Quantum error correction architecture for qudit stabilizer codes

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Quantum communication channels benefit from nonbinary entanglement-assisted stabilizer codes using preshared entangled states for achieving better error correction capability compared to those that do not use preshared entanglement, making them indispensable for realizing large-scale quantum computing and communication systems over qudits. We provide a previously unreported design architecture of the syndrome computation unit for qudit stabilizer codes based on classical additive codes using discrete Fourier transform gates, ADD gates, and multiplication gates. The proposed syndrome computation circuit architectures are necessary toward the implementable realization of such entanglement-assisted and -unassisted qudit stabilizer codes within the quantum transceiver system. We further provide an equivalent design architecture of the syndrome computation unit that decomposes into two syndrome computation units based on X errors and Z errors separately for entanglement-assisted and -unassisted qudit CSS codes. The proposed quantum error correction architectures are useful for building high-density coded quantum memories for archival quantum storage.

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I. INTRODUCTION

The ability to transfer and store quantum information reliably is of prime importance in quantum communication and storage channels. The quantum states in the communication and storage channels decohere on interacting with the environment, leading to the loss of quantum information that we view as quantum errors. Using quantum error mitigation or correction techniques that involve performing additional measurements or adding redundant information, quantum errors can be reduced or corrected, respectively. Redundancy introduced in the quantum information by quantum error correction codes (QECCs) protect the information from errors; hence, they are vital for building reliable quantum systems [1].

Gottesman [2] proposed the stabilizer framework to construct quantum codes over qubits. The code design involves constructing an Abelian subgroup of the Pauli group called the *stabilizer group* [3,4]. The code is the set of all simultaneous $+1$-eigenstates of the stabilizer group elements. Brun *et al.* [5] constructed quantum codes over qubits from non-Abelian subgroups of the Pauli group by extending their elements to a higher-dimensional space through quantum operators to form an Abelian group and constructing a stabilizer code from this Abelian group. The qubits used for extension are assumed to be maintained error free at the receiver end throughout. These codes from non-Abelian groups require preshared entangled states between the transmitter and the receiver; hence, they are called entanglement-assisted (EA) codes. Using the EA framework, EA codes with better error correction capability compared to the stabilizer codes that do not use preshared entanglement can be obtained [6].

Higher-dimensional qudits have an increased sensitivity to eavesdropping and a decreased sensitivity to noise compared to qubits, improving the robustness and reliability of the quantum system [7,8]. Qudits can be implemented by utilizing different degrees of freedom of a single photon or using multiple energy levels of atoms or ions [9,10]. These qudits can be manipulated using programmable filters and modulators or using microwave pulses [9–11].

Ashikhmin and Knill [12] generalized the stabilizer framework for qubits to qudits of dimension p^m for construction of entanglement-unassisted stabilizer codes, where p is prime and $k \in \mathbb{Z}^+$. Ketkar *et al.* [13] provided described the theory of qudit stabilizer codes over finite fields. Luo *et al.* [14] generalized the construction of EA stabilizer codes to qudits of prime dimension. Galindo *et al.* [15] proposed the EA stabilizer code construction similar to CSS codes over qudits ($q = p^m$) from classical linear codes but not classical additive codes.

A coding-theoretic framework for the construction of EA stabilizer codes over qudits of dimension $q = p^m$ was proposed in Ref. [16] that are analogous to classical additive codes[1] over $\mathbb{F}_{p^m}$. The well-known frameworks of binary EA stabilizer codes [5], entanglement-unassisted qudit stabilizer codes [12], and EA qudit stabilizer codes based on classical linear codes [14,15] are special cases of this framework. QECCs can be constructed from well-known classical codes, such as Reed-Solomon codes [17], tensor product codes [18], etc., using the results in Ref. [16]. These codes can be used for encoding higher-dimensional quantum states [11] for applications within quantum communication systems for transmitting or storing more information.

Error correction using an EA stabilizer code involves first computing the *syndrome*, which is a function of the

[1]Classical additive codes are more generalized compared to classical linear codes. All classical linear codes are additive codes, but a classical additive code need not be a linear code.

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erroneous codeword that characterizes the error and is independent of the codeword. From the syndrome, the error is deduced and the inverse error operation is applied to obtain the codeword when the error is correctable. The syndrome computation circuit architectures[2] for qubit stabilizer codes and qudits stabilizer codes based on linear codes have been provided in Ref. [19, Fig. 10.16] and Ref. [20], respectively. However, the explicit form of the syndrome computation circuit architectures for stabilizer codes over qudits of dimension p^m based on classical additive codes are previously unreported. The design of circuit architectures is absolutely necessary toward the implementable realization of these codes, motivating the work in this article. In this paper, we provide syndrome computation circuit architectures for entanglement-assisted and -unassisted stabilizer codes over qudits of dimension p^m based on classical additive codes using discrete Fourier transforms (DFT) over finite fields, ADD gates, and multiplication gates. The error correction circuit architectures involve computing the syndrome using the syndrome computation unit, deducing the error based on the syndrome, and applying the inverse operation to recover the information. The syndrome computation circuits for entanglement-unassisted qubit and qudit stabilizer codes based on classical linear codes provided in Ref. [19, Fig. 10.16] and Ref. [20], respectively, are special cases of the syndrome computation units provided in this paper obtained by considering $q = 2$ and $q = p^m$ with the stabilizer codes based on classical linear codes, respectively.

We also provide an equivalent syndrome computation architecture, useful for correction of X and Z errors separately for entanglement-assisted and entanglement-unassisted CSS codes. The syndrome computation circuit architectures for entanglement-unassisted qubit stabilizer codes provided in Ref. [19, Fig. 10.17] is a special case of our circuit architecture.

We note that the stabilizers of nonbinary stabilizer codes are not Hermitian operators. Thus, the syndrome cannot be extracted by measuring all observables in the stabilizers of nonbinary stabilizer codes, unlike the qubit case [21]. Thus, the proposed syndrome computation circuit is vital within the framework of channel encoding for quantum communication and storage. Quantum memories with embedded QECCs over qudits proposed in this work, implemented using the qudit gates in Ref. [9], could be used for archival quantum storage.

This paper is organized as follows: In Sec. II, we briefly review the qudit states and operators representation and entanglement-assisted and -unassisted qudit stabilizer codes required toward the rest of the paper. In Sec. III, we discuss the flow of information in coded quantum systems. In Sec. V, we briefly discuss the error correction procedure for qudit stabilizer codes. In Sec. V, we design the syndrome computation circuits for entanglement-unassisted and -assisted qudit stabilizer codes. In Sec. VI, we provide an application of the proposed error correction circuits for archival quantum storage, followed by conclusions in Sec. VII.

[2]We use the term circuit architecture as we provide the circuit at the gate level. A circuit would require the knowledge of the physical component and techniques used to implement them.

II. QUDIT STABILIZER CODES

Let us consider a qudit of prime power dimension, i.e., $q = p^m$, where p is prime and $m \in \mathbb{Z}^+$, as stabilizer codes use finite fields [22]. We represent a qudit as a normalized superposition of the p^m basis states $\{|\beta\rangle_{p^m}\}_{\beta\in\mathbb{F}_{p^m}}$, i.e., $|\psi\rangle = \sum_{\beta\in\mathbb{F}_{p^m}} a_\beta|\beta\rangle_{p^m}$, where $a_\beta \in \mathbb{C}$ and $\sum_{\beta\in\mathbb{F}_{p^m}} |a_\beta|^2 = 1$. We also represent the qubit by $[a_\beta] \in \mathbb{C}^{p^m}$. Let the representation of $\beta \in \mathbb{F}_{p^m}$ using the polynomial basis $\{1, \alpha, \dots, \alpha^{m-1}\}$ be $\beta = b_{m-1}\alpha^{m-1} + \dots + b_1\alpha + b_0$, where $b_{m-1}, \dots, b_1, b_0 \in \mathbb{F}_p$, then $|\beta\rangle_{p^m} = |b_{m-1}\rangle_p \otimes \dots \otimes |b_1\rangle_p \otimes |b_0\rangle_p$ [23]. Each component $|b_i\rangle_p$ is called a *subqudit* of the qudit $|\beta\rangle_{p^m}$, where $i \in \{0, 1, \dots, m-1\}$.

We define the *field trace* of an element $\beta \in \mathbb{F}_{p^m}$ as the $\mathbb{F}_p$-linear function $\mathrm{Tr}_{p^m/p}(\beta) = \sum_{i=0}^{m-1} \beta^{p^i} \in \mathbb{F}_p$. The quantum operators on qudits of dimension p^m belong to the unitary group $\mathrm{U}(p^m)$ and can be represented using the following unitary basis of $\mathbb{C}^{p^m\times p^m}$:

$$\mathcal{G}_{p^m}^{(b)} = \{\mathrm{X}^{(p^m)}(\beta)\mathrm{Z}^{(p^m)}(\gamma)|\beta, \gamma \in \mathbb{F}_{p^m}\}, \tag{1}$$

where $\mathrm{X}^{(p^m)}(\beta)|\theta\rangle_{p^m} = |\beta+\theta\rangle_{p^m}\ \forall\theta \in \mathbb{F}_{p^m}$, $\mathrm{Z}^{(p^m)}(\gamma)|\theta\rangle_{p^m} = \omega^{\mathrm{Tr}_{p^m/p}(\gamma\theta)}|\theta\rangle_{p^m}\ \forall\theta \in \mathbb{F}_{p^m}$, and $\omega = e^{\frac{i2\pi}{p}}$. We represent the basis operators $\mathrm{X}^{(p^m)}(\beta)\mathrm{Z}^{(p^m)}(\gamma)$ by its *field isomorphism* $[\beta|\gamma]_{p^m}$ over $\mathbb{F}_{p^m}$, where the subscript p^m denotes that its elements belong to $\mathbb{F}_{p^m}$.

Similarly to the case of the basis states, we obtain [23]:

$$\mathrm{X}^{(p^m)}(\beta) = \bigotimes_{i=0}^{m-1}\mathrm{X}^{(p)}(b_{(m-1-i)}), \tag{2}$$

$$\mathrm{Z}^{(p^m)}(\gamma) = \bigotimes_{i=0}^{m-1}\mathrm{Z}^{(p)}[\mathrm{Tr}_{p^m/p}(\gamma\alpha^{(m-1-i)})], \tag{3}$$

where $\beta, \gamma \in \mathbb{F}_{p^m}$ and the $\mathbb{F}_p$-ary representation of β is $[b_{(m-1)}\cdots b_0]$; hence, $\mathrm{X}^{(p^m)}(\beta)\mathrm{Z}^{(p^m)}(\gamma) \equiv [\beta|\gamma]_{p^m} \equiv [b_{(m-1)}\cdots b_0|\mathrm{Tr}_{p^m/p}(\gamma\alpha^{(m-1)})\cdots\mathrm{Tr}_{p^m/p}(\gamma)]_p$. We note that an operation can be performed on one subqudit physically by performing an appropriate operation over the qudit in which the subqudit physically resides.

In this paper, we use the following notations to denote $\mathrm{X}^{(q)}(\cdot)$ or $\mathrm{Z}^{(q)}(\cdot)$ operating only on the g^{th} qudit or subqudit among the n qudits or subqudits:

$$\mathrm{X}_{(g,n)}^{(q)}(\cdot) := \mathrm{I}_q^{\otimes(g-1)} \otimes \mathrm{X}^{(q)}(\cdot) \otimes \mathrm{I}_q^{\otimes(n-g)}, \tag{4}$$

$$\mathrm{Z}_{(g,n)}^{(q)}(\cdot) := \mathrm{I}_q^{\otimes(g-1)} \otimes \mathrm{Z}^{(q)}(\cdot) \otimes \mathrm{I}_q^{\otimes(n-g)}, \tag{5}$$

where I_q is the identity operator on a qudit ($q = p^m$) or a subqudit ($q = p$). Let $\mathrm{P}^{(q)}(i, j) := |i\rangle_q\langle j|_q$ be the projector with $i, j \in \mathbb{F}_q$. Then the operator with the projector $\mathrm{P}^{(q)}(i, j)$ operating on the g^{th} qudit or subqudit out of the n qudits or subqudits is represented by

$$\mathrm{P}_{(g,n)}^{(q)}(i, j) := \mathrm{I}_q^{\otimes(g-1)} \otimes \mathrm{P}^{(q)}(i, j) \otimes \mathrm{I}_q^{\otimes(n-g)}.$$

Two n qudit basis operators commute as follows [13]:

$$\left(\bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)\right)\left(\bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\theta_i)\mathrm{Z}^{(p^m)}(\nu_i)\right)$$
$$= \omega^s \left(\bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\theta_i)\mathrm{Z}^{(p^m)}(\nu_i)\right)\left(\bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)\right),$$

where s is the symplectic product of the basis operators.

$$s = \mathrm{Symp}(O_1, O_2) := \mathrm{Tr}_{p^m/p}\left(\sum_{i=1}^{n} \gamma_i\theta_i - \sum_{i=1}^{n} \beta_i\nu_i\right). \quad (6)$$

A. Entanglement-unassisted qudit stabilizer codes

Let $\mathcal{G}_{p^m}$ be the group generated by $\mathcal{G}_{p^m}^{(b)}$ when $p \neq 2$ and generated by $\mathcal{G}_{p^m}^{(b)}$ and iI_q when $p = 2$ [24]. The stabilizer code is obtained from an Abelian subgroup S of $\mathcal{G}_{p^m}^{\otimes n}$ as the set of all quantum states stabilized by operators in S. As each nonidentity element in $\mathcal{G}_{p^m}^{\otimes n}$ has p distinct eigenvalues, namely $\{\omega^l\}_{l=0}^{p-1}$, when S contains p^ρ elements, the code dimension is $p^{(mn-\rho)}$ [12]. From an Abelian subgroup S of $\mathcal{G}_{p^m}^{\otimes n}$ that has ρ minimal stabilizer generators and does not contain any zero weight nonidentity element, a $((n, p^{(mn-\rho)}, d))_{p^m}$ stabilizer code is obtained [13], where d is the minimum weight of the elements in $\mathcal{N}_S \backslash S$. $\mathcal{N}_S \backslash S$ form the set of undetected errors of the code that transforms one codeword to another.

The minimal generators $\{S_i \equiv [\mathbf{a}_i|\mathbf{b}_i]\}_{i=1}^{\rho}$ of S are compactly represented by the check matrix $\mathcal{H}_S = [\mathcal{H}_\mathrm{X}|\mathcal{H}_\mathrm{Z}]_{p^m}$, where $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{F}_{p^m}^n$, $\mathcal{H}_\mathrm{X} = [\mathbf{a}_i]_{i=1}^{\rho}$, and $\mathcal{H}_\mathrm{Z} = [\mathbf{b}_i]_{i=1}^{\rho}$. When the stabilizer code is obtained from an Abelian subgroup S of $\mathcal{G}_{p^m}^{\otimes n}$, it does not require preshared entangled states between the encoder and the decoder; hence, it is called an entanglement-unassisted stabilizer code.

B. Entanglement-assisted qudit stabilizer codes

Quantum codes are constructed from a non-Abelian subgroup $\mathcal{R}$ of $\mathcal{G}_{p^m}^{\otimes n}$ by extending the elements of $\mathcal{R}$ with a few subqudit operators that form an Abelian group S, from which a stabilizer quantum code is obtained. The extensions of the elements of $\mathcal{R}$ are designed such that the phase in the commutativity relations between the operators added for extension of elements of $\mathcal{R}$ cancel out the phase in the commutativity relations between the elements of $\mathcal{R}$. The stabilizer codes obtained by considering the Abelian group S as the stabilizer group requires preshared entangled states between the transmitter and the receiver during encoding [25,26]; hence, they are called entanglement-assisted stabilizer codes. The subqudits used for extension of the elements of $\mathcal{R}$ correspond to the receiver end subqudits of the preshared entangled states and are assumed to be maintained error free throughout.

Let n'_e be the minimum number of preshared entangled subqudit pairs required between the encoder and the decoder. We note that we use n'_e to denote the number of entangled subqudits states and n_e to denote the number of entangled qudits states. An $((n, K, d; n'_e))_{p^m}$ quantum code has a codelength n, code dimension K, where $K = p^l$ for some $l \in \mathbb{Z}^+$, minimum distance d, and requires at least n'_e preshared entangled subqudit pairs to construct the code. The subscript p^m in the notation denotes that the code is over qudits of dimension p^m.

Let $\mathcal{R}$ be generated by minimal generators $R_1, R_2, \ldots, R_\rho$ that form a noncommuting set. Let $S_1, S_2, \ldots, S_\rho$ be the operators obtained by extending $R_1, R_2, \ldots, R_\rho$ by n'_e subqudit operators such that they form an Abelian group. One such extension procedure is provided in Ref. [16]. The group S generated by $S_1, \ldots, S_\rho$ is an Abelian subgroup of $\mathcal{G}_{p^m}^{\otimes n} \otimes \mathcal{G}_p^{\otimes n'_e}$ with p^ρ elements. An $((n, p^{(mn+n'_e-\rho)}, d; n'_e))$ EA stabilizer code is obtained based on a non-Abelian subgroup $\mathcal{R}$ of $\mathcal{G}_{p^m}^{\otimes n}$ that has ρ minimal generators over n qudits [16], where n'_e is based on the commutativity relations between elements of $\mathcal{R}$. The minimum distance d of the code is obtained from the normalizer $\mathcal{N}_\mathcal{R}$ of $\mathcal{R}$ as the receiver end subqudits are considered to be error free.

The quantum code $\mathcal{Q}_\mathcal{C}$ based on $\mathcal{R}$ is given by

$$\mathcal{Q}_\mathcal{C} = \{|\psi\rangle \in \mathbb{C}^{p^{(mn+n'_e)}} | \mathcal{T}^{(E)}(R_i)|\psi\rangle = |\psi\rangle \forall\, i \in \{1, \ldots, \rho\}\}$$

where $\mathcal{T}^{(E)}(R_i) = S_i$ is obtained by extending R_i.

Encoding procedures for EA qudit stabilizer codes based on classical linear and additive codes have been provided in Ref. [15] and Ref. [26], respectively. EA qudit stabilizer codes can be encoded by operating Clifford operators such as ADD_p, DFT_p, etc., on the message $|\phi\rangle$, a few ancilla subqudits, and n'_e entangled subqudit pairs, where $\mathrm{ADD}_p = \sum_{x,y \in \mathbb{F}_p} |x\rangle\langle x| \otimes |x+y\rangle\langle y|$ and

$$\mathrm{DFT}_p = \frac{1}{\sqrt{p}} \sum_{i,\, l \in \mathbb{F}_p} \omega^{il} |i\rangle_p \langle l|_p. \quad (7)$$

Remark 1. In this paper, we assume that the n'_e receiver end subqudits of the n'_e preshared entangled subqudit pairs are implemented as parts of qudits and stored in registers at the receiver. We also assume that the receiver has the knowledge about the location of the receiver end subqudits in the register that correspond to a particular codeword as these subqudits are required for computing the syndrome during error correction.

III. QUANTUM COMMUNICATION PROTOCOL

In coded quantum systems, the information to be transmitted or stored is encoded by adding redundancy in the quantum information through the use of entanglement-assisted or entanglement-unassisted stabilizer codes. When the encoded quantum information is sent through a quantum communication channel, it interacts with the environment leading to quantum decoherence, which we view as an error. This error is corrected using the error correction circuit at the receiver end, followed by decoding to obtain the codeword.

In Fig. 1, we illustrate the flow of quantum information through a coded quantum system. For entanglement-assisted stabilizer codes, initially, the maximally entangled subqudit pairs are generated and distributed between the transmitter and the receiver, where they are stored in quantum registers. We consider the stabilizer codes to have ρ minimal stabilizer generators.

A. Entangled pair generation and encoding

We consider the maximally entangled state to be $|\eta\rangle = \sum_{i=0}^{p-1} \frac{1}{\sqrt{p}} |i\rangle_p |p-i\rangle_p$ [26]. The state $|\eta\rangle$ is generated by

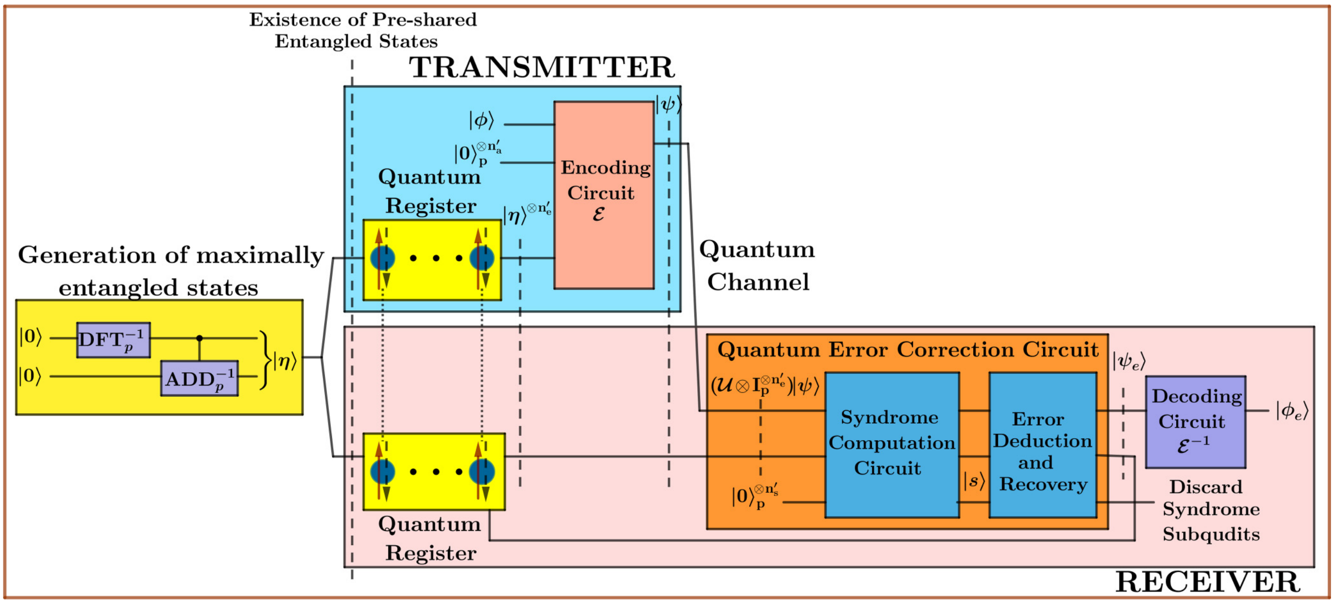

FIG. 1. Information flow in coded quantum systems: The maximally entangled state $|\eta\rangle$ is generated by performing DFT_p^{-1} and ADD_p^{-1} operations on $|00\rangle_p$, where $|00\rangle_p$ are two subqudits implemented with two different sets of qudits. One set of qudits is sent to the transmitter and the other to the receiver, where they are stored in quantum registers. In the transmitter, using encoding operation $\mathcal{E}$, the n'_m subqudit unencoded quantum information $|\phi\rangle$ is encoded to the codeword $|\psi\rangle$ using n'_a ancilla subqudits in state $|0\rangle_p$ and the transmitter end subqudits of n'_e maximally entangled states. The transmitter end qudits of $|\psi\rangle$ are transmitted through a quantum communication or storage channel that introduces an error $\mathcal{U}$. Error correction is performed over $(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n'_e})|\psi\rangle$ using n'_s syndrome subqudits initially in state $|0\rangle_p$ by computing the syndrome $|s\rangle$ using the syndrome computation circuit, followed by error deduction and recovery. The decoding operation $\mathcal{E}^{-1}$ is performed on the first n qudits of the codeword to obtain the estimated unencoded quantum information $|\phi_e\rangle$.

performing the DFT_p^{-1} operation on the first subqudit of $|00\rangle_p$, followed by the $\mathrm{ADD}_p^{-1}(1,2)$ operation [26] as shown in Fig. 1. The two subqudits in each maximally entangled pair are implemented with two different sets of qudits. One set of these qudits is sent to the transmitter, while the other set is sent to the receiver, where they are stored in quantum registers. These preshared entangled subqudits between the transmitter and the receiver end are used to encode the quantum information using EA stabilizer codes.

When the quantum information $|\phi\rangle$ needs to be transmitted or stored, an encoding operation $\mathcal{E}$ is performed on $|\phi\rangle$, with n'_a ancilla subqudits in state $|0\rangle_p$, and the transmitter end subqudits of the n'_e preshared entangled pairs to obtain the state over the n transmitter end qudits, along with the n'_e receiver end subqudits to be the codeword $|\psi\rangle$, where $n'_a = \rho - 2n'_e$.

B. Quantum channel and error model

The n transmitter end qudits of $|\psi\rangle$ are transmitted or stored, which we view as passing them through a quantum transmission or storage channel. We consider the quantum channel to be a quantum depolarizing channel that introduces each nonidentity error with probability $p_d/(p^{2m}-1)$ and no error occurs with probability $(1-p_d)$, where p_d is the depolarizing probability. We assume the receiver end subqudits are at the receiver end throughout and are maintained error free. We also assume noise-free operations in the maximally entangled state generation circuit, in the transmitter, and the receiver.

C. Error correction

Let $\mathcal{U}$ be the error introduced by the communication or storage channel. The state of the codeword subqudits at the receiver is $(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n'_e})|\psi\rangle$, where $\mathrm{I}_p^{\otimes n'_e}$ depicts that the receiver end subqudits are maintained error free.

To correct the error, n'_s syndrome subqudits, initially in state $|0\rangle_p$ are obtained from a quantum register, where $n'_s = \rho$. The erroneous codeword $(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n'_e})|\psi\rangle$ along with the syndrome subqudits are passed through the error correction circuit to obtain the estimated codeword $|\psi_e\rangle$ and the syndrome subqudits in state $|s\rangle$, where $|s\rangle$ corresponds to the eigenvalue-based syndrome.

The error correction procedure comprises two steps, namely (a) syndrome computation and (b) error deduction and recovery. The syndrome computation procedure involves computing the eigenvalue-based syndrome state $|s\rangle$ using the stabilizers S_is of the code. Using the syndrome state $|s\rangle$ obtained, the error is deduced and the inverse error operation is performed to recover the codeword. The error deduction and recovery circuit consists of control-based gates to apply the inverse error operator by using the syndrome subqudits as the control subqudits. This circuit is based on the code used. Finally, the decoding operation, i.e., the inverse of the encoding operation, is performed on the first n qudits of $|\psi_e\rangle$ to obtain the estimated quantum information $|\phi_e\rangle$. When the error $\mathcal{U}$ is a correctable error of the EA stabilizer code, $|\psi_e\rangle = |\psi\rangle$ and $|\phi_e\rangle = |\phi\rangle$.

In quantum memories, the n'_e receiver end codeword subqudits are stored back in the quantum register for the quantum memories to use for error correction in the next period. In quantum communication systems, these n'_e subqudits are discarded.

For a coded quantum system that uses entanglement-unassisted stabilizer codes, the flow of information is similar to the flow illustrated in Fig. 1, except that there is no circuit for the generation of the maximally entangled states and no preshared entangled pairs are involved, i.e., $n'_e = 0$.

The encoding circuit architectures for entanglement-unassisted and entanglement-assisted qudit stabilizer codes[3] based on classical additive codes is provided in Ref. [26]. In this article, we provide the architectures of the syndrome computation circuit for the entanglement-assisted and entanglement-unassisted stabilizer codes. The error deduction and recovery circuit is based on a specific code; hence, we do not provide it explicitly in this article.

IV. ERROR CORRECTION WITH QUDIT STABILIZER CODES

In this section, we briefly discuss the error correction procedure, which comprises the following two steps: (a) the syndrome computation and (b) the error deduction and recovery.

There has not been any prior work that provides the syndrome computation circuit schematic[4] for stabilizer codes over qudits based on classical additive codes; hence, motivating this work as part of this article.

We represent the error by a unitary operator $\mathcal{U}$. At the receiver, the erroneous information is corrected using the syndrome based error correction procedure. The syndrome is obtained based on the eigenvalues of the stabilizer generators considering the erroneous state as the eigenstate. For any basis error B from $\mathcal{G}_{p^m}^{\otimes n}$, $\omega^{\text{Symp}(S_i,B)}$ is an eigenvalue of the erroneous state $B|\psi\rangle$, for $i \in \{1,\ldots,\rho\}$, as $S_iB|\psi\rangle = \omega^{\text{Symp}(S_i,B)}BS_i|\psi\rangle = \omega^{\text{Symp}(S_i,B)}B|\psi\rangle$. The syndrome comprises the values $\text{Symp}(S_i, B)$ for all S_i's. We note that these values belong to $\mathbb{F}_p$. The eigenvalues correspond to the phase involved in the commutative relations of the stabilizer generators with the basis operators.

As the normalizer $\mathcal{N}_S$ is a subgroup of $\mathcal{G}_{p^m}$, cosets of $\mathcal{N}_S$ are formed in $\mathcal{G}_p^{\otimes(mn+n'_e)}$. Two elements G_1 and G_2 in the same coset of the normalizer $\mathcal{N}_S$ are related as $G_2 = G_1N$, where $N \in \mathcal{N}_S$. As $N \in \mathcal{N}_S$, we note that $S_iN = NS_i$ for any stabilizer S_i. The elements in the same coset correspond to the same set of eigenvalues and the same syndrome as $S_iG_2|\psi\rangle = S_iG_1N|\psi\rangle = \omega^{\text{Symp}(S_i,G_1)}G_1S_iN|\psi\rangle = \omega^{\text{Symp}(S_i,G_1)}G_1NS_i|\psi\rangle = \omega^{\text{Symp}(S_i,G_1)}G_1N|\psi\rangle = \omega^{\text{Symp}(S_i,G_1)}G_2|\psi\rangle$. Similarly, elements from different cosets[5] have different syndrome.

[3]The encoding circuit idea for entanglement-assisted qudit stabilizer codes based on classical linear codes is provided in Ref. [15].

[4]The syndrome computation circuit is an important part of the error correction circuit as the error correction of stabilizer codes is based on the eigenvalue-based syndrome; hence, they are vital toward the practical implementation of coded quantum communication and storage systems.

[5]Neglecting phase, $\mathcal{G}_{p^m}$ contains p^ρ cosets of the normalizer $\mathcal{N}_S$.

In stabilizer codes, the coset representative is chosen as the least weight operator in the coset as it corresponds to the most probable error in the coset. To correct the error based on the syndrome, the error correction technique for all the errors having the same syndrome should be the same. For stabilizer codes, we note that the set of correctable errors are the errors of weight less than or equal to $t = \lfloor (d-1)/2 \rfloor$, where d is the minimum distance of the code [2].

A. Error correction routine

For error correction, we introduce $n'_s = \rho$ ancilla subqudits termed as the syndrome subqudits. Let $\mathcal{U} = \sum_{\mathbf{g},\mathbf{h}\in\mathbb{F}_{p^m}^n} a_{\mathbf{gh}} \bigotimes_{i=1}^n \mathrm{X}^{(p^m)}(g_i)\mathrm{Z}^{(p^m)}(h_i)$, where $a_{\mathbf{gh}} \in \mathbb{C}$, $\mathbf{g} = [g_i]_{i=1}^n$ and $\mathbf{h} = [h_i]_{i=1}^n$, be the unitary error. For the entanglement-assisted stabilizer code, as receiver end subqudits are maintained error free throughout, the error on the $(mn + n'_e)$ subqudits of the codeword is $(\mathcal{U} \bigotimes \mathrm{I}_p^{\bigotimes n'_e})$. For the entanglement-unassisted stabilizer code, $n'_e = 0$. Thus, the state of the erroneous codeword along with the ancilla subqudits for the stabilizer code is as follows:

$$\begin{aligned}&\big(\mathcal{U} \otimes \mathrm{I}_p^{\otimes n'_e}\big)|\psi\rangle|0\rangle_p^{\otimes n'_s}\\&\quad = \left[\left(\sum_{\mathbf{g},\mathbf{h}\in\mathbb{F}_{p^m}^n} a_{\mathbf{gh}} \bigotimes_{i=1}^n \mathrm{X}^{(p^m)}(g_i)\mathrm{Z}^{(p^m)}(h_i)\right)\otimes \mathrm{I}_p^{\otimes n'_e}\right]|\psi\rangle|0\rangle_p^{\otimes n'_s}.\end{aligned} \tag{8}$$

Later in this section, in Lemma 1, we propose the syndrome computation operator $\mathcal{S}'$. When $\mathcal{S}'$ is performed on the state $(\mathcal{U} \otimes \mathrm{I}_p^{\otimes n'_e})|\psi\rangle|0\rangle_p^{\otimes n'_s}$ in Eq. (8), the state $|0\rangle_p^{\otimes n'_s}$ transforms to the syndrome state $|\boldsymbol{s}_{\mathbf{gh}}\rangle$, where $\boldsymbol{s}_{\mathbf{gh}}$ is the n'_e-symbol syndrome over $\mathbb{F}_p$ based on the basis error $(\bigotimes_{i=1}^n \mathrm{X}^{(p^m)}(g_i)\mathrm{Z}^{(p^m)}(h_i) \otimes \mathrm{I}_p^{\otimes n'_e})$. The resultant state obtained is as follows:

$$\begin{aligned}&\mathcal{S}'\big(\mathcal{U} \otimes \mathrm{I}_p^{\otimes n'_e}\big)|\psi\rangle|0\rangle_p^{\otimes n'_s}\\&\quad = \sum_{\mathbf{g},\mathbf{h}\in\mathbb{F}_{p^m}^n} a_{\mathbf{gh}}\left(\bigotimes_{i=1}^n \mathrm{X}^{(p^m)}(g_i)\mathrm{Z}^{(p^m)}(h_i)\otimes \mathrm{I}_p^{\otimes n'_e}\right)|\psi\rangle|\boldsymbol{s}_{\mathbf{gh}}\rangle.\end{aligned} \tag{9}$$

On performing measurement on the syndrome subqudits in $\mathcal{S}'(\mathcal{U} \otimes \mathrm{I}_p^{\otimes n'_e})|\psi\rangle|0\rangle_p^{\otimes n'_s}$ in Eq. (9), the state of the syndrome subqudits collapse based on the measurement outcome to a state $|\boldsymbol{s}_{\mathbf{gh}}\rangle$ for some $\mathbf{g}, \mathbf{h} \in \mathrm{F}_{p^m}^n$. As only errors in the same coset have the same syndrome, only the terms corresponding to the errors in the particular coset corresponding to the syndrome remain. For entanglement-assisted stabilizer codes, as the error performs nonidentity operation on the first n qudits of the codeword, we consider the cosets of $\mathcal{N}_{\mathcal{R}}$ instead of the cosets of $\mathcal{N}_S$. We note that for entanglement-unassisted stabilizer code, $\mathcal{N}_{\mathcal{R}} = \mathcal{N}_S$.

Let $Q_l = [\mathbf{q_{lx}}|\mathbf{q_{lz}}]$ be the coset representative of the particular coset of $\mathcal{N}_R$ whose syndrome $\boldsymbol{s}_{\mathbf{q_{lx}q_{lz}}}$ is obtained as the measurement outcome. Let $\mathcal{V}$ be the projective measurement operator over the syndrome subqudits. Then, by representing the other elements in the coset as Q_lN, where $N \in \mathcal{N}_{\mathcal{R}}$, from

Eq. (9), we obtain

$$\mathcal{VS}'\big(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle|0\rangle_p^{\otimes n_s'} = \left(\sum_{N\in\mathcal{N}_{\mathcal{R}}} a_{NQ_l}\big(Q_l N\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle\right)\big|s_{\mathbf{q}_{lx}\mathbf{q}_{lz}}\big\rangle, \qquad (10)$$

where $a_{NQ_l} = a_{\mathbf{gh}}$ such that $NQ_l \equiv [\mathbf{g}|\mathbf{h}]$.

The state $\mathcal{VS}'(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'})|\psi\rangle|0\rangle_p^{\otimes n_s'}$ in Eq. (10) is the state obtained after performing the syndrome computation operator, followed by the measurement on the syndrome subqudits. We note that the total number of possible syndromes are p^ρ as the syndrome vector $s_{\mathbf{gh}}$ is obtained from the eigenvalues with respect to the ρ stabilizer generators and the eigenvalues belong to $\mathbb{F}_p$.

We next consider a unitary operator $\mathcal{W} = \sum_{f=1}^{p^\rho} Q_f^\dagger \otimes \mathrm{I}_p^{\otimes n_e'} \otimes |s_{\mathbf{q}_{fx}\mathbf{q}_{fz}}\rangle\langle s_{\mathbf{q}_{fx}\mathbf{q}_{fz}}|$ that performs the inverse operation $Q_l^\dagger$ on the state $\mathcal{VS}'(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'})|\psi\rangle|0\rangle_p^{\otimes n_s'}$ in Eq. (10) to correct the error Q_l based on the syndrome $s_{\mathbf{q}_{lx}\mathbf{q}_{lz}}$. Applying $\mathcal{W}$ to $\mathcal{VS}'(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'})|\psi\rangle|0\rangle_p^{\otimes n_s'}$ in Eq. (10), we obtain

$$\mathcal{WVS}'\big(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle|0\rangle_p^{\otimes n_s'} = \sum_{N\in\mathcal{N}_{\mathcal{R}}} a_{NQ}\big(N\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle\big|s_{\mathbf{q}_{lx}\mathbf{q}_{lz}}\big\rangle. \qquad (11)$$

When $N\in\mathcal{N}_{\mathcal{R}}$, the operators having the form $(N\otimes \mathrm{I}_p^{\otimes n_e'})$ belong to $\mathcal{N}_S$. The only correctable errors in $\mathcal{N}_S$ are the operators in S; hence, the coefficients a_{NQ}s are nonzero only when $(N\otimes \mathrm{I}_p^{\otimes n_e'})\in S$. Thus, from Eq. (11), we obtain

$$\begin{aligned}\mathcal{WVS}'\big(\mathcal{U}\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle|0\rangle_p^{\otimes n_s'} &= \sum_{N:(N\otimes \mathrm{I}_p^{\otimes n_e'})\in S} a_{NQ}\big(N\otimes \mathrm{I}_p^{\otimes n_e'}\big)|\psi\rangle\big|s_{\mathbf{q}_{lx}\mathbf{q}_{lz}}\big\rangle\\ &= \sum_{N:(N\otimes \mathrm{I}_p^{\otimes n_e'})\in S} a_{NQ}|\psi\rangle\big|s_{\mathbf{q}_{lx}\mathbf{q}_{lz}}\big\rangle.\end{aligned}$$

Assuming $\sum_{N:(N\otimes \mathrm{I}_p^{\otimes n_e'})\in S} a_{NQ}$ to be a global phase factor, we have corrected the error to obtain the codeword $|\psi\rangle$.

We next design the syndrome computation operator $\mathcal{S}'$ for entanglement-unassisted and -assisted qudit stabilizer codes. Let $E := (Q_l N\otimes \mathrm{I}_p^{\otimes n_e'})$. The syndrome computation operator $\mathcal{S}'$ acts on the erroneous codeword and the syndrome subqudits, and transforms $E|\psi\rangle|0\rangle_p^{\otimes\rho}$ to $E|\psi\rangle|s_E\rangle$, where $|s_E\rangle$ is the syndrome corresponding to E, a basis operator. Let $S_iE = \omega^{l_i}ES_i\,\forall\, i\in\{1,\ldots,\rho\}$. As the syndrome is obtained from the eigenvalues of the stabilizer generators S_is with respect to the erroneous state $E|\psi\rangle$, the syndrome is $|s_E\rangle = |l_1 l_2\ldots l_\rho\rangle$. In general, we refer to $|s_E\rangle$ as $|s\rangle$.

B. Error deduction and recovery

The error deduction and recovery step involves deducing the error based on the syndrome and applying the inverse operation of the error to obtain the codeword. This step could involve one of the following two procedures:

(a) Perform measurement on the syndrome subqudits, classically deduce the error, and apply its inverse operation, or

(b) Perform control-based operations like controlled-X and controlled-Z operations to perform the inverse operation of the error based on the syndrome.

V. SYNDROME COMPUTATION CIRCUIT ARCHITECTURES

In this section, we provide the explicit form of the syndrome computation circuit schematic for the entanglement-unassisted and -assisted qudit stabilizer code based on classical additive codes. We note that the syndrome computation circuit schematic provided in this Section works for all qudit stabilizer codes based on classical additive codes over finite fields.

A. Utility of the DFT in syndrome computation

We first provide the idea to obtain the syndrome computation operator. The syndrome is based on the eigenvalue of the minimal stabilizer generators with respect to the erroneous state $E|\psi\rangle$.

Let S_1 be a minimal stabilizer generator whose eigenvalue we need to obtain with respect to the eigenvector $E|\psi\rangle$. Let $S_1E|\psi\rangle = \omega^i E|\psi\rangle$. We first introduce a syndrome subqudit in state $|0\rangle_p$. We perform DFT_p operation on the syndrome subqudit to obtain $|s_I\rangle = \mathrm{DFT}_p|0\rangle = \frac{1}{\sqrt{p}}\sum_{l\in\mathbb{F}_p}|l\rangle_p$. The state of the erroneous state $E|\psi\rangle$ along with the syndrome subqudits is

$$E|\psi\rangle|s_I\rangle = \big(\mathrm{I}_p^{\otimes mn}\otimes \mathrm{DFT}_p\big)(E|\psi\rangle|0\rangle_p). \qquad (12)$$

We next choose the following operator $\mathcal{S}_1' = \sum_{j\in\mathbb{F}_p} S_1^j\otimes |j\rangle_p\langle j|$ to compute the syndrome with respect to S_1, where S_1^js act on the codeword subqudits and $|j\rangle_p\langle j|$ acts on the syndrome subqudit. We note that the eigenvalue of S_1^j with respect to the eigenstate $E|\psi\rangle$ is ω^{ij} as $S_1^jE|\psi\rangle = \omega^{ij}E|\psi\rangle$. Hence, using $\mathcal{S}_1'$, we obtain a superposition of $|j\rangle$s with coefficients being the corresponding eigenvalue ω^{ij} of S_1^j with respect to $E|\psi\rangle$, i.e.,

$$\mathcal{S}_1'(E|\psi\rangle|s_I\rangle) = E|\psi\rangle\otimes\frac{1}{\sqrt{p}}\sum_{j\in\mathbb{F}_p}\omega^{ij}|j\rangle_p. \qquad (13)$$

When DFT_p operates on the state $|i\rangle_p$, where $i\in\mathbb{F}_p$, it produces the superposition of the states as follows:

$$\mathrm{DFT}_p|i\rangle_p = \frac{1}{\sqrt{p}}\sum_{j\in\mathbb{F}_p}\omega^{ij}|j\rangle_p, \quad \text{where} \quad i\in\mathbb{F}_p. \qquad (14)$$

Substituting Eq. (14) in Eq. (13), we obtain

$$\mathcal{S}_1'(E|\psi\rangle|s_I\rangle) = E|\psi\rangle\otimes \mathrm{DFT}_p|i\rangle_p. \qquad (15)$$

On performing $\mathrm{DFT}_p^\dagger$ operation on the syndrome subqudit of $\mathcal{S}_1'(E|\psi\rangle|s_I\rangle)$ in Eq. (15), we obtain

$$\big(\mathrm{I}_p^{\otimes mn}\otimes \mathrm{DFT}_p^\dagger\big)\mathcal{S}_1'(E|\psi\rangle|s_I\rangle) = E|\psi\rangle\otimes|i\rangle_p. \qquad (16)$$

Thus, using the DFT_p, $\mathcal{S}_1'$, and $\mathrm{DFT}_p^\dagger$ operators, we have transferred the eigenvalue based information, i.e., the value of i in $S_1E|\psi\rangle = \omega^iE|\psi\rangle$, to the syndrome subqudit $|i\rangle_p$. From Equs. (12), (15), and (16), the operators $(\mathrm{I}_p^{\otimes mn}\otimes \mathrm{DFT}_p)$, $\mathcal{S}_1'$, and

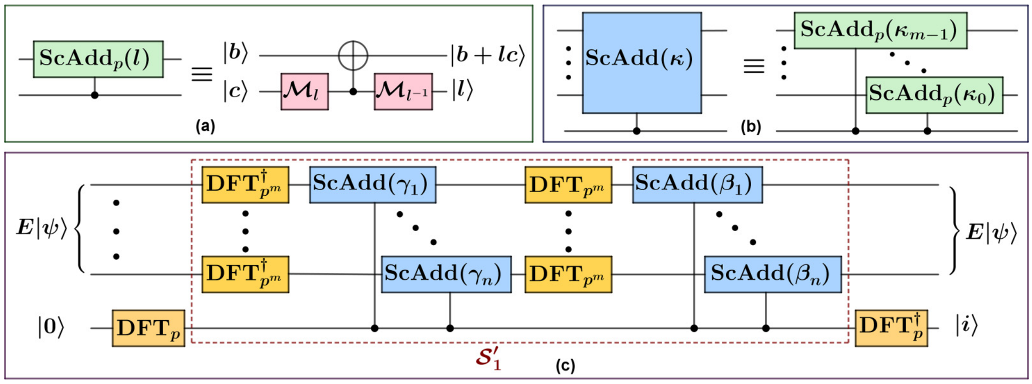

FIG. 2. The syndrome computation operator based on stabilizer S_1: (a) The scaled addition operation $\text{ScAdd}_p(l)$ over subqudits is obtained using the multiplication gates M_l and $\text{M}_{l^{-1}}$ and the ADD_p gate, where $l \in \mathbb{F}_p$. (b) Using the scaled addition operators over subqudits, a scaled addition operator $\text{ScAdd}(\kappa)$ over a qudit is obtained, where $\kappa = \sum_{j=0}^{m-1} \kappa_j \alpha^j$. For $\text{ScAdd}(\kappa)$ the control subqudit is a single subqudit and the target is either k subqudits or a qudit. (c) Using DFT operations over qudits and subqudits, and the scaled addition operators over qudits with the syndrome subqudit as the control and the codeword qudits as the target, the syndrome computation is performed. Post syndrome computation, the syndrome qudits are in state $|i\rangle$, where $i = \text{Symp}(S_1, E)$.

$(\text{I}_p^{\otimes mn} \otimes \text{DFT}_p^\dagger)$ used to transform $E|\psi\rangle|0\rangle_p$ to $E|\psi\rangle \otimes |i\rangle_p$ are invertible; hence there is a bijection between the eigenvalue ω^i corresponding to $E|\psi\rangle$ and the state $|i\rangle_p$ obtained in $(E|\psi\rangle) \otimes |i\rangle_p$.

Using standard basis measurements on the qudit that comprises the syndrome subqudit, the value of i is obtained. We note that the measurement is not performed on the codeword subqudits.

1. Implementation of the operator S_1'

The operator $\mathcal{S}_1'$ can be implemented using one and two qudit operators,[6] similar to the operator shown in Ref. [20]. In Ref. [20], the operators are over qudits, while in this paper, the operators are over subqudits.

Let $S_1 = \bigotimes_{i=1}^n \text{X}^{(p^m)}(\beta_i)\text{Z}^{(p^m)}(\gamma_i)$, then, we obtain

$$\mathcal{S}_1' = \left(\sum_{j\in\mathbb{F}_p} \bigotimes_{i=1}^n \text{X}^{(p^m)}(j\beta_i) \otimes |j\rangle_p\langle j|\right)\left(\text{DFT}_{p^m}^{\otimes n} \otimes \text{I}_p\right) \times \left(\sum_{j\in\mathbb{F}_p} \bigotimes_{i=1}^n \text{X}^{(p^m)}(j\gamma_i) \otimes |j\rangle_p\langle j|\right)\left((\text{DFT}_{p^m}^\dagger)^{\otimes n} \otimes \text{I}_p\right), \tag{17}$$

where

$$\sum_{j\in\mathbb{F}_p} \bigotimes_{i=1}^n \text{X}^{(p^m)}(j\beta_i) \otimes |j\rangle_p\langle j| = \prod_{i=1}^n \sum_{j\in\mathbb{F}_p} \text{X}_{(i,n)}^{(p^m)}(j\beta_i) \otimes |j\rangle_p\langle j|. \tag{18}$$

$\sum_{j\in\mathbb{F}_p} \text{X}_{(i,n)}^{(p^m)}(j\beta_i) \otimes |j\rangle_p\langle j|$ are scaled addition operations with the syndrome subqudits as the control and codeword qudits as the target. Obtaining the scaled addition operations using multiplication operations and addition operations and obtaining the syndrome computation operation using multiplication, addition, and DFT operations are shown in Fig. 2. The detailed derivations are provided in the Supplemental Material [27].

We note that the operator $\mathcal{S}_1'$ based on the scaled addition operators extracts the information of the eigenvalue of S_1 with respect to $E|\psi\rangle$ and transfers it to the syndrome subqudits. The DFT_p operation and its inverse transform the information in the power of ω, i.e., i in ω^i into the measurement basis state $|i\rangle$, where $i \in \{0, \dots, p-1\}$.

[6] A one or two subqudit operator can be represented as a one or two qudit operator by tensoring with identity operators appropriately and using Eqs. (2) and (3).

B. Syndrome computation operator for entanglement-unassisted qudit stabilizer code

In this subsection, we provide the syndrome computation circuit schematic for the entanglement-unassisted stabilizer code.

We next provide the syndrome computation operator in Lemma 1. We use the operation DFT_p in Lemma 1 to first obtain the superposition of all the basis states as shown in Eq. (14). Then, we apply the stabilizer generators to obtain the eigenvalues, followed by applying the $\text{DFT}^\dagger$ operator and performing the standard basis measurement.

We will use the following two relations of a function $f(i, r)$ in Lemma 1 to obtain the syndrome computation operator:

$$\bigotimes_{i=1}^{\rho} \sum_{r\in\mathbb{F}_p} f(i, r) = \sum_{r_1,\dots,r_\rho\in\mathbb{F}_p} \left(\bigotimes_{i=1}^{\rho} f(i, r_i)\right), \tag{19}$$

$$\prod_{i=1}^{\rho} \sum_{r\in\mathbb{F}_p} f(i, r) = \sum_{r_1,\dots,r_\rho\in\mathbb{F}_p} \left(\prod_{i=1}^{\rho} f(i, r_i)\right). \tag{20}$$

We define an operator $\mathcal{S}_\mathcal{D}$ as follows:

$$\mathcal{S}_\mathcal{D} = \sum_{j_1,\dots,j_\rho\in\mathbb{F}_p} S_1^{j_1} S_2^{j_2} \dots S_\rho^{j_\rho} \otimes |j_1 j_2 \dots j_\rho\rangle\langle j_1 j_2 \dots j_\rho|. \tag{21}$$

We use the subscript $\mathcal{D}$ as it corresponds to the syndrome computation operator in the DFT basis.

Lemma 1. The syndrome computation operator $\mathcal{S}'$ corresponding to the stabilizer generators $S_1, \ldots, S_\rho$ is

$$\mathcal{S}' = \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\mathcal{S}_{\mathcal{D}}\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right).$$

Proof. Let us consider the basis error operator E such that $S_i E = \omega^{l_i} E S_i \, \forall \, i \in \{1, \ldots, \rho\}$. For codeword $|\psi\rangle$, let the erroneous codeword be $E|\psi\rangle$. We note that the syndrome obtained based on the eigenvalues ω^{l_i}s is $[l_1\, l_2\, \ldots\, l_\rho]$. We show that $\mathcal{S}'$ is the syndrome computation operator by proving that $\mathcal{S}'(E|\psi\rangle|0\rangle_p^{\otimes\rho}) = E|\psi\rangle|l_1 l_2 \ldots l_\rho\rangle$, where $|l_1 l_2 \ldots l_\rho\rangle$ is the syndrome state. We first simplify the operator $\mathcal{S}'$ as follows:

$$\mathcal{S}' = \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p} S_1^{j_1} S_2^{j_2} \ldots S_\rho^{j_\rho} \otimes |\boldsymbol{j}\rangle\langle\boldsymbol{j}|\right)$$
$$\times \left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right), \quad \text{where } \boldsymbol{j} = [j_1 \ldots j_\rho]$$
$$= \sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p} S_1^{j_1} S_2^{j_2} \ldots S_\rho^{j_\rho} \otimes \left(\bigotimes_{i=1}^{\rho} \mathrm{DFT}_p^\dagger |j_i\rangle\langle j_i| \mathrm{DFT}_p\right). \tag{22}$$

We simplify $\mathrm{DFT}_p^\dagger|j\rangle\langle j|\mathrm{DFT}_p = \mathrm{DFT}_p^\dagger|j\rangle(\mathrm{DFT}_p^\dagger|j\rangle)^\dagger$ in Eq. (22) by using $\mathrm{DFT}_p^\dagger|j\rangle = (\frac{1}{\sqrt{p}}\sum_{s\in\mathbb{F}_p}\omega^{-sj}|s\rangle)$ similarly to $\mathrm{DFT}_p|j\rangle$ in Eq. (14) to obtain

$$\mathrm{DFT}_p^\dagger|j\rangle\langle j|\mathrm{DFT}_p = \frac{1}{p}\sum_{s,t\in\mathbb{F}_p}\omega^{(t-s)j}|s\rangle\langle t|.$$

Let us consider $r = s - t$. Thus, $s = t + r$ and

$$\mathrm{DFT}_p^\dagger|j\rangle\langle j|\mathrm{DFT}_p$$
$$= \frac{1}{p}\sum_{r\in\mathbb{F}_p}\omega^{-rj}\sum_{t\in\mathbb{F}_p}|t+r\rangle\langle t|,$$
$$= \frac{1}{p}\sum_{r\in\mathbb{F}_p}\omega^{-rj}\mathrm{X}^{(p)}(r). \text{ (From definition of } \mathrm{X}^{(p)}(r)) \tag{23}$$

Substituting Eq. (23) in Eq. (22), we obtain

$$\mathcal{S}' = \sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p} S_1^{j_1} S_2^{j_2} \ldots S_\rho^{j_\rho} \otimes \left(\bigotimes_{i=1}^{\rho} \frac{1}{p}\sum_{r\in\mathbb{F}_p}\omega^{-rj_i}\mathrm{X}^{(p)}(r)\right),$$

When $\mathcal{S}'$ operates on $E|\psi\rangle|0\rangle_p^{\otimes\rho}$, we obtain

$$\mathcal{S}'\left(E|\psi\rangle|0\rangle_p^{\otimes\rho}\right)$$
$$= \frac{1}{p^\rho}\sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p} S_1^{j_1} S_2^{j_2} \ldots S_\rho^{j_\rho} E|\psi\rangle \otimes \left(\bigotimes_{i=1}^{\rho}\sum_{r\in\mathbb{F}_p}\omega^{-rj_i}|r\rangle_p\right). \tag{24}$$

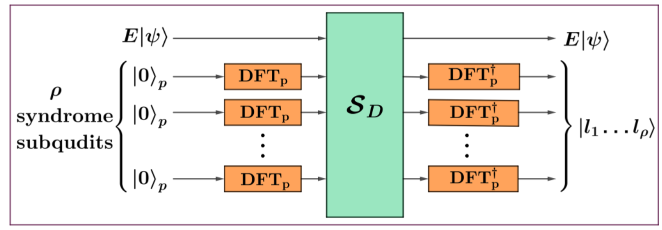

FIG. 3. Syndrome computation circuit schematic for qudit stabilizer codes: The DFT_p operations are first performed on the ρ syndrome subqudits, followed by $\mathcal{S}_{\mathcal{D}}$ on the codeword subqudits along with the syndrome subqudits. Finally, the $\mathrm{DFT}_p^\dagger$ operation is performed on the ρ syndrome subqudits to obtain the syndrome $|l_1 \ldots l_\rho\rangle$ over the syndrome subqudits.

For $i \in \{1, \ldots, \rho\}$, as $S_i E = \omega^{l_i} E S_i$, $S_i^{j_i} E = \omega^{j_i l_i} E S_i^{j_i}$, and $S_i|\psi\rangle = |\psi\rangle$, we obtain

$$S_1^{j_1} S_2^{j_2} \ldots S_\rho^{j_\rho} E|\psi\rangle = \omega^{\sum_{g=1}^{\rho} j_g l_g} E|\psi\rangle. \tag{25}$$

The sum of roots of unity is 0, i.e., $\sum_{j\in\mathbb{F}_p}\omega^{jy} = p\delta_{y,0}$. Substituting $\bigotimes_{i=1}^{\rho}\sum_{r\in\mathbb{F}_p}\omega^{-rj_i}|r\rangle_p = \sum_{r_1,\ldots,r_\rho\in\mathbb{F}_p}\bigotimes_{i=1}^{\rho}\omega^{-r_i j_i}|r_i\rangle_p$ obtained using Eqs. (19) and (25) in Eq. (24),

$$\mathcal{S}'\left(E|\psi\rangle|0\rangle_p^{\otimes\rho}\right)$$
$$= \frac{1}{p^\rho}\sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p}\omega^{\sum_{g=1}^{\rho} j_g l_g}E|\psi\rangle \otimes \sum_{r_1,\ldots,r_\rho\in\mathbb{F}_p}\left(\bigotimes_{i=1}^{\rho}\omega^{-r_i j_i}|r_i\rangle_p\right),$$
$$= E|\psi\rangle \otimes \frac{1}{p^\rho}\sum_{r_1,\ldots,r_\rho\in\mathbb{F}_p}\left(\bigotimes_{i=1}^{\rho}|r_i\rangle_p\right)\sum_{j_1,\ldots,j_\rho \in \mathbb{F}_p}\prod_{i=1}^{\rho}\omega^{j_i(l_i-r_i)},$$
$$= E|\psi\rangle \otimes \frac{1}{p^\rho}\sum_{r_1,\ldots,r_\rho\in\mathbb{F}_p}\left(\bigotimes_{i=1}^{\rho}|r_i\rangle_p\right)\prod_{i=1}^{\rho}\sum_{j_i\in\mathbb{F}_p}\omega^{j_i(l_i-r_i)},$$
$$\text{[From Eq. (20)]}$$
$$= E|\psi\rangle \otimes \frac{1}{p^\rho}\sum_{r_1,\ldots,r_\rho\in\mathbb{F}_p}\left(\bigotimes_{i=1}^{\rho}|r_i\rangle_p\right)\prod_{i=1}^{\rho}p\delta_{l_i,r_i},$$
$$= E|\psi\rangle \otimes |l_1 l_2 \ldots l_\rho\rangle = E|\psi\rangle|l_1 l_2 \ldots l_\rho\rangle. \tag{26}$$

■

From Lemma 1, we obtained the syndrome computation operator that involves transformations $S_1^{j_1} \ldots S_\rho^{j_\rho}$ on the received state based on the control subqudits being the syndrome subqudits in state $|j_1 \ldots j_\rho\rangle$.

Based on the syndrome computation operator $\mathcal{S}'$ from Lemma 1, we provide the syndrome computation circuit schematic in Fig. 3.

The syndrome computation circuit for qubit stabilizer codes and qudit stabilizer codes based on $\mathbb{F}_{p^m}$-linear codes provided in Ref. [19] and Ref. [20] are special cases of the syndrome computation circuit provided in Lemma 1 by considering $p^m = 2$ and by considering the stabilizers based on classical linear codes,[7] respectively. We note that $\mathcal{S}_{\mathcal{D}} = \prod_{i=1}^{\rho} \mathcal{S}'_i$, where $\mathcal{S}'_i$s are based on S_is, similarly to $\mathcal{S}'_1$ in Sec. V A [27].

1. Equivalent circuit schematic for syndrome computation

We next propose an equivalent circuit schematic that performs the transformations on the syndrome subqudits with the received state being the control subqudits.

Let $\mathcal{P} : \mathbb{F}_{p^m} \to \mathbb{F}_{p^m}$ be a transformation such that

$$\mathcal{P}(\beta) = \sum_{g=0}^{m-1} \mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)})\alpha^{(m-1-g)}. \tag{27}$$

We note that the transformation $\mathcal{P}$ is based on the expansion of $\mathrm{Z}^{(p^m)}(\cdot)$ over $\mathrm{Z}^{(p)}(\cdot)$ in Eq. (3). The transformation $\mathcal{P}$ is unique [16].

Let the stabilizer generators be $S_y = \bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_{yi})\mathrm{Z}^{(p^m)}(\gamma_{yi})$; then the syndrome computation operator $\mathcal{S}'$ can also be written as follows:

$$\mathcal{S}' = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l,y)\right)\right], \tag{28}$$

where, for some $\beta = b_0 + b_1\alpha + \dots + b_{(m-1)}\alpha^{(m-1)} \in \mathbb{F}_{p^m}$ with $b_0, \dots, b_{(m-1)} \in \mathbb{F}_p$,

$$\mathcal{X}'_\beta(i,y) = \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}^{(p)}_{(i,n)}(\zeta,\zeta) \otimes \mathrm{X}^{(p)}_{(y,\rho)}(j_{(m-1)}b_{(m-1)} + \dots + j_0 b_0),$$

$$\mathcal{X}'_{\mathcal{P}(\beta)}(l,y) = \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}^{(p)}_{(l,n)}(\zeta,\zeta) \otimes \mathrm{X}^{(p)}_{(y,\rho)}[j_{(m-1)}\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1)}) + \dots + j_0\mathrm{Tr}_{p^m/p}(\beta\alpha^0)], \quad \text{where} \quad \zeta = \sum_{l=0}^{m-1} j_l\alpha^l$$

We prove Eq. (28) through a series of lemmas, i.e., Lemmas 4–7 in Appendix A, with the final result in Lemma 7. In Fig. 4, we illustrate that $\mathcal{X}'_\beta(i,y)$ and $\mathcal{X}'_{\mathcal{P}(\beta)}(l,y)$ can be implemented using scaled addition gates[8] provided in Fig. 2, that are further implemented using ADD_p gates and multiplication gates. The detailed derivations are provided in the supplemental material [27].

We note that the form of $\mathcal{S}'$ in Eq. (28) contains only the DFT_p and $\mathrm{DFT}_p^\dagger$ operations over the codeword qudits along with the codeword qudits being the control qudits in the $\mathcal{X}'_{\beta_i}(i,y)$ and $\mathcal{X}'_{\mathcal{P}(\gamma_l)}(l,y)$ operations. However, in $\mathcal{S}'$ in Lemma 1, the codeword qudits are considered to be the target qudits for the operation $\mathcal{S}_D$.

When the probability of an error occurring on the control qudits of the quantum gate is less compared to that on the target qudit, the form of the syndrome computation operator in Eq. (28) is preferred over the form in Lemma 1 for building the circuit, especially for CSS codes as shown in Fig. 5. We note that our equivalent circuit is analogous to the equivalent circuit provided in Ref. [19, Figure 10.17] for stabilizer codes over qubits.

Let $\mathcal{X}(\boldsymbol{\beta}_y, y) := \prod_{i=1}^{n} \mathcal{X}'_{\beta_{yi}}(i,y)$ and $\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y) := \prod_{l=1}^{n} \mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l,y)$. The operators $\mathcal{X}(\boldsymbol{\beta}_y, y)$ and $\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)$ are control-$\mathrm{X}^{(p)}$ operators with control qudits as the codeword qudits and target subqudits as the syndrome subqudits. Using the operators $\mathcal{X}(\boldsymbol{\beta}_y, y)$ and $\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)$, we provide the equivalent syndrome computation circuit schematic in Fig. 4, based on the following expression obtained from Eq. (28):

$$\mathcal{S}' = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes\rho}\right)\mathcal{X}(\boldsymbol{\beta}_y, y) \times \left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)\right].$$

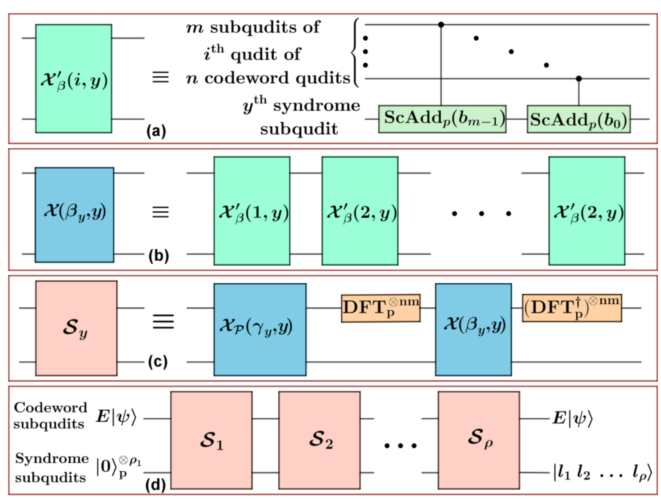

FIG. 4. Equivalent syndrome computation circuit schematic for qudit stabilizer codes. (a) The operator $\mathcal{X}'_\beta$ is computed using scaled addition operations from Fig. 2 based on b_is, where $\beta = \sum_{i=0}^{m-1} b_i\alpha^i$. (b) The operator $\mathcal{X}(\beta_y, y)$ is obtained from a series of $\mathcal{X}'_\beta$ operations. $\mathcal{X}_{\mathcal{P}}(\gamma_y, y)$ can be obtained similarly. (c) The operator $\mathcal{S}_y$ is obtained from $\mathcal{X}(\beta_y, y)$, $\mathcal{X}_{\mathcal{P}}(\gamma_y, y)$ and DFT_p operations. (d) The syndrome computation circuit involves a series of $\mathcal{S}_y$ operations, where $y \in \{1, \dots, \rho\}$, on the erroneous state $E|\psi\rangle$ along with the syndrome subqudits in state $|0\rangle_p$ each to obtain $E|\psi\rangle$ and the syndrome state $|l_1\, l_2\, \dots\, l_\rho\rangle$.

[7]We note that the syndrome computation circuit provided in Lemma 1 can be obtained using the syndrome computation circuit in Ref. [20] for stabilizer codes of dimension p and the relations in Eqs. (2) and (3).

[8]The interested reader must note that these scaled addition gates consider the codeword subqudits are the control subqudits and the syndrome subqudits are the target subqudits, unlike the operators in Lemma 1 [27].

We note that the blocks corresponding to $((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})\mathcal{X}(\boldsymbol{\beta}_y, y)(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$ and $\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)$ should not be exchanged. This is because $\mathrm{X}^{(p^m)}(\cdot)$ and $\mathrm{Z}^{(p^m)}(\cdot)$ operators need not commute with each other and will not give the desired syndrome.

For the CSS codes, the check matrix $\mathcal{H}_S$ is of the following form [17]:

$$\mathcal{H}_S = \left[\begin{array}{c|c} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{array}\right], \tag{29}$$

where B_1 and B_2 are $(m\rho_1 \times n)$ and $(m\rho_2 \times n)$ matrices, ρ_1 and ρ_2 are integers, and $m\rho_1 + m\rho_2 = \rho$.

From Eq. (29), for the CSS code, for $y \in \{1, \ldots, m\rho_1\}$, $\gamma_{yl} = 0$ for all $l \in \{1, \ldots, n\}$ and for $y \in \{m\rho_1 + 1, \ldots, \rho\}$, $\beta_{yi} = 0$ for all $i \in \{1, \ldots, n\}$; hence, from Eq. (28), the syndrome computation operator $\mathcal{S}'$ is

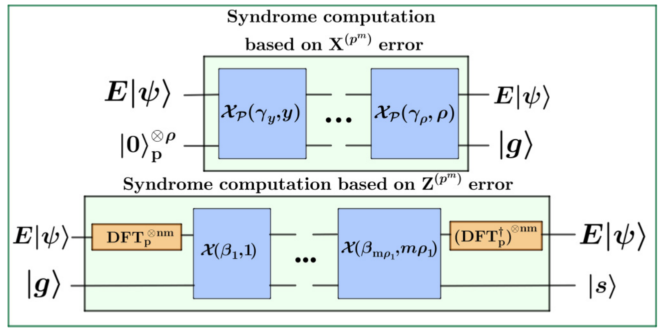

FIG. 5. Equivalent syndrome computation circuit schematic for CSS codes over qudits: The syndrome computation circuit involves two parts: (a) Syndrome computation operator for the $\mathrm{X}^{(p^m)}$ errors using a series of $\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)$ operations, where $y \in \{m\rho_1 + 1, \ldots, \rho\}$; and (b) Syndrome computation operator for the $\mathrm{Z}^{(p^m)}$ errors using DFT_p operations, a series of $\mathcal{X}(\boldsymbol{\beta}_y, y)$ operations, where $y \in \{1, \ldots, m\rho_1\}$, and $\mathrm{DFT}_p^\dagger$ operations. After syndrome computation, the syndrome subqudits are in syndrome state $|\boldsymbol{s}\rangle$, while the codeword subqudits remain in state $E|\psi\rangle$.

$$\begin{aligned}
\mathcal{S}' &= \prod_{y=1}^{\rho}\left[\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l,y)\right)\right],\\
&= \left(\prod_{y=1}^{m\rho_1}\left[\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\right]\right)\left[\prod_{y=k\rho_1+1}^{\rho}\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l,y)\right)\right],\\
&= \left[\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{y=1}^{m\rho_1}\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\right]\left[\prod_{y=m\rho_1+1}^{\rho}\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l,y)\right)\right].
\end{aligned} \tag{30}$$

where the last step is obtained from Eq. (A14) in Appendix A as $\mathrm{DFT}_p\mathrm{DFT}_p^\dagger = \mathrm{I}_p$. From Eq. (30), we obtain

$$\mathcal{S}' = \left[\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{y=1}^{m\rho_1}\mathcal{X}(\boldsymbol{\beta}_y, y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)\right]\left(\prod_{y=m\rho_1+1}^{\rho}\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)\right). \tag{31}$$

We note that the blocks corresponding to $(\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})(\prod_{y=1}^{m\rho_1}\mathcal{X}(\boldsymbol{\beta}_y, y))(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$ and $\prod_{y=m\rho_1+1}^{\rho}\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y)$ can be exchanged as the stabilizer generators either consist $\mathrm{X}^{(p^m)}(\cdot)$ operators or $\mathrm{Z}^{(p^m)}(\cdot)$ operators.

From Eq. (31), the syndrome computation circuit schematic can be viewed to have two parts,[9] namely, one based on $\mathcal{X}'_{\beta_{yi}}(i, y)$ with DFT_p and its inverse, and the other based on $\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l, y)$, as shown in Fig. 5. The component $[((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})(\prod_{y=1}^{m\rho_1}\mathcal{X}(\boldsymbol{\beta}_y, y))(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})]$ of $\mathcal{S}'$ in Eq. (31) corresponds to the syndrome computation operator with respect to the $\mathrm{Z}^{(p^m)}$ error as it corresponds to the $\mathrm{X}^{(p^m)}$ stabilizer from Lemma 4 in Appendix A.

Similarly, from Lemma 5 in Appendix A, the component $(\prod_{y=m\rho_1+1}^{\rho}\mathcal{X}_{\mathcal{P}}(\boldsymbol{\gamma}_y, y))$ of $\mathcal{S}'$ in Eq. (31) corresponds to the syndrome computation operators with respect to the $\mathrm{X}^{(p^m)}$ error.

Let the CSS code be constructed from two classical codes $C_1[n, k_1, d_1]$ and $C_2[n, k_2, d_2]$ over $\mathbb{F}_{p^m}$ with parity check matrices H_1 and H_2 that satisfy the dual-containing criteria $C_1^\perp \subset C_2$. The check matrix of the CSS code is $\mathcal{H}_S = \left[\begin{smallmatrix} H_1\otimes \boldsymbol{a} & \mathbf{0} \\ \mathbf{0} & H_2\otimes\boldsymbol{a}\end{smallmatrix}\right]$, where $\boldsymbol{a} = [1\,\alpha\,\ldots\,\alpha^{m-1}]$. Using the stabilizers in the check matrix $\mathcal{H}_S$ and based on the symplectic product in Eq. (6), the syndrome can be obtained using the operator in Eq. (31). Alternatively, we note that the number of stabilizer generators is a multiple of m for CSS codes. Thus, for error $E \equiv [\boldsymbol{e}_{\mathbf{X}}|\boldsymbol{e}_{\mathbf{Z}}]$, the syndromes $|\boldsymbol{s}_{\mathbf{X}}\rangle = |H_2\boldsymbol{e}_{\mathbf{X}}^{\mathrm{T}}\rangle$ and $|\boldsymbol{s}_{\mathbf{Z}}\rangle = |H_1\boldsymbol{e}_{\mathbf{Z}}^{\mathrm{T}}\rangle$ based on the X and Z errors can be obtained over ρ_1 and ρ_2 qudits, where $\rho_i = n - k_i$ for $i = \{1, 2\}$. Let $H_1 = [\beta_{yi}]$ and $H_2 = [\gamma_{yi}]$. The syndrome computation

[9]We note that such a decomposition of the syndrome computation circuit into two parts can be obtained for any qudit stabilizer code whose check matrix has the form in Eq. (29).

operator used to obtain $|\boldsymbol{s}_{\mathbf{X}}\rangle = |H_2\boldsymbol{e}_{\mathbf{X}}^{\mathrm{T}}\rangle$ and $|\boldsymbol{s}_{\mathbf{Z}}\rangle = |H_1\boldsymbol{e}_{\mathbf{Z}}^{\mathrm{T}}\rangle$ is

$$\mathcal{S}' = \left[\left((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_{p^m}^{\otimes\sigma}\right)\left(\prod_{y=1}^{\rho_1}\mathcal{B}(\boldsymbol{\beta}_y, y)\right)\left(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_{p^m}^{\otimes\sigma}\right)\right]\left(\prod_{y=\rho_1+1}^{\sigma}\mathcal{B}(\boldsymbol{\gamma}_y, y)\right), \tag{32}$$

where $\mathrm{DFT}_{p^m} = \frac{1}{\sqrt{p^m}}\sum_{\theta\kappa\in\mathbb{F}_{p^m}}\omega^{-\mathrm{Tr}_{p^m/p}(\theta\kappa)}|\theta\rangle\langle\kappa|$, $\sigma = \rho_1 + \rho_2$, and

$$\mathcal{B}(\boldsymbol{\beta}_y, y) := \prod_{i=1}^{n}\mathcal{B}'_{\beta_{yi}}(i, y), \quad \text{where} \quad \mathcal{B}'_{\beta_{yi}}(i, y) = \sum_{j\in\mathbb{F}_{p^m}}\mathrm{P}_{(i,n)}^{(p^m)}(j, j)\otimes \mathrm{X}_{(y,\sigma)}^{(p^m)}(j\beta_{yi}). \tag{33}$$

The proof of Eq. (32) has been provided in Appendix B. We note that $\mathcal{B}'_{\beta_{yi}}(i, y)$ is a scaled addition operator over qudits as provided in Fig. 2, where the codeword qudits are the control qudits and the syndrome qudits are the target qudits [27]. The circuit schematic for the syndrome computation operator is similar to Fig. 5 by replacing $\mathcal{X}(\beta_i, j)$ and $\mathcal{X}(\gamma_i, j)$ by $\mathcal{B}(\beta_y, y)$ and $\mathcal{B}(\gamma_y, y)$. In the Supplemental Material [27], we have provided an example of syndrome computation operator for qudit CSS codes that can be decomposed into two parts, one based on $\mathrm{X}^{(p^m)}$ error and the other based on $\mathrm{Z}^{(p^m)}$ error.

Remark 2. As the syndrome computation operator $\mathcal{S}'$ is a unitary operator and does not involve measurement to obtain the syndrome, we can obtain the syndrome entangled to the basis operator without measurement. Using controlled basis operations with the syndrome subqudits as control and the received state as target, we can correct each basis operator to obtain the codeword $|\psi\rangle$.

C. Syndrome computation operator for entanglement-assisted qudit stabilizer code

Consider an EA stabilizer code defined by check operators $R_1, \ldots, R_\rho$. Let these check operators be extended to stabilizer generators $S_1, \ldots, S_\rho$. Let n'_e be the number of subqudits used for the extension of check operators. The error correction procedure for the EA stabilizer code constructed from these stabilizer generators is similar to the procedure for the entanglement-unassisted stabilizer codes.

The syndrome computation operator is performed on the received subqudits and the entangled subqudits present at the receiver end. Although the receiver end entangled subqudits are assumed to be error free, we need to consider them because the encoded state $|\psi\rangle$ is stabilized by the stabilizers S_is, and not by the check operators R_is. We provide the syndrome computation operator $\mathcal{S}'$ and its equivalent form for the entanglement-assisted stabilizer code.

In Lemma 2, we provide the syndrome computation operator for the EA stabilizer code. As the EA stabilizer code is also a stabilizer code, the syndrome computation operator is similar to the operator $\mathcal{S}'$ in Lemma 1. The only difference between the syndrome computation operators in Lemmas 1 and 2 is that the syndrome computation operator in Lemma 1 operates on $(mn + \rho)$ subqudits, while the operator in Lemma 2 operates on $(mn + n'_e + \rho)$ subqudits.

Lemma 2. The syndrome computation operator $\mathcal{S}'$ corresponding to the stabilizer generators $S_1, \ldots, S_\rho$ is

$$\mathcal{S}' = \left(\mathrm{I}_p^{\otimes(nm+n'_e)}\otimes(\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\mathcal{S}_{\mathcal{D}}\left(\mathrm{I}_p^{\otimes(nm+n'_e)}\otimes \mathrm{DFT}_p^{\otimes\rho}\right).$$

Proof. Let us consider the basis error operator $E \in \mathcal{G}_p^{\otimes(nm+n'_e)}$. As the receiver-end entangled subqudits are considered error free, $E = E' \otimes \mathrm{I}_p^{\otimes n'_e}$, where $E' \in \mathcal{G}_{p^m}^{\otimes n}$. From Eqs. (2) and (3), we note that $\mathcal{G}_{p^m}^{\otimes n} = \mathcal{G}_p^{\otimes nm}$.

Let $S_iE = \omega^{l_i}ES_i\,\forall\, i \in \{1, \ldots, \rho\}$. For codeword $|\psi\rangle$, let the erroneous codeword be $E|\psi\rangle$. We show that $\mathcal{S}'$ is the syndrome computation operator by proving that $\mathcal{S}'(E|\psi\rangle|0\rangle_p^{\otimes\rho}) = E|\psi\rangle|l_1l_2\ldots l_\rho\rangle$ as the syndrome obtained from the eigenvalues of S_is with respect to $E|\psi\rangle$ is $[l_1l_2\ldots l_\rho]$.

We first simplify the operator $\mathcal{S}'$ as follows:

$$\begin{aligned}\mathcal{S}' &= \left(\mathrm{I}_p^{\otimes(nm+n'_e)}\otimes(\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\sum_{j_1,\ldots,j_\rho\in\mathbb{F}_p}S_1^{j_1}S_2^{j_2}\ldots S_\rho^{j_\rho}\otimes|j_1j_2\ldots j_\rho\rangle\langle j_1j_2\ldots j_\rho|\right)\left(\mathrm{I}_p^{\otimes(nm+n'_e)}\otimes\mathrm{DFT}_p^{\otimes\rho}\right),\\ &= \sum_{j_1,\ldots,j_\rho\in\mathbb{F}_p}S_1^{j_1}S_2^{j_2}\ldots S_\rho^{j_\rho}\otimes\left(\bigotimes_{i=1}^{\rho}\mathrm{DFT}_p^\dagger|j_i\rangle\langle j_i|\mathrm{DFT}_p\right).\end{aligned} \tag{34}$$

We note that S_is are defined over $(nm + n'_e)$ subqudits and $E|\psi\rangle$ is a $(nm + n'_e)$-subqudit state. We also note that Eq. (34) is same as Eq. (22) but defined for a different dimensional quantum system. From Eqs. (22) and (26), we obtain $\mathcal{S}'(E|\psi\rangle|0\rangle_p^{\otimes\rho}) = E|\psi\rangle|l_1l_2\ldots l_\rho\rangle$. ■

Similarly to the syndrome computation operator for the entanglement-unassisted stabilizer codes in Eq. (28), in Lemma 3, we obtain an equivalent form for the syndrome computation operator of the entanglement-assisted stabilizer code.

Lemma 3. Let $\{S_y\}_{y=1}^{\rho}$ be the stabilizer generators of the entanglement-assisted qudit stabilizer code, where $S_y = \bigotimes_{i=1}^{nm+n'_e} \mathrm{X}^{(p)}(b_{yi})\mathrm{Z}^{(p)}(g_{yi})$. The syndrome computation operator $\mathcal{S}'$ has the following equivalent form:

$$\mathcal{S}' = \prod_{y=1}^{\rho}\left(\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{nm+n'_e} \mathcal{X}'_{b_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{l=1}^{nm+n'_e} \mathcal{X}'_{g_{yi}}(l,y)\right),$$

where

$$\mathcal{X}'_{b_{yi}}(s,t) = \sum_j \mathrm{P}^{(p)}_{(s,nm+n'_e)}(j,j) \otimes \mathrm{X}^{(p)}_{(t,\rho)}(jb_{yi}).$$

Proof. Based on the relation of the quantum basis states and quantum operators of quantum systems with $q = p^m$ with those of quantum systems with $q = p$ provided in Eqs. (2) and (3), we mathematically represent the codeword and the stabilizers S_ms as the quantum states and operators in quantum system with $q = p$. We note that this is only a mathematical representation. Physically, they still corresponds to quantum systems with $q = p^m$. In Eq. (28), substituting $k = 1$ and $S_y = \bigotimes_{i=1}^{nm+n'_e} \mathrm{X}^{(p)}(b_{yi})\mathrm{Z}^{(p)}(g_{yi})$ and length of the codeword as $(nm + n'_e)$ (here m need not be 1), we obtain

$$\mathcal{S}' = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{nm+n'_e} \mathcal{X}'_{b_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{l=1}^{nm+n'_e} \mathcal{X}'_{\mathcal{P}(g_{yi})}(l,y)\right)\right]. \tag{35}$$

From the definition of $\mathcal{P}$ in Eq. (27), when $g_{yi} \in \mathbb{F}_p$, $\mathcal{P}(g_{yi}) = \sum_{s=0}^{m-1} \mathrm{Tr}_{p^m/p}(g_{yi}\alpha^{m-1-s})\alpha^{m-1-s}$. As we consider $m = 1$ while using Eq. (28), we obtain $\mathcal{P}(g_{yi}) = \sum_{s=0}^{m-1} \mathrm{Tr}_{p^m/p}(g_{yi}\alpha^{m-1-s})\alpha^{m-1-s} = \mathrm{Tr}_{p^m/p}(g_{yi}) = g_{yi}$ as $g_{yi} \in \mathbb{F}_p$. Substituting $\mathcal{P}(g_{yi}) = g_{yi}$ in Eq. (35),

$$\mathcal{S}' = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{i=1}^{nm+n'_e} \mathcal{X}'_{b_{yi}}(i,y)\right)\left(\mathrm{DFT}_p^{\otimes(nm+n'_e)} \otimes \mathrm{I}_p^{\otimes\rho}\right)\left(\prod_{l=1}^{nm+n'_e} \mathcal{X}'_{g_{yi}}(l,y)\right)\right]. \qquad \blacksquare$$

The syndrome computation circuit schematics of the entanglement-assisted stabilizer codes are similar to the circuit schematics provided in Figs. 3 and 4, except that the erroneous state $E|\psi\rangle$ is over $(mn + n'_e)$ subqudits. The error correction procedure involves syndrome computation using $\mathcal{S}'$ followed by correction of the error E based on the syndrome. The correction of the error is achieved using control-based operations, where the syndrome subqudits are the control subqudits and E^{-1} is performed on the codeword subqudits.

VI. APPLICATIONS TOWARD QUDIT CODED ARCHIVAL QUANTUM STORAGE

Using the proposed encoding and syndrome computation circuit schematic, coded quantum memories for archival quantum storage can be built. A three-dimensional quantum memory can be viewed as an array of atoms trapped in lattices, where the atoms correspond to the qudits. The quantum information is initially encoded into a codeword $|\psi\rangle$ and stored in the quantum memory. The receiver end subqudits of a codeword are stored as the message information in another codeword; hence, they are maintained error free. In archival quantum storage as quantum data are rarely accessed, we propose using a quantum sensing circuit to periodically monitor the qudits within the quantum memory and perform error correction on them.

Prior to quantum error correction, let the state of the codeword qudits be in an erroneous state $E|\psi\rangle$, where E is the error that has occurred on the codeword qudits. The codeword qudits in state $E|\psi\rangle$ along with n'_s subqudits in state $|0\rangle$ are passed through the syndrome computation operation to obtain the codeword qudits unchanged in state $E|\psi\rangle$ and the syndrome subqudits in the syndrome state $|s\rangle$. When the code used is an entanglement-assisted stabilizer code, the syndrome computation circuit also requires the n'_e receiver end subqudits stored as a message in another codeword.

The error deduction and recovery circuit deduces the error and recovers the codeword by either of the following two methods: (a) Perform measurement on the syndrome subqudits in state $|s\rangle$, classically deduce the error, and then perform the inverse quantum error operation on the state $E|\psi\rangle$ to obtain the codeword or (b) using control-based operations with the syndrome subqudits as the control and the codeword qudits in state $E|\psi\rangle$ as the target, correct the error based on the syndrome, and obtain the codeword in state $|\psi\rangle$. The codeword qudits in state $|\psi\rangle$ are stored back into the quantum memory. This coded quantum memory could be used within the context of archival quantum storage.

VII. CONCLUSION

Practical quantum coded systems require syndrome computation circuits toward error deduction and recovery. The proposed syndrome computation architectures are essential for realizing entanglement-unassisted and entanglement-assisted qudit coded systems using a few preshared entangled states within the quantum transceiver system. We provided an equivalent syndrome computation circuit for realizing a two-level error correction circuit. We illustrated the implementation of proposed syndrome computation architectures using one and two qudit gates. Finally, we provided an application where coded quantum states could be used within the context of archival quantum storage.

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APPENDIX A: PROOF OF EQUIVALENT SYNDROME COMPUTATION OPERATOR

To prove Eq. (28), we first note that the stabilizers belong to $\mathcal{G}_{p^m}^{\otimes n}$ whose generators are of the form $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$ and $\mathrm{Z}_{(s,n)}^{(p^m)}(\beta)$, where $\beta \in \mathbb{F}_{p^m}$ and $s \in \{1, \dots, n\}$. Thus, we first prove the Eq. (28) for these generators $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$ and $\mathrm{Z}_{(s,n)}^{(p^m)}(\beta)$ in Lemma 4 and Lemma 5, respectively. Using Lemmas 4 and 5, we prove Eq. (28) for stabilizer generators S_is of arbitrary form in Lemma 7.

1. Equivalent syndrome computation operator for stabilizer of form $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$

We first prove Eq. (28) for the stabilizer generator of the form $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$ in Lemma 4.

Lemma 4. Let $\beta = b_0 + b_1\alpha + \cdots + b_{(m-1)}\alpha^{(m-1)}$, where $b_0, b_1, \dots, b_{(m-1)} \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_{p^m}$. Then, $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho})\mathcal{X}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}) = ((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})\mathcal{X}'_\beta(s,t)(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$, where

$$\mathcal{X}_\beta(s,t) = \sum_{j\in\mathbb{F}_p} \mathrm{X}_{(s,n)}^{(p^m)}(j\beta) \otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j),$$

$$\mathcal{X}'_\beta(s,t) = \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}_{(s,n)}^{(p^m)}(\zeta,\zeta) \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0)$$

and $\zeta = \sum_{l=0}^{m-1} j_l\alpha^l$.

Proof. We note that the operator $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho})\mathcal{X}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho})$ is equal to the operator $\mathcal{S}'$ in Lemma 1 when the t^{th} stabilizer generator S_t is considered to be $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$ and all other stabilizer generators S_is are considered to be the identity operator. Thus, to show that Eq. (28) is satisfied when one stabilizer generator is of the form $\mathrm{X}_{(s,n)}^{(p^m)}(\beta)$ with all other stabilizer generators being identity operators, we need to prove that $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho})\mathcal{X}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}) = ((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})\mathcal{X}'_\beta(s,t)(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$.

To obtain the result, we first simplify $((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})\mathcal{X}'_\beta(s,t)(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$ as follows:

$$\begin{aligned}&\big((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\big)\mathcal{X}'_\beta(s,t)\big(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\big)\\ &\quad= \big((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\big)\left(\sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}_{(s,n)}^{(p^m)}(\zeta,\zeta) \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0)\right)\big(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\big),\end{aligned} \tag{A1}$$

where $\zeta = \sum_{l=0}^{m-1} j_l\alpha^l$. We note that the operator $\sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}_{(s,n)}^{(p^m)}(\zeta,\zeta) \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0)$ in Eq. (A1) performs only identity operator on the n codeword qudits, except for the sth qudit. Thus, the $(\mathrm{DFT}_p^{\otimes m})^\dagger$ operator in $((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho})$ on each of the first n qudits except the sth qudit cancels the $\mathrm{DFT}_p^{\otimes m}$ operator in $(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho})$. Thus, from Eq. (A1), we obtain

$$\begin{aligned}&\big((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\big)\mathcal{X}'_\beta(s,t)\big(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\big)\\ &\quad= \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \big((\mathrm{DFT}_p^{\otimes m})^\dagger \mathrm{P}^{(p^m)}(\zeta,\zeta)\mathrm{DFT}_p^{\otimes m}\big)_{(s,n)} \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0),\\ &\quad= \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \left[\bigotimes_{i=0}^{m-1}\left(\frac{1}{p}\sum_{r\in\mathbb{F}_p}\omega^{-j_{(m-1-i)}r}\mathrm{X}^{(p)}(r)\right)\right]_{(s,n)} \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0), \quad \text{[from Eq. (23)]}\\ &\quad= \frac{1}{p^m}\sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p}\left(\sum_{r_0,\dots,r_{(m-1)}\in\mathbb{F}_p}\bigotimes_{i=0}^{m-1}\omega^{-j_{(m-1-i)}r_i}\mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}b_{(m-1)} + \cdots + j_0b_0), \quad \text{[from Eq. (19)]}\\ &\quad= \frac{1}{p^m}\sum_{r_0,\dots,r_{(m-1)}\in\mathbb{F}_p}\sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p}\omega^{\sum_{g=0}^{m-1}-j_{(m-1-g)}r_g}\left(\bigotimes_{i=0}^{m-1}\mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \left[\prod_{g=0}^{m-1}\mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1-g)}b_{(m-1-g)})\right], (\because \mathrm{X}^{(p)}(\gamma)\mathrm{X}^{(p)}(\zeta)=\mathrm{X}^{(p)}(\gamma+\zeta))\end{aligned}$$

$$= \frac{1}{p^m} \sum_{r_0,\ldots,r_{(m-1)}\in\mathbb{F}_p} \left(\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \sum_{j_0,\ldots,j_{(m-1)}\in\mathbb{F}_p} \left[\prod_{g=0}^{m-1} \omega^{-j_{(m-1-g)} r_g} \mathrm{X}^{(p)}_{(t,\rho)}(j_{(m-1-g)} b_{(m-1-g)})\right],$$

$$= \sum_{r_0,\ldots,r_{(m-1)}\in\mathbb{F}_p} \left(\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \prod_{g=0}^{m-1} \left[\frac{1}{p}\sum_{j\in\mathbb{F}_p} \omega^{-jr_g} \mathrm{X}^{(p)}_{(t,\rho)}(jb_{(m-1-g)})\right], \quad \text{[from Eq. (20)].} \tag{A2}$$

We change the variable in the summation from j to $l = jb_{(m-1-g)}$. Due to the closure property of $\mathbb{F}_p$, the summation over all j in $\mathbb{F}_p$ changes to the summation over all l in $\mathbb{F}_p$. From Eq. (A2), we obtain

$$\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\mathcal{X}'_\beta(s,t)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)$$

$$= \sum_{r_0,\ldots,r_{(m-1)}\in\mathbb{F}_p} \left(\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \prod_{g=0}^{m-1} \left(\frac{1}{p}\sum_{l\in\mathbb{F}_p} \omega^{-lb^{-1}_{(m-1-g)} r_g} \mathrm{X}^{(p)}_{(t,\rho)}(l)\right),$$

$$= \sum_{r_0,\ldots,r_{(m-1)}\in\mathbb{F}_p} \left(\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \prod_{g=0}^{m-1} \left(\mathrm{DFT}_p^\dagger |b^{-1}_{(m-1-g)} r_g\rangle\langle b^{-1}_{(m-1-g)} r_g| \mathrm{DFT}_p\right)_{(t,\rho)}, \tag{A3}$$

where the last step follows from Eq. (23).

We simplify $\prod_{g=0}^{m-1} \mathrm{DFT}_p^\dagger |b^{-1}_{(m-1-g)} r_g\rangle\langle b^{-1}_{(m-1-g)} r_g|\mathrm{DFT}_p$ in Eq. (A3) as follows:

$$\prod_{g=0}^{m-1} \mathrm{DFT}_p^\dagger |b^{-1}_{(m-1-g)} r_g\rangle\langle b^{-1}_{(m-1-g)} r_g| \mathrm{DFT}_p = \mathrm{DFT}_p^\dagger |b^{-1}_{(m-1)} r_0\rangle\langle b^{-1}_{(m-1)} r_0| \mathrm{DFT}_p \mathrm{DFT}_p^\dagger |b^{-1}_{(m-2)} r_1\rangle$$

$$\langle b^{-1}_{m-2} r_1| \mathrm{DFT}_p \ldots \mathrm{DFT}_p^\dagger |b_0^{-1} r_{(m-1)}\rangle\langle b_0^{-1} r_{(m-1)}| \mathrm{DFT}_p, = \mathrm{DFT}_p^\dagger |b^{-1}_{(m-1)} r_0\rangle\langle b_0^{-1} r_{(m-1)}| \mathrm{DFT}_p \delta_{b^{-1}_{(m-1)} r_0, b^{-1}_{(m-2)} r_1} \ldots \delta_{b_1^{-1} r_{m-2}, b_0^{-1} r_{(m-1)}},$$

$$\Rightarrow r_1 = b_{(m-2)} b^{-1}_{(m-1)} r_0,$$

$$\Rightarrow r_2 = b_{(m-3)} b^{-1}_{(m-2)} r_1 = b_{(m-3)} b^{-1}_{(m-1)} r_0,$$

$$\text{In general,} \quad r_i = b_{(m-1-i)} b^{-1}_{(m-1)} r_0 \, \forall\, i \in \{0, \ldots, (m-1)\}. \tag{A4}$$

Substituting Eqs. (A4) and $r_i = b_{(m-1-i)} b^{-1}_{(m-1)} r_0$ in Eq. (A3), we obtain

$$\left((\mathrm{DFT}_p^{\otimes nm})^\dagger \otimes \mathrm{I}_p^{\otimes\rho}\right)\mathcal{X}'_\beta(s,t)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes\rho}\right)$$

$$= \sum_{r_0,\ldots,r_{(m-1)}\in\mathbb{F}_p} \left(\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(r_i)\right)_{(s,n)} \otimes \left(\mathrm{DFT}_p^\dagger |b^{-1}_{(m-1)} r_0\rangle\langle b_0^{-1} r_{(m-1)}| \mathrm{DFT}_p\right)_{(t,\rho)} \delta_{b^{-1}_{(m-1)} r_0, b^{-1}_{(m-2)} r_1} \ldots \delta_{b_1^{-1} r_{(m-2)}, b_0^{-1} r_{(m-1)}},$$

$$= \sum_{j\in\mathbb{F}_p} \left[\bigotimes_{i=0}^{m-1} \mathrm{X}^{(p)}(jb_{(m-1-i)})\right]_{(s,n)} \otimes \left(\mathrm{DFT}_p^\dagger |j\rangle\langle j| \mathrm{DFT}_p\right)_{(t,\rho)}, \text{ where } j = b^{-1}_{(m-1)} r_0,$$

$$= \sum_{j\in\mathbb{F}_p} \mathrm{X}^{(p^m)}_{(s,n)}(j\beta) \otimes (\mathrm{DFT}_p^\dagger |j\rangle\langle j|\mathrm{DFT}_p)_{(t,\rho)}, \quad \text{[from Eqs. (2) and (3)]}$$

$$= \sum_{j\in\mathbb{F}_p} \mathrm{X}^{(p^m)}_{(s,n)}(j\beta) \otimes ((\mathrm{DFT}_p^\dagger)^{\otimes\rho})\mathrm{P}^{(p)}_{(t,\rho)}(j,j)\mathrm{DFT}_p^{\otimes\rho}, \ (\because \mathrm{DFT}_p^\dagger \mathrm{DFT}_p = \mathrm{I}_p)$$

$$= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\sum_{j\in\mathbb{F}_p} \mathrm{X}^{(p^m)}_{(s,n)}(j\beta) \otimes \mathrm{P}^{(p)}_{(t,\rho)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right),$$

$$= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\mathcal{X}_\beta(s,t)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right). \qquad \blacksquare$$

2. Equivalent syndrome computation operator for stabilizer of form $\mathrm{Z}^{(p^m)}_{(s,n)}(\beta)$

We next prove Eq. (28) for the stabilizer generator of the form $\mathrm{Z}^{(p^m)}_{(s,n)}(\beta)$ in Lemma 5, where $\beta \in \mathbb{F}_{p^m}$.

We note that the transformation $\mathcal{P}(\beta) = \sum_{g=0}^{m-1} \mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)})\alpha^{(m-1-g)}$ is based on the expansion of $\mathrm{Z}^{(p^m)}(\cdot)$ over $\mathrm{Z}^{(p)}(\cdot)$ in Eq. (3) and is used in the syndrome computation operator as Lemma 5 is based on the stabilizer of form $\mathrm{Z}^{(p^m)}_{(s,n)}(\beta)$.

Lemma 5. Let $\beta = b_0 + b_1\alpha + \cdots + b_{(m-1)}\alpha^{(m-1)}$, where $b_0, b_1, \dots, b_{(m-1)} \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_{p^m}$. Then, $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho})\mathcal{Z}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}) = \mathcal{X}'_{\mathcal{P}(\beta)}(s,t)$, where

$$\mathcal{Z}_\beta(s,t) = \sum_{j\in\mathbb{F}_p} \mathrm{Z}_{(s,n)}^{(p^m)}(j\beta) \otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j),$$

$$\mathcal{X}'_{\mathcal{P}(\beta)}(s,t) = \sum_{j_0,\dots,j_{(m-1)}\in\mathbb{F}_p} \mathrm{P}_{(s,n)}^{(p^m)}(\zeta,\zeta) \otimes \mathrm{X}_{(t,\rho)}^{(p)}(j_{(m-1)}\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1)}) + \cdots + j_0\mathrm{Tr}_{p^m/p}(\beta\alpha^0)),$$

where $\zeta = \sum_{l=0}^{m-1} j_l\alpha^l$.

Proof. We note that the operator $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho})\mathcal{Z}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho})$ is equal to the operator $\mathcal{S}'$ in Lemma 1 when the t^{th} stabilizer generator S_t is considered to be $\mathrm{Z}_{(s,n)}^{(p^m)}(\beta)$ and all other stabilizer generators S_is are considered to be the identity operator. Thus, to show that Eq. (28) is satisfied when one stabilizer generator is of the form $\mathrm{Z}_{(s,n)}^{(p^m)}(\beta)$ with all other stabilizer generators being identity operators, we need to prove that $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho})\mathcal{Z}_\beta(s,t)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}) = \mathcal{X}'_{\mathcal{P}(\beta)}(s,t)$.

To obtain the result, we first prove that $\mathrm{Z}^{(p)}(b) = \mathrm{DFT}_p\mathrm{X}^{(p)}(b)\mathrm{DFT}_p^\dagger$. From the definition of DFT_p in Eq. (7),

$$\begin{aligned}\mathrm{DFT}_p\mathrm{X}^{(p)}(b)\mathrm{DFT}_p^\dagger &= \left(\frac{1}{\sqrt{p}}\sum_{r,s\in\mathbb{F}_p}\omega^{rs}|r\rangle\langle s|\right)\left(\sum_{t\in\mathbb{F}_p}|t+b\rangle\langle t|\right)\left(\frac{1}{\sqrt{p}}\sum_{u,v\in\mathbb{F}_p}\omega^{-uv}|u\rangle\langle v|\right),\\ &= \frac{1}{p}\sum_{r,t,v\in\mathbb{F}_p}\omega^{(r(t+b)-tv)}|r\rangle\langle v| = \frac{1}{p}\sum_{r,v\in\mathbb{F}_p}\omega^{rb}|r\rangle\langle v|\sum_{t\in\mathbb{F}_p}\omega^{(t(r-v))}\\ &= \frac{1}{p}\sum_{r,v\in\mathbb{F}_p}\omega^{rb}|r\rangle\langle v|p\delta_{r,v} = \sum_{r\in\mathbb{F}_p}\omega^{rb}|r\rangle\langle r| = \mathrm{Z}^{(p)}(b).\end{aligned} \tag{A5}$$

Thus, for $\beta = b_0 + b_1\alpha + \cdots + b_{(m-1)}\alpha^{(m-1)}$ and $j \in \mathbb{F}_p$,

$$\begin{aligned}\mathrm{Z}^{(p^m)}(j\beta) &= \bigotimes_{g=0}^{m-1}\mathrm{Z}^{(p)}(\mathrm{Tr}_{p^m/p}(j\beta\alpha^{(m-1-g)})), \quad \text{[from Eq. (3)]}\\ &= \bigotimes_{g=0}^{m-1}\mathrm{Z}^{(p)}(j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)})), \quad (\because \mathrm{Tr}_{p^m/p} \text{ is } \mathbb{F}_p\text{-linear})\\ &= \bigotimes_{g=0}^{m-1}\mathrm{DFT}_p\mathrm{X}^{(p)}(j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)}))\mathrm{DFT}_p^\dagger, \quad \text{[from Eq. (A5)]}\\ &= \mathrm{DFT}_p^{\otimes m}\left[\bigotimes_{g=0}^{m-1}\mathrm{X}^{(p)}(j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)}))\right](\mathrm{DFT}_p^\dagger)^{\otimes m}.\end{aligned} \tag{A6}$$

Considering the stabilizer generator as $\mathrm{Z}_{(s,n)}^{(p^m)}(\beta)$, the syndrome computation operator $(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho})\mathcal{Z}_\beta(s,t)\,(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho})$ based on $\mathcal{S}'$ in Lemma 1 is simplified as

$$\begin{aligned}&\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho}\right)\mathcal{Z}_\beta(s,t)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right)\\ &= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho}\right)\left(\sum_{j\in\mathbb{F}_p}\mathrm{Z}_{(s,n)}^{(p^m)}(j\beta) \otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right),\\ &= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho}\right)\left(\sum_{j\in\mathbb{F}_p}\left(\mathrm{DFT}_p^{\otimes m}\left[\bigotimes_{g=0}^{m-1}\mathrm{X}^{(p)}(j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)}))\right](\mathrm{DFT}_p^\dagger)^{\otimes m}\right)_{(s,n)} \otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right),\\ &\quad \text{[from Eq. A6)]}\\ &= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes \rho}\right)\left(\mathrm{DFT}_p^{\otimes mn} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\sum_{j\in\mathbb{F}_p}\left[\bigotimes_{g=0}^{m-1}\mathrm{X}^{(p)}(j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)}))\right]_{(s,n)} \otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j)\right)\end{aligned}$$

$$\times\ \big((\mathrm{DFT}_p^{\dagger})^{\otimes mn}\otimes \mathrm{I}_p^{\otimes\rho}\big)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big),\ \ (\because \mathrm{DFT}_p\mathrm{DFT}_p^{\dagger}=\mathrm{I}_p)$$

$$=\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\big(\mathrm{DFT}_p^{\otimes mn}\otimes \mathrm{I}_p^{\otimes\rho}\big)\left(\sum_{j\in\mathbb{F}_p}\mathrm{X}_{(s,n)}^{(p^m)}\left[\sum_{g=0}^{m-1} j\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)})\alpha^{(m-1-g)}\right]\otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j)\right)$$

$$\times\ \big((\mathrm{DFT}_p^{\dagger})^{\otimes mn}\otimes \mathrm{I}_p^{\otimes\rho}\big)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big),\ \ \text{(from Eq. (2)]}$$

$$=\big(\mathrm{DFT}_p^{\otimes mn}\otimes \mathrm{I}_p^{\otimes\rho}\big)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\left(\sum_{j\in\mathbb{F}_p}\mathrm{X}_{(s,n)}^{(p^m)}(j\mathcal{P}(\beta))\otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j)\right)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big)\big((\mathrm{DFT}_p^{\dagger})^{\otimes m}\otimes \mathrm{I}_p^{\otimes\rho}\big), \tag{A7}$$

where $\mathcal{P}:\mathbb{F}_{p^m}\to\mathbb{F}_{p^m}$ is a transformation of field element $\beta\in\mathbb{F}_{p^m}$ to $\mathcal{P}(\beta)=\sum_{g=0}^{m-1}\mathrm{Tr}_{p^m/p}(\beta\alpha^{(m-1-g)})\alpha^{(m-1-g)}$.

Using Lemma 4, we replace $(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho})(\sum_{j\in\mathbb{F}_p}\mathrm{X}_{(s,n)}^{(p^m)}(j\mathcal{P}(\beta))\otimes \mathrm{P}_{(t,\rho)}^{(p)}(j,j))\ (\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho})$ with $((\mathrm{DFT}_p^{\otimes nm})^{\dagger}\otimes \mathrm{I}_p^{\otimes\rho})\mathcal{X}'_{\beta}(s,t)(\mathrm{DFT}_p^{\otimes nm}\otimes \mathrm{I}_p^{\otimes\rho})$ in Eq. (A7) to obtain

$$\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\mathcal{Z}_{\beta}(s,t)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big)=\big(\mathrm{DFT}_p^{\otimes mn}\otimes \mathrm{I}_p^{\otimes\rho}\big)\big((\mathrm{DFT}_p^{\otimes nm})^{\dagger}\otimes \mathrm{I}_p^{\otimes\rho}\big)\mathcal{X}'_{\mathcal{P}(\beta)}(s,t)$$
$$\big(\mathrm{DFT}_p^{\otimes nm}\otimes \mathrm{I}_p^{\otimes\rho}\big)\big((\mathrm{DFT}_p^{\dagger})^{\otimes k}\otimes \mathrm{I}_p^{\otimes\rho}\big),=\mathcal{X}'_{\mathcal{P}(\beta)}(s,t).\ (\because \mathrm{DFT}_p\mathrm{DFT}_p^{\dagger}=\mathrm{I}_p).$$

■

3. Equivalent syndrome computation operator for stabilizer of form $S_y=\otimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)$

As every stabilizer generator is a basis operator, it is of the form $S_y=\bigotimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)$. Using Lemmas 4 and 5, we next prove Eq. (28) for the stabilizer generator of the form S_y in Lemma 6, where $\beta_i,\gamma_i\in\mathbb{F}_{p^m}$. We note that, we consider the other $(\rho-1)$ stabilizer generators to be identity operators.

Lemma 6. Let $S_y=\bigotimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)$ for $\beta_1,\dots,\beta_n,\ \gamma_1,\dots,\gamma_n\in\mathbb{F}_{p^m}$. Let $\mathcal{S}_y=(\mathrm{I}_{p^m}^{\otimes n}\otimes\ (\mathrm{DFT}_p^{\dagger})^{\otimes\rho})\ (\sum_{j\in\mathbb{F}_p}S_y^j\ \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j))(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho})$, then the following relations hold true:

(a) $\mathcal{S}_y=((\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho})\prod_{i=1}^{n}\mathcal{X}_{\beta_i}(i,y)(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}))\ ((\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho})\prod_{l=1}^{n}\mathcal{Z}_{\gamma_l}(l,y)(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}))$.

(b) $\mathcal{S}_y=((\mathrm{DFT}_p^{\otimes nm})^{\dagger}\otimes \mathrm{I}_p^{\otimes\rho})(\prod_{i=1}^{n}\mathcal{X}'_{\beta_i}(i,y))\ (\mathrm{DFT}_p^{\otimes nm}\otimes \mathrm{I}_p^{\otimes\rho})(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_l)}(l,y))$.

Proof. Let $S_y=\bigotimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)$. We note that the operator $\mathcal{S}_y=(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho})\ (\sum_{j\in\mathbb{F}_p}S_y^j\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j))$ $(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho})$ is equal to the operator $\mathcal{S}'$ in Lemma 1 when the yth stabilizer generator is S_y and all other stabilizer generators S_is are considered to be the identity operator. Thus, to show that Eq. (28) is satisfied when one stabilizer generator is of the form $S_y=\bigotimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)$ with all other stabilizer generators being identity operators, we need to prove that $\mathcal{S}_y=((\mathrm{DFT}_p^{\otimes nm})^{\dagger}\otimes \mathrm{I}_p^{\otimes\rho})(\prod_{i=1}^{n}\mathcal{X}'_{\beta_i}(i,y))(\mathrm{DFT}_p^{\otimes nm}\otimes \mathrm{I}_p^{\otimes\rho})(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_l)}(l,y))$.

To prove the result, we first simplify $\mathcal{S}_y$ as follows:

$$\mathcal{S}_y=\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\left(\sum_{j\in\mathbb{F}_p}S_y^j\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big),$$
$$=\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\left[\sum_{j\in\mathbb{F}_p}\left(\bigotimes_{i=1}^{n}\mathrm{X}^{(p^m)}(\beta_i)\mathrm{Z}^{(p^m)}(\gamma_i)\right)^{j}\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right]\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big),$$
$$=\big(\mathrm{I}_{p^m}^{\otimes n}\otimes (\mathrm{DFT}_p^{\dagger})^{\otimes\rho}\big)\left(\sum_{j\in\mathbb{F}_p}\prod_{i=1}^{n}\mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i)\prod_{l=1}^{n}\mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l)\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\big(\mathrm{I}_{p^m}^{\otimes n}\otimes \mathrm{DFT}_p^{\otimes\rho}\big), \tag{A8}$$

where the last step is obtained from properties $\mathrm{X}^{(p^m)}(\zeta)\mathrm{X}^{(p^m)}(\xi)=\mathrm{X}^{(p^m)}(\zeta+\xi)$ and $\mathrm{Z}^{(p^m)}(\zeta)\mathrm{Z}^{(p^m)}(\xi)=\mathrm{Z}^{(p^m)}(\zeta+\xi)$.

We next show that the sum of products $(\sum_{j\in\mathbb{F}_p}\prod_{i=1}^{n}\mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i)\prod_{l=1}^{n}\mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l)\otimes\ \mathrm{P}_{(y,\rho)}^{(p)}(j,j))$ in Eq. (A8) can be written as product of sums.

We first show that $\prod_{i=1}^{n}\sum_{j\in\mathbb{F}_p}\mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i)\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)=\sum_{j\in\mathbb{F}_p}\prod_{i=1}^{n}\mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i)\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)$ as follows:

$$\prod_{i=1}^{n}\sum_{j\in\mathbb{F}_p}\mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i)\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)=\sum_{j_1,\dots,j_n\in\mathbb{F}_p}\prod_{i=1}^{n}\mathrm{X}_{(i,n)}^{(p^m)}(j_i\beta_i)\otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j),\quad \text{[from Eq. (20)]} \tag{A9}$$
$$=\sum_{j_1,\dots,j_n\in\mathbb{F}_p}\left(\prod_{i=1}^{n}\mathrm{X}_{(i,n)}^{(p^m)}(j_i\beta_i)\right)\otimes\left(\prod_{i=1}^{n}\mathrm{P}^{(p)}(j_i,j_i)\right)_{(y,\rho)},$$

$$= \sum_{j_1,\ldots,j_n\in\mathbb{F}_p} \left(\prod_{i=1}^{n} \mathrm{X}_{(i,n)}^{(p^m)}(j_i\beta_i)\right) \otimes \left(|j_1\rangle \prod_{l=1}^{j_{n-1}} \langle j_l|j_{l+1}\rangle\langle j_n|\right)_{(y,\rho)},$$

$$= \sum_{j\in\mathbb{F}_p} \prod_{i=1}^{n} \mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j), \tag{A10}$$

where $j = j_1 = \cdots = j_n$. Similarly to Eq. (A10),

$$\prod_{l=1}^{n} \sum_{j\in\mathbb{F}_p} \mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j) = \sum_{j\in\mathbb{F}_p} \prod_{l=1}^{n} \mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j). \tag{A11}$$

Multiplying Eqs. (A10) and (A11), we obtain

$$\left(\prod_{i=1}^{n} \sum_{j_x\in\mathbb{F}_p} \mathrm{X}_{(i,n)}^{(p^m)}(j_x\beta_i) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j_x,j_x)\right)\left(\prod_{l=1}^{n} \sum_{j_z\in\mathbb{F}_p} \mathrm{Z}_{(l,n)}^{(p^m)}(j_z\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j_z,j_z)\right)$$

$$= \left(\sum_{j_x\in\mathbb{F}_p} \prod_{i=1}^{n} \mathrm{X}_{(i,n)}^{(p^m)}(j_x\beta_i) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j_x,j_x)\right)\left(\sum_{j_z\in\mathbb{F}_p} \prod_{l=1}^{n} \mathrm{Z}_{(l,n)}^{(p^m)}(j_z\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j_z,j_z)\right),$$

$$= \sum_{j_x,j_z\in\mathbb{F}_p} \prod_{i=1}^{n} \mathrm{X}_{(i,n)}^{(p^m)}(j_x\beta_i) \prod_{l=1}^{n} \mathrm{Z}_{(l,n)}^{(p^m)}(j_z\gamma_l) \otimes (|j_x\rangle\langle j_x|j_z\rangle\langle j_z|)_{(y,\rho)},$$

$$= \sum_{j\in\mathbb{F}_p} \prod_{i=1}^{n} \mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i) \prod_{l=1}^{n} \mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j), \tag{A12}$$

where $j = j_x = j_z$. Substituting Eq. (A12) in Eq. (A8), we obtain

$$\mathcal{S}_m = \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\prod_{i=1}^{n} \sum_{j\in\mathbb{F}_p} \mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\left(\prod_{l=1}^{n} \sum_{j\in\mathbb{F}_p} \mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right). \tag{A13}$$

Let us consider $K = (\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho})$, then $K^\dagger = (\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho})$. We know that $KK^\dagger = \mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\rho}$ as $\mathrm{DFT}_p\mathrm{DFT}_p^\dagger = \mathrm{I}_p$. For operator T_is that have the same dimension as K, where $i \in \{1,\ldots,l\}$, we obtain

$$K^\dagger T_1 \ldots T_l K = K^\dagger T_1 K K^\dagger T_2 K \ldots K^\dagger T_l K = \prod_{i=1}^{l} K^\dagger T_i K. \tag{A14}$$

Based on Eq. (A14), from Eq. (A13), we obtain

$$\mathcal{S}_y = \left(\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\prod_{i=1}^{n} \sum_{j\in\mathbb{F}_p} \mathrm{X}_{(i,n)}^{(p^m)}(j\beta_i) \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right)$$

$$\times \left[\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\left(\prod_{l=1}^{n} \sum_{j\in\mathbb{F}_p} \mathrm{Z}_{(l,n)}^{(p^m)}(j\gamma_l), \otimes \mathrm{P}_{(y,\rho)}^{(p)}(j,j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right], \quad \text{(from Lemmas 4 and 5)}$$

$$= \left[\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\prod_{i=1}^{n} \mathcal{X}_{\beta_i}(i,y)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right]\left[\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\prod_{l=1}^{n} \mathcal{Z}_{\gamma_l}(l,y)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right],$$

$$= \left[\prod_{i=1}^{n}\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\mathcal{X}_{\beta_i}(i,y)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right]\left[\prod_{l=1}^{n}\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^\dagger)^{\otimes\rho}\right)\mathcal{Z}_{\gamma_l}(l,y)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes\rho}\right)\right], \tag{A15}$$

where the last step follows from Eq. (A14).

From Lemmas 4 and 5, we obtain

$$\mathcal{S}_m = \left[\prod_{i=1}^{n}\left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\mathcal{X}'_{\beta_i}(i, y)\right)\right]\left[\prod_{l=1}^{n}\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\mathcal{X}'_{\mathcal{P}(\gamma_l)}(l, y)\right)\right],$$

$$= \left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_i}(i, y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_l)}(l, y)\right), \tag{A16}$$

where the last step is obtained from Eq. (A14). ■

4. Equivalent syndrome computation operator for stabilizer generators S_is

Let $\{S_y = \bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_{yi})\mathrm{Z}^{(p^m)}(\gamma_{yi})\}_{y=1}^{\rho}$ be the set of ρ stabilizer generators of the stabilizer code over qudits. Using Lemma 6, we next prove Eq. (28) in Lemma 7 by considering the ρ stabilizer generators $S_1, \ldots, S_\rho$.

Lemma 7. Let $S_y = \bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_{yi})\mathrm{Z}^{(p^m)}(\gamma_{yi})$, where $y \in \{1, \ldots, \rho\}$. The syndrome computation operator $\mathcal{S}'$ has the following equivalent form:

$$\mathcal{S}' = \prod_{y=1}^{\rho}\mathcal{S}_y = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i, y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l, y)\right)\right].$$

Proof. From Lemma 6, we compute the product of all $\mathcal{S}_y$ operators, where $y \in \{1, \ldots, \rho\}$, as follows:

$$\prod_{y=1}^{\rho}\mathcal{S}_y = \prod_{y=1}^{\rho}\left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^{\dagger})^{\otimes \rho}\right)\left(\sum_{j\in\mathbb{F}_p} S_y^j \otimes \mathrm{P}^{(p)}_{(y,\rho)}(j, j)\right)\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right),$$

$$= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^{\dagger})^{\otimes \rho}\right)\left[\prod_{y=1}^{\rho}\left(\sum_{j\in\mathbb{F}_p} S_y^j \otimes \mathrm{P}^{(p)}_{(y,\rho)}(j, j)\right)\right]\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right)[\text{from Eq. (A14)}]$$

$$= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^{\dagger})^{\otimes \rho}\right)\left[\sum_{j_1,\ldots,j_\rho\in\mathbb{F}_p}\prod_{y=1}^{\rho}\left(S_y^{j_y} \otimes \mathrm{P}^{(p)}_{(y,\rho)}(j_y, j_y)\right)\right]\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right)[\text{from Eq. (20)}]$$

$$= \left(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_p^{\dagger})^{\otimes \rho}\right)\left[\sum_{j_1,\ldots,j_\rho\in\mathbb{F}_p} S_1^{j_1}\ldots S_\rho^{j_\rho} \otimes |j_1\ldots j_\rho\rangle\langle j_1\ldots j_\rho|\right]\left(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_p^{\otimes \rho}\right) = \mathcal{S}'. \tag{A17}$$

Let $S_y = \bigotimes_{i=1}^{n} \mathrm{X}^{(p^m)}(\beta_{yi})\mathrm{Z}^{(p^m)}(\gamma_{yi})$. Using Lemma 6 and Eq. (A17), we obtain

$$\mathcal{S}' = \prod_{y=1}^{\rho}\mathcal{S}_y = \prod_{y=1}^{\rho}\left[\left(\left(\mathrm{DFT}_p^{\otimes nm}\right)^{\dagger} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{i=1}^{n}\mathcal{X}'_{\beta_{yi}}(i, y)\right)\left(\mathrm{DFT}_p^{\otimes nm} \otimes \mathrm{I}_p^{\otimes \rho}\right)\left(\prod_{l=1}^{n}\mathcal{X}'_{\mathcal{P}(\gamma_{yl})}(l, y)\right)\right]. \quad ■$$

Through Lemma 7, we proved Eq. (28) and the proposed syndrome computation operator performs only few DFT_p and DFT_p^{-1} operations on the codeword subqudits and performs rest of the operations on the syndrome subqudits considering the codeword subqudits as the control subqudits.

APPENDIX B: PROOF OF EQUIVALENT SYNDROME COMPUTATION OPERATOR FOR CSS CODES

Let $C_1[n, k_1, d_1]$ and $C_2[n, k_2, d_2]$ be two classical codes over $\mathbb{F}_{p^m}$ with parity check matrices H_1 and H_2 that satisfy the dual-containing criteria $C_1^{\perp} \subset C_2$. Let $\rho_i = n - k_i$, for $i = \{1, 2\}$. The check matrix of the CSS code obtained from C_1 and C_2 is given by [17]

$$\mathcal{H} = \left[\begin{array}{c|c} H_1 & \\ \alpha H_1 & \mathbf{0} \\ \vdots & \\ \alpha^{m-1}H_1 & \\ & H_2 \\ \mathbf{0} & \alpha H_2 \\ & \vdots \\ & \alpha^{m-1}H_2 \end{array}\right]. \tag{B1}$$

Rearranging the rows of the check matrix, we obtain

$$\mathcal{H}_S = \left[\begin{array}{c|c} H_1 \otimes \boldsymbol{a} & \mathbf{0} \\ \mathbf{0} & H_2 \otimes \boldsymbol{a} \end{array}\right], \tag{B2}$$

where $\boldsymbol{a} = [1\,\alpha\,\ldots\,\alpha^{m-1}]$.

Let the $H_1 = [\beta_{ij}]$ and $H_2 = [\gamma_{ij}]$, then, from (B2), for $y \in \{1, \ldots, \rho\}$ and $\rho = m(\rho_1 + \rho_2)$, the stabilizer generator S_y is

$$S_y = \begin{cases} \bigotimes_{i=1}^n \mathrm{X}^{(p^m)}(\alpha^{g(y)}\beta_{l_1(y)i}) & 1 \leqslant y \leqslant m\rho_1 \\ \bigotimes_{i=1}^n \mathrm{Z}^{(p^m)}(\alpha^{g(y)}\gamma_{l_2(y)i}) & m\rho_1 \leqslant y \leqslant \rho \end{cases}, \tag{B3}$$

where $g(y) = y \bmod m$, $l_1(y) = (\lfloor (y-1)/m \rfloor + 1)$, $l_2(y) = \lfloor (y - 1 - m\rho_1)/m \rfloor + 1$. Using the stabilizers S_ys and the trace-based symplectic product in Eq. (6), the syndrome computation operator can be obtained from Eq. (31), the syndrome can be obtained.

Let $\sigma = \rho_1 + \rho_2$. Alternatively, we use a series of controlled qudit operations of form $\mathcal{B}(\boldsymbol{\beta}_y, y) = \prod_{i=1}^n \sum_{j\in\mathbb{F}_{p^m}} \mathrm{P}_{(i,n)}^{(p^m)}(j, j) \otimes \mathrm{X}_{(y,\rho)}^{(p^m)}(j\beta_{yi})$ and DFT_{p^m} gates to compute the syndrome as provided in Eq. (32).

We show that we can obtain the syndrome state with respect to one row of the matrix H_1 using the block of form $\mathcal{B}(\boldsymbol{\beta}_y, y)$ along with DFT_{p^m} gates. The proof for obtaining the ρ qudit syndrome based on all stabilizers can be obtained similarly using the concepts in Appendix A. The syndrome based on the row $[\beta_{y1}\,\ldots\,\beta_{yn}]$ of H_1 is computed as follows:

$$\begin{aligned}
\big((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_p^{\otimes\sigma}\big)\mathcal{B}(\boldsymbol{\beta}_y, y)\big(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\sigma}\big) &= \big((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_p^{\otimes\sigma}\big)\prod_{i=1}^n \mathcal{B}'_{\beta_{yi}}(i, y)\big(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\sigma}\big) \\
&= \prod_{i=1}^n \big((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_p^{\otimes\sigma}\big)\mathcal{B}'_{\beta_{yi}}(i, y)\big(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\sigma}\big) \\
&= \prod_{i=1}^n \sum_{\zeta\in\mathbb{F}_{p^m}} \big((\mathrm{DFT}_{p^m})^\dagger \mathrm{P}^{(p^m)}(\zeta, \zeta)\mathrm{DFT}_{p^m}\big)_{(i,n)} \otimes \mathrm{X}_{(y,\sigma)}^{(p)}(\zeta\beta_{yi}), \\
&= \prod_{i=1}^n \sum_{\zeta\in\mathbb{F}_{p^m}} \left(\frac{1}{p^m}\sum_{r\in\mathbb{F}_{p^m}} \omega^{-\mathrm{Tr}_{p^m/p}(r\zeta)}\mathrm{X}^{(p^m)}(r)\right)_{(i,n)} \otimes \mathrm{X}_{(y,\sigma)}^{(p^m)}(\zeta\beta_{yi}), \\
&\quad [\text{from Eq. (23)}] \\
&= \prod_{i=1}^n \frac{1}{p^m}\sum_{r\in\mathbb{F}_{p^m}} \mathrm{X}_{(i,n)}^{(p^m)}(r) \otimes \sum_{\zeta\in\mathbb{F}_{p^m}} \big(\omega^{-\mathrm{Tr}_{p^m/p}(\zeta r)}\mathrm{X}_{(y,\sigma)}^{(p^m)}(\zeta\beta_{yi})\big),
\end{aligned} \tag{B4}$$

We change the variable in the summation from β_{yi} to $\xi = \zeta\beta_{yi}$. Due to the closure property of $\mathbb{F}_{p^m}$, the summation over all ζ in $\mathbb{F}_{p^m}$ changes to the summation over all ξ in $\mathbb{F}_{p^m}$. From Eq. (B4), we obtain

$$\begin{aligned}
\big((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_p^{\otimes\sigma}\big)\mathcal{B}(\boldsymbol{\beta}_y, y)\big(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\sigma}\big) &= \prod_{i=1}^n \sum_{r\in\mathbb{F}_{p^m}} \mathrm{X}_{(i,n)}^{(p^m)}(r) \otimes \left(\frac{1}{p^m}\sum_{\xi\in\mathbb{F}_{p^m}} \omega^{-\mathrm{Tr}_{(p^m/p)}(\xi\beta_{yi}^{-1}r)}\mathrm{X}_{(y,\sigma)}^{(p^m)}(\xi)\right), \\
&= \prod_{i=1}^n \sum_{r\in\mathbb{F}_{p^m}} \mathrm{X}_{(i,n)}^{(p^m)}(r) \otimes \big(\mathrm{DFT}_{p^m}^\dagger\big|\beta_{yi}^{-1}r\big\rangle\big\langle\beta_{yi}^{-1}r\big|\mathrm{DFT}_{p^m}\big)_{(y,\sigma)}, \quad [\text{from Eq. (23)}] \\
&= \prod_{i=1}^n \sum_{\kappa\in\mathbb{F}_{p^m}} \mathrm{X}_{(i,n)}^{(p^m)}(\beta_{yi}\kappa) \otimes \big(\mathrm{DFT}_{p^m}^\dagger|\kappa\rangle\langle\kappa|\mathrm{DFT}_{p^m}\big)_{(y,\sigma)}, \quad \text{where } \kappa = \beta_{yi}^{-1}r \\
&= \prod_{i=1}^n \big(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_{p^m}^\dagger)^{\otimes\sigma}\big)\left(\sum_{\kappa\in\mathbb{F}_{p^m}} \mathrm{X}_{(i,n)}^{(p^m)}(\kappa\beta_{yi}) \otimes \mathrm{P}_{(y,\sigma)}^{(p^m)}(\kappa, \kappa)\right)\big(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_{p^m}^{\otimes\sigma}\big), \\
&= \prod_{i=1}^n \big(\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_{p^m}^\dagger)^{\otimes\sigma}\big)\mathcal{B}_{\beta_{yi}}(i, y)\big(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_{p^m}^{\otimes\sigma}\big),
\end{aligned} \tag{B5}$$

which is similar to the syndrome computation operator in Lemma 1. The proof that the operator $\prod_{i=1}^n (\mathrm{I}_{p^m}^{\otimes n} \otimes (\mathrm{DFT}_{p^m}^\dagger)^{\otimes\sigma})\mathcal{B}_{\beta_{yi}}(i, y)(\mathrm{I}_{p^m}^{\otimes n} \otimes \mathrm{DFT}_{p^m}^{\otimes\sigma})$ is the syndrome computation operator for qudit stabilizer code based on $\mathbb{F}_{p^m}$-linear code has also been provided in Ref. [20]. Thus, from Eq. (B5), $((\mathrm{DFT}_{p^m}^{\otimes n})^\dagger \otimes \mathrm{I}_p^{\otimes\sigma})\mathcal{B}(\boldsymbol{\beta}_y, y)(\mathrm{DFT}_{p^m}^{\otimes n} \otimes \mathrm{I}_p^{\otimes\sigma})$ is the syndrome computation operator to compute the syndrome state based on one row of H_1. Similarly, the syndrome computation operator based on all the rows of H_1 and H_2 can be obtained using $\mathcal{S}'$ in Eq. (32).

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