Collisionless sound of bosonic superfluids in lower dimensions

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The superfluidity of low-temperature bosons is well established in the collisional regime. In the collisionless regime, however, the presence of superfluidity is not yet fully clarified, in particular in lower spatial dimensions. Here, we compare the Vlasov-Landau equation, which does not take into account the superfluid nature of the bosonic system, with the Andreev-Khalatnikov equations, which instead explicitly contain a superfluid velocity. We show that recent experimental data of the sound mode in a two-dimensional collisionless Bose gas of ^{87}Rb atoms are in good agreement with both theories, but the sound damping is better reproduced by the Andreev-Khalatnikov equations below the Berezinskii-Kosterlitz-Thouless critical temperature T_c, while above T_c the Vlasov-Landau results are closer to the experimental ones. For one-dimensional bosonic fluids, where experimental data are not yet available, we find larger differences between the sound velocities predicted by the two transport theories and, also in this case, the existence of a superfluid velocity reduces the sound damping.

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I. INTRODUCTION

According to Landau [1], the liquid helium below the critical temperature is characterized by a superfluid component and a normal component. This idea was inspired by similar models used for superconductors [2] and superfluids [3]. In the standard hydrodynamic treatment of a neutral superfluid [4–6] the normal component is supposed to be in the collisional regime. The very special case of the collisionless superfluid helium-4, where the normal component is in the collisionless regime was analyzed by Andreev and Khalatnikov [7]. In the collisionless regime [5,8] the dimensionless parameter $\omega\tau_c$ is such that $\omega\tau_c \gg 1$, where τ_c is the collision time of quasiparticles [8] and ω is the frequency of a generic macroscopic oscillation traveling along the fluid. Usually τ_c grows by decreasing the temperature T, and at extremely low temperatures one expects that collisionless phenomena dominate the dynamics of superfluids and, more generally, the dynamics of quantum liquids. Indeed, the Andreev and Khalatnikov [7] collisionless approach is in full agreement with experimental measurements [9] of the sound velocity of helium-4 for the temperature below 0.4 K. In general, depending on size and density, the system can be in the collisionless regime also far from zero temperature [4–8]. Actually, natural systems as ionized plasmas do exist which, due to the velocity dependence of the collision frequency, become collisionless in the opposite regime of very high temperature [10].

The interest in collisionless superfluids has been renewed by a recent experiment [11], where the sound mode was measured in a uniform quasi-two-dimensional (2D) Bose gas made of ^{87}Rb atoms. The experimental data of the speed of sound are in good agreement with theoretical results [12,13] based on the Vlasov-Landau equation [14,15] (which is substantially equivalent to the random-phase approximation [16]) for neutral collisionless bosons. There are, however, some discrepancies between the experimental data of sound damping and the prediction of the Vlasov-Landau equation [12]. Very recently it has been shown [17] that the second sound of modified two-fluid hydrodynamic equations, which incorporate the dynamics of the quantized vortices, reproduce quite well the experimental sound velocity of Ref. [11]. However, in this dynamical Kosterlitz-Thouless theory [17] there is a fitting parameter in the dielectric function which makes this theory not really predictive. In Refs. [12,13] the superfluid nature of the system is not taken into account: The superfluid velocity $\mathbf{v}_s(\mathbf{r}, t)$ does not appear and the phase-space distribution $f(\mathbf{r}, \mathbf{p}, t)$ of particles is used instead of the phase-space distribution $f_{qp}(\mathbf{r}, \mathbf{p}, t)$ of quasiparticles.

In this paper we investigate the collisionless sound mode of bosonic quantum gases both in two and one spatial dimensions. We compare the Vlasov-Landau equation, which does not take into account the superfluid nature of the neutral bosonic system, with the Andreev-Khalatnikov equations [7], which instead explicitly contain a superfluid velocity. We find that the behavior of the speed of sound obtained with the two approaches is similar but the experimental data of sound damping [11] in a 2D collisionless Bose gas are closer to the theoretical predictions based on Andreev-Khalatnikov equations, below the Berezinskii-Kosterlitz-Thouless critical temperature T_c [18,19]. In 1D the superfluidity is much more elusive [20], but it could be experimentally found at low temperature for finite-size systems where phase slips are inhibited [21]. For the collisionless 1D Bose gas we show that the speed of sound predicted by the two transport theories is quite different. The damping rates of the sound velocities are instead very close to each other, but also in this 1D case the presence of a superfluid velocity suppresses the sound damping.

II. VLASOV-LANDAU THEORY OF NEUTRAL COLLISIONLESS BOSONS

The equilibrium distribution of a weakly interacting gas of D-dimensional neutral bosons, each of them with mass m, is given by

$$f_0(p) = \frac{1}{e^{\beta\left(\frac{p^2}{2m}+gn_0-\mu\right)} - 1}, \tag{1}$$

where μ is the chemical potential, fixed by the condition $n_0 = \int dV_{\mathbf{p}}\, f_0(p)$ with n_0 the total number density at equilibrium, $dV_{\mathbf{p}} = d^D\mathbf{p}/(2\pi\hbar)^D$, and $p = |\mathbf{p}|$. Here, we assume a weakly interacting bosonic gas with zero-range interaction of strength g. Notice that, because n_0 is constant, introducing the effective chemical potential $\tilde{\mu} = \mu - gn_0$, $f_0(p)$ can also be interpreted as the distribution of noninteracting bosons.

The interaction strength g appears also in the out-of-equilibrium mean-field external potential $U_{\text{mf}}(\mathbf{r}, t) = g\int dV_{\mathbf{p}}\, f(\mathbf{r}, \mathbf{p}, t)$, where $f(\mathbf{r}, \mathbf{p}, t)$ is the out-of-equilibrium distribution function, which is driven by the following mean-field collisionless Vlasov-Landau equation,

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla - \nabla U_{\text{mf}}(\mathbf{r}, t)\cdot\nabla_{\mathbf{p}}\right) f(\mathbf{r}, \mathbf{p}, t) = 0, \tag{2}$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$ and $\nabla_{\mathbf{p}} = (\partial_{p_x}, \partial_{p_y}, \partial_{p_z})$. As previously stressed, the equilibrium interaction term gn_0 is not essential in Eq. (1) because it can be absorbed in the definition of μ. Instead, the nonequilibrium interaction term $gn(\mathbf{r}, t)$ with $n(\mathbf{r}, t) = \int dV_{\mathbf{p}}\, f(\mathbf{r}, \mathbf{p}, t)$ is crucial in the Vlasov-Landau equation (2). We observe that in the three-dimensional case one must use $2gn(\mathbf{r}, t)$ above T_c because the exchange term in the thermal component is responsible for doubling the value of the density fluctuations [12]. For two-dimensional bosonic systems the absence of the factor 2 is justified not only close to zero temperature but also above the Berezinskii-Kosterlitz-Thouless transition due to the persistence of a quasicondensate regime [22,23].

Linearized Vlasov-Landau equation

Starting from the Vlasov-Landau equation (2) and setting

$$f(\mathbf{r}, \mathbf{p}, t) = f_0(p) + \tilde{f}(\mathbf{p})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{3}$$

where $f_0(p)$ is the equilibrium distribution and the plane-wave fluctuations with amplitude $\tilde{f}(\mathbf{p})$ are supposed to be small with respect to the equilibrium distribution, we get the following linearized equation,

$$(\omega - \mathbf{p}\cdot\mathbf{k})\tilde{f}(\mathbf{p}) + g\int dV_{\mathbf{p}'}\tilde{f}(\mathbf{p}')\mathbf{k}\cdot\nabla_{\mathbf{p}} f_0(p) = 0. \tag{4}$$

From this expression one gets an implicity formula for the collisionless (zero-sound) velocity $u_0 = \omega/k$, namely

$$1 - g\int dV_{\mathbf{p}} \frac{\nabla_{\mathbf{p}} f_0(p)\cdot\mathbf{n}}{\mathbf{p}\cdot\mathbf{n} - u_0} = 0, \tag{5}$$

where $\mathbf{n} = \mathbf{k}/k$ with $k = |\mathbf{k}|$. Thus, linearizing Eq. (2) around the equilibrium configuration one obtains a plane-wave solution with frequency ω and wave vector $\mathbf{k}$ such that $\omega = u_0 k$, where u_0 is the speed of sound and $k = |\mathbf{k}|$. The determination of this complex quantity u_0 requires nontrivial integrations in the complex domain of Eq. (5) [24]. For analytical and numerical details, see Appendix A. In general, the frequency ω and, correspondingly, the velocity u_0 are complex numbers: The real parts represent the actual propagation frequency/speed, whereas the imaginary part is the damping rate.

III. ANDREEV-KHALATNIKOV THEORY OF NEUTRAL COLLISIONLESS SUPERFLUIDS

Let us now consider a D-dimensional collisionless superfluid made of identical bosonic particles of mass m. At thermal equilibrium the system is characterized by the total mass density $\rho_0 = \rho_{s0} + \rho_{n0}$ where ρ_{s0} is the superfluid mass density and ρ_{n0} is the normal mass density. At fixed ρ_0 both ρ_{s0} and ρ_{n0} depend on the absolute temperature T. In particular, the normal mass density ρ_{n0} can be obtained from the equilibrium distribution $f_{\text{qp},0}(p)$ of quasiparticles [1] as $\rho_{n0} = -\frac{1}{D}\int dV_{\mathbf{p}}\, p^2 \frac{df_0(p)}{dE}$ with $p = |\mathbf{p}|$ and

$$f_{\text{qp},0}(p) = \frac{1}{e^{\beta E[p,\rho_0]} - 1}, \tag{6}$$

where $\beta = 1/(k_B T)$ with k_B the Boltzmann constant and $E(p)$ is the spectrum of quasiparticles. Here, we assume the Bogoliubov spectrum [25] of a weakly interacting bosonic gas with zero-range interaction of strength g, given by

$$E[p, \rho_0] = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + \frac{2g}{m}\rho_0\right)}. \tag{7}$$

Notice that, in the most general case, the Bogoliubov spectrum (7) has a temperature dependence [26], which is not included in our approach.

Within the Andreev and Khalatnikov theory [5,7,8], the collisionless superfluid is characterized by three dynamical variables: the phase-space distribution of quasiparticles $f_{\text{qp}}(\mathbf{r}, \mathbf{p}, t)$, the local mass density $\rho(\mathbf{r}, t)$, and the superfluid velocity $\mathbf{v}_s(\mathbf{r}, t)$. There are three coupled partial differential equations. One is the collisionless Vlasov-Landau equation for the distribution of quasiparticles,

$$\left(\frac{\partial}{\partial t} + \nabla_{\mathbf{p}}(E[p, \rho(\mathbf{r}, t)] + \mathbf{v}_s(\mathbf{r}, t)\cdot\mathbf{p})\cdot\nabla \right.$$
$$\left. -\nabla(E[p, \rho(\mathbf{r}, t)] + \mathbf{v}_s(\mathbf{r}, t)\cdot\mathbf{p})\cdot\nabla_{\mathbf{p}}\right) f_{\text{qp}}(\mathbf{r}, \mathbf{p}, t) = 0, \tag{8}$$

where the term $\mathbf{v}_s(\mathbf{r}, t)\cdot\mathbf{p}$ in Eq. (8) is due to the fact that the energy of quasiparticles is obtained in a frame of reference at rest, in which the superfluid velocity is $\mathbf{v}_s(\mathbf{r}, t)$ [8]. There is also the equation of continuity,

$$\frac{\partial\rho(\mathbf{r}, t)}{\partial t} + \nabla\cdot\left(\rho(\mathbf{r}, t)\mathbf{v}_s(\mathbf{r}, t) + \int dV_{\mathbf{p}}\,\mathbf{p}\, f_{\text{qp}}(\mathbf{r}, \mathbf{p}, t)\right) = 0, \tag{9}$$

and it is important to observe that in front of $\mathbf{v}_s(\mathbf{r},t)$ it appears $\rho(\mathbf{r},t)$. Finally, there is an equation for the superfluid velocity $\mathbf{v}_s(\mathbf{r},t)$, which reads

$$\frac{\partial \mathbf{v}_s(\mathbf{r},t)}{\partial t}+\nabla\left[\frac{1}{2}v_s(\mathbf{r},t)^2+\frac{\mu_0[\rho(\mathbf{r},t)]}{m}+\int dV_{\mathbf{p}}\,\frac{\partial E[p,\rho(\mathbf{r},t)]}{\partial \rho}f_{\text{qp}}(\mathbf{r},\mathbf{p},t)\right]=\mathbf{0}, \quad (10)$$

where

$$E[p,\rho(\mathbf{r},t)]=\sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m}+\frac{2g}{m}\rho(\mathbf{r},t)\right)}, \quad (11)$$

and μ_0 the chemical potential of the system at zero temperature (i.e., $T=0$). The Landau-Vlasov equation (2) can be formally recovered from Eq. (8) setting $v_s(\mathbf{r},t)=0$ and expanding Eq. (11) for $p^2/(2m)\gg(2g/m)\rho(\mathbf{r},t)$. In this regime the mean-field force of Eq. (8) is $-\nabla E[p,\rho(\mathbf{r},t)]\simeq -g\nabla n(\mathbf{r},t)$ with $n(\mathbf{r},t)=\rho(\mathbf{r},t)/m$.

Linearized Andreev-Khalatnikov equations

Similarly to the linearized Vlasov-Landau equation, also the linearized Andreev-Khalatnikov equations around the equilibrium configuration admit plane-wave solutions with frequency ω and wave vector $\mathbf{k}$ such that $\omega=u_0k$ with u_0 the corresponding speed of sound. We linearize the Andreev-Khalatnikov equations setting

$$f_{\text{qp}}(\mathbf{r},\mathbf{p},t)=f_{\text{qp},0}(p)+\tilde{f}_{\text{qp}}(\mathbf{p})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (12)$$

$$\rho(\mathbf{r},t)=\rho_0+\tilde{\rho}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (13)$$

$$\mathbf{v}_s(\mathbf{r},t)=\mathbf{0}+\tilde{\mathbf{v}}_s\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (14)$$

where the plane-wave fluctuations are supposed to be small with respect to the equilibrium quantities. It follows that the linearized equations of motion are given by

$$[\omega-\nabla_{\mathbf{p}}E(p)\cdot\mathbf{k}]\tilde{f}_{\text{qp}}(\mathbf{p})+\nabla_{\mathbf{p}}E(p)\cdot\mathbf{k}\frac{df_{\text{qp},0}(p)}{dE}\times\left(\frac{dE(p)}{d\rho_0}\tilde{\rho}+\mathbf{p}\cdot\tilde{\mathbf{v}}_s\right)=0, \quad (15)$$

$$\omega\,\tilde{\rho}-\rho_0\,k\,\tilde{v}_s-k\int dV_{\mathbf{p}}\,p\,\tilde{f}_{\text{qp}}(\mathbf{p})=0, \quad (16)$$

$$-\omega\,\tilde{v}_s+k\left(\frac{1}{\rho_0}\frac{dP_0}{d\rho_0}+\int dV_{\mathbf{p}}\tilde{f}_{\text{qp},0}(\mathbf{p})\frac{d^2E(p)}{d\rho_0^2}\right)\tilde{\rho}+k\int dV_{\mathbf{p}}\frac{dE(p)}{d\rho_0}\tilde{f}_{\text{qp}}(\mathbf{p})=0, \quad (17)$$

where P_0 is the pressure at zero temperature. Equations (16) and (17) contain respectively the terms $\int dV_{\mathbf{p}}\,p\,\tilde{f}_{\text{qp}}$ and $\int dV_{\mathbf{p}}\tilde{f}_{\text{qp},0}\frac{d^2E(p)}{d\rho_0^2}$. Both terms may be computed from Eq. (15); thus any dependence from $\tilde{f}_{\text{qp},0}$ disappears from Eqs. (16) and (17), which become a set of two linear homogeneous equations for the two variables $\tilde{v}_s$, $\tilde{\rho}$. The condition of vanishing determinant of the above set of linear equations yields the dispersion curve

$$(\mathcal{A}-u_0)^2-\left(\mathcal{C}+c_T^2\right)(1+\mathcal{B})=0, \quad (18)$$

where, as before, $u_0=\omega/k$,

$$c_T^2=\frac{dP_0}{d\rho_0}+\rho_0\int dV_{\mathbf{p}}\tilde{f}_{\text{qp},0}(\mathbf{p})\frac{d^2E(p)}{d\rho_0^2}, \quad (19)$$

and

$$\mathcal{A}=\int dV_p p\frac{\partial f_0}{\partial p}\frac{\partial E}{\partial \rho_0}\frac{1}{\frac{\partial E}{\partial p}-u_0}, \quad (20)$$

$$\mathcal{B}=\int dV_p p^2\frac{\partial f_0}{\partial p}\frac{1}{\frac{\partial E}{\partial p}-u_0}, \quad (21)$$

$$\mathcal{C}=\int dV_p\frac{\partial f_0}{\partial p}\left(\frac{\partial E}{\partial \rho_0}\right)^2\frac{1}{\frac{\partial E}{\partial p}-u_0}. \quad (22)$$

Analytical and numerical details on the derivation and solution of Eq. (18) are discussed in Appendix B.

IV. COLLISIONLESS SOUND AND ITS DAMPING

We now discuss the numerical results of the collisionless sound we obtain by solving the linearized Landau-Vlasov equation and the linearized Andreev-Khalatnikov equations. It is important to stress that, to investigate the low-temperature properties of 2D helium-4, in Refs. [5,7,8] a phononlike spectrum was used. Here, we employ the full Bogoliubov expression.

In Fig. 1 we report our numerical solutions of the speed of sound $u_0=c_R-ic_I$ in the 2D case, with $i=\sqrt{-1}$ the imaginary unit. Dashed curves are obtained by using the Vlasov-Landau equation while solid curves are produced by adopting the Andreev-Khalatnikov equations. In the figure there are also, as solid red circles, the experimental data of Ref. [11] obtained with a collisionless Bose gas of ^{87}Rb atoms. In the figure, the quantities are plotted versus the scaled temperature T/T_c, with T_c the Berezinskii-Kosterlitz-Thouless critical temperature [18,19] predicted at thermal equilibrium for 2D interacting superfluid bosons [22,23]. The superfluid-to-normal Kosterlitz-Thouless phase transition occurs due to the unbinding of vortex-antivortex pairs, whose number strongly increases close to the critical temperature T_c. The presence of vortices with quantized circulation is strictly related to the existence of a superfluid velocity $\mathbf{v}_s(\mathbf{r},t)$, which must satisfy the equation $\mathbf{v}_s(\mathbf{r},t)=(\hbar/m)\nabla\phi(\mathbf{r},t)$ with $\phi(\mathbf{r},t)$ the angle of the phase of a complex order parameter [27]. As previously stressed, the Vlasov-Landau equation does not include a superfluid velocity. Instead, the Andreev-Khalatnikov equations take into account the superfluid velocity but not the formation of quantized vortices nor the presence of a complex order parameter associated with the quasicondensate [21–23]. Thus, one can expect that below T_c the 2D Bose gas follows the Andreev-Khalatnikov while above T_c the 2D bosonic system is better described by the Vlasov-Landau equation.

In the upper panel of Fig. 1 we plot the real part of the scaled speed of sound c_R/c_B, with $c_B=\sqrt{gn_0/m}$ the Bogoliubov sound velocity. Remarkably, the experimental data (solid circles) are very well reproduced, both below and above T_c, by the Vlasov-Landau equation (dashed curve) but also by the Andreev-Khalatnikov equations (solid curve). At very low temperature T the two curves of the two theories practically

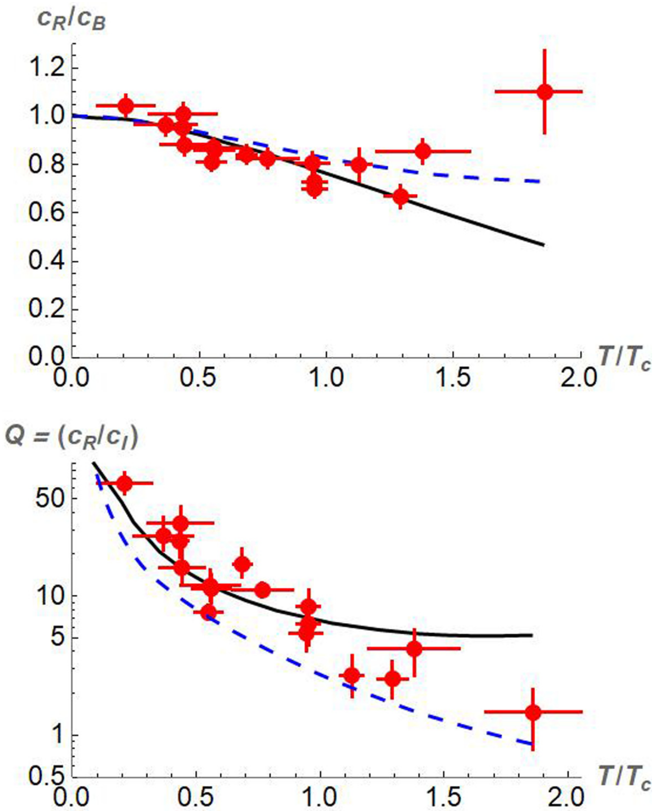

FIG. 1. Results from the numerical solution of the dispersion equation $u_0 = c_R - ic_I$ vs temperature in the 2D case. Upper panel: The normalized speed of sound c_R/c_B as a function of the normalized temperature T/T_c, where $c_B = \sqrt{gn_0/m}$ is the Bogoliubov velocity, $T_c = 2\pi\hbar^2 n_0/\{mk_B \ln[380\hbar^2/(mg)]\}$ is the Berezinskii-Kosterlitz-Thouless critical temperature, and n_0 is the 2D number density at equilibrium. Lower panel: $Q = c_R/c_I$ quality factor of the sound damping. To compare the two transport theories with the experiment of Ref. [11] we choose $g = 0.16\hbar^2/m$. The blue dashed curve is the result of the Vlasov-Landau theory; the black solid curve the result of the Andreev-Khalatnikov theory. Red dots are measured data of Ref. [11].

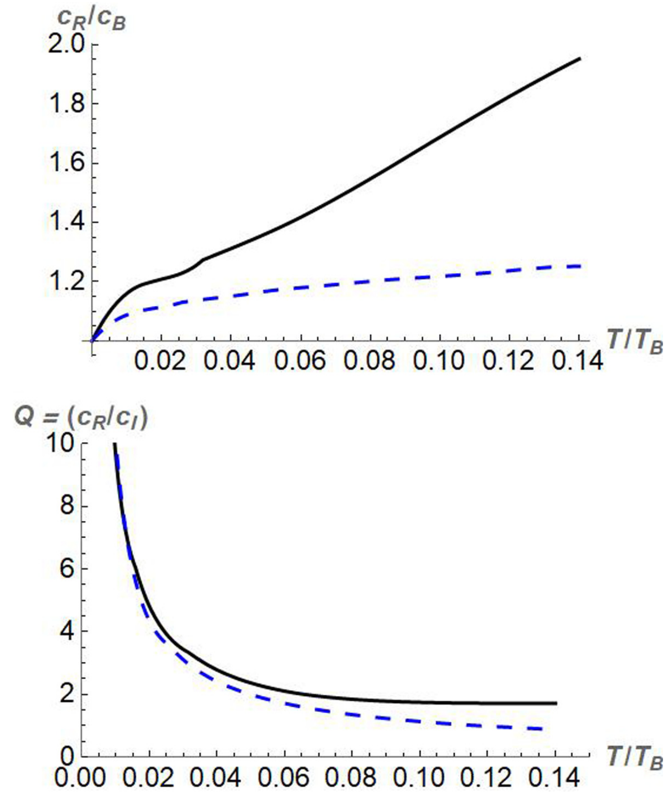

FIG. 2. Results from numerical solution of the dispersion equation $u_0 = c_R - ic_I$ vs temperature in the 1D case. Upper panel: The normalized speed of sound c_R/c_B as a function of the normalized temperature T/T_B, where $c_B = \sqrt{gn_0/m}$ is the Bogobliubov velocity, $T_B = 2\pi n_0^2\hbar^2/(mk_B)$ is the degeneration temperature, and n_0 the 1D number density at equilibrium. We choose $g = 0.16\hbar^2 n_0/m$. Lower panel: Q quality factor of the sound damping: $Q = c_R/c_I$. The blue dashed curve is the result of the Vlasov-Landau theory; the black solid curve is obtained using the Andreev-Khalatnikov theory.

superimpose. In the lower panel of Fig. 1 there is instead the quality factor $Q = c_R/c_I$ of the sound damping, namely the ratio between the real and the imaginary part of the sound velocity $u_0 = c_R - ic_I$. For this quality factor Q, the Andreev-Khalatnikov theory (solid curve) is in much better agreement with the experimental results (solid circles) with respect to the Vlasov-Landau theory (dashed curve) up to the critical temperature T_c. Above the critical temperature T_c it seems that the quality factor Q can be better reproduced by the Vlasov-Landau equation. Notice that in 2D the damping of the collisionless mode was investigated also in Ref. [28] by using a time-dependent Hartree-Fock-Bogoliubov approach, which practically gives the same results of the linearized Vlasov-Landau equation [12,16].

We investigate also the 1D weakly interacting Bose gas in the collisionless regime. Unfortunately, experimental data in this configuration are not yet available. Thus, our 1D predictions can be a strong benchmark for future experiments and also for forthcoming theoretical investigations. Strictly speaking, in the thermodynamic limit and with $T > 0$, for a 1D weakly interacting Bose gas there is neither Bose-Einstein condensation nor superfluidity [20,21,27]. However, a finite 1D system of spatial size L is effectively superfluid if $k_B T \ll E_\phi/\ln(L/\xi)$, where $E_\phi \simeq \hbar^2 n_0/(m\xi)$ is the energy needed to create a phase slip (topological defect, also known as a black soliton) and $\xi = \hbar/\sqrt{2mgn_0}$ is the healing length [21].

In Fig. 2 we show our numerical results for the complex speed of sound $u_0 = c_R - ic_I$ of the 1D bosonic system obtained by solving the Vlasov-Landau equation (dashed curves) and the Andreev-Khalatnikov equations (solid curves). The quantities are plotted as a function of the scaled temperature T/T_B where $T_B = 2\pi n_0^2\hbar^2/(mk_B)$ is the temperature of Bose degeneracy, where the 1D thermal de Broglie wavelength $\lambda_T = \hbar\sqrt{2\pi/(mk_B T)}$ becomes equal to the average distance n_0^{-1} between bosons, with n_0 the equilibrium 1D number density. As clearly reported in the upper panel of Fig. 2, contrary to the 2D case, in 1D the real part c_R of the sound velocity u_0 increases by increasing the temperature T. However, the Andreev-Khalatnikov theory predicts a much larger slope. Indeed, this suggests that in 1D the determination of this slope can be experimentally used to the determine the superfluid nature of the Bose gas. We have also found that, while in 2D the isothermal velocity c_T of Eq. (19) at low temperature is close to the real part of u_0 obtained by solving Eq. (18), in the 1D system this is not the case.

In the lower panel we plot the quality factor $Q = c_s/c_I$ of the sound damping: The two theoretical curves are very close each other. This result implies that in 1D the damping is not very useful to discriminate between the two transport theories.

It is important to stress that marked differences shown in Figs. 1 and 2 are also due to the fact that the scaled temperature is in units of T_c in 2D and in units of T_B in 1D. A quite complicated analytical expression for the sound velocity u_0 of the Vlasov-Landau equation (2) can be derived if $f_0(p) \simeq k_B T/(p^2/2m - \tilde{\mu})$, i.e., under the condition $|\tilde{\mu}| \ll k_B T \ll k_B T_B$. In this way, in 2D one finds [12,13] that the real part of u_0 decreases by increasing the temperature T, while in 1D we obtain the opposite, in very good agreement with our numerical results. As discussed in Appendix A, the 2D Vlasov-Landau equation can be reduced to an effective 1D equation but with an effective 1D Bose-Einstein distribution which is quite different with respect the one of the strictly 1D case. Clearly, the behavior of u_0 vs T crucially depends on the considered Bose-Einstein distribution.

V. CONCLUSIONS

We have analyzed the collisionless sound mode of a 2D weakly interacting bosonic fluid, where recent experimental data are available [11], but also the collisionless sound mode of the 1D bosonic fluid, where experimental data are not yet available. We have compared two theories: the Vlasov-Landau equation versus the Andreev-Khalatnikov equations. The Andreev-Khalatnikov equations are more sophisticated because, contrary to the Vlasov-Landau equation, they also take into account the presence of a superfluid velocity. Our 2D theoretical results, also confronted with the experimental data, strongly suggest that below the critical temperature of the superfluid-to-normal transition the bosonic fluid is better described by the Andreev-Khalatnikov theory, while above the critical temperature the Vlasov-Landau theory seems more reliable. For the collisionless 1D Bose gas, our calculations show that the real part of the sound velocity grows by increasing the temperature and its slope determines the superfluid nature of the system. This prediction, as well as the reduction of sound damping due to the superfluid velocity, can be very useful for forthcoming theoretical and experimental investigations of collisionless superfluids.

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APPENDIX A: NUMERICAL PROCEDURE FOR THE VLASOV-LANDAU EQUATION

In the linearized Vlasov-Landau equation (5) there is the relevant quantity,

$$\int d^D\mathbf{p}\frac{\nabla_p f_0 \cdot \mathbf{n}}{\mathbf{p}\cdot\mathbf{n} - u_0}. \tag{A1}$$

By choosing $\mathbf{n}$ parallel to the x axis, this expression simplifies to

$$\int d^D\mathbf{p}\frac{\partial_{p_x} f_0}{p_x - u_0}. \tag{A2}$$

In dimension $D = 2$ it is straightforward to note that

$$\int d^2\mathbf{p}\frac{\partial_{p_x} f_0}{p_x - u_0} = \int dp_x \partial_{p_x}\left(\int dp_y f_0\right)\frac{1}{p_x - u_0}. \tag{A3}$$

Thus, both in dimensions one and two, ultimately one has to deal with one-dimensional integrals. The integral operator comes from an inverse Laplace transform, hence the path of integration is defined in the complex p plane. The recipe for choosing the right path was given by Landau [15], and may be found in several recent references, e.g., Refs. [10,24]. Here, we provide just the results. The integral (A3) writes as the sum of an integral along the real axis plus a contribution coming from poles in the complex plane:

$$\int_{\infty}^{+\infty} dp_x \partial_{p_x}\left(\int dp_y f_0\right)\frac{1}{p_x - u_0} + \mathcal{J}. \tag{A4}$$

If $\mathrm{Im}(u_0) > 0$, then $\mathcal{J} = 0$. Conversely, if $\mathrm{Im}(u_0) < 0$, we have

$$\mathcal{J} = 2\pi i\, \partial_{p_x} f_x(p_x = u_0), \tag{A5}$$

with

$$f_x(p_x) = \int dp_y\, p_y\, f_0(p_x, p_y). \tag{A6}$$

APPENDIX B: NUMERICAL PROCEDURE FOR THE ANDREEV-KHALATNIKOV EQUATIONS

In the Andreev-Khalatnikov theory one has to deal with several integrals of the kind

$$\int dp \frac{F(p)}{\partial_p E(p) - u_0}, \tag{B1}$$

where we have dropped the x subscript for convenience. $F(p)$ is one of the functions appearing in Eq. (22). Since $E(p)$, as defined in (11), is a nonlinear function of p, the recipe of Eqs. (A4) and (A5) needs some modifications. Let $\bar{p}$ be a root of the function

$$\mathcal{D}(p) = \partial_p E(p) - u_0 : \mathcal{D}(\bar{p}) = 0, \tag{B2}$$

namely

$$\mathcal{D}(\bar{p}) = 0. \tag{B3}$$

Then, we may expand $\mathcal{D}(p)$ around $p = \bar{p}$:

$$\mathcal{D} \simeq (p - \bar{p})\partial_p^2 E(\bar{p}). \tag{B4}$$

Ultimately, therefore, the integrals (B1) are evaluated as

$$\int dp \frac{F(p)}{\partial_p E(p) - u_0} = \int_{-\infty}^{+\infty} dp \frac{F(p)}{\partial_p E - u_0} + \mathcal{J}'. \tag{B5}$$

This time we get

$$\mathcal{J}' = 2\pi i \frac{F(\bar{p})}{\partial_p^2 E(\bar{p})}, \quad \mathrm{Im}(\bar{p}) < 0. \tag{B6}$$

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