

Classical communication through quantum causal structures

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(Received 6 January 2021; accepted 17 March 2021; published 12 April 2021)

Quantum mechanics allows operations to be in indefinite causal order, leading to advantages in computational and communication tasks. However, basic notions such as communication capacity have not yet been formalized. In light of this, through the process matrix formalism, we formulate different classical capacities for a bipartite quantum process. For one-way communication, we find that general quantum processes satisfy a generalization of the Holevo bound, i.e., we can send, at most, one bit per qubit. The result also holds for processes with indefinite causal order and with shared entanglement between the sender and the receiver. We further extend known bounds to classical channel capacities, showing that general processes cannot outperform causally separable ones. Next, we study bidirectional communication through a causally separable process. Our result shows that a bidirectional protocol cannot exceed the limit of a one-way communication protocol. Finally, we generalize this result to a multiparty broadcast communication protocol through a definite ordered process.

DOI: [10.1103/PhysRevA.103.042606](https://doi.org/10.1103/PhysRevA.103.042606)

I. INTRODUCTION

One of the key questions in quantum information is the rate at which a quantum channel can transmit classical information, as quantified by the *classical capacity* of the channel [1,2]. Holevo's seminal result [3], and following works [4,5], provide upper bounds on the classical capacity, showing that each qubit can communicate, at most, one bit of classical information.

In a typical quantum communication protocol, the parties act in a fixed order. However, more general situations are possible, where causal order might be uncertain or even not defined. A practical example can be a distributed system, such as the internet, where different nodes communicate with each other. In such systems, local clocks can suffer from random errors and delays, leading to uncertainty in the ordering of the events [6]. Even more radically, recent developments have shown possibilities of *indefinite* causal structure, i.e., scenarios where the lack of order between the parties cannot be reduced to classical ignorance [7,8]. From a foundational point of view, this is relevant, for example, in quantum gravity scenarios, where quantum superposition of spacetimes can result in an indefinite causal order of events [9–11]. Pragmatically, quantum control of causal order has been proposed as a resource for computation and communication [7,12–20], with several experimental implementations already performed [21–27].

In light of the foundational and applied relevance, it is important to understand how general quantum causal structure affects classical communication. In particular, given the possibility to violate causal inequalities [8,28–30] and the advantage in certain communication tasks [13,14,16], one may

wonder whether an indefinite causal structure can augment the classical communication capacity and possibly exceed the Holevo bound [4,5]. However, despite much work on various communication protocols, the notion of classical capacities in situations where the communicating parties themselves are indefinitely ordered has not yet been developed. One of the challenges is the fact that the definition of capacity involves asymptotically many independent uses of the communication resource but, for indefinite causal order, the notion of repeated use turns out to be problematic [31,32].

We address this gap through the process matrix formalism [8]. We introduce an asymptotic setting appropriate to a communication scenario without predefined causal order. This allows us to develop expressions for the asymptotic capacity of a process, under different encoding and decoding settings, reducing to analog expressions for quantum channels. We find that in each case, the classical communication from a sender to a receiver cannot exceed what is achievable in a definite causal order. In particular, the Holevo bound extends to general processes: the classical capacity cannot exceed one bit per received qubit. This is despite the fact that unlike in conventional channels, the receiver has access to an output system that can signal back to the sender, which constitutes an extra communication resource (for example, for the violation of causal inequalities). We also explore two-party communication protocols when causal order is definite but unknown (probabilistic). In such situations, the total bidirectional communication cannot exceed the maximum one-way communication—again, at most one bit per qubit in either direction. This extends to a similar bound for communication between multiple parties in a definite (but possibly probabilistic) causal order.

We present the work as follows. In Sec. II, we give an introduction to classical communication through quantum channels. In Sec. III, we introduce the framework of the process matrix. In Sec. IV, we introduce asymptotic setting for

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processes, and subsequently we define classical capacities of a process and develop a bound for one-way communication. In Sec. V, we develop a bound for a bidirectional communication protocol. We then generalize the bound for a multiparty broadcast communication protocol.

II. CLASSICAL COMMUNICATION THROUGH A QUANTUM CHANNEL

Let us first review how one can use ordinary quantum channels to send classical information [2,33]. In a one-way communication protocol, Alice has a classical message m , prepared according to some probability distribution $P(m)$, and encodes it into a quantum state ρ_m . She then sends it to Bob through a noisy quantum channel \mathcal{N} . Upon receiving the state, Bob extracts the message by using a positive operator-valued measure (POVM) $\{E_{m'}\}_{m'}$, where $E_{m'} \geq 0$, $\sum_{m'} E_{m'} = \mathbb{1}$. Here, m' denotes the measurement outcome. The conditional probability of Bob receiving a message m' given that Alice sends the message m is

$$p(m'|m) = \text{Tr}[E_{m'}\mathcal{N}(\rho_m)]. \quad (1)$$

The probability of error for a particular message m is

$$p_e(m) = p(m' \neq m|m) = 1 - p(m|m). \quad (2)$$

The goal of the protocol is to minimize the maximal probability of error, $p_e^* := \max_m p_e(m)$. An *asymptotic setting* is a scenario where Alice can use n copies of the channel to send a k -bit message $m \in \{0, 1\}^k$, where both k and n can be arbitrarily large. In other words, she encodes k bits into an ($n \geq k$)-bit message $X_m^{(n)} \in \{0, 1\}^n$ and subsequently an n -qubit state $\rho_m^{(n)}$, and then sends each qubit through an independent copy of the channel. The classical capacity of the quantum channel \mathcal{N} is defined as the maximal rate $C = k/n$ such that asymptotically, $n \rightarrow \infty$, one can achieve noiseless communication, $p_e^* \rightarrow 0$ [2,34].

Different encoding and decoding strategies can lead to different asymptotic settings, resulting in different classical communication capacities for a channel \mathcal{N} , which we review below. A quantification of classical communication possible through a channel \mathcal{N} is given by the *Holevo quantity* [3], defined as

$$\chi(\mathcal{N}) := \max_{p(m), \rho_m} S\left[\sum p(m)\mathcal{N}(\rho_m)\right] - \sum p(m)S(\mathcal{N}(\rho_m)). \quad (3)$$

Here, $S(\cdot)$ is the von Neumann entropy. Having introduced the Holevo quantity for a channel, it is interesting to see how this quantity is related to different classical capacities corresponding to different asymptotic configurations of channels. We discuss it below.

Product encoding and product decoding. Let the input quantum state be a product state of the form $\rho_m^{(n)} = \otimes_{i=1}^n \rho_i$ and the measurement operation $E_{m'}^{(n)} = \otimes_{i=1}^n E_i$, with each E_i acting on the qubit $\mathcal{N}(\rho_i)$. Let us consider the measurement result producing an n -bit string $Y_{m'}^{(n)} \in \{0, 1\}^n$ corresponding to the message m' . In the asymptotic setting, $n \rightarrow \infty$, the capacity in this setup is given by the conventional definition of classical capacity obtained by maximizing the regularized mutual information, $I(Y_{m'}^{(n)}:X_m^{(n)})/n$, between Alice's input and Bob's

output over the input probability distribution, the encoded quantum states, and the decoding measurement operators. The central idea of Shannon's capacity formula is that the mutual information $I(Y_{m'}^{(n)}:X_m^{(n)})$ is additive. Thus, the corresponding capacity is called *product capacity*, $C^{(1)}(\mathcal{N})$. This capacity is determined by the single use of the channel \mathcal{N} , with the optimized mutual information $I(Y_{m'}^{(1)}:X_m^{(1)})$ corresponding to the single-copy input and output variables $X_m^{(1)}$ and $Y_{m'}^{(1)}$, respectively, i.e.,

$$\begin{aligned} C^{(1)}(\mathcal{N}) &= \lim_{n \rightarrow \infty} \max_{p(m), \rho_m^{(n)}, E_{m'}^{(n)}} \frac{I(Y_{m'}^{(n)}:X_m^{(n)})}{n} \\ &= \max_{p(m), \rho_m^{(1)}, E_{m'}^{(1)}} I(Y_{m'}^{(1)}:X_m^{(1)}). \end{aligned} \quad (4)$$

Holevo's theorem [3] states that this quantity is upper bounded by

$$C^{(1)}(\mathcal{N}) \leq \chi(\mathcal{N}), \quad (5)$$

with the $\chi(\mathcal{N})$ defined in Eq. (3). Hereinafter, for the sake of clarity, we are going to represent $I(Y_{m'}^{(1)}:X_m^{(1)})$ with $I(m':m)$.

Product encoding and joint decoding. The difference with the previous case is that Bob can perform a joint measurement on the n -qubit system. The *Holevo-Schumacher-Westmoreland (HSW) capacity*, $C(\mathcal{N})$, associated with this strategy is simply equal to the Holevo quantity [4,5],

$$C(\mathcal{N}) = \chi(\mathcal{N}). \quad (6)$$

It is worth mentioning that the HSW capacity for some channels can be strictly greater than one-shot capacity, i.e., $C(\mathcal{N}) > C^{(1)}(\mathcal{N})$. This is also known as *superadditivity*.

Joint encoding and joint decoding. Here, Alice uses an entangled n -qubit state to encode the information and Bob performs a joint measurement on his output. The capacity associated with this strategy is given by the *regularized Holevo quantity* [3–5],

$$C^\infty(\mathcal{N}) = \chi_{\text{reg}}(\mathcal{N}), \quad (7)$$

with

$$\chi_{\text{reg}}(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{\chi(\mathcal{N}^{\otimes n})}{n}. \quad (8)$$

It has been shown in [35] that the capacity $C^\infty(\mathcal{N})$ can be strictly greater than the HSW capacity: $C^\infty(\mathcal{N}) > C(\mathcal{N})$. Thus, in general, we can write $C^\infty(\mathcal{N}) \geq C(\mathcal{N}) \geq C^{(1)}(\mathcal{N})$.

The Holevo quantity $\chi(\mathcal{N})$, and consequently the regularized Holevo quantity $\chi_{\text{reg}}(\mathcal{N})$, are further upper bounded by $\log_2(d)$, where d is the output dimension of the channel \mathcal{N} . With this, we summarize a sequence of inequalities:

$$\begin{aligned} I(m':m) &\leq C^{(1)}(\mathcal{N}) \leq C(\mathcal{N}) = \chi(\mathcal{N}) \\ &\leq C^\infty(\mathcal{N}) = \chi_{\text{reg}}(\mathcal{N}) \leq \log_2(d), \end{aligned} \quad (9)$$

with $I(m':m)$ being the unoptimized mutual information between Alice's input m and Bob's output m' . A consequence of this chain of inequalities is that for any communication setting, a d -dimensional quantum channel cannot transfer more than $\log_2(d)$ bits. In other words, quantum systems can carry, at most, one bit per qubit.

III. THE PROCESS FRAMEWORK

In conventional quantum communication protocols, the communicating parties act in a well-defined order. However, quantum mechanics allows for possibilities where the order between the communicating parties is unknown or even indefinite [7,9]. This possibility can be modeled within the so-called process matrix formalism [8,36,37]. Consider a situation involving two parties, Alice and Bob, each acting in a local laboratory. In each run of the experiment, each of them receives a quantum system in their respective laboratories, performs some operation on it, and sends it out [38]; Alice's (Bob's) input and output systems will be denoted by A_I (B_I) and A_O (B_O), respectively. Most generally, each party's operation consists in letting the system of interest interact with an additional "probe" system. The parties can use their probes to input information (by preparing them in arbitrary states) and to extract information, through arbitrary measurements. The most general operation is, therefore, a completely positive (CP) map $\mathcal{M} : X_I \otimes X'_I \rightarrow X_O \otimes X'_O$, where, for $X = A, B, X'_I, X'_O$ denote the additional system and we use the system's label to represent the corresponding state space.

It is convenient to represent CP maps as positive-semidefinite matrices, $M \in X_I \otimes X'_I \otimes X_O \otimes X'_O$, using the Choi isomorphism [39],

$$M^{X_I X'_I X_O X'_O} = \sum_{i,j=1}^{d_{X_I} d_{X'_I}} |i\rangle\langle j|^{X_I X'_I} \otimes \mathcal{M}(|i\rangle\langle j|^{X_I X'_I}). \quad (10)$$

Here, the set $\{|i\rangle\}$ represents an orthonormal basis in $X_I \otimes X'_I$ and d_X represents the dimension of X . If the map \mathcal{M} is completely positive and trace preserving (CPTP), then the Choi representation gives an additional constraint,

$$\text{Tr}_{X_O X'_O} M^{X_I X'_I X_O X'_O} = \mathbb{1}^{X_I X'_I}. \quad (11)$$

A CPTP map (also called *channel*) represents an operation that can be performed with probability one, while a CP, trace-nonincreasing map is generally the conditional transformation corresponding to a particular outcome of a measurement.

The resource connecting the two communicating parties is described by the *process matrix* $W^{A_I A_O B_I B_O}$. This encodes the background process that governs how the systems on which the parties act relate to each other, be it a shared state, a channel from one to the other, or more general scenarios. We can view the process as a higher-order map, transforming input quantum maps (the local operations) to output quantum maps (a channel acting on the probe system). The process matrix W has to satisfy a set of constraints:

$$W \geq 0, \quad (12)$$

$$\text{Tr} W = d_{A_O} d_{B_O}, \quad (13)$$

$$B_I B_O W = A_O B_I B_O W, \quad (14)$$

$$A_I A_O W = B_O A_I A_O W, \quad (15)$$

$$W =_{A_O} W +_{B_O} W -_{A_O B_O} W. \quad (16)$$

Here, ${}_x W := \mathbb{1}^x / d_x \otimes \text{Tr}_x W$ is the "trace-and-replace" notation [36], which discards subsystem x and replaces it with the normalized identity. Here appropriate reordering of the tensor factors is implied.

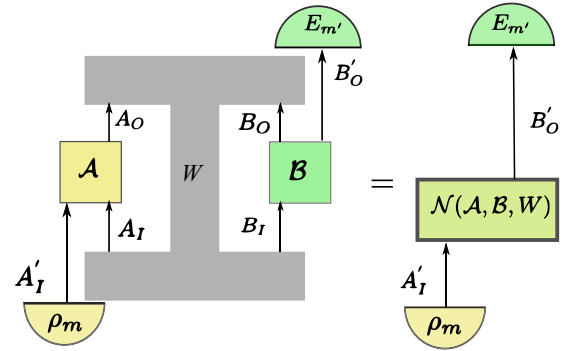


FIG. 1. A process $W^{A_I A_O B_I B_O}$ with two CPTP maps $A^{A'_I A_I A_O}$ and $B^{B_I B_O B'_O}$ forms a new channel $\mathcal{N}(A, B, W)$, as in Eq. (18), with input system A'_I and output system B'_O . Alice can use this channel to communicate to Bob by encoding the quantum state ρ_m at her input system and Bob performing a POVM measurement $E_{m'}$ at his output system.

To compose two CP maps, as well as a CP map and the process, we use the link product denoted by " $*$ " [40,41]. For two positive-semidefinite operators P and Q with the respective Hilbert spaces \mathcal{P} and \mathcal{Q} , the composition $P * Q$ is given as

$$P * Q := \text{Tr}_{\mathcal{P} \cap \mathcal{Q}} [(\mathbb{1}^{\mathcal{P} \setminus \mathcal{Q}} \otimes P^{T_{\mathcal{P} \cap \mathcal{Q}}})(Q \otimes \mathbb{1}^{\mathcal{Q} \setminus \mathcal{P}})]. \quad (17)$$

Here, \mathcal{P} and \mathcal{Q} are the Hilbert spaces associated with P and Q , and the superscript " $T_{\mathcal{P} \cap \mathcal{Q}}$ " represents partial transpose on the shared Hilbert spaces. Note that if the operators do not share Hilbert spaces, $\mathcal{P} \cap \mathcal{Q} = \emptyset$, then the link product reduces to the tensor product, $P * Q = P \otimes Q$. In contrast, when the operators do not have any nonoverlapping spaces, $\mathcal{P} \setminus \mathcal{Q} \cup \mathcal{Q} \setminus \mathcal{P} = \emptyset$, the link product yields a scalar number, $P * Q = \text{Tr}(P^T Q)$.

The link product generalizes the state transformation, composition of channels, and state measurement; a state ρ transformed by a channel \mathcal{N} is given by $\mathcal{N}(\rho) = \rho * N$, where N is the Choi representation of \mathcal{N} . A sequential composition $\mathcal{M}_1 \circ \mathcal{M}_2 : X_1 \rightarrow X_3$, with the Choi representation $M_3^{X_1 X_3}$, is given by $M_3^{X_1 X_3} = M_1^{X_1 X_2} * M_2^{X_2 X_3}$. Here we assume the constituent channels to be $\mathcal{M}_1 : X_1 \rightarrow X_2$ and $\mathcal{M}_2 : X_2 \rightarrow X_3$ with the Choi representations M_1 and M_2 , respectively. Finally, we can rewrite the state measurement in Eq. (1) as $p(m'|m) = E_{m'}^T * \mathcal{N}(\rho_m) = E_{m'}^T * \rho_m * N$. Note that the Choi representation of a POVM element $E_{m'}$ is its transpose $E_{m'}^T$.

The link product also captures the action of the process on the local operations. In our case, the action of W on Alice's and Bob's local operations \mathcal{M}^A and \mathcal{M}^B is given by $(M^A * M^B) * W$. Here, M^A (M^B) is the Choi representation of the map \mathcal{M}^A (\mathcal{M}^B).

IV. ONE-DIRECTIONAL COMMUNICATION THROUGH AN INDEFINITELY ORDERED PROCESS

In this section, we introduce our classical communication protocol through an arbitrary process, as shown in Fig. 1. In this protocol, both Alice and Bob can use some quantum channels \mathcal{A} and \mathcal{B} . Alice's channel \mathcal{A} has $A'_I A_I$ as input and A_O as output, while Bob's channel \mathcal{B} has B_I as input and $B_O B'_O$ as output.

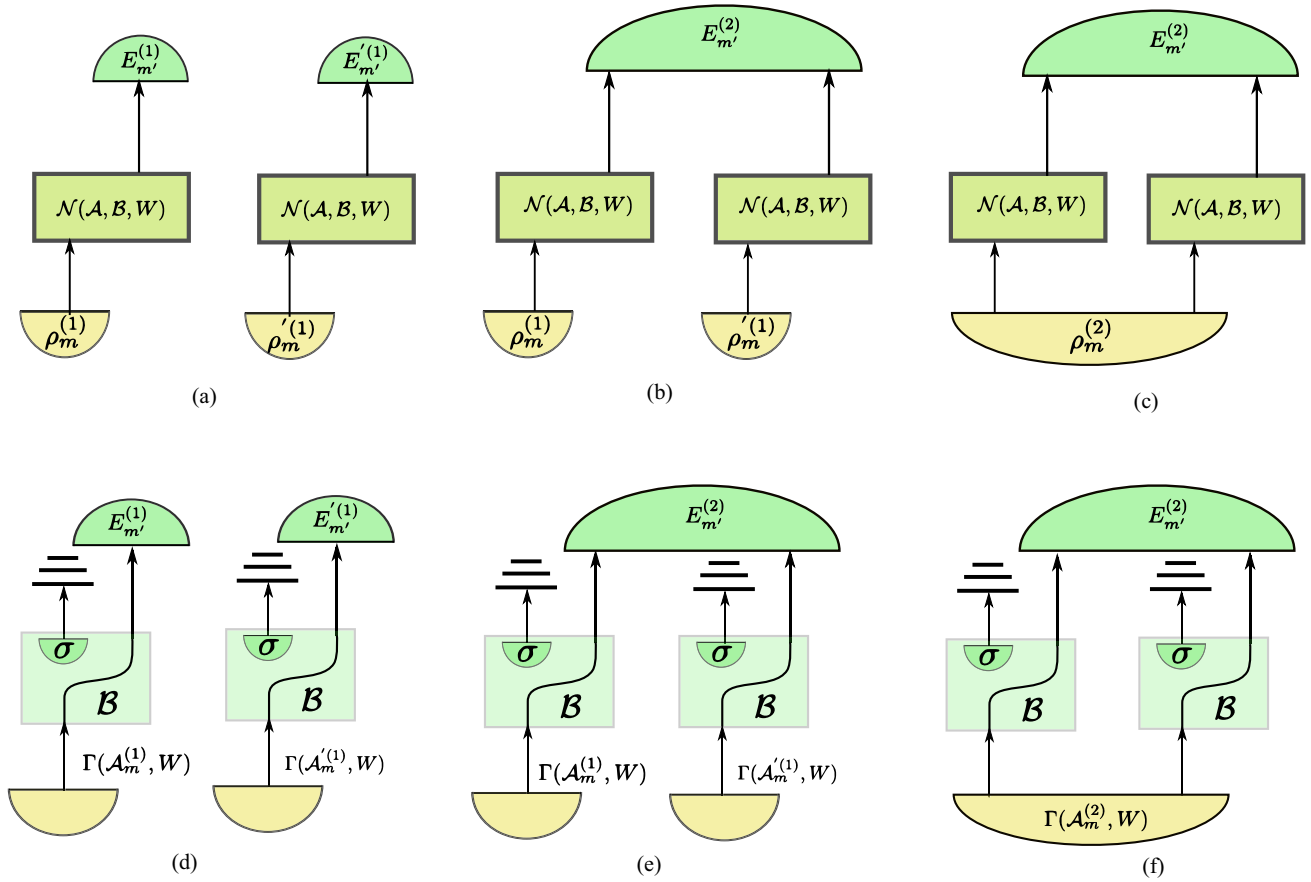


FIG. 2. Encoding and decoding schemes. Here we show two copies of process W . (a)–(c) Product encoding–product decoding, product encoding–joint decoding, and joint encoding–joint decoding, respectively. Product encoding is achieved using the joint state $\rho_m^{(1)} \otimes \rho_m'^{(1)}$ and joint encoding is achieved using the entangled state $\rho_m^{(2)}$. Similarly, we use $E_m'^{(1)} \otimes E_m'^{(1)}$ for product decoding and $E_m'^{(2)}$ for joint decoding. (d)–(f) The simplifications due to Theorems 1 and 2. Relevant labeling of the Hilbert spaces is described in the text.

The process matrix $W^{A_I A_O B_I B_O}$ acting on these channels forms a new quantum channel $\mathcal{N}(A, B, W)$ with input quantum system A'_I and output quantum system B'_O , as shown in Fig. 1. The Choi representation of this new channel is

$$\begin{aligned} N(A, B, W)^{A'_I B'_O} &:= (A^{A'_I A_I A_O} * B^{B_I B_O B'_O}) * W^{A_I A_O B_I B_O} \\ &= (A^{A'_I A_I A_O} \otimes B^{B_I B_O B'_O}) * W^{A_I A_O B_I B_O}. \end{aligned} \quad (18)$$

Here, A and B are the Choi representations of the quantum channels \mathcal{A} and \mathcal{B} , respectively.

It is worth stressing that turning a process into a channel does not yet provide an estimate of the communication capacity. This is because the channel's input and output, A'_I and B'_O , can have arbitrary dimensions, which, in principle, could encode an arbitrary amount of information. For example, Bob could have a single-qubit B_I and B_O and a two-qubit B'_O . One of the two qubits is simply B_I , while the other can be a qubit entangled with B_O (that is to say, he sends through his output part of an entangled state while measuring the other part together with B_I). For a process with indefinite causal order, Bob's output is not discarded and one could expect that the correlations between B_O and Alice's systems could allow her to encode two bits into B'_O , exceeding the Holevo bound. As we will see, this is in fact not possible, as the receiver's output turns out to be useless in a one-way communication scenario,

even for processes with indefinite causal order. Before we can get to this, however, we have to define appropriate asymptotic settings for general processes.

A. Asymptotic setting

Similar to the conventional classical communication through quantum channels, we need to introduce a notion of asymptotic setting for the process, namely, to formalize the notion of repeated use of independent copies of a process. The goal turns out to be nontrivial as one can construct asymptotic settings by allowing joint local operations across different copies of processes [31,32], resulting in nontrivial constraints on the admissible operations and processes [42,43]. For example, Alice could feed the output of her first channel to her second one. This, however, would require extra knowledge about the causal relations between the different uses of the process and, for a process with bidirectional signaling, it would be incompatible with Bob sending his second output to the first input. As we are investigating causal structures as communication resources, we assume that all available causal relations are encoded in the process itself, which leads to the asymptotic setting, first introduced in Ref. [19], where only product operations across different parties are allowed.

Our choice of asymptotic setting results in a set of independent channels, $\mathcal{N}_j = \mathcal{N}(A_j, B_j, W)$, as shown in Fig. 2.

Here, $\mathcal{A}_j, \mathcal{B}_j$ are the local operations performed by Alice and Bob, respectively. Alice encodes her message m in a quantum state $\rho_m^{(n)} \in \otimes_{j=1}^n A_I^{j'}$ and sends the state to Bob through the channels $\{\mathcal{N}_j\}$. After receiving the transformed state, Bob performs a POVM on his quantum system, $\otimes_{j=1}^n B_O^{j'}$. With this, we conceptualize a protocol for one-way communication from Alice to Bob ($A \rightarrow B$) in the following way:

Definition 1. Given a bipartite processes matrix W^{AB} , we define an $A \rightarrow B$ protocol with n uses of W as

- (i) a set of local operations $\{\mathcal{A}_j, \mathcal{B}_j\}_{j=1}^n$, where

$$\begin{aligned} \mathcal{A}_j &: A_I^{j'} \otimes A_I^j \rightarrow A_O^j, \\ \mathcal{B}_j &: B_I^j \rightarrow B_O^j \otimes B_O^{j'} \end{aligned}$$

are CPTP maps;

- (ii) a state encoding $m \mapsto \rho_m^{(n)} \in \otimes_{j=1}^n A_I^{j'}$, where $\rho_m^{(n)} \geq 0$ and $\text{Tr} \rho_m^{(n)} = 1$;

- (iii) a decoding POVM $\{E_{m'}^{(n)}\}_{m'}$, where $E_{m'}^{(n)} \geq 0$ and $\sum_{m'} E_{m'}^{(n)} = \mathbb{1}$.

Such a protocol produces a classical channel described by the conditional probabilities

$$P(m'|m) = \text{Tr} \left[E_{m'}^{(n)} \bigotimes_{j=1}^n \mathcal{N}_j(\rho_m^{(n)}) \right]. \quad (19)$$

We say that two protocols for the same process W are *equivalent* if they produce the same conditional probabilities $P(m'|m)$.

In general, the ancillary spaces $A_I^{j'}, B_O^{j'}$ need not be isomorphic for different j . However, we can always embed each of them into a space that is isomorphic to one of the highest dimension. In the following, we assume that all spaces are of equal dimension and are identified through a choice of canonical basis.

Note that when we consider multiple copies of processes, in general, the local CPTP maps of Alice and Bob need not be identical. Thus the asymptotic setting of the quantum process results in multiple copies of different channels, also known as a *nonstationary asymptotic setting*. Formulating communication capacity of such a setup poses a nontrivial challenge [44,45]. To alleviate this issue, we employ a scheme to make the channels stationary. Specifically, we replace the local operations $\{\mathcal{A}_j\}$ and $\{\mathcal{B}_j\}$ with fixed local operations $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$, respectively, with the support of additional local CPTP maps \mathcal{E}_j and \mathcal{F}_j , where $\mathcal{A}_j = \mathcal{A} * \mathcal{E}_j$ and $\mathcal{B}_j = \mathcal{F}_j * \mathcal{B}$. Thus we have multiple independent and identical copies of the channel $\mathcal{N} = \mathcal{N}(\mathcal{A}, \mathcal{B}, W)$. The feasibility of this approach is due to the fact that in an $A \rightarrow B$ protocol, Bob's output system B_O can be discarded, i.e., in such a protocol a process matrix W can be replaced by ${}_{B_O}W$, as shown in Fig. 3 and in Refs. [7,8,46,47]. As we are going to use this fact multiple times, we formulate it as a lemma and prove it below for completeness.

Lemma 1. If Alice has trivial ancillary output ($d_{A_O'} = 1$), i.e., $\mathcal{A} : A_I' \otimes A_I \rightarrow A_O$, we can replace W with ${}_{B_O}W$,

$$\mathcal{N}(\mathcal{A}, \mathcal{B}, W) = \mathcal{N}(\mathcal{A}, \mathcal{B}, {}_{B_O}W). \quad (20)$$

Proof. It is sufficient to show $A * W = A * {}_{B_O}W$. Using condition (16), we can write $A * W = A * {}_{A_O}W + A * {}_{B_O}W - A * {}_{A_O B_O}W$. As the second and third terms are already in the

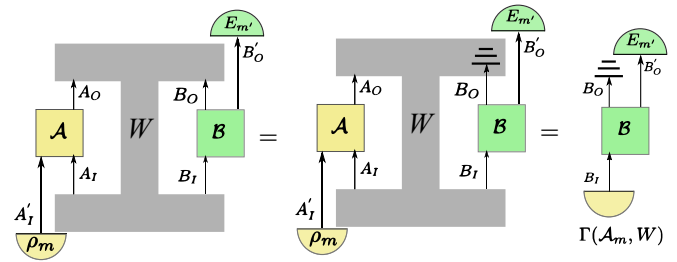


FIG. 3. Pictorial depiction of Lemma 1. A one-way communication from Alice to Bob through a channel $\mathcal{N}(\mathcal{A}, \mathcal{B}, W)$ simplifies to a channel $\mathcal{N}(\mathcal{A}, \mathcal{B}, {}_{B_O}W)$. The rightmost picture shows further simplification to a quantum state $\Gamma(\mathcal{A}_m, W)$ with the system B_O being discarded, as in Eq. (39).

desired form, we only need to look at the first term,

$$\begin{aligned} A * {}_{A_O}W &= \text{Tr}_{A_I A_O} \left[A^{A_I A_I A_O} \left(\frac{\mathbb{1}^{A_O}}{d_{A_O}^A} \otimes \text{Tr}_{A_O} W \right) \right] \\ &= \frac{1}{d_{A_O}^A} \text{Tr}_{A_I} \left[(\text{Tr}_{A_O} A^{A_I A_I A_O}) (\text{Tr}_{A_O} W) \right] \\ &= \frac{1}{d_{A_O}^A} \text{Tr}_{A_I} \left[\mathbb{1}^{A_I A_I} (\text{Tr}_{A_O} W) \right] \\ &= \frac{\mathbb{1}^{A_I}}{d_{A_O}^A} \otimes \text{Tr}_{A_I A_O} W = \frac{\mathbb{1}^{A_I}}{d_{A_O}^A} \otimes \text{Tr}_{A_I A_O} ({}_{B_O}W), \end{aligned} \quad (21)$$

where we used $\text{Tr}_{A_O} A^{A_I A_I A_O} = \mathbb{1}^{A_I A_I}$ (because $A^{A_I A_I A_O}$ is CPTP) in the third line and Eq. (15) in the last line. ■

This lemma allows us to replace Bob's operation \mathcal{B}_j by $\sigma^{B_O} \otimes (\text{Tr}_{B_O} \mathcal{B}_j)^{B_I B_O}$, with σ^{B_O} being an arbitrary state. For the encoding operation \mathcal{A}_j , on the other hand, we extend the input system to make it a controlled operation, while treating the control state as Alice's extended encoded message. Thus we present the following theorem.

Theorem 1. Every $A \rightarrow B$ protocol is equivalent to one with fixed local operations,

$$\bar{\mathcal{A}} : A_I^{j'} \otimes A_I^j \rightarrow A_O^j, \quad (22)$$

$$\bar{\mathcal{B}} : B_I^j \rightarrow B_O^j \otimes B_O^{j'}, \quad (23)$$

state encoding,

$$\bar{\rho}_m^{(n)} = \bigotimes_{j=1}^n \mathcal{E}_j(\rho_m^{(n)}), \quad (24)$$

and decoding POVM,

$$\bar{E}_{m'}^{(n)} = \bigotimes_{j=1}^n \mathcal{F}_j^\dagger(E_{m'}^{(n)}), \quad (25)$$

where $\mathcal{E}_j : A_I^{j'} \rightarrow A_I^{j'}$, $\mathcal{F}_j : B_O^{j'} \rightarrow B_O^{j'}$ are CPTP and \mathcal{F}^\dagger denotes the Hilbert-Schmidt adjoint, defined through $\text{Tr}[A^\dagger \mathcal{F}(B)] = \text{Tr}[\mathcal{F}^\dagger(A^\dagger)B]$.

Proof. (See Fig. 4 for a pictorial representation of the proof.) Let us start with the decoding. Since Alice only performs CPTP maps with no ancillary output, we can apply Lemma 1 and replace the process matrix W with ${}_{B_O}W$, which is equal to identity on B_O . This implies that any $A \rightarrow B$

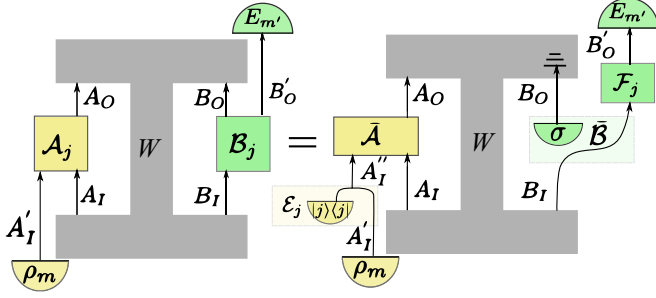


FIG. 4. Pictorial depiction of Theorem 1. We convert a nonstationary channel $\mathcal{N}(\mathcal{A}_j, \mathcal{B}_j, W)$ to a stationary channel $\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)$. Due to Lemma 1, Bob's system B_O can be set to a fixed state σ and corresponding operation $\bar{\mathcal{B}} = \sigma^{B'_O} \otimes \mathcal{I}^{B'_I \rightarrow B''_O}$. Alice's operation, on the other hand, can be extended to a controlled CPTP map $\bar{\mathcal{A}}$ as described in Eq. (27).

protocol is equivalent to one where we replace the local operations \mathcal{B}_j with $\sigma^{B'_O} \otimes \text{Tr}_{B'_O} \mathcal{B}_j$ for some arbitrary state σ . Choosing the space B'_O isomorphic to B''_I , we see that the original combination of local operations \mathcal{B}_j and decoding POVM is equivalent to performing the fixed operation $\bar{\mathcal{B}} = \sigma^{B'_O} \otimes \mathcal{I}^{B'_I \rightarrow B''_O}$ in each laboratory, and decoding POVM as in Eq. (25), with $\mathcal{F}_j = (\text{Tr}_{B'_O} \mathcal{B}_j \circ \mathcal{I}^{B'_I \rightarrow B''_O})^{B'_O}$.

Now for the encoding side, we set $A''_I = \mathcal{X}_j \otimes A'_I$ and define the controlled operation $\bar{\mathcal{A}} : \mathcal{X}_j \otimes A'_I \rightarrow A''_O$ as

$$\bar{\mathcal{A}}(\sigma \otimes \rho) = \sum_{j=1}^n |j\rangle\langle j| \mathcal{A}_j(\rho)^{A'_O}. \quad (26)$$

Here, each \mathcal{X}_j is a state space isomorphic to the operator space $\mathcal{L}(\mathbb{C}^n)$. The operation $\bar{\mathcal{A}}$ is manifestly CPTP. For canonical basis states in \mathcal{X}_j , this map gives

$$\bar{\mathcal{A}}(|j\rangle\langle j| \otimes \rho) = \mathcal{A}_j(\rho), \quad (27)$$

so the choice of local operation can be encoded into a choice of initial state, expanding the original encoding state as in Eq. (24), with the maps $\mathcal{E}_j : A''_I \rightarrow A''_O$ defined as

$$\mathcal{E}_j(\rho^{A''_I}) = (|j\rangle\langle j| \otimes \rho)^{A''_O}. \quad (28)$$

■

The relevance of this theorem is twofold. First, it shows that any protocol involving a different choice of local operations can be reproduced by fixing the local operations once and for all. This means that an asymptotic setting for processes can always be mapped to an asymptotic setting where the same channel is used n times, $\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)^{\otimes n}$. We call a protocol of this type *stationary*. Second, state encoding and decoding POVMs of an arbitrary protocol transform into those of a stationary one through product maps, given by Eqs. (24) and (25). This means that the transformation preserves the nature of the asymptotic setting, viz., joint or product encoding or decoding. From now on, we will represent $\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\rho}_m^{(n)}$, and $\bar{E}_m^{(n)}$ without the bar on top.

B. Classical capacities of a quantum process

Holevo quantity for a process. Having introduced a stationary protocol with an asymptotic setting of the channel $\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)$, as shown in Theorem 1, we can define the corresponding Holevo quantity for a process W as

$$\chi(W)^{A \rightarrow B} := \max_{\mathcal{A}, \mathcal{B}} \chi[\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)]. \quad (29)$$

We also introduce the n th extension $\chi(W^{\otimes n})^{A \rightarrow B}$ of the above quantity as

$$\chi(W^{\otimes n})^{A \rightarrow B} := \max_{\mathcal{A}, \mathcal{B}} \chi[\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)^{\otimes n}]. \quad (30)$$

Communication capacity for a process. We can associate different communication capacities to an arbitrary process as

$$C^\sharp(W)^{A \rightarrow B} = \max_{\mathcal{A}, \mathcal{B}} C^\sharp(\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)r), \quad (31)$$

where $C^\sharp = C^{(1)}, C, C^\infty$. Here, $C^{(1)}(W)^{A \rightarrow B}$ represents the product encoding–product decoding capacity, as in Eq. (4), $C(W)^{A \rightarrow B}$ represents product encoding–joint decoding capacity, as in Eq. (6), and, finally, $C^\infty(W)^{A \rightarrow B}$ represents joint encoding–joint decoding capacity, as in Eq. (8).

We can relate the Holevo quantity for a process to different $C^\sharp(W)^{A \rightarrow B}$ capacities. We show this in the following lemma.

Lemma 2. Different capacities associated with an arbitrary process W are related to the Holevo quantity $\chi(W)^{A \rightarrow B}$ in the following way.

Product encoding–product decoding:

$$C^{(1)}(W)^{A \rightarrow B} \leq \chi(W)^{A \rightarrow B}. \quad (32)$$

Product encoding–joint decoding:

$$C(W)^{A \rightarrow B} = \chi(W)^{A \rightarrow B}. \quad (33)$$

Joint encoding–joint decoding:

$$C^\infty(W)^{A \rightarrow B} = \lim_{n \rightarrow \infty} \frac{\chi(W^{\otimes n})}{n}. \quad (34)$$

Proof. Product encoding–product decoding: Using Eqs. (31), (5), and (29), we can write

$$\begin{aligned} C^{(1)}(W)^{A \rightarrow B} &= \max_{\mathcal{A}, \mathcal{B}} C^{(1)}(\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)) \\ &\leq \max_{\mathcal{A}, \mathcal{B}} \chi(\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)) \\ &= \chi(W)^{A \rightarrow B}. \end{aligned} \quad (35)$$

Product encoding–joint decoding: Using Eqs. (31), (6), and (29), we can write

$$\begin{aligned} C(W)^{A \rightarrow B} &= \max_{\mathcal{A}, \mathcal{B}} C(\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)) \\ &= \max_{\mathcal{A}, \mathcal{B}} \chi(\mathcal{N}(\bar{\mathcal{A}}, \bar{\mathcal{B}}, W)) \\ &= \chi(W)^{A \rightarrow B}. \end{aligned} \quad (36)$$

Joint encoding–joint decoding: Using Eqs. (31), (7), (8), and (30), we can write

$$\begin{aligned}
 C^\infty(W)^{A \rightarrow B} &= \max_{\mathcal{A}, \mathcal{B}} C^\infty(\mathcal{N}(\mathcal{A}, \mathcal{B}, W)) \\
 &= \max_{\mathcal{A}, \mathcal{B}} \chi_{\text{reg}}(\mathcal{N}(\mathcal{A}, \mathcal{B}, W)) \\
 &= \max_{\mathcal{A}, \mathcal{B}} \lim_{n \rightarrow \infty} \frac{\chi[\mathcal{N}(\mathcal{A}, \mathcal{B}, W)^{\otimes n}]}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{\chi(W^{\otimes n})}{n}. \tag{37}
 \end{aligned}$$

C. Bounds on the classical capacities of a quantum process

Although we have been able to reduce the classical capacities of processes to that of channels, our results so far do not provide an upper bound on how much information can be transmitted through a process. This is because the channel $\mathcal{N}(\mathcal{A}, \mathcal{B}, W)$ can have arbitrary input and output dimensions.

To establish a bound, we first describe our protocol from a slightly different point of view. With Alice’s input ensemble $\{p(m), \rho_m\}$, we can introduce a concatenation of ρ_m with the channel \mathcal{A} as $A_m^{A_I A_O} = \rho_m^{A_I} * A^{A_I A_I A_O}$, with A_m being the Choi representation of the resulting CPTP map \mathcal{A}_m . Similarly, we can combine Bob’s channel \mathcal{B} and POVM operation $\{E_{m'}\}$ to describe a set of CP maps, $\{B_{m'}^{B_I B_O} = B^{B_I B_O B'_O} * (E_{m'}^T)^{B'_O}\}_{m'}$, where $\sum_{m'} B_{m'}^{B_I B_O}$ is a CPTP map and $(E_{m'}^T)^{B'_O}$ is the Choi representation of $E_{m'}$. The superscript “ T ,” denoting the transpose operator, is due to definition (10), according to which the Choi of a measurement operator $E_{m'}$ is its transpose, $E_{m'}^T$.

With this in mind, we present two theorems, which apply, respectively, to the product and joint encoding scenarios.

Theorem 2. In a one-way communication scenario, the optimization of the Holevo quantity of a process W can be simplified as

$$\begin{aligned}
 \chi(W)^{A \rightarrow B} &= \max_{\rho_m, \mathcal{A}, p(m)} S\left[\sum p(m)\Gamma(A_m, W)\right] \\
 &\quad - \sum p(m)S[\Gamma(A_m, W)]. \tag{38}
 \end{aligned}$$

Here, the $\Gamma(A_m, W) := A_m * \text{Tr}_{B_O} W / d_{B_O}$ is a map that transforms the Choi representation of the CPTP operation \mathcal{A}_m and the process matrix W to a quantum state on Bob’s input space B_I .

Proof. The reduced process [36] on which Bob applies his CPTP map $B_{m'}^{B_I B_O}$ is described by $A_m^{A_I A_O} * W^{A_I A_O B_I B_O}$. Now, using Lemma 1, we can write

$$\begin{aligned}
 A_m^{A_I A_O} * W^{A_I A_O B_I B_O} &= \mathbb{1}^{B_O} \otimes \left(A_m^{A_I A_O} * \frac{\text{Tr}_{B_O} W^{A_I A_O B_I}}{d_{B_O}} \right) \\
 &= \mathbb{1}^{B_O} \otimes \Gamma(A_m, W)^{B_I}, \tag{39}
 \end{aligned}$$

where $\Gamma(A_m, W) := A_m * \text{Tr}_{B_O} W / d_{B_O}$. In other words, as shown in Fig. 3, we can simplify Alice’s CPTP map and the process to a quantum state $\Gamma(A_m, W)$ in the Hilbert space B_I , with Bob’s output at B_O being discarded. The maximum classical information that can be encoded in the ensemble $\{p(m), \Gamma(A_m, W)\}$ is given by Eq. (38) [3], where we only need to optimize over the free parameters $p(m)$, ρ_m , and \mathcal{A} .

This implies that one does not need to optimize over Bob’s operation to obtain the Holevo quantity for the process. ■

A direct consequence of this theorem is that $\chi(W)^{A \rightarrow B} \leq \log_2(d_{B_I})$ because we have reduced the Holevo quantity of a process to that of an ensemble of states in B_I . In turn, this allows us to establish a bound on the product encoding capacities, i.e., $C^{(1)}(W)^{A \rightarrow B}$ and $C(W)^{A \rightarrow B}$, according to Eq. (35) and Eq. (36), respectively. However, for joint encoding schemes, we need to evaluate the regularized Holevo quantity for the optimum channel $\mathcal{N}(\mathcal{A}, \mathcal{B}, W)$, as shown in Eq. (34). In light of this, we introduce the following theorem that bounds the capacity, $C^\infty(W)^{A \rightarrow B}$.

Theorem 3. The joint encoding capacity for a process, $C^\infty(W)^{A \rightarrow B}$, is limited to Alice’s joint CPTP map $\mathcal{A}_m^{(n)} = \rho_m^{(n)} * \mathcal{A}^{\otimes n}$ with $n \rightarrow \infty$, and the distribution $p(m)$,

$$\begin{aligned}
 C^\infty(W)^{A \rightarrow B} &= \lim_{n \rightarrow \infty} \max_{\rho_m^{(n)}, \mathcal{A}, p(m)} \frac{1}{n} \left\{ S\left[\sum p(m)\Gamma(\mathcal{A}_m^{(n)}, W)\right] \right. \\
 &\quad \left. - \sum p(m)S[\Gamma(\mathcal{A}_m^{(n)}, W)] \right\}. \tag{40}
 \end{aligned}$$

Here the map Γ transforms the joint CPTP map $\mathcal{A}_m^{(n)}$ and the process W to an entangled state at Bob’s input space $\otimes_{j=1}^n B_I^j$.

Proof. First, we apply Lemma 1 to each copy of the channel $\mathcal{N}(\mathcal{A}, \mathcal{B}, W)$ and replace it with $\mathcal{N}(\mathcal{A}, \mathcal{B}, B_O^j W)$. Then, in a joint encoding scheme, we apply $\otimes_j \mathcal{N}(\mathcal{A}, \mathcal{B}, B_O^j W)$ to a (possibly entangled) joint state $\rho_m^{(n)}$. Combining this joint state with the n copies of Alice’s operation \mathcal{A} , we obtain a joint CPTP map $\mathcal{A}_m^{(n)} : \otimes_j A_I^j \rightarrow \otimes_j A_O^j$, with Choi representation $A_m^{(n)} = \rho_m^{(n)} * \mathcal{A}^{\otimes n}$. Plugging $A_m^{(n)}$ into the n copies $B_O^j W$, we get $A_m^{(n)} * (\otimes_j B_O^j W) = \mathbb{1}^{\otimes_{j=1}^n B'_O} \otimes \Gamma(\mathcal{A}_m^{(n)}, W)$, where $\Gamma(\mathcal{A}_m^{(n)}, W) \in \otimes_{j=1}^n B_I^j$ is a (possibly entangled) state, defined as

$$\Gamma(\mathcal{A}_m^{(n)}, W) = \frac{A_m^{(n)} * (\text{Tr}_{\otimes_{j=1}^n B'_O} W^{\otimes n})}{\prod_{j=1}^n d_{B'_O}}. \tag{41}$$

One can extend this setup to $n \rightarrow \infty$ and achieve a joint state at Bob’s input Hilbert space $\otimes_{j=1}^\infty B_I^j$. Similar to Theorem 2, we calculate the maximum amount of classical information encoded in the ensemble $\{p(m), \Gamma(\mathcal{A}_m^{(n)}, W)\}$ and regularize it to obtain the joint encoding capacity $C^\infty(W)^{A \rightarrow B}$, where the free parameters are, of course, $p(m)$, $\rho_m^{(n)}$, and \mathcal{A} . Thus we obtain Eq. (40). ■

Corollary 1. The capacity $C^\infty(W)^{A \rightarrow B}$ is upper bounded by the logarithm of the dimension of Bob’s input Hilbert space, i.e., $C^\infty(W)^{A \rightarrow B} \leq \log_2(d_{B_I})$.

Proof. This is the consequence of Holevo’s theorem [3]. The information content of the ensemble $\{p(m), \Gamma(\mathcal{A}_m^{(n)}, W)\}$ cannot exceed the logarithm of the dimension of $\Gamma(\mathcal{A}_m^{(n)}, W)$, i.e., $n \log_2(d_{B_I})$. Regularizing this quantity proves the corollary. ■

Now we summarize our results. If we consider Alice’s input message m and Bob’s output message m' , we can introduce a chain of inequalities for different classical capacities of the

process W ,

$$\begin{aligned} I(m' : m) &\leq C^{(1)}(W)^{A \rightarrow B} \leq C(W)^{A \rightarrow B} = \chi(W)^{A \rightarrow B} \\ &\leq C^\infty(W)^{A \rightarrow B} \leq \log_2(d_{B_I}). \end{aligned} \quad (42)$$

One can write a similar chain of inequalities for a communication protocol from Bob to Alice. Note that this inequality holds even for a process W that contains shared entanglement between Alice's and Bob's input Hilbert spaces. This does not contradict the higher capacity achievable in an entanglement-assisted communication protocol, such as superdense coding [48], because, when applying the inequalities in Eq. (42), one has to consider the total dimension of Bob's input Hilbert space, which consists of the part of the shared entangled state in Bob's possession and the quantum state that Alice communicates to him.

V. BROADCAST COMMUNICATION

Having established the notion of one-way communication through a process, we proceed to explore scenarios where all communicating parties can transmit and receive information.

Two-party communication. Let us first consider the two-party situation, where Alice (Bob) sends the message $m(k)$ and Bob (Alice) receives the message $m'(k')$. The possibility to violate causal inequalities indicates that indefinite causal order can indeed provide an advantage in some two-party games [8,28]; however, it is unclear if this advantage results in a communication enhancement. To address this question, it is necessary to find limits on two-way communication for causally separable processes. In this section, we address this question.

There are at least two ways a process can be used as a resource for bidirectional communication, depending on whether Alice's and Bob's instruments are fixed or if they are chosen depending on the direction of communication that is attempted. In the first case, the parties produce a single probability distribution $P(m', k'|m, k)$ from the process, and one looks for communication in the marginals $P(m'|m)$, $P(k'|k)$. In the second case, the parties can generate different probability distributions depending on who is sending and who is receiving. The one-directional capacities for the first case are upper bounded by those in the second case, as the best instrument to receive a message might differ from the best to send a message. In this section, we will be mostly concerned with the second case.

Let us then consider a scenario where the order between Alice's and Bob's local operations is determined based on a random outcome. We represent a process where Alice can signal to Bob, but not the other way around, by $W^{A \prec B} = {}_{B_O}W^{A \prec B}$, and the reversed direction of signaling by $W^{B \prec A} = {}_{A_O}W^{B \prec A}$. The process matrix W_{sep} in this case is a convex combination of $W^{B \prec A}$ and $W^{A \prec B}$ [36],

$$W_{\text{sep}} = \lambda W^{B \prec A} + (1 - \lambda) W^{A \prec B}, \quad (43)$$

where $0 \leq \lambda \leq 1$ is the probability for Bob to be first. We call such a process a *causally separable process* [8]. We investigate a scenario where both Alice and Bob are trying to send information to each other through the background process W_{sep} . A reasonable attempt to quantify this bidirec-

tional communication is to evaluate the sum of two product capacities, $C^{(1)}(W)^{A \rightarrow B}$ and $C^{(1)}(W)^{B \rightarrow A}$. We investigate this quantity and evaluate an operationally significant upper bound from the perspective of the classical capacity of the process.

Theorem 4. For a bidirectional communication protocol through a causally separable process, defined in Eq. (43), the following inequality holds:

$$\begin{aligned} C^{(1)}(W_{\text{sep}})^{A \rightarrow B} + C^{(1)}(W_{\text{sep}})^{B \rightarrow A} \\ \leq \lambda \log_2(d_{A_I}) + (1 - \lambda) \log_2(d_{B_I}). \end{aligned} \quad (44)$$

Proof. Considering a fixed input probability distribution $P(a)$, the following linear relationship among marginal conditional probabilities holds for a causally separable process [8]:

$$P(a'|a)_{W_{\text{sep}}} = \lambda P(a'|a)_{W^{B \prec A}} + (1 - \lambda) P(a'|a)_{W^{A \prec B}}. \quad (45)$$

In our protocol, we consider $a \in \{m, k\}$ to be the inputs and $a' \in \{m', k'\}$ to be the outputs. Let us consider $A \rightarrow B$ communication. Consequently, we can write

$$\begin{aligned} C^{(1)}(W_{\text{sep}})^{A \rightarrow B} &= \max I(m' : m) \\ &\leq \max [\lambda I(m' : m)_{W^{B \prec A}} + (1 - \lambda) I(m' : m)_{W^{A \prec B}}] \\ &= (1 - \lambda) \max I(m' : m)_{W^{A \prec B}} \\ &\leq (1 - \lambda) \log_2(d_{B_I}). \end{aligned} \quad (46)$$

Here, the first equation is due to Eqs. (4) and (31). The maximization is taken over Alice's and Bob's local operations, their message ensembles, and the POVM operations. The first inequality is due to the fact that mutual information $I(a' : a)$ is a convex function of $p(a'|a)$ for a fixed input probability distribution $p(a)$ [49]. We obtain the second equality because, for a definite ordered scenario $B \prec A$, output m' of Bob's local laboratory becomes independent of Alice's input. This makes $I(m' : m)_{W^{B \prec A}} = 0$. The final inequality is due to Eq. (42). We apply a similar set of reasoning to obtain a bound for $B \rightarrow A$ communication to obtain

$$C^{(1)}(W_{\text{sep}})^{B \rightarrow A} \leq \lambda \log_2(d_{A_I}). \quad (47)$$

Adding Eqs. (46) and (47), we find

$$\begin{aligned} C^{(1)}(W_{\text{sep}})^{A \rightarrow B} + C^{(1)}(W_{\text{sep}})^{B \rightarrow A} \\ \leq \lambda \log_2(d_{A_I}) + (1 - \lambda) \log_2(d_{B_I}). \end{aligned} \quad (48)$$

For the particular case $d_{A_I} = d_{B_I} = d$, we see that the sum of two product capacities is upper bounded by $\log_2(d)$. In other words, the total communication in causally separable processes can be no more than maximal one-way communication. We note that a weaker version of this inequality holds for the scenario where the parties' instruments are fixed regardless of the attempted direction of communication. In this case, the single-shot capacities coincide with the mutual information obtained from a single conditional probability distribution $P(m', k'|m, k)$, resulting in the inequality $I(m' : m) + I(k' : k) \leq \log_2(d)$. This is an example of an *entropic causal inequality*, first considered in Ref. [50]. Remarkably, no violation of this inequality is known, and our own numerical search also did not reveal any violation of Eq. (44). This

suggests that the bound on the total bidirectional communication might hold for general processes.

We note that the bound we established applies to all the quantum processes for which a physical interpretation is known. For example, in a process with coherent control of causal order, such as the quantum switch [40], tracing out the control leads to a separable bipartite process, to which the bound applies. More generally, it has been shown that any bipartite processes that admit a unitary extension are causally separable [51,52].

Multipart communication. The above-mentioned protocol can be extended to multiple parties. In that case, each party tries to communicate his or her information to the remaining parties. Similarly to above, we consider a process for N parties, $A^{(1)}, A^{(2)}, \dots, A^{(N)}$, that can be written as a probabilistic mixture of permutations of different causal order,

$$W_{\text{sep}}^N = \sum_{\sigma} q_{\sigma} W^{\sigma}. \tag{49}$$

Here, σ denotes the different permutations of the communicating parties and q_{σ} denotes the probability of occurrence of each permutation. Although this is not the most general process with definite causal order [37,53], it is one of particular interest as it represents a scenario where the order among parties can be set by external, random variables, but is independent of the parties' actions.

Motivated by the previous section, we intend to find an upper bound to the quantity $\sum_{i,j} C^{(1)}(W_{\text{sep}}^N)^{i \rightarrow j}$. Here, $i \rightarrow j$ refers to signaling from the party $A^{(i)}$ to the party $A^{(j)}$ and $C^{(1)}(W_{\text{sep}}^N)^{i \rightarrow j}$ is the one-way capacity defined for the bipartite reduced process for $A^{(i)}$ and $A^{(j)}$, obtained by fixing CPTP maps for all other parties, and maximized over the other parties' CPTP maps.

Theorem 5. If the dimensions of all the input Hilbert spaces of the communicating parties are equal (d), then

$$\sum_{i,j} C^{(1)}(W_{\text{sep}}^N)^{i \rightarrow j} \leq \frac{N(N-1)}{2} \log_2(d). \tag{50}$$

Proof. We can write the conditional probability $P(\vec{m}'|\vec{m}) = \sum_{\sigma} q_{\sigma} P_{\sigma}(\vec{m}'|\vec{m})$. We can write the marginals $P(m'_j|m_i) = \sum_{\sigma} q_{\sigma} P_{\sigma}(m'_j|m_i) \forall i, j$. By the convexity of mutual information and the inequalities introduced in Eq. (42),

$$\begin{aligned} C^{(1)}(W_{\text{sep}}^N)^{i \rightarrow j} &\leq \max_{\sigma} \sum_{\sigma} q_{\sigma} I_{\sigma}(m'_j : m_i) \\ &= \max_{\forall \{i,j\} | \sigma(i) < \sigma(j)} \sum_{\sigma} q_{\sigma} I_{\sigma}(m'_j : m_i) \\ &\leq \sum_{\forall \{i,j\} | \sigma(i) < \sigma(j)} q_{\sigma} \log_2(d_{A_j}) \\ &= \sum_{\forall \{i,j\} | \sigma(i) < \sigma(j)} q_{\sigma} \log_2(d). \end{aligned} \tag{51}$$

The maximization is taken over all communicating parties' local operations, their message ensembles, and the POVM operations. The first inequality is due to the convexity of mutual information relative to mixtures of conditional probabilities (as in the bipartite case). The first equality follows from the fact that if the party $\sigma(j)$ is in the causal past of the party $\sigma(i)$, then $I(m'_j;m_i) = 0$. d_{A_j} is the dimension of the input

Hilbert space of the party A_j . The second equality follows because of our assumption of all the dimensions of the input Hilbert spaces being equal. Now, it is easy to see that the n th party has total $n-1$ parties in his or her causal past. Therefore, considering each party trying to communicate with the remaining $N-1$ parties, the total number of available channels is $N(N-1) - \sum_{n=1}^N (n-1) = N(N-1)/2$. This results in

$$\sum_{i,j} C^{(1)}(W_{\text{sep}}^N)^{i \rightarrow j} \leq \frac{N(N-1)}{2} \log_2(d). \tag{52}$$

The key property that leads to the above bounds is the convexity of mutual information under probabilistic mixtures of classical channels. With this in mind, we see that the above results can be extended directly to the product encoding–joint decoding setting, replacing the product capacity $C^{(1)}$ with the HSW capacity C . Indeed, we have seen that C is given by the (maximized) Holevo quantity χ of the one-way channel generated by a process and, just like mutual information, χ is convex over the probabilistic mixture of channels. It remains an open question whether higher total transmission rates can be achieved in a joint encoding setting.

VI. CONCLUSION

We have formalized classical communication through a general quantum causal structure, which may be probabilistic or indefinite. We have defined the Holevo quantity as well as different classical capacities for an arbitrary process and established relationships among them. We have found that for one-way communication, the various capacities can be reduced to those of ordinary channels, up to an optimization over the operations performed in local laboratories. We have further shown that for one-way communication, the classical capacity of a process cannot exceed the Holevo bound—at most, one classical bit per received qubit—even in the case of indefinite causal order.

Next, we have quantified bidirectional and, more generally, broadcast communication protocols for processes with definite but classically uncertain causal order. We have demonstrated that the total amount of communication between two parties cannot exceed the maximal one-way capacity in a fixed causal order, with a similar bound extending to multipartite broadcast communication. One can ask whether a process with an indefinite causal structure can violate these bounds. We have answered this negatively for coherent control of causal order, as in the quantum switch [40]. It is an open question whether a more general process can violate the bounds. As we have not found any violation, it is an interesting possibility that the bounds we have found might constitute a universal limit to the total communication possible in any process.

ACKNOWLEDGMENTS

K.G. thanks the organizers of the Quantum Information Structure of Spacetime 2020 workshop for providing an opportunity to present this talk. We are grateful to Alastair Abbott, Ding Jia, Nitica Sakharwade, Marco Tomamichel,

and Magdalena Zych for helpful discussions. This work has been supported by the Australian Research Council (ARC) by Centre of Excellence for Engineered Quantum Systems (EQUS, Grant No. CE170100009), F.C. acknowledges sup-

port through an Australian Research Council Discovery Early Career Researcher Award (Grant No. DE170100712). K.G. is financially supported through a RTP scholarship from the University of Queensland.

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