Woods, Silva, and Oppenheim in 2019, with their model of quasi-ideal clock states [3]. It consists of a Salecker-Wigner-Peres clock, with the states of interest no longer being the pointer states but, rather, a superposition of them, given by

$$|\psi(k_0)\rangle = \sum_{k\in S_d(k_0)} A e^{-\frac{\pi}{\sigma^2}(k-k_0)^2} e^{i2\pi n_0(k-k_0)/d}\,|\theta_k\rangle\,, \tag{35}$$

where k_0 is a real number related to the parameter in the Schrödinger equation, $k_0 = td/\tau$, σ is a Gaussian standard deviation, and n_0 is associated with the mean energy of the state, $\langle H\rangle_{\psi(k_0)} = n_0\omega$. A is a normalization factor, and $S_d(k_0)$ is given by

$$S_d(k_0) := \begin{cases} \{k \in \mathbb{Z}\,| -\frac{d}{2} \leqslant k - k_0 < \frac{d}{2}\}, & \text{even } d;\\ \{k \in \mathbb{Z} + \frac{1}{2}\,| -\frac{d}{2} \leqslant k - k_0 < \frac{d}{2}\}, & \text{odd } d. \end{cases} \tag{36}$$

It is very interesting to see that the expectation value of the time operator given by Eq. (33) for these states covaries with the external time t, as shown in Fig. 3. There are two relevant results for these quasi-ideal states that we enunciate below:

Theorem 2 (quasi-continuity). Let $\mathcal{H}$ be the Hilbert space of a Salecker-Wigner-Peres clock, with H being its Hamiltonian and T being the time operator. Let $|\psi(k_0)\rangle$ be a quasi-ideal clock state. Then, for any $t \in \mathbb{R}$,

$$e^{-iH_C t}\,|\psi(k_0)\rangle = \sum_{k\in S_d(k_0+td/\tau)} A e^{-\frac{\pi}{\sigma^2}(k-k_0+td/\tau)^2} \times e^{i2\pi n_0(k-k_0+td/\tau)}\,|\theta_k\rangle + |\epsilon\rangle\,, \tag{37}$$

with

$$|\langle\theta_k|\epsilon\rangle| \leqslant O\big(t\,\text{poly}(d)e^{-\frac{\pi d}{4}}\big), \quad d \to \infty. \tag{38}$$

This means that, for a clock size that is large enough, the Hamiltonian H_C generates a continuous shift in the quasi-ideal clock states up to a vanishing error.

Theorem 3 (quasi-canonical commutation). Let $\mathcal{H}$ be a Hilbert space of a Salecker-Wigner-Peres clock, with H_C, T_C, and $|\psi(k_0)\rangle$ the previously defined operators and state. Then

$$[T_C, H_C]\,|\psi(k_0)\rangle = i\,|\psi(k_0)\rangle + |\epsilon_c\rangle\,, \tag{39}$$

with

$$|\langle\epsilon_c|\epsilon_c\rangle|^2 \leqslant O(\text{poly}(d)e^{-\frac{\pi d}{4}}), \quad d \to \infty. \tag{40}$$

With this theorem, the statistics of a canonical pair are recovered for quasi-ideal clock states, evading Pauli's argument and still keeping track of time in a satisfactory way. These results must be enough to ensure the applicability of the Woods-Silva-Oppenheim clock in our WFS model.

IV. WIGNER'S FRIEND SCENARIO WITH A QUASI-IDEAL CLOCK

Now we propose a Wigner's friend scenario. It is comprised of a single laboratory, in which Alice is going to detect the spin of a spin-$\frac{1}{2}$ particle on the z axis. The state is initially prepared in a state

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S), \tag{41}$$

and Alice is going to perform her measurement σ_z^S with respect to a classical and ideal clock at time t_z. At a further time t_W on her clock, Wigner, who is outside the laboratory and unaware of how time is passing inside the laboratory, is going to perform his measurement, a projection over

$$|\mathrm{ok}\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle_S \otimes |\uparrow\rangle_A + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle_S \otimes |\downarrow\rangle_A\,, \tag{42}$$

so from Alice's perspective, Wigner's measurement after her measurement will occur with conditional probabilities

$$P_A(\mathrm{ok}|t_W,\uparrow) = \cos^2\left(\frac{\theta}{2}\right), \tag{43}$$

$$P_A(\mathrm{ok}|t_W,\downarrow) = \sin^2\left(\frac{\theta}{2}\right). \tag{44}$$

From Wigner's perspective, there is a unitary evolution being carried on inside the laboratory, and at the time T_W on his SWP clock, he performs his projection. The system is prepared initially in the state

$$\rho_{AS} = |\perp\rangle\langle\perp|_A \otimes |+\rangle\langle+|_S\,, \tag{45}$$

and the clock is prepared in the quasi-ideal clock state

$$\rho_C = |\psi(0)\rangle\langle\psi(0)|\,. \tag{46}$$

Since we are equipped with the local asymmetric states only, we must perform a G-twirling operation over the product state $\rho = \rho_S \otimes \rho_C$. The global Hamiltonian, however, is not complete, for only H_C is known [Eq. (29)]. We must therefore search for a reasonable Hamiltonian capable of describing the evolution

$$|+\rangle_S \otimes |\perp\rangle_A \to \frac{1}{\sqrt{2}}(|\uparrow\rangle_S|\uparrow\rangle_A + |\downarrow\rangle_S|\downarrow\rangle_A). \tag{47}$$

For this, we present three possible unitaries. The first is an instantaneous transition, given by

$$U_t = \begin{pmatrix} 1-\Theta(\Delta t) & 0 & -i\Theta(\Delta t) & 0 & 0 & 0 \\ 0 & 1-\Theta(\Delta t) & 0 & 0 & 0 & -i\Theta(\Delta t) \\ -i\Theta(\Delta t) & 0 & 1-\Theta(\Delta t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i\Theta(\Delta t) & 0 & 0 & 0 & 1-\Theta(\Delta t) \end{pmatrix}, \tag{48}$$

where the basis adopted is given by

$$\{|\perp\rangle_A|\uparrow(\downarrow)\rangle_S\,, |\uparrow\rangle_A|\uparrow(\downarrow)\rangle_S\,, |\downarrow\rangle_A|\uparrow(\downarrow)\rangle_S\}, \tag{49}$$

and $\Theta(\Delta t) = \Theta(t - t_z)$ is the Heaviside step function. However, this description of the evolution inside the laboratory might be claimed to be too unrealistic, and we could also work with the analytical version of the step function given by hyperbolic functions,

$$U_t = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1-\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & -i\frac{1+\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & 0 & 0 \\ 0 & \frac{1-\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & 0 & 0 & -i\frac{1+\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} \\ -i\frac{1+\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & \frac{1-\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & -i\frac{1+\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} & 0 & 0 & 0 & \frac{1-\tanh(\omega_0\Delta t)}{\sqrt{1+\tanh^2(\omega_0\Delta t)}} \end{pmatrix}. \tag{50}$$

Here, ω_0 is a parameter quantifying how fast the transition happens. Another possible option for the transition is the following, which we call the periodic transition,

$$U_t = \begin{pmatrix} \cos\omega_0 t & 0 & -i\sin\omega_0 t & 0 & 0 & 0 \\ 0 & \cos\omega_0 t & 0 & 0 & 0 & -i\sin\omega_0 t \\ -i\sin\omega_0 t & 0 & \cos\omega_0 t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i\sin\omega_0 t & 0 & 0 & 0 & \cos\omega_0 t \end{pmatrix} \tag{51}$$

where ω_0 again quantifies how fast the transition happens. However, this periodic evolution must be halted at a specific time $T = (m + \frac{1}{2})\frac{\pi}{\omega_0}$, with $m \in \mathbb{Z}$, to effectively represent Eq. (47). Otherwise, the measurement either is not completed or is already being undone. This model of periodic transition has already been used in dynamical models for projective measurements [21] and is adopted as the ruling evolution, since it has a simple generating Hamiltonian given by

$$H_{AS} = \omega_0(|\perp\uparrow\rangle\langle\uparrow\uparrow|_{AS} + |\perp\downarrow\rangle\langle\downarrow\downarrow|_{AS} + \mathrm{H.c.}). \tag{52}$$

We are now capable of describing the global Hamiltonian, given by

$$H = H_{AS} \otimes \mathbb{I}_C + \mathbb{I}_{AS} \otimes H_C\,, \tag{53}$$

and thus performing G-twirling over the state

$$\rho = |\perp\rangle\langle\perp|_A \otimes |+\rangle\langle+|_S \otimes |\psi(0)\rangle\langle\psi(0)|_C . \tag{54}$$

The relevant quantities in our analysis are given by the differences in probabilities predicted by Alice and Wigner for the outcome "ok," represented by

$$\Delta_0 = P_A(\text{ok}|t_z, \uparrow) - P_W(\text{ok}|T_W), \tag{55}$$

$$\Delta_1 = P_A(\text{ok}|t_z, \downarrow) - P_W(\text{ok}|T_W). \tag{56}$$

The paradox vanishes whenever

$$\Delta_0 = \Delta_1 = 0, \tag{57}$$

a constraint equation we define as our *consistency condition*. Particularly, if quantum mechanics is a consistent theory, then Eq. (57) should be satisfied for all $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ that characterize Wigner's detection "ok."

V. RESULTS

A. Satisfying the consistency condition

It is interesting to see that analytical calculations lead us to two relational density operators for the laboratory (see the Appendix), given approximations ($\sigma \geqslant \sqrt{d}$ and $d \to \infty$). The first one is simply a statistical mixture,

$$\rho_{SA}(K) = \frac{1}{2}|\perp\rangle\langle\perp|_A \otimes |+\rangle\langle+|_S + \frac{1}{2}|\Phi_+\rangle\langle\Phi_+|, \tag{58}$$

where $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_S + |\downarrow\rangle_A|\downarrow\rangle_S)$. However, for specific energies ω_0 of the laboratory given by

$$\omega_0 = \frac{q}{2}\omega, \quad q \in \mathbb{Z}, \tag{59}$$

where ω is the SWP clock frequency, the relational state is described as

$$\rho_{SA}(K) = \frac{1}{4}\begin{pmatrix} 1+\mathcal{R}(K) & 1+\mathcal{R}(K) & i\mathcal{Q}(K) & 0 & 0 & i\mathcal{Q}(K) \\ 1+\mathcal{R}(K) & 1+\mathcal{R}(K) & i\mathcal{Q}(K) & 0 & 0 & i\mathcal{Q}(K) \\ -i\mathcal{Q}(K) & -i\mathcal{Q}(K) & 1-\mathcal{R}(K) & 0 & 0 & 1-\mathcal{R}(K) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i\mathcal{Q}(K) & -i\mathcal{Q}(K) & 1-\mathcal{R}(K) & 0 & 0 & 1-\mathcal{R}(K) \end{pmatrix}, \tag{60}$$

where the functions $\mathcal{R}(K)$ and $\mathcal{Q}(K)$ preserve both the periodic behavior of the transition and the proportion $\frac{\omega_0}{\omega} = \frac{q}{2}$ of the resonant behavior. Explicitly, they can be represented as

$$\mathcal{R}(K) = e^{-\Gamma^2}\frac{\text{Re}\{\text{erf}[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} + i\Gamma]\}}{\text{erf}[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}]}\cos\left(\frac{2\pi q}{d}K\right), \tag{61}$$

$$\mathcal{Q}(K) = e^{-\Gamma^2}\frac{\text{Re}\{\text{erf}[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2} + i\Gamma]\}}{\text{erf}[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}]}\sin\left(\frac{2\pi q}{d}K\right), \tag{62}$$

with

$$\Gamma = \sqrt{2\pi}\frac{\sigma}{d}\frac{\omega_0}{\omega} = \sqrt{\frac{\pi}{2}}\frac{\sigma}{d}q, \tag{63}$$

erf[x] being the error function and K being related to the time detected by Wigner on his clock, $K = \frac{Td}{\tau}$.

According to the discussion in Sec. IV, Wigner must perform his measurement at a time $T_W = (m + \frac{1}{2})\frac{\pi}{\omega_0}$ in order to detect the von Neumann measurement in the laboratory. This time is related to a pointer state $K_W = \frac{T_W d}{\tau} = (m + \frac{1}{2})\frac{d\pi}{\omega_0 \tau}$, and for resonant evolution,

$$K_W = \left(m + \frac{1}{2}\right)\frac{d}{q}, \quad m, q \in \mathbb{Z}, \tag{64}$$

which can explain the source of the resonant behavior. If Wigner is supposed to halt the evolution at a time T_W, this time must be an eigenvalue of his clock, i.e., it must be related to a pointer $|\theta_K\rangle$ on his clock. But since these pointers can be associated only with integer (for even d) or half-integer (for odd d) numbers, by definition, the transition must be characterized by a resonant frequency. If not, then Wigner is not allowed to perform its measurement at the proper instant and will always do it over a state in which the measurement has not yet been completed or is already being undone, resulting in the mixed relative state given by Eq. (58).

At the specific pointer K_W, the resonant relative state is given by

$$\rho_{SA}(K_W) = \frac{1}{4}\begin{pmatrix} 1-\mathcal{R}(0) & 1-\mathcal{R}(0) & 0 & 0 & 0 & 0 \\ 1-\mathcal{R}(0) & 1-\mathcal{R}(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathcal{R}(0) & 0 & 0 & 1+\mathcal{R}(0) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mathcal{R}(0) & 0 & 0 & 1+\mathcal{R}(0) \end{pmatrix}, \tag{65}$$

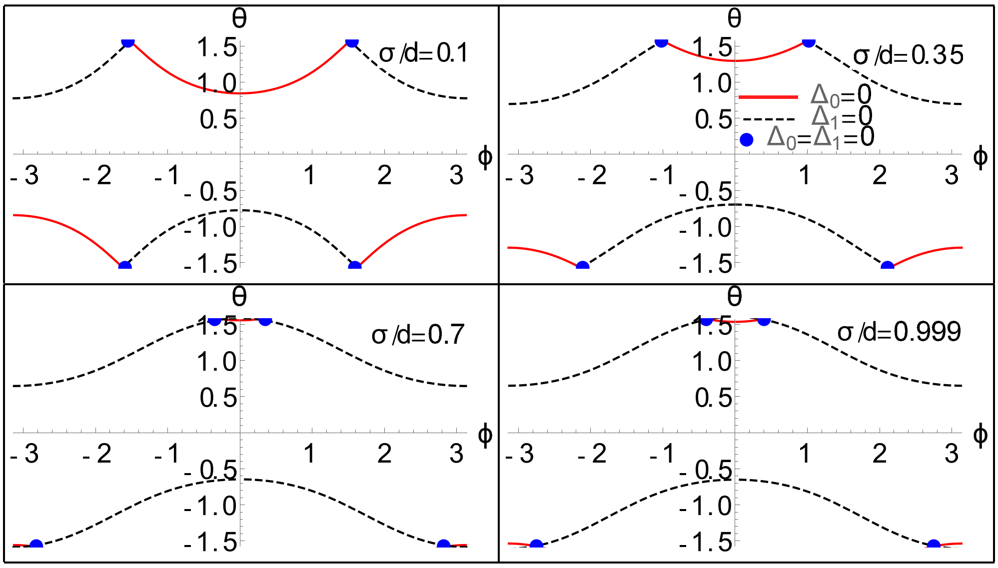

FIG. 4. Values of θ and ϕ (in rad) of Wigner's projection $|\mathrm{ok}\rangle$, for which Δ_0 (solid red lines) and Δ_1 (dashed black lines) are null, for different ratios $\frac{\sigma}{d}$, $d \gg 1$, $\omega_0 = \frac{1}{2}\omega$. Small blue circles refer to measurements for which the consistency condition is satisfied.

since $\mathcal{Q}(K_W) = 0$ and $\mathcal{R}(K_W) = -\mathcal{R}(0)$. From this, it is possible to obtain Wigner's conditional probability $P_W(\mathrm{ok}|T_W) = \mathrm{Tr}\{|\mathrm{ok}\rangle\langle\mathrm{ok}|\,\rho_{SA}(K_W)\}$, leading to differences in the prediction given by

$$\Delta_{0(1)} = \frac{1 \pm \cos\theta}{2} - \frac{1}{4}\left(1 + e^{-(q\sqrt{\frac{\pi}{2}}\frac{\sigma}{d})^2}\frac{\mathrm{Re}\{\mathrm{erf}[\sqrt{\frac{\pi}{2}}\frac{d}{\sigma} + iq\sqrt{\frac{\pi}{2}}\frac{\sigma}{d}]\}}{\mathrm{erf}[\sqrt{\frac{\pi}{2}}\frac{d}{\sigma}]}\right)(1 + \sin\theta\cos\phi). \tag{66}$$

In Fig. 4 it is possible to see the values of θ and ϕ for which both Δ_0 and Δ_1 are null. The consistency condition [Eq. (57)] is satisfied whenever a solid red and a dashed black line cross (small blue circles). The paradox evidently vanishes for very specific measurements $|\mathrm{ok}\rangle$ performed by Wigner. Note that, for $\frac{\sigma}{d} \to 0$ (within the restrictions needed for our calculations to be valid, i.e., $\sigma \geqslant \sqrt{d}$), the only values of θ and ϕ for which the consistency condition is satisfied are related to the observables

$$|\mathrm{ok}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_S \pm i|\downarrow\rangle_A|\downarrow\rangle_S), \tag{67}$$

precisely the same observables that rule out the paradox for the original WFS introduced in Sec. II. It is interesting to see that for a clock state with Gaussian spread $\sigma = \sqrt{d}$ (called a *symmetric state*), if d is large enough, it is possible to recover the same scenario as the one associated with a shared classical clock.

For a ratio $\frac{\sigma}{d} \to 1$, otherwise, the relative state becomes closer to the mixed relative state given by Eq. (58), and the only observable that Wigner can measure without raising a paradox approaches to

$$|\mathrm{ok}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_S + |\downarrow\rangle_A|\downarrow\rangle_S) = |\Phi_+\rangle. \tag{68}$$

It is convenient to know that the uncertainty in the clock state preparation can control which observable is allowed to be measured, and even if this model is not capable of ruling out the paradox for any observable "ok," this might indicate that the right choice of evolution in the laboratory associated with the quasi-ideal clock states could eventually catalyze the reduced state observed by Alice.

B. Analysis of Wigner's measurements

From a resource-theoretic point of view, it is important to analyze which measurements are allowed for Wigner. If time is being internalized in the sense that Wigner should no longer be aware of how time is passing in the laboratory, his measurements might be such that no resource is generated (in this case, the asymmetry with respect to time evolution). For that purpose, we look at the invariant subspaces of $\mathcal{H} = \mathcal{H}_{SA} \otimes \mathcal{H}_C$ with respect to the global time evolution, the so-called *charge sectors* [16,18].

The clock Hamiltonian is already diagonalized, and by diagonalizing the laboratory Hamiltonian, we can work in the basis $\{|-\omega_{0\uparrow(\downarrow)}, n\rangle, |0_{\uparrow(\downarrow)}, n\rangle, |+\omega_0\uparrow(\downarrow), n\rangle\}$, where

$$|\pm\omega_{0\uparrow}\rangle_{SA} = \frac{1}{\sqrt{2}}(|\perp\uparrow\rangle_{SA} \pm |\uparrow\uparrow\rangle_{SA}), \tag{69}$$

$$|\pm\omega_{0\downarrow}\rangle_{SA} = \frac{1}{\sqrt{2}}(|\perp\downarrow\rangle_{SA} \pm |\downarrow\downarrow\rangle_{SA}), \tag{70}$$

$$|0_\uparrow\rangle_{SA} = |\uparrow\downarrow\rangle_{SA}, \quad |0_\downarrow\rangle_{SA} = |\downarrow\uparrow\rangle_{SA}, \tag{71}$$

are eigenstates related to the eigenvalues $\{\pm\omega_0, 0\}$ of the laboratory Hamiltonian.

The charge sectors are comprised of subspaces of $\mathcal{H}$ associated with the same eigenvalues of the global Hamiltonian H. In the first moment, each charge sector will have dimension

2, associated with the numbers $n\omega - \omega_0$, $n\omega$, or $n\omega + \omega_0$, for $n \in [0, d-1]$. However, when $\omega_0 = \frac{q}{2}\omega$, $q \in \mathbb{Z}$, some of these eigenvalues coincide with each other, ensuring nontrivial charge sectors of higher dimension.

For odd q, such as the one adopted in Fig. 4, the charge sectors associated with $n\omega$ remain two-dimensional, while the one associated with $n\omega + \omega_0$ is four-dimensional and generated by the basis

$$\{|{+\omega_0}_{\uparrow(\downarrow)}, n\rangle\,, |{-\omega_0}_{\uparrow(\downarrow)}, n+q\rangle\} \tag{72}$$

for $0 \leqslant |n+q| \leqslant d-1$. For even q, the nontrivial charge sector associated with this number is six-dimensional and generated by the basis

$$\{|{+\omega_0}_{\uparrow(\downarrow)}, n\rangle\,, |0_{\uparrow(\downarrow)}, n+q\rangle\,, |{-\omega_0}_{\uparrow(\downarrow)}, n+2q\rangle\} \tag{73}$$

for $0 \leqslant |n+2q| \leqslant d-1$. We can see then why the resonance emerges, since it is the only regime in which nontrivial charge sectors occur. Larger charge sectors can protect more information against the dephasing provoked by the G-twirling operation [18].

If Wigner is allowed to perform only symmetric operations, i.e., operations that preserve the asymmetry of the global state, then he should only perform measurements that cannot transfer information from one charge sector to another [22,23]. For our definition of $|\mathrm{ok}\rangle$, the projector Π_{ok} is written in the diagonal laboratory basis as

$$\Pi_{\mathrm{ok}} = \frac{1}{2}\begin{pmatrix} A & O & -A \\ O & O & O \\ -A & O & A \end{pmatrix}, \tag{74}$$

where

$$A = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}, \tag{75}$$

and O is a 2×2 null matrix. Then writing down the operator $\Pi_{\mathrm{ok}} \otimes \mathbb{I}_C$ in the diagonal global basis, e.g., for $d = 3$ and $q = 1$, we are led to

$$\Pi_{ok} \otimes \mathbb{I}_C = \frac{1}{2}\left(\begin{array}{ccccccccc} \boxed{A} & O & -A & & & & & & \\ O & \boxed{O} & O & & & & & & \\ -A & O & A & & & & & & \\ & & & A & O & -A & & & \\ & & & O & \boxed{O} & O & & & \\ & & & -A & O & A & & & \\ & & & & & & A & O & -A \\ & & & & & & O & \boxed{O} & O \\ & & & & & & -A & O & \boxed{A} \end{array}\right), \tag{76}$$

which allows for information flow between charge sectors (rectangles in the matrix). However, if we impose this restriction on Wigner measurements, then he would also be forbidden from measuring time, since the projector over a pointer state, given in this basis as

$$\mathbb{I}_{SA} \otimes \Pi_K^C = \mathbb{I}_{SA} \otimes \frac{1}{d}\sum_{n,n'=0}^{d} e^{-i2\pi(n-n')k/d}\,|n\rangle\,\langle n'|\,, \tag{77}$$

also allows for information flow between charge sectors. There is thus no point in restricting Wigner's measurement to free operations only, since he would then be forbidden from consulting his clock. It is crucial to consider, however, that no asymmetry monotone can work in such a scenario, given that Wigner is free to generate resources at his will.

One might further ask if the consistency condition would be satisfied for time-symmetric local operations on the laboratory, even with asymmetric operations performed locally on the clock. Such operations are convex combinations of the projectors

$$|\mathrm{ok}^{\pm}_{\mathrm{sym}}\rangle = \frac{1}{\sqrt{2}}(|\xi\rangle_S \otimes |\perp\rangle_A \pm |\mathrm{ok}\rangle_{SA}), \tag{78}$$

where

$$|\xi\rangle_S = \cos\frac{\theta}{2}\,|\uparrow\rangle_S + e^{i\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle_S\,. \tag{79}$$

Even for such measurements, it is easy to see that the consistency condition is not satisfied for all (θ, ϕ). If Wigner is performing a projection over one of the two $\mathrm{ok}_{\mathrm{sym}}$'s, then

$$\Delta_{0(1)} = \frac{1}{4}\left[\frac{1}{2} \pm \cos\theta - \cos\phi\sin\theta\right] \tag{80}$$

and the uncertainty σ of the quasi-ideal clock state plays no role in the statistics of these symmetric measurements. Wigner agrees with Alice's predictions only when $(\theta, \phi) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{3})$, and the paradox thus persists.

VI. CONCLUSIONS

This work aimed to study the consequences of the insertion of a feasible quantum clock in a Wigner's friend scenario. By working with a Salecker-Wigner-Peres clock [19,20] and Woods, Silva, and Oppenheim's quasi-ideal clock states [3] it was possible to internalize time in a WFS in a Page-Wootters formalism. The choice of a periodic model of transition to rule the laboratory dynamics [21] led to interesting results, indicating that the paradox still does not vanish for any measurement made by Wigner but, rather, for very specific observables controlled by the ratios $\frac{\omega_0}{\omega}$ between the laboratory dynamics and the SWP clock frequency, and $\frac{\sigma}{d}$ of the clock states' Gaussian spread. An analysis of how the process of time internalization affects Wigner's possible measurements indicates that restricting them to symmetric operations implies forbidding Wigner from consulting his clock. Furthermore, the paradox persists even for time-symmetric operations over the laboratory.

This result might imply that the quasi-ideal clock state, with its intrinsic uncertainty σ, is not enough to trigger the desired decoherent behavior that Alice observes when performing her measurement. Indeed, decoherence can be considered as part of the definition of a measurement [9], and the insertion of uncontrolled degrees of freedom might be unavoidable.

Another possible claim is that the consistency condition imposed by the constraint in Eq. (57) is unnecessary. Since Alice and Wigner have access to different parts of the global state, it is not reasonable to demand that they should predict the same probability distributions. The subjectivity of objective measurements due to different reference frames has recently been shown [24].

Monogamy between Alice and the spin-$\frac{1}{2}$ particle might be preventing the clock from fully accessing the laboratory dynamics and, thus, stealing any coherence [25,26]. There is also a theorem ensuring the possibility of a catalytic conversion from the entangled state to the reduced state through the insertion of a clock [27].

Finally, there is the need to test other models for the laboratory dynamics. Particularly, investigating the entanglement in the so-called *internal states* [18] and how quantifiers such as shared asymmetry [18,28,29] would act in this sort of scenario could shed light on this fundamental problem.

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APPENDIX: *G*-TWIRLING OVER THE GLOBAL STATE

The noninteracting Hamiltonian generates a global evolution given by

$$U_t = e^{-iH_{AS}t} \otimes e^{-iH_C t}. \tag{A1}$$

The G-twirling operation will therefore be an integration whose integrand is the state

$$U_t \rho U_t^\dagger = \rho_{SA}(t) \otimes \rho_C(t). \tag{A2}$$

The time-evolved laboratory state is explicitly given by

$$\rho_{AS}(t) = \frac{1}{2}\begin{pmatrix} \cos^2\omega_0 t & \cos^2\omega_0 t & \frac{i}{2}\sin 2\omega_0 t & 0 & 0 & \frac{i}{2}\sin 2\omega_0 t \\ \cos^2\omega_0 t & \cos^2\omega_0 t & \frac{i}{2}\sin 2\omega_0 t & 0 & 0 & \frac{i}{2}\sin 2\omega_0 t \\ -\frac{i}{2}\sin 2\omega_0 t & -\frac{i}{2}\sin 2\omega_0 t & \sin^2\omega_0 t & 0 & 0 & \sin^2\omega_0 t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2}\sin 2\omega_0 t & -\frac{i}{2}\sin 2\omega_0 t & \sin^2\omega_0 t & 0 & 0 & \sin^2\omega_0 t \end{pmatrix}, \tag{A3}$$

while for $d \to \infty$, the time-evolved clock state is given by

$$\rho_C(t) \approx |\psi(td/\tau)\rangle\langle\psi(td/\tau)|, \tag{A4}$$

up to an exponentially vanishing error. We can therefore work with integrations of the terms of $U_t\rho U_t^\dagger$, which will be products of one of three basis functions, 1, $e^{\pm i2\omega_0 t}$, and the time-dependent term in Eq. (A4). Explicitly, it can be written as

$$\rho_C(t) \approx \sum_{k,k' \in S_d(td/\tau)} |A|^2 e^{-\frac{\pi}{\sigma^2}(k-td/\tau)^2} e^{-\frac{\pi}{\sigma^2}(k'-td/\tau)^2} e^{i2\pi n_0(k-k')/d} |\theta_k\rangle\langle\theta_{k'}|. \tag{A5}$$

A result of Woods, Silva, and Oppenheim [3] ensures that, if $\sigma \geqslant \sqrt{d}$, then $|A|$ is nearly constant in time, and within this range of σ we are allowed to take the normalizing factor out of the integral. The summation limits, however, still depend on time, and in the first moment, there is no way the integral and the summation commute.

However, an analysis of the behavior of $S_d(td/\tau)$ with respect to t leads to the result

$$S_d(td/\tau) = S_d(n), \quad t \in \left(\frac{\tau}{d}(n-1), \frac{\tau}{d}n\right], \quad n \in \mathbb{Z}, \tag{A6}$$

which allows us to write the G-twirling operation as

$$\mathcal{G}\begin{bmatrix} 1 \\ e^{\pm i2\omega_0 t}\end{bmatrix} = \lim_{N\to\infty} \frac{|A|^2\sigma d}{2\sqrt{2}\tau N} \sum_{n=-N}^{N} \sum_{k,k'\in S_d(n)} e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0(k-k')/d} |\theta_k\rangle\langle\theta_{k'}| \int_{\frac{\tau}{d}(n-1)}^{\frac{\tau}{d}n} e^{-\frac{\pi}{\sigma^2}\left(\frac{\sqrt{2}d}{\tau}t - \frac{k+k'}{\sqrt{2}}\right)^2} \begin{pmatrix} 1 \\ e^{\pm i2\omega_0 t}\end{pmatrix} dt. \tag{A7}$$

For the basic function 1, this integration leads to

$$\begin{aligned}\mathcal{G}[1] = \lim_{N\to\infty} \frac{|A|^2\sigma^2}{8N} \sum_{n=-N}^{N} \sum_{k,k'\in S_d(n)} |\theta_k\rangle\langle\theta_{k'}| e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0(k-k')/d} \\ \times \left\{\mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n - \frac{k+k'}{2}\right)\right] - \mathrm{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-1-\frac{k+k'}{2}\right)\right]\right\},\end{aligned} \tag{A8}$$

while for $e^{\pm i2\omega_0 t}$, it leads to

$$\mathcal{G}[e^{\pm i2\omega_0 t}] = \lim_{N\to\infty} \frac{|A|^2\sigma e^{-\Gamma^2}}{4\sqrt{2}d} \sum_{n=-N}^{N} \sum_{k,k'\in S_d(n)} |\theta_k\rangle\langle\theta_{k'}| \, e^{-\frac{\pi}{2\sigma^2}(k-k')^2} e^{i2\pi n_0(k-k')/d} e^{\pm i\frac{\sqrt{2\pi}}{\sigma}\Gamma(k+k')} \times\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-\frac{k+k'}{2}\right)\pm i\Gamma\right] - \text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\left(n-1-\frac{k+k'}{2}\right)\pm i\Gamma\right]\right\}. \tag{A9}$$

Since we are not interested directly in the global symmetric state $\mathcal{G}[\rho]$, but rather in the relational state given by Eq. (24), we can start to project these results over a specific pointer state $|\theta_K\rangle$. One must remember, however, to take into account not only the term associated with $|\theta_K\rangle$ but every other element corresponding to $|\theta_{K+md}\rangle$, $m\in\mathbb{Z}$, since the pointer basis is infinitely degenerated. For $\mathcal{G}[1]$ and $\mathcal{G}[e^{\pm i2\omega_0 t}$, this analysis leads to

$$\Pi_K^C\mathcal{G}[1]\Pi_K^C = \lim_{N\to\infty}\frac{|A|^2\sigma^2}{8N}\frac{2N}{d}2\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right] = \frac{|A|^2\sigma^2}{2d}\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\right], \tag{A10}$$

$$\Pi_K^C\mathcal{G}[e^{\pm i2\omega_0 t}]\Pi_K^C = \lim_{N\to\infty}\frac{|A|^2\sigma^2 e^{-\Gamma^2}}{4N} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma K}\text{Re}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\pm i\Gamma\right]\right\}\sum_{m=-N/d}^{N/d} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma md}. \tag{A11}$$

Note that the limit in Eq. (A11) goes to 0 unless each term in the summation is equal to 1. This will happen whenever

$$\frac{\sqrt{2\pi}}{\sigma}\Gamma d = q\pi \iff \omega_0\tau = q\pi, \quad q\in\mathbb{Z}, \tag{A12}$$

which means that

$$\frac{\omega_0}{\omega} = \frac{q}{2}, \quad q\in\mathbb{Z}. \tag{A13}$$

In this case, we can rewrite Eq. (A11) as

$$\Pi_K^C\mathcal{G}[e^{\pm i2\omega_0 t}]\Pi_K^C = \begin{cases}\frac{|A|^2\sigma^2 e^{-\Gamma^2}}{2d} e^{\pm i2\frac{\sqrt{2\pi}}{\sigma}\Gamma K}\text{Re}\left\{\text{erf}\left[\frac{\sqrt{2\pi}}{\sigma}\frac{d}{2}\pm i\Gamma\right]\right\} & \text{if} \quad \omega_0 = \frac{q}{2}\omega, \\ 0 & \text{otherwise}.\end{cases} \tag{A14}$$

Therefore, we can finally describe every entry of the relative laboratory state as a linear combination of the symmetric functions $\mathcal{G}[1]$ and $\mathcal{G}[e^{\pm i2\omega_0 t}]$, such that

$$\Pi_K^C\mathcal{G}[\cos^2\omega_0 t]\Pi_K^C = \frac{1}{2}\Pi_K^C\mathcal{G}[1]\Pi_K^C + \frac{1}{4}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C + \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C), \tag{A15}$$

$$\Pi_K^C\mathcal{G}[\sin^2\omega_0 t]\Pi_K^C = \frac{1}{2}\Pi_K^C\mathcal{G}[1]\Pi_K^C - \frac{1}{4}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C + \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C), \tag{A16}$$

$$\Pi_K^C\mathcal{G}[\sin 2\omega_0 t]\Pi_K^C = \frac{1}{2i}(\Pi_K^C\mathcal{G}[e^{i2\omega_0 t}]\Pi_K^C - \Pi_K^C\mathcal{G}[e^{-i2\omega_0 t}]\Pi_K^C), \tag{A17}$$

finally obtaining the relative state

$$\rho_{\text{FS}}^W(K) = \text{Tr}_C\left\{\frac{(\mathbb{I}_{\text{FS}}\otimes\Pi_K^C)\mathcal{G}[\rho](\mathbb{I}_{\text{FS}}\otimes\Pi_K^C)}{\text{Tr}\{(\mathbb{I}_{\text{FS}}\otimes\Pi_K^C)\mathcal{G}[\rho]\}}\right\} = \frac{1}{4}\begin{pmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&1\end{pmatrix}, \tag{A18}$$

if $\frac{\omega_0}{\omega}\neq\frac{q}{2}$, or

$$\rho(K) = \frac{1}{4}\begin{pmatrix}1+\mathcal{R}(K)&1+\mathcal{R}(K)&i\mathcal{Q}(K)&0&0&i\mathcal{Q}(K)\\1+\mathcal{R}(K)&1+\mathcal{R}(K)&i\mathcal{Q}(K)&0&0&i\mathcal{Q}(K)\\-i\mathcal{Q}(K)&-i\mathcal{Q}(K)&1-\mathcal{R}(K)&0&0&1-\mathcal{R}(K)\\0&0&0&0&0&0\\0&0&0&0&0&0\\-i\mathcal{Q}(K)&-i\mathcal{Q}(K)&1-\mathcal{R}(K)&0&0&1-\mathcal{R}(K)\end{pmatrix}, \tag{A19}$$

if $\frac{\omega_0}{\omega}=\frac{q}{2}$, with $\mathcal{R}(K)$ and $\mathcal{Q}(K)$ already defined by Eqs. (61) and (62).

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